Analysis Comprehensive Examination

Pure Mathematics, University of Waterloo

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ANSWER ALL QUESTIONS

1. Evaluate $\int_0^\infty \frac{x^2}{1+x^4}\,dx$. Justify your answer.

2. Suppose that $0 < a_1 < b_1$ are real numbers. Define a_n and b_n, $n \geq 1$ by $a_{n+1} = (a_n b_n)^{1/2}$ and $b_{n+1} = \frac{1}{2}(a_n+b_n)$. Set $I_n = [a_n, b_n]$ for $n \geq 1$. Prove that the intersection of all the I_n is nonempty.

3. If A is a subset of a topological space X, the boundary of A is defined as $\partial A = \bar{A} \setminus \operatorname{int} A$. Prove that X is connected if and only if every non-empty proper closed subset of X has non-empty boundary.

4. Let $f(z)$ be an analytic function defined on an open disc $D \subseteq \mathbb{C}$. Let a be a positive real number. Prove that if $|f'(z)| = a$ for all z in D, then f can be extended to an analytic automorphism of $\mathbb{C}$ (i.e. a bijective analytic map with an analytic inverse.)

5. If $f(x)$ is a real valued continuous function on $[0,1]$ such that $\int_0^1 f(x)x^n\,dx = 0$, $n = 0,1,2,\ldots$, show that $f(x)$ is identically zero.

6. Recall that a filter on X is <u>principal</u> if it has the form $\mathcal{F} = \{S \subseteq X: S \supseteq S_0\}$ for some subset $S_0 \subseteq X$.

   a) Prove that every filter on a finite set is principal.

   b) Prove that there is an ultra-filter on $\mathbb{N}$ which is not principal.

7. Let $f: \mathbb{R} \to \mathbb{R}$ be Lebesgue measureable. Show there exists a Borel function g such that $f = g$ almost everywhere.

8. Let $f(z) = e^z - z$. Let $A = \{a \in \mathbb{C}: f(z) = a$ for infinitely many $z\}$. Prove that A is dense in $\mathbb{C}$.

   Hint: Consider $g(z) = f(1/z)$.