UNIVERSITY OF WATERLOO

DEPARTMENT OF PURE MATHEMATICS

ANALYSIS & TOPOLOGY COMPREHENSIVE EXAM

May 22, 1986

REAL & COMPLEX ANALYSIS

TIME: 3 Hours

1. Suppose $\{a_k\}_{k=0}^{\infty}$ is a sequence of complex numbers which converges to a. Prove that $\lim\limits_{x\to+\infty} e^{-x} \sum\limits_{k=0}^{\infty} \frac{a_k x^k}{k!} = a$.

2. Use the Weierstrass approximation theorem and the fact that $X = \{f\colon [0,1] \to \mathbb{R} \mid f$ is continuous and nowhere differentiable$\}$ is nonempty to prove that X is dense in $C[0,1]$ with respect to the uniform norm.

3. Prove that if f and g are entire functions and $|f(z)| \le |g(z)|$ for all $z \in \mathbb{C}$ then there exists $c \in \mathbb{C}$ such that $f(z) = cg(z)$ for all $z \in \mathbb{C}$.

4. Exhibit a one-to-one conformal mapping of $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0$ and $\operatorname{Im} z > 0\}$ onto $\{z \in \mathbb{C} \mid |z| < 1\}$.

5. Use contour integration to find $\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{1+x^2}\,dx$ for $\lambda > 0$. Find the Fourier transform of φ if $\varphi(x) = \frac{1}{1+x^2}$, $x \in \mathbb{R}$.

6. Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Suppose $f_n\colon D \to \mathbb{C}$ is analytic for each $n = 1,2,\ldots$ and suppose that for each compact $K \subseteq D$ there exists $M_K > 0$ such that $|f_n(z)| \le M_K$ for all $z \in K$ and all $n = 1,2,\ldots$. Prove that there exists a subsequence of $\{f_n\}_{n=1}^{\infty}$ which converges uniformly on each compact subset of D to a function $f\colon D \to \mathbb{C}$, and prove that f analytic on D. HINT: Consider Cauchy's integral formula's for the derivatives.

TOPOLOGY

1. Let (X,ρ) be a metric space. For $\phi \neq A \subseteq X$ let $\rho_A(x) = \inf\{\rho(x,y) \mid y \in A\}$.

   (a) Prove that ρ_A is continuous whenever $\phi \neq A \subseteq X$.

   (b) Prove that if $\phi \neq A \subseteq X$ then $\rho_A(x) = 0$ if and only if $x \in \bar{A}$ — the closure of A.

   (c) Prove that if A and B are disjoint closed subsets of X then there exists a continuous $\varphi\colon X \to [0,1]$ such that

   $\varphi(x) = 0$ if and only if $x \in A$ and

   $\varphi(x) = 1$ if and only if $x \in B$.

2. Suppose X is a complete metric space and $f\colon (0,1) \to X$ is uniformly continuous. Prove that $\lim\limits_{t\to 0+} f(t)$ exists.

3. Let X be a normal space, F a subspace of X, and f a continuous real function defined on F whose values lie in the closed interval $[a,b]$. Does f have a continuous extension f_0 defined on all of X whose values are in $[a,b]$? Explain.

4. (a) Define the concepts of first countable, second countable, and separability for a topological space.

   (b) Show that a second countable space is separable.

   (c) Show that for a metric space the concepts of second countable and separability are equivalent.

   (d) Give an example of a metric space which is not second countable. Give details.

5. (a) Explain what is meant by saying that a topological space is metrizable.

   (b) Give a sufficient condition that a normal space be metrizable.

   (c) Is your condition in 4.(b) also necessary? Explain.

6. (a) Define the concept of topological isomorphism of two normed linear spaces.

   (b) Prove: If X and Y are normed linear spaces, they are topologically isomorphic if and only if there exists a linear operator T with domain X and range Y and positive constants m, M such that

$$m\|x\| \leq \|Tx\| \leq M\|x\|$$

   for every x in X.

   (c) Prove: Let X be a linear space, and suppose two norms $\|x\|_1$ and $\|x\|_2$ are defined on X. These norms define the same topology on X if and only if there exist positive constants m and M such that

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1 \quad .$$