DEPARTMENT OF PURE MATHEMATICS

ALGEBRA COMPREHENSIVE EXAM

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Time: 3 hours

Instructions: Do *seven* questions, including at least one of the three questions in each of the four parts. Provide proofs for all substantive details except where specifically directed otherwise. As for 'fundamental' theorems, quote the theorem without proof, except of course in any place where the proof of a basic result is explicitly asked for. Parts (a), (b), ... of questions are often related, but not always.

FIELDS

F1. (a) Give an example (with justification) of a finite extension of fields which is not simple.

(b) Then calculate the automorphism group of that extension.

(c) Show that $\sqrt{3} + e^{2\pi i/3}$ generates the splitting field of $(x^2 - 3)(x^3 - 1)$ over $\mathbf{Q}$.

F2. (a) State (without proof) the best theorem you know which relates geometric constructibility by straightedge and compass to field-theoretic properties.

(b) Give an example (with proof) of a positive real number, r, which is the root of an irreducible rational polynomial of degree 4, but such that it is impossible to construct, with straightedge and compass (starting as usual with two points distance 1 'unit' apart), a line interval of length r. (If you have a workable idea, but details become problematic, describing how one might obtain such an example will be worth reasonable part marks.)

F3. (a) Let F be a field of characteristic $p > 0$. With $\alpha \in F$, prove that $x^p - \alpha$ is irreducible in $F[x]$ if its splitting field over F has degree greater than 1 over F.

(b) Obtain a formula for counting the number of monic irreducibles of each degree in $\mathbf{Z}_p[x]$, where p is a prime. (You may use without proof any general facts needed about finite fields)

GROUPS

G1. (a) Prove that two finite abelian groups with the same number of elements of each order are isomorphic.

(b) Prove that if G is a finite group in which any two elements of coprime order commute, then G is isomorphic to the direct product of its Sylow subgroups.

(c) Show that the group of inner automorphisms of a group G is a normal subgroup of the automorphism group, $\mathrm{AUT}(G)$, of G.

G2. (a) Prove that the group of rational numbers under addition has only two automorphisms of finite order.

(b) Show that a finite abelian group which has exactly two automorphisms must be isomorphic to one of the cyclic groups C_3 , C_4 or C_6 .

G3. (a) Express the number of conjugates in G of a subgroup H in terms of $N_G H$, the normalizer of H in G. Prove your claim without using (unless you prove them) any general theorems about group actions.

(b) Prove that a simple group cannot have order equal to the product of three distinct primes.

LINEAR ALGEBRA

L1. (a) Give an example of two square complex matrices which have the same minimal and the same characteristic polynomials, but are not similar. What is the smallest size that such matrices can be?

(b) Can two real square matrices, which are not similar over the reals, be similar when regarded as complex matrices? Justify.

L2. (a) Give a proof (carefully stating any set theoretic principle used) of the existence of a basis for an arbitrary vector space.

(b) Is every minimal spanning set for a vector space V (with respect to inclusion) necessarily a basis for V? Justify.

(c) Give an example of a vector space V and a chain (with respect to set inclusion) of spanning sets for V such that the intersection of all the sets in that chain is *not* a spanning set for V.

L3. (In this question, you may use without proof any theorem giving a basis for a tensor product of vector spaces.)

(a) Let $\{v_1, \cdots, v_n\}$ be a linearly independent set in the vector space V. Show that the element $\sum_{1\le i\le n} v_i \otimes v_i$, (of $V \bigotimes V$) cannot be expressed as a sum of fewer than "n" elements of the form $v \otimes w$.

(b) Assuming V and W are finite-dimensional and each splits as the direct sum of the eigenspaces of linear operators

$$T : V \to V \quad \text{and} \quad S : W \to W \quad \text{respectively ,}$$

prove that the same splitting property holds for $V \bigotimes W$ with respect to $T \bigotimes S$.

RINGS

R1. In this problem, $J(R)$ denotes the Jacobson radical of R.

(a) Show that if R is a subring of T, and if T is left Artinian, then $J(T) \cap R \subseteq J(R)$.

(b) With the same assumptions as in (a), show that if T is commutative, then $J(T) \cap R = J(R)$.

R is Artinian to

(c) Give an example to show that the commutativity assumption in (b) cannot be dropped.

(d) How many solutions does the equation $x^2 = -1$ have in the real division algebra of quaternions?

R2. Let $\bar{u}$ and $\bar{v}$ be the images of u and v respectively in $R := \mathbf{C}[u,v]/ < u^2 + v^2 - 1 >$, where $\mathbf{C}$ is the field of complex numbers.

(a) Prove that R is a Noetherian integral domain.

(b) Show that the ideal $\bar{u}R$ is *not* a prime ideal of R.

(c) Prove that $M \cap \mathbf{C}[\bar{u}]$ is a maximal ideal of $\mathbf{C}[\bar{u}]$ whenever M is a maximal ideal of R.

(d) Prove that if F is a field with a subring R, and if F is integral over R, then R is also a field.

R3. (a) Use the Artin-Wedderburn theorem to show that there are, up to isomorphism, exactly two R which are Artinian semiprimitive algebras over the complex numbers with the following two properties:

I. $\dim_{\mathbf{C}} R = 39$.

II. *R has 4 irreducible modules, up to isomorphism.*

(b) Which of the R in part (a) also satisfies the third condition below?

III. *As a module over itself, R is a direct sum of 11 minimal left ideals (that is, a composition series for ${}_R R$ has length 11).*

(c) Give an example of a $\mathbf{Z}$-module which is Artinian but not Noetherian.

(d) Prove that if R is a PID such that every Artinian R-module is Noetherian, then R is a field.