PURE MATHEMATICS DEPARTMENT

ANALYSIS AND TOPOLOGY COMPREHENSIVE EXAMINATION

Answer all questions. May, 1989

1. Let X be any non-empty set and let S be a family of subsets of X such that every finite collection of members of S has non-empty intersection. In this case we say that S has the finite intersection property which we abbreviate to f.i.p. . Prove that there exists a family M of subsets of X with the f.i.p. such that $S \subseteq M$ and such that M is not a proper subfamily of any family of subsets of X having the f.i.p.

2. a) Prove that if X is a compact topological space then each closed subset of X is compact.

   b) Show that a compact subset of a Hausdorff space is closed.

3. a) State the Baire Category Theorem.

   b) Let X be a complete metric space and let $E \subseteq X$ be a perfect set (i.e. a non-empty, closed set in which every point is an accumulation point.) Prove that E is uncountable.

   c) The Cantor set is an example of such a set E. Describe its construction and prove it is perfect.

continued on page 2

4. Let $f\colon [0,2\pi) \to \mathbb{C}$ be continuous. Suppose that the Fourier coefficients $\hat{f}(n)$ of f satisfy $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$. Let N be a positve integer. Show that $\frac{1}{N} \sum_{m=1}^{N} f(\frac{m}{N}) = \sum_{r=-\infty}^{\infty} \hat{f}(rN)$.

5. Let f be a Lebesgue measurable function on $[0,1]$. Show that the map $\phi\colon [1,\infty] \to [0,\infty]$ given by $\phi(p) = \|f\|_p$ is increasing and left continuous at any p such that $\phi(p) < \infty$.

6. a) Let U be a non-empty open subset of $\mathbb{R}^2$ with finite Lebesgue measure $m(U)$. Show that $m\{x \in U: \text{dist}(x, \text{boundary } U) \le t\} \to 0$ as $t \to 0$.

   b) Show that this may fail if U has infinite measure.

7. Does there exist a function f which is analytic for $|z| < 1$ and which satisfies $|f(z)| \ge 1/(1-|z|)$ for $|z| < 1$? Justify your answer.

8. Find an analytic isomorphism f (i.e. an analytic map with an analytic inverse) from

$$A = \{z \in \mathbb{C} \mid \text{Re}(z) < -2 \text{ or } \text{Re}(z) \ge -2 \text{ and } \text{Im}(z) \ne 0\}$$

to $\qquad U = \{z \in \mathbb{C} \mid (z) < 1\}$.

continued on page 3

9. a) Let D be an open connected subset of $\mathbb{C}$ and let $(f_n)_{n=1}^{\infty}$ be a sequence of functions which are analytic and non-zero in D. If the sequence converges to a function f uniformly on compact subsets of D prove that either f has no zeros in D or f is identically zero on D.

b) Give an example where the latter possibility occurs, (i.e. f is identically zero).