Department of Pure Mathematics

Algebra Comprehensive Examination

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Read the instructions carefully.

Do one question from each of the four sections, plus two other questions. If you attempt more than two others, the best two will count.

Rings

1. Let R be the ring $\mathbf{C}[x, y, z]/I$, where I is the ideal generated by the element $xy - z^2$.

   (a) Show that R is a Noetherian ring.

   (b) Show that R is a domain. [Hint: Show that it is isomorphic to a subring of the field $\mathbf{C}(x, y, \sqrt{xy})$.]

   (c) Show that R is not a unique factorization domain.

2. Let $R = \mathbf{Z}[i]$ be the ring of Gaussian integers.

   (a) Find a single generator for the ideal $I = (2 + i, 3 - i)$ in R.

   (b) Let $M = R^2$ be a free R-module of rank two. Define an R-module homomorphism $\phi\colon M \to R$ by
   $$\phi(a, b) = (2 + i)a - (3 - i)b$$
   Find a minimal set of generators for the kernel of ϕ.

   (c) How many elements does the R-module $R/\phi(M)$ contain? Prove your answer.

Fields

3. Let F be the Galois closure of the field $K = \mathbf{Q}(\sqrt{2} + \sqrt[3]{2})$.

   (a) Prove that $\sqrt{2} \in F$, $\sqrt[3]{2} \in F$, and $\sqrt[6]{2} \in F$.

   (b) Prove that $\zeta_3 \in F$, where ζ_3 is a primitive cube root of unity.

   (c) Prove that $F = \mathbf{Q}(\zeta_3, \sqrt[6]{2})$, and compute the Galois group of $F/\mathbf{Q}$.

4. Let F be the field with three elements, and let $K = F(i)$ be the field with nine elements, where $i^2 = -1$.

   (a) The set of nonzero elements of K form a cyclic group. Find a generator for this group.

   (b) Let N and T denote the norm and trace maps, respectively, from K to F. Compute $N(a + bi)$ and $T(a + bi)$ for an arbitrary element $a + bi \in K$.

   (c) Compute the minimal polynomial over F for an arbitrary element $a + bi \in K$.

Groups

5. (a) What does it mean for a group G to act on a set X ?

   (b) Let the finite group G acts on the finite set X and let a be any element of X. Prove that the order of the stabilizer of the element a times the number of elements in the orbit of a is equal to the order of G.

   (c) Prove *Burnside's Theorem* that the number of orbits in a finite set X under the action of a finite group G is

$$N = \frac{1}{|G|} \sum_{g \in G} |X_g|.$$

   where $|X_g|$ is the number of elements of X fixed by $g \in G$.

6. (a) What is a Sylow p-subgroup of a finite group G ?

   (b) Find all the groups of order $2p$ when p is an odd prime.

Linear Algebra

7. The minimal polynomial of a linear operator T on a finite-dimensional vector space is the monic polynomial $m(x)$ of lowest degree such that $m(T) = 0$.

   (a) State, without proof, the Cayley-Hamilton Theorem.

   (b) Prove that the minimal polynomial divides the characteristic polynomial.

   (c) Show that the minimal polynomial and the characteristic polynomial have the same zeros.

   (d) Let V be the real vector space of dimension 3 consisting of the zero polynomial and all real polynomials of degree less than or equal to 2. Find the minimal and characteristic polynomials of the differentiation operator on V.

8. (a) Find the Jordan form of $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -6 & 12 \\ 0 & -4 & 8 \end{pmatrix}$.

   (b) If B is a complex $n \times n$ matrix that is similar to B^2, determine the possible eigenvalues and blocks of the Jordan form of B.