Algebra Comprehensive Exam

May 2009

There are nine questions on this exam. Students should attempt six questions, including at least one from each of the four sections.

1 Rings

1. Let $R = \mathbb{Z}[\sqrt{2}]$. Let $x = a - \sqrt{2}$ for some $a \in \mathbb{Z}$, and assume that $p = a^2 - 2$ is prime.

(a) Prove that R/xR is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

(b) Prove that if p is odd, then R/x^2R is isomorphic to $\mathbb{Z}/p^2\mathbb{Z}$.

(c) Prove that if $p = 2$, then R/x^2R is not isomorphic to $\mathbb{Z}/p^2\mathbb{Z} = \mathbb{Z}/4\mathbb{Z}$.

2. (a) State and prove the Hilbert Basis Theorem.

(b) Demonstrate this explicitly by finding a finite set of generators for the following ideal:

$$I =< x^p - y^q z^r \mid p, q, r, \text{ prime } >\subset \mathbb{Z}[x, y, z]$$

(c) Prove that every monomial $x^a y^b z^c$ in $Z[x, y, z]$ is congruent modulo I to one of the four following monomials:

$$yz^n,\ y^n z,\ xyz^n,\ xy^n z$$

for some integer n (possibly zero).

2 Fields

3. Let p be a prime, and let $\mathbb{F}_p$ be the field with p elements. Let $L = \mathbb{F}_p(x, y)$ be the field of rational functions in two variables with coefficients in $\mathbb{F}_p$, and let $K = \mathbb{F}_p(x^p, y^p)$.

(a) Prove that $[L : K] = p^2$, and that for any element $\alpha \in L$, we have $[K(\alpha) : K] = p$.

(b) Prove that L/K is not a simple extension.

(c) Prove that there are infinitely many different fields M such that $K \subsetneq M \subsetneq L$.

4. Let K be a field, and let $F = K(\frac{x^3}{x+1}) \subset K(x)$.

(a) Find a polynomial $p(T) \in F[T]$ such that $p(x) = 0$.

(b) Compute the degree $[K(x) : F]$.

(c) Find a field M and two finite extension fields L_1 and L_2 such that L_1 is isomorphic to L_2, but $[L_1 : M] \neq [L_2 : M]$.

Questions on Groups

5. Let $G = D_4 =< x, y \mid yx = xy^{-1},\ x^2 = 1,\ y^4 = 1 >$ be the dihedral group on four letters.

(a) Prove that the centre of G has two elements.

(b) Define a map $\phi\colon G \to G$ by $\phi(y^n) = y^n$ and $\phi(xy^n) = xy^{n+1}$. Prove that ϕ is an automorphism of G.

(c) Prove that the homomorphism ϕ from part (b) is not an inner automorphism of G. That is, prove that there is no element g of G such that for all $h \in G$, we have $\phi(h) = g^{-1}hg$.

6. (a) Let $A, D \in S_{2n}$ be products of n disjoint transpositions. Write AD

as a product of disjoint cycles. For any integer d, show that AD contains an even number of cycles of length d.

(b) Let $A, D \in S_8$ be such that $A(1) = 3$ and $AD = (1, 5, 2)(3, 8, 6)$. Find A and D.

7. Let $G = \mathbb{Z} \times (\mathbb{Z}/6\mathbb{Z})$ be a finitely generated abelian group.

(a) Find all proper subgroups and quotient groups of G.

(b) Show that every increasing chain of proper subgroups of G is finite.

(c) Show that for every positive integer n, there is an increasing chain of proper subgroups of G of length at least n.

3 Linear Algebra

8. A symmetric matrix A is positive definite if $\mathbf{x}^t A\mathbf{x} > 0$ for all nonzero vectors $\mathbf{x}$. Let $A = (A_{ij})$ be a positive definite matrix.

(a) Show that A is non-singular.

(b) Prove that $A_{ii} > 0$ for all i.

(c) Prove that $\max_{k \neq j} |A_{kj}| \leq \max_i |A_{ii}$.

(d) Prove that if $i \neq j$, then $(a_{ij})^2 \leq a_{ii}a_{jj}$.

9. Let M be an m by m matrix with complex entries.

(a) Assume that $M^n = 0$ for some integer $n > 1$. Prove that $n \leq m$.

(b) Assume that $M^n = I$ for some integer $n > 1$. Prove that M is diagonalizable.