Department of Pure Mathematics

Algebra Comprehensive Examination

January 24, 2006

3 hours

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Instructions: Answer **seven** of the following eight questions. If you answer all eight, clearly indicate which question you do *not* want marked.

Linear and Multilinear Algebra

1. Let

$$A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{pmatrix}.$$

   (a) Compute the Jordan canonical form J of A.

   (b) Find a matrix Q such that $J = Q^{-1}AQ$.

2. Recall that an $n \times n$ matrix A is called *nilpotent* if $A^k = 0$ for some $k \geq 1$.

   (a) Let A be an $n \times n$ matrix over $\mathbb{C}$. Prove that A is nilpotent if and only if its characteristic polynomial is $p(t) = t^n$.

   (b) Let N_1, N_2 be 6×6 nilpotent matrices over $\mathbb{C}$. Suppose that N_1 and N_2 have the same minimal polynomial and the same nullity. Prove that N_1 and N_2 are similar. Also show that this is *not* true for 7×7 nilpotent matrices over $\mathbb{C}$.

   (c) Recall that the *trace* of an $n \times n$ matrix A, written $\mathrm{tr}(A)$, is defined by

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

   Prove that if B is an $n \times n$ matrix over $\mathbb{C}$ and $\mathrm{tr}(B^k) = 0$ for all $k \geq 1$, then B is nilpotent.

Group Theory

3. (a) Show that if g is an element of a finite group G, then the number of conjugates of g is a divisor of $|G|$.

   (b) Prove that if G is a group of order p^n, where p is prime and $n \geq 1$, then the center of G is nontrivial.

4. (a) Prove that if p, q are primes with $p > q$ and $p-1$ not divisible by q, then every group of order pq is abelian.
   (b) Give an example of a nonabelian group of order pq for some primes p, q with $p > q$ and where $p-1$ *is* divisible by q.
   (c) Prove that there is no simple group of order 56.

Ring Theory (All rings are rings with unity.)

5. (a) Show that a ring R is left Noetherian (i.e., R satisfies the Ascending Chain Condition for left ideals) if and only if every left ideal of R is finitely generated.
   (b) Let
$$S = \{ \begin{bmatrix} n & r \\ 0 & s \end{bmatrix} : n \in \mathbb{Z},\ r, s \in \mathbb{Q}\},$$
   a subring of $M_{2\times 2}(\mathbb{Q})$. Prove that S is *not* left Noetherian.

6. Recall that a nonzero, nonunit element p in an integral domain D is *irreducible* if it cannot be written as a product of two nonunits of D; and is *prime* if $p|ab$ implies $p|a$ or $p|b$ whenever $a, b \in D$.

   Let $\mathbb{Z}[\sqrt{-5}]$ be the ring $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \,:\, a, b \in \mathbb{Z}\}$.
   (a) Prove that the element 2 is irreducible but not prime in $\mathbb{Z}[\sqrt{-5}]$. (Hint: $2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$.)
   (b) Prove that the ideal $\langle 2,\ 1+\sqrt{-5}\rangle$ of $\mathbb{Z}[\sqrt{-5}]$ generated by $\{2,\ 1+\sqrt{-5}\}$ is not principal.

Field Theory

7. Let E be the splitting field of $x^3 - 7$ over $\mathbb{Q}$.
   (a) Compute $[E : \mathbb{Q}]$. Justify your answer.
   (b) Prove that $\mathrm{Gal}_{\mathbb{Q}}(E) \cong S_3$.
   (c) Write down the lattice of the subgroups of $\mathrm{Gal}_{\mathbb{Q}}(E)$. Justify your answer.
   (d) Write down the lattice of the corresponding intermediate fields of $E/\mathbb{Q}$. Justify your answer.

8. Let E/F be an algebraic field extension. We say E/F is a *normal extension* if for any irreducible polynomial $p(x) \in F[x]$, either $p(x)$ has no root in E or $p(x)$ has all roots in E. Let $\alpha \in \mathbb{R}$ satisfying $\alpha^4 = 11$ and let $\beta = (1+i)\alpha$.
   (a) Show that $\beta^4 = -44$.
   (b) Prove that the minimal polynomial of β over $\mathbb{Q}$ is $x^4 + 44$.
   (c) Prove that $i \notin \mathbb{Q}(\beta)$.
   (d) Prove that the extension $\mathbb{Q}(\beta)/\mathbb{Q}$ is not normal.