Department of Pure Mathematics

Algebra Comprehensive Examination

June 14, 2004

3 hours

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Instructions: Answer six questions (the best six counted) and at least one from each of the pairs $\{\mathbf{1},\mathbf{2}\}$, $\{\mathbf{3},\mathbf{4}\}$, $\{\mathbf{5},\mathbf{6}\}$, $\{\mathbf{7},\mathbf{8}\}$. Note that all rings are rings with unity.

Notations: $\mathbb{C}$ — complex numbers, $\mathbb{R}$ — real numbers, $\mathbb{Q}$ — rational numbers, $\mathbb{Z}$ — integers, $\mathbb{N}$ — natural numbers, $GF(p^n)$ — the Galois field of order p^n.

1. (a) Find all eigenvalues of the matrix
$$A = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix}.$$
(b) Define the standard inner product $\langle , \rangle$ on the vector space $V = \mathbb{C}^n$ over $\mathbb{C}$ as follows. For any two vectors $\mathbf{v} = (v_1, \ldots, v_n), \mathbf{w} = (w_1, \ldots, w_n) \in V, v_1, \ldots, v_n, w_1, \ldots, w_n \in \mathbb{C}$,
$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^{n} v_i \overline{w_i},$$
where $^{-}$ is the complex conjugate. A linear transformation T from V to V is called *Hermitian* if for all $\mathbf{v}, \mathbf{w} \in V$, $\langle T(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, T(\mathbf{w}) \rangle$. Prove that all eigenvalues of a Hermitian linear transformation T are real.
(c) Prove that if T is a Hermitian linear transformation on V, then there is an orthonormal basis in which that matrix of T is diagonal.

2. (a) Suppose that $x^4(x-1)^5$ is the characteristic polynomial of a complex matrix T and its minimal polynomial is $x^2(x-1)^3$. Give a complete list of all possible Jordan forms for T such that no two of the matrices in your list are similar.
(b) Let A be an $n \times n$ complex matrix. Consider the linear operator T defined on the space $M_n(\mathbb{C})$ of all $n \times n$ complex matrices by the rule $T(B) = AB - BA$. Prove that the rank of this operator is at most $n^2 - n$.

3. (a) State all three Sylow theorems.
(b) Use the fact that a group of order 30 has a subgroup of order 15 to show that there are precisely four isomorphism classes of groups of order 30.
(c) Describe, up to isomorphism, the Sylow-7 subgroups on S_{14}, the permutation group of 14 elements. How many Sylow-7 subgroups are there?

4. (a) Let A be an abelian group with a subgroup B isomorphic to $(\mathbb{Q}, +)$, the rationals under addition. Prove that B is a direct summand of A (i.e., there exist a subgroup C such that $B \times C \cong A$.) (Hint: you will have to use Zorn's lemma)
(b) Let $\mathbb{F}$ be the free group generated by a and b. Describe $\mathbb{F}/\mathbb{F}'$, where $\mathbb{F}'$ is the commutator subgroup of $\mathbb{F}$.
(c) Prove that a^2b^2 is **not** a conjugate of $abab$ in $\mathbb{F}$.

5. (a) How many isomorphism classes of non-commutative semi-primitive(semi-simple) $\mathbb{C}$-algebras of dimension 9 are there? Explain. Answer the same question with $\mathbb{C}$ replaced by $\mathbb{R}$.

(b) Let

$$A = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & \gamma \\ 0 & 0 & \beta \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{Z}_3 \right\}.$$

(i) Describe the Jacobson radical $J(A)$ of A.
(ii) Describe $A/J(A)$. (up to isomorphism)
(iii) Describe the group of units of A.

6. (a) Let $\mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\}$ be the ring of Gaussian integers. Prove that 7 is an irreducible element in $\mathbb{Z}[i]$.

(b) Let $\mathcal{P}$ be a *nonzero* prime ideal of $\mathbb{Z}[\sqrt{10}]$. Prove that $\mathcal{P} \cap \mathbb{Z}$ is a *nonzero* prime ideal of $\mathbb{Z}$.

(c) Prove the following statement (Gauss' Lemma):

Let R be a unique factorization domain and $f(x), g(x) \in R[x]$. Assume that a prime p in R divides all coefficients of $f(x)g(x)$. Prove that the prime p divides all coefficients of $f(x)$ or of $g(x)$.

7. (a) State the condition that two positive integers m and n must satisfy if the finite field $GF(p^m)$ is isomorphic to a subfield of $GF(p^n)$, where p is a prime. Prove one direction.

(b) State Eisenstein's criterion and use your statement to prove that the polynomial

$$\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1,$$

(where p is a prime number), is irreducible over $\mathbb{Q}$. (Hint: $\Phi_p(x) = (x^p - 1)/(x - 1)$ and prove that $\Phi_p(x + 1)$ is irreducible)

(c) Let p be a prime number. Describe the Galois group (up to isomorphism) of the p-th cyclotomic field $\mathbb{Q}(\zeta_p)$ over $\mathbb{Q}$, where $\zeta_p = e^{2\pi i/p}$.

8. Let K be the splitting field of $x^7 - 2$ over $\mathbb{Q}$.

(a) Prove that K is generated over $\mathbb{Q}$ by the 7th root of 2 and a primitive 7th root $\zeta = e^{2\pi i/7}$ of unity.

(b) Prove that $[K : \mathbb{Q}] = 42$. (Hint: #7(b))

(c) Prove that the Galois groups of K over $\mathbb{Q}$ is isomorphic to the group of invertible 2×2 matrices with entries in $GF(7)$ of the form $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$, and describe the actions of the elements $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ the generators explicitly.

9. For each of the following, describe an example(s) of the object stated or give a proof that they cannot exist.

(1) A field with 8 elements.
(2) A nonabelian group with 21 elements.
(3) A noncommutative ring with 14 elements.
(4) Two non-isomorphic groups each with 35 elements.
(5) A nonabelian simple group with 69 elements.
(6) A ring with no zero-divisors with 198 elements.