

UNIVERSITY OF WATERLOO
DEPARTMENT OF PURE MATHEMATICS

ANALYSIS & TOPOLOGY COMPREHENSIVE EXAM

OCT. 22, 1986

TIME: 3 Hours

Candidates are expected to demonstrate competence in the three areas of real analysis, complex analysis, and topology.

REAL ANALYSIS

1. Suppose $\{a_k\}_{k=0}^{\infty}$ is a sequence of complex numbers which converges to a .

Prove that
$$\lim_{x \rightarrow +\infty} e^{-x} \sum_{k=0}^{\infty} \frac{a_k x^k}{k!} = a.$$

2. Use the Weierstrass approximation theorem and the fact that $X = \{f: [0,1] \rightarrow \mathbb{R} \mid f \text{ is continuous and nowhere differentiable}\}$ is nonempty to prove that X is dense in $C[0,1]$ with respect to the uniform norm.

3. Let (X, ρ) be a metric space. For $\emptyset \neq A \subseteq X$ let $\rho_A(x) = \inf\{\rho(x, y) \mid y \in A\}$.

(a) Prove that ρ_A is continuous.

(b) Prove that $\rho_A(x) = 0$ if and only if x is in the closure of A .

(c) Prove that if A and B are disjoint closed subsets of X then there exists a continuous $\varphi: X \rightarrow [0,1]$ such that

$$\varphi(x) = 0 \quad \text{if and only if } x \in A \quad \text{and}$$

$$\varphi(x) = 1 \quad \text{if and only if } x \in B.$$

4. (a) Define the concept of topological isomorphism (linear homeomorphism) of two normed linear spaces.

4. (b) Prove: If X and Y are normed linear spaces, they are topologically isomorphic if and only if there exists a linear operator T with domain X and range Y and positive constants m, M such that

$$m\|x\| \leq \|Tx\| \leq M\|x\|$$

for every x in X .

- (c) Prove: Let X be a linear space, and suppose two norms $\|x\|_1$ and $\|x\|_2$ are defined on X . These norms define the same topology on X if and only if there exist positive constants m and M such that

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1 .$$

5. (a) Suppose E is a Lebesgue measurable subset of \mathbb{R} , $0 < \lambda(E) < \infty$ (where λ denotes Lebesgue measure), and $\epsilon > 0$. Prove that there exists a continuous function $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ with compact support such that $\lambda(\{x \in \mathbb{R} \mid \chi_E(x) \neq \varphi(x)\}) < \epsilon$.

- (b) Suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ is Lebesgue integrable and $\epsilon > 0$. Prove that there exists a continuous function $g: \mathbb{R} \rightarrow \mathbb{R}$ with compact support such that $\int_{-\infty}^{\infty} |f(x) - g(x)| dx < \epsilon$.

COMPLEX ANALYSIS

1. Prove that if f and g are entire functions and $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$ then there exists $c \in \mathbb{C}$ such that $f(z) = cg(z)$ for all $z \in \mathbb{C}$.

2. Exhibit a one-to-one conformal mapping of $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0 \text{ and } \operatorname{Im} z > 0\}$ onto $\{z \in \mathbb{C} \mid |z| < 1\}$.

3. Use contour integration to find $\int_{-\infty-i}^{\infty-i} \frac{e^{i\lambda x}}{1+x^2} dx$ for $\lambda > 0$. Find the Fourier transform of φ if $\varphi(x) = \frac{1}{1+x^2}$, $x \in \mathbb{R}$.

4. Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Suppose $f_n : D \rightarrow \mathbb{C}$ is analytic for each $n = 1, 2, \dots$ and suppose that for each compact $K \subseteq D$ there exists $M_K > 0$ such that $|f_n(z)| \leq M_K$ for all $z \in K$ and all $n = 1, 2, \dots$. Prove that there exists a subsequence of $\{f_n\}_{n=1}^{\infty}$ which converges uniformly on each compact subset of D to a function $f : D \rightarrow \mathbb{C}$, and prove that f is analytic on D .

HINT: Consider Cauchy's integral formula's for the derivatives.

TOPOLOGY

1. Suppose X is a complete metric space and $f : (0,1) \rightarrow X$ is uniformly continuous. Prove that $\lim_{t \rightarrow 0^+} f(t)$ exists.

2. Let X be a normal space, F a subspace of X , and f a continuous real function defined on F whose values lie in the closed interval $[a,b]$. Does f have a continuous extension f_0 defined on all of X whose values are in $[a,b]$? Explain.

3. (a) Define the concepts of first countable, second countable, and separability for a topological space.
 - (b) Show that a second countable space is separable.
 - (c) Show that for a metric space the concepts of second countable and separability are equivalent.
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4. (a) Explain what is meant by saying that a topological space is metrizable.
 - (b) Give a sufficient condition that a normal space be metrizable.
 - (c) Is your condition in 4.(b) also necessary? Explain.