Department of Pure Mathematics

Algebra Comprehensive Examination

1pm-4pm, June 10, 2008

3 hours

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Attempt six questions with at least one from each of the four sections. Parts (a), (b), (c),... etc, of a question are often but not always related.

Linear Algebra

1. (a) Let A be an $n \times n$ matrix with coefficients in an algebraically closed field F. Show that A has an eigenvalue. Give a counterexample if F is not algebraically closed.

   (b) Let A and B be $n \times n$ commuting matrices with entries in an algebraically closed field F; i.e., $AB = BA$. If
   $$V_\lambda = \{v \in F^n | Av = \lambda v\}$$
   is the λ-eigenspace of A, show that V_λ is B-invariant; i.e.,
   $$B(V_\lambda) \subset V_\lambda.$$

   (c) Let A and B be $n \times n$ commuting matrices with entries in an algebraically closed field F. Show that A and B are simultaneously triangularizable; i.e., there is an $n \times n$ invertible matrix Q such that $Q^{-1}AQ$ and $Q^{-1}BQ$ are upper triangular.

2. (a) Let N_1 and N_2 be 6×6 nilpotent matrices over an algebraically closed field F. Suppose that N_1 and N_2 have the same minimal polynomial and the same nullity. Prove that N_1 and N_2 are similar. Show that this is not true for 7×7 nilpotent matrices.

   (b) Let n be a positive integer, $n \geq 2$, and let N be an $n \times n$ nilpotent matrix over an algebraically closed field F such that $N^n = 0$, but $N^{n-1} \neq 0$. Prove that N has no square root; i.e., there is no $n \times n$ matrix A such that $A^2 = N$.

Groups

3. Let p and q be distinct primes, $p < q$.

   (a) If $p \nmid (q-1)$, show that any group of order pq must be cyclic.

   (b) If $p | (q-1)$, show that there exists only one non-abelian group of order pq.

4. Let $G = \mathrm{GL}(2, \mathbb{R})$ be the general linear group of all 2×2 invertible matrices with real entries, $N = \mathrm{SL}(2, \mathbb{R})$ be the special linear subgroup of matrices with determinant 1, and G' be the commutator subgroup of G; i.e., the subgroup of G generated by all elements of the form $ABA^{-1}B^{-1}$ for all $A, B \in G$.
   (a) Show that $N \supset G'$,
   (b) Show that G' contains all elementary matrices of the form
   $$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix},$$
   where b and c are real numbers, and all diagonal matrices
   $$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},$$
   where $a \neq 0$.

   (Hint: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.)
   (c) Show that $N = G'$.

Rings

5. (a) Prove that if R is an integral domain then the polynomial ring $R[x_1, \dots, x_n]$ is an integral domain.
   (b) Let R be an integral domain. Prove that R is a field if and only if the polynomial ring $R[x]$ is a principal ideal domain.
   (c) Show that $y^n - x^2y - x$ is irreducible in $\mathbb{C}[x, y]$, for $n \geq 2$.

6. (a) Find all the simple $\mathbb{Z}$-modules.
   (b) Recall that a ring R is (left) *primitive* if there exists a simple faithful left R-module. Show that a commutative primitive ring is a field.
   (c) Show that "commutative" is necessary in part (b) by considering $R = \mathrm{Mat}_n(F)$, the ring of $n \times n$ matrices over a field F.

Fields

7. Suppose $f : F \to K$ is an isomorphism of fields, F^{alg} and K^{alg} are algebraic closures of F and K respectively, and $a \in F^{\mathrm{alg}}$.
   (a) Define the *minimal polynomial* of a over F.
   (b) Prove that there exists $b \in K^{\mathrm{alg}}$ such that f extends to an isomorphism $g : F(a) \to K(b)$ with $g(a) = b$.
   (c) Prove that f extends to an isomorphism between F^{alg} and K^{alg}.

8. (a) Show that a finite separable field extension has only finitely many intermediate field extensions.
   (b) Find c such that $\mathbb{Q}(c) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Justify your solution.
   (c) Suppose F is an infinite field of characteristic $p \neq 0$, and $E = F(a, b)$ is an extension of degree p^2 with $a^p, b^p \in F$. Find infinitely many intermediate extensions between F and E.