Department of Pure Mathematics

Algebra Comprehensive Examination

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Do any six questions. But do at least one from each part.

Part I

1. (a) State the first, second, and third Sylow theorems.
   (b) Show that groups of order 45 must be abelian. Hence find all groups of order 45 up to isomorphism.
   (c) Let H be a subgroup of a group G. Let $N_G(H) = \{g \in G;\ g^{-1}Hg = H\}$. Suppose H is a Sylow–p–subgroup of a finite group G. Show that $N_G(N_G(H)) = N_G(H)$.

2. (a) Define nilpotent groups.
   (b) Let G be a nilpotent group and H be a proper subgroup of G. Show that $H \underset{\neq}{\subset} N_G(H)$.
   (c) Let G be a finite nilpotent group. Show that G is the direct product of its Sylow–p–subgroups. (Use 1c) and 2b)).

Part II

3. Let R be a commutative ring with identity.
   (a) Define (i) Euclidean domain.
   (ii) Principal ideal domain.
   Show that every Euclidean domain is a principal ideal domain.
   (b) Define (i) prime ideal of a ring.
   (ii) maximal ideal of a ring.
   Show that every maximal ideal is a prime ideal.

4. (a) Give examples of rings R and R-modules M such that
   (i) M is Artinian but not Noetherian
   (ii) M is Noetherian but not Artinian
   (iii) M is both Artinian and Noetherian
   (iv) M is neither Artinian nor Noetherian.
   (b) Show that an R–module is Noetherian if and only if every submodule of M is finitely generated.
   (c) Let R be a ring. Define the Jacobson radical $J(R)$ to be the set of all elements of R which annihilate all the irreducible R–modules. Show that if R is Artinian then $J(R)$ is a nilpotent ideal.

Part III

5. Let K be any field and α transcendental over K,

   (a) Show that $\{\alpha^k | k \in \mathbb{Z}\}$ is a linearly independent set over K.

   (b) Is this a basis of $K(\alpha)$ over K? Explain.

   (c) Suppose E is a finite simple extension of K. Prove that there are only finitely many fields F such that $K \subset F \subset E$.

6. (a) Express the splitting field of $f(x) = x^4 + 1$ as a simple extension of $\mathbb{Q}$.

   (b) Let G be the Galois group of $t^p - 1$ over $\mathbb{Q}$ when p is prime. Describe G up to isomorphism.

   (c) If E is a finite simple algebraic extension of a field k, then prove that

$$\left| G\left(\frac{E}{K}\right) \right| \leq [E : K].$$

Part IV

7. (a) Let x, y be two real variables. If A is any $n \times n$ matrix with all entries in the set $\{x, y\}$ then prove that

$$\det A = (x - y)^{n-1}(Px + (-1)^{n-1}Qy)$$

   where P, Q are integers defined by

$$P = \det A\big|_{x=1,y=0} \quad \text{and} \quad Q = \det A\big|_{x=0,y=1}.$$

   (b) (i) Find the Jordan canonical form for the matrices

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

   (ii) Are A and B similar? Explain.

8. Let V be an n–dimensional vector space over a field F. V^* denotes the vector space of functionals $f : V \to F$.

   (a) Prove that for every non–zero $f \in V^*, V \cong \ker f \bigoplus F$.

   (b) If U and V are finite–dimensional vector spaces prove that $(U \bigotimes V)^* \cong U^* \bigotimes V^*$.

   (c) Is (b) still true if the restriction to finite–dimensions is lifted? Explain.