Department of Pure Mathematics
Algebra Comprehensive Examination
9am–noon, May 20, 2005

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Instructions: There are 4 sections in this exam, each section with 3 problems. Solve at least 1 and at most 2 problems from each section. Attempt at least 6 problems overall.

Group Theory

1. (a) Suppose that N and M are two normal subgroups of a group G. Also suppose that $N \cap M = \{1\}$. Prove that for any $n \in N$, $m \in M$, we have $nm = mn$.

(b) Let the group H be generated by elements x and y that satisfy $x^5y^3 = x^8y^5 = 1$. Determine whether H is the trivial group or not.

2. (a) State Sylow's theorem. If p is a prime, what does Sylow's theorem say about the number of p-Sylow subgroups of a finite group G?

(b) Let p and q be primes satisfying $p < q$. Let G be a group of order pq. If p does not divide $q-1$, prove that G is cyclic.

(c) Prove that any group of order 15 is cyclic.

3. For an integer $n \geq 3$, the dihedral group D_n of order $2n$ is given by

$$D_n = \langle\, a, b \mid a^n = b^2 = 1,\, ba = a^{-1}b \,\rangle.$$

(a) Show that every element of D_n can be expressed as $a^r b^s$ for some non-negative integers r, s.

(b) Determine the center of D_n.

(c) When n is odd, how many Sylow 2-subgroups does D_n have?

Linear Algebra

4. Let $\mathbb{R}^3$ be equipped with the standard inner product, and let $v = (a, b, c) \in \mathbb{R}^3$ be a vector of length 1.

(a) Let W be the plane in $\mathbb{R}^3$ defined by $ax + by + cz = 0$. Let T be the orthogonal projection of $\mathbb{R}^3$ onto W. Find the 3×3 matrix representative of T with respect to the standard basis of $\mathbb{R}^3$.

(b) Let S be the linear transformation of $\mathbb{R}^3$ which is the rotation by $180°$ about v. Find the 3×3 matrix representative of S with respect to the standard basis of $\mathbb{R}^3$.

5. Let $A = [a_{ij}]$ be an $n \times n$ matrix of complex numbers satisfying

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \qquad \text{for } 1 \leq i \leq n.$$

Suppose $Ax = 0$, where $x = (x_1, x_2, \ldots, x_n)^t \in \mathbb{C}^n$.

(a) Show that $a_{ii}x_i = -\sum_{j \neq i} a_{ij}x_j$ for each i.

(b) Let $M = \max\limits_{1 \leq k \leq n} |x_k|$. Prove that $M = 0$.

(c) Show that the matrix A is invertible.

6. Consider the 6×6 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

(a) Show that $(A - I)^3 = O$.

(b) Find the Jordan canonical form of A.

Ring Theory

7. (a) State the Chinese Remainder Theorem for commutative rings.

(b) Let A be a 2-dimensional unital associative algebra over a field F. Prove that A is either a field or isomorphic (as an F-algebra) to $F \times F$ or $F[x]/(x^2)$, where x is an indeterminate.

(c) Let R be an integral domain. A *prime* element a of R is an element, which is neither 0 nor a unit, that satisfies

$$a|xy \implies a|x \text{ or } a|y$$

for $x, y \in R$. If $a \in R$ is a prime element, is it true that a is also prime in the polynomial ring $R[x]$? Justify your answer.

8. Let $R = M_n(D)$ be the $n \times n$ matrix ring over a division ring D.

(a) Prove that R is simple, i.e., it has no nonzero proper two-sided ideals.

(b) Give an example of an irreducible left R-module, V, and prove that any irreducible left R-module is isomorphic to V.

(c) Let W be a direct sum of k copies of an irreducible left R-module V. Prove that W is a free R-module if and only if n divides k.

9. Let R be a nonzero commutative ring, A an $m \times n$ matrix with coefficients in R, and let $\varphi : R^n \to R^m$ be the left multiplication by A. Prove that the following are equivalent:

(i) φ is surjective.

(ii) A has a right inverse, i.e., there is a matrix B with coefficients in R such that $AB = I_m$.

(iii) The determinants of the $m \times m$ submatrices of A generate the unit ideal of R.

Field Theory

10. Let $E = \mathbb{Q}(\sqrt{r}, \sqrt{s})$ where $r, s \in \mathbb{Z}$ and none of r, s, rs is a square of an integer.

(a) Prove that the fields $\mathbb{Q}(\sqrt{r})$ and $\mathbb{Q}(\sqrt{s})$ are not isomorphic.

(b) Prove that $[E : \mathbb{Q}] = 4$.

(c) Describe the Galois group of the field extension $E/\mathbb{Q}$.

(d) How many quadratic extensions of $\mathbb{Q}$ does E contain?

11. Let $\zeta \in \mathbb{C}$ be a primitive 11-th root of unity and $E = \mathbb{Q}(\zeta)$.

(a) Describe the Galois group of $E/\mathbb{Q}$.

(b) If $\alpha = \zeta + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^9$, prove that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2$ and find the minimal polynomial of α over $\mathbb{Q}$.

(c) Find $\beta \in E$ such that $[\mathbb{Q}(\beta) : \mathbb{Q}] = 5$ and find its minimal polynomial over $\mathbb{Q}$.

12. Let $q = p^n$, p a prime, $n \geq 1$ integer, and let F be a finite field of order q and $\mathbb{Z}_p$ its prime subfield. Let $\varphi : F \to F$ be the automorphism of F defined by $\varphi(\xi) = \xi^p$.

(a) If $c_0, c_1, \ldots, c_{n-1} \in F$ and

$$\sum_{k=0}^{n-1} c_k \varphi^k(\xi) = 0, \quad \forall \xi \in F,$$

then it is a fact that $c_0 = c_1 = \cdots = c_{n-1} = 0$. Do you know a general theorem that contains the stated fact as a special case? If so, then state the theorem.

(b) Verify that φ is a linear operator when F is viewed as a $\mathbb{Z}_p$-vector space.

(c) Prove that $x^n - 1$ is the minimal polynomial of this operator.

(d) Use a theorem from Linear Algebra to show that there is an $\alpha \in F$ such that

$$\{\alpha, \alpha^p, \alpha^{p^2}, \ldots, \alpha^{p^{n-1}}\}$$

is a basis of F over $\mathbb{Z}_p$. State this theorem from Linear Algebra.