Department of Pure Mathematics

Algebra Comprehensive Examination

June 2003

Set by P. Hoffman and F.C.Y. Tang

Do any six questions, including at least one from each of the four sections, $\{1., 2.\}$, $\{3., 4.\}$, $\{5., 6., 7.\}$, $\{8., 9.\}$ **.**

Groups

1.(a) Prove that any group of order p^2, where p is a prime, is abelian.

(b) Find all groups of order 45 (up to isomorphism).

2.(a) Define the terms *nilpotent group* and *solvable group.*

(b) Give an example, with proof, of a solvable group which is not nilpotent.

(c) Let G be a finite group such that every maximal subgroup in G is normal. Show that G is nilpotent.

Fields

3.(a) Let F be a field of characteristic $p > 0$. Consider the polynomial $f(x) = x^p - \alpha \quad \in \quad F[x]$. Prove : either $f(x)$ is irreducible in $F[x]$, or F is the splitting field of $f(x)$ over F.

(b) For a field F and any $g \in F[x]$, show that any two splitting fields of g over F are isomorphic.

4.(a) Show that, if α is a root of $x^3 + x^2 - 2x - 1 \in \mathbf{Q}[x]$, then $\alpha^2 - 2$ is also a root.

(b) Find the Galois group over $\mathbf{Q}$ of $(x^5 - 1)(x^3 + x^2 - 2x - 1)$. Determine all normal extensions of $\mathbf{Q}$ in the splitting field of $(x^5 - 1)(x^3 + x^2 - 2x - 1)$ over $\mathbf{Q}$.

Rings

5.(a) Let R be a commutative ring with 1. For $r \in R$, define what is meant by

(i) *r is irreducible in R*, and

(ii) *r is prime in R* .

Give an example of an irreducible element which is not prime.

(b) Give an example, with proof, of a domain which is not a unique factorization domain.

(c) Show that every principal ideal domain is a unique factorization domain.

Do 6. or 7. or neither, but not both.

6.(a) Define the term *simple ring*. Show that the ring of $n \times n$ matrices over a field is simple.

(b) Let R be a simple ring with 1. Show that $\mathrm{char}R$ is either 0 or a prime.

7.(a) Show that any finite dimensional algebra with 1 over a field F is algebraic over F.

(b) Prove that any non-commutative algebraic division algebra over $\mathbf{R}$ is central; that is, it has $\mathbf{R}$ as its center, where $\mathbf{R}$ is identified with the subalgebra generated by the identity element.

Linear Algebra

8. Let T be a linear operator on a finite-dimensional vector space V over a field F.

(a) Suppose that $(x-3)^5(x-5)^4$ is the characteristic polynomial of T, and that its minimal polynomial is $(x-3)^2(x-5)^2$. Give a complete list of possible Jordan forms for T, no two of the matrices in your list being similar.

(b) Prove that, if W is a T-invariant subspace of V, and $\bar{T}$ is the induced linear operator on $\bar{V} := V/W$, then the minimal polynomial of $\bar{T}$ divides that of T in $F[x]$.

9.(a) Show that (group) conjugation defines a group action of the unitary group, $U(n)$, on the set of all $n \times n$ hermitian complex matrices.

(b) Consider 2×2 hermitian complex matrices M for which $M^5 = 4M$. Determine how many orbits under the action of $U(2)$ in part (a) can contain such M.

(c) Let V be a complex inner product space, and let $T : V \to V$ be a unitary operator. Suppose that W is a T-invariant subspace. Show that $W^\perp$ is also a T-invariant subspace.