UNIVERSITY OF WATERLOO

WATERLOO ONTARIO

Analysis Comprehensive Examination

Department of Pure Mathematics May 8, 1984

THREE HOURS

MARKS ANSWER ALL QUESTIONS (Total marks 100)

20 1. a) In what sense is $\int_0^\infty \frac{\sin x}{x}\,dx$ defined? Does it make sense as a Lebesgue integral? Explain.

b) Evaluate this integral, and justify your methods.

12 2. Let Ω_0 be the set of ordinals less than the first uncountable ordinal endowed with the order topology.

a) Prove that Ω_0 is sequentially compact, but not compact.

b) Prove that every continuous function on Ω_0 is bounded and attains its supremum.

10 3. Find an explicit conformal map of the region

$$S = \{z: |z| < 1, \operatorname{Im} z > 0\}$$

onto the unit disc D.

20 4. a) Let A be a subset of the real line with positive Lebesgue measure. Show that $A-A = \{x-y: x,y \in A\}$ contains a neighbourhood of 0.

b) Let $f: \mathbb{R} \to \mathbb{R}$ be a function satisfying

(1) $f(x+y) = f(x)+f(y)$.

Show that if f is not continuous, then f is not bounded on any open interval.

c) Combine (a) and (b) to prove that every measurable solution of (1) is continuous. Hence find all measurable solutions.

MARKS

10 5. a) State Picard's Theorem.

b) Let $f(z)$ be an entire function such that

$$f(z+1) = f(z) \quad \text{for all } z \text{ in } \mathbb{C}.$$

Prove that there is a point z_0 in $\mathbb{C}$ such that $f(z_0) = z_0$.

10 6. Let $f(x)$ be an odd C^1-function on $[-\pi,\pi]$. Prove that $\|f\|_2 \le \|f'\|_2$ where

$$\|g\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x)|^2 dx\right)^{1/2}.$$

When is this inequality sharp?

10 7. Let I be the unit interval, and let $\mathbb{N}$ be the positive integers.

a) Show that $I^{\mathbb{N}}$ with the product topology is metrizable.

b) Show that I^{I} with the product topology is not metrizable.

8 8. Is the set $P = \{p(z) = \sum_{i=0}^{n} a_i z^i : a_i \in \mathbb{C}\}$ of polynomials of a complex variable dense in $C(T)$, the space of continuous complex valued functions on the unit circle? Explain.