# Department of Pure Mathematics

## Algebra Comprehensive Examination

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Prepared by D.Ž. Djoković and J.W. Lawrence

**Instructions:** Answer six questions including {1,2} (required questions) and at least one from each of the pairs {3,4}, {5,6}, and {7,8}.

**Notation**

$\mathbf{C}$ – complex numbers, $\mathbf{Q}$ – rational numbers,
Tr = trace of operators.

1. (a) Let $U, V, W$ be finite–dimensional vector spaces over a field $F$, and $S : V \to W$ and $T : U \to V$ linear transformations. Prove that

$$\dim \ker(ST) \leq \dim \ker(S) + \dim \ker(T)\,.$$

   (b) Let $\{e_1, e_2, \ldots, e_n\}$ , $n > 3$, be a basis of a real vector space. For which $n$ is

$$\{e_1 + e_2 + e_3, e_2 + e_3 + e_4, \ \ldots, e_{n-1} + e_n + e_1, e_n + e_1 + e_2\}$$

   also a basis?

2. (a) Determine the number of isomorphism classes of abelian groups of order $2^6 3^5$.
   (b) Prove that the alternating group $A_n$ is generated by its 3-cycles.
   (c) Prove that a group of order 12 either has a subgroup of order 6 or has a normal subgroup of order 4.

3. (a) Define what is meant by an irreducible (left) $R$–module.
   (b) Show that the endomorphism ring of an irreducible module is a division ring (i.e. prove Schur's Lemma).
   (c) Let $R = M_2(\mathbf{C}) \times M_2(\mathbf{C})$ and let $e_1 = \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right)$,
   $e_2 = \left( \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right)$ and $e_3 = \left( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right)$.
   Prove that, as $R$–module, $Re_1$ is isomorphic to $Re_2$ but is not isomorphic to $Re_3$.

4. Recall that an integral domain $R$ is *integrally closed* if every element in the field of fractions of $R$ which is a root of a monic polynomial in $R[x]$ is in fact in $R$.
   (a) Prove that a unique factorization domain (UFD) is integrally closed.
   (b) Give an example of an integrally closed domain which is not a UFD.
   (Give a brief justification.)
   (c) Is every integral domain integrally closed? Why?

5. (a) If $\alpha = \sqrt{2} + \sqrt[3]{3}$ show that $\mathbf{Q}(\alpha) = \mathbf{Q}(\sqrt{2}, \sqrt[3]{3})$.

   (b) Find the Galois groups over $\mathbf{Q}$ of the polynomials:

      i) $x^4 + x^2 + 1$,

      ii) $x^3 - 3x + 1$.

   (c) Let $E/\mathbf{Q}$ be a Galois extension whose Galois group is the symmetric group $S_3$. Is it true that $E$ is the splitting field of an irreducible cubic polynomial over $\mathbf{Q}$?

6. (a) State the main theorem of Galois theory.

   (b) Let $E/F$ be a finite Galois extension with Galois group $G$. If $H$ is a subgroup of $G$, prove that there exists $a \in E$ such that

$$H = \{\phi \in G : \phi(a) = a\}.$$

   (c) Assume that $f \in \mathbf{Q}[X]$ has degree 4 and that its Galois group (over $\mathbf{Q}$) is the symmetric group $S_4$. If $a \in \mathbf{C}$ is a root of $f$, prove that $\mathbf{Q}(a)$ contains no quadratic extensions of $\mathbf{Q}$.

7. Let $V$ and $W$ be vector spaces, over a field $F$, of dimensions $m$ and $n$, respectively. If $S : V \to V$ and $T : W \to W$ are linear operators, then there is a unique linear operator $S \otimes T : V \otimes W \to V \otimes W$ such that

$$(S \otimes T)(v \otimes w) = S(v) \otimes T(w),\ \forall v \in V,\ \forall w \in W.$$

   (a) Show that, given bases for $V$ and $W$, one can select a basis for $V \otimes W$ such that

$$\mathrm{Mat}\,(S \otimes T) = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ \vdots & & & \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{pmatrix}$$

   where $A = (a_{ij}) = \mathrm{Mat}(S)$ and $B = \mathrm{Mat}(T)$.
   (Mat denotes the matrix of a linear operator with respect to the given basis.)

   (b) Prove that $\mathrm{Tr}\,(S \otimes T) = \mathrm{Tr}\,(S) \cdot \mathrm{Tr}\,(T)$.

   (c) Prove that $\det(S \otimes T) = (\det S)^n (\det T)^m$.

8. Let $U$ and $V$ be finite–dimensional vector spaces over a field $F$, $\mathrm{char}\,(F) \neq 2$. Let $f : U \times U \to F$ and $g : V \times V \to F$ be nondegenerate bilinear forms with $f$ symmetric and $g$ skew–symmetric. Finally let $W = \mathrm{Hom}_F(U, V)$ (the space of all linear maps from $U$ to $V$).

   (a) For $w \in W$, show that there exists a unique $w^* \in \mathrm{Hom}_F(V, U)$ such that

   $$g(w(u), v) = f(u, w^*(v))\,,\ \forall\, u \in U, \forall\, v \in V\,.$$

   (b) Show that the map $W \to \mathrm{Hom}_F(V, U)$ sending $w$ to $w^*$ is an isomorphism of vector spaces.

   (c) If $L$ is a linear operator on $U$ satisfying

   $$f(L(x), y) + f(x, L(y)) = 0,\ \ \forall\, x, y \in U\,,$$

   prove that $\mathrm{Tr}\,(L) = 0$.

   (d) Prove that $h : W \times W \to F$ defined by

   $$h(w, w') = \mathrm{Tr}\,(w^* \circ w')$$

   is a nondegenerate skew-symmetric bilinear form.