UNIVERSITY OF WATERLOO

WATERLOO ONTARIO

COMPREHENSIVE EXAMINATION IN ANALYSIS

DEPARTMENT OF PURE MATHEMATICS

MAY 12, 1983

Answer all questions. Question 2 is worth 16 marks; all the remaining questions are worth 14 marks.

1. A partial order on a set A is a binary relation $\leq$ that satisfies

   (i) for all $a \in A$, $a \leq a$ (the reflexive property)

   (ii) for all $a,b \in A$, if $a \leq b$ and $b \leq a$, then $a = b$, (the antisymmetric property), and

   (iii) for all $a,b,c \in A$, if $a \leq b$ and $b \leq c$, then $a \leq c$ (the transitive property).

   Furthermore the partial order is called linear provided that for any $a,b \in A$, we must have $a \leq b$ or $b \leq a$.

   (a) Suppose $\leq$ is a partial order on A and suppose d,e are two fixed elements of A which are not comparable; that is, $d \not\leq e$ and $e \not\leq d$. Define a new relation $\leq^*$ on A as follows:

   $$a \leq^* b \text{ if and only if } a \leq b \text{ or } (a \leq d \text{ and } e \leq b)$$

   Show that $d \leq^* e$, and that $\leq^*$ is another partial order on A which extends $\leq$.

   (b) State Zorn's lemma.

   (c) Apply Zorn's lemma and part (a) above to show that any partial order on A can be extended to a linear order on A.

2. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers defined recursively by

   $$a_0 = 0,\ a_1 = 1,\ a_{n+2} = \sqrt{2}\, a_{n+1} - a_n;\ n = 0,1,2,\dots . \qquad a_n$$

   Let R be the radius of convergence of

   $$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

2. (cont'd)

(a) Prove that $|a_n| \le (2\sqrt{2})^n$ for $n \ge 0$ and prove that $R > 0$.

(b) Verify that $(z^2-\sqrt{2}z+1)f(z) = z$, for all complex z such that $|z| < R$.

(c) Let ξ_1, ξ_2 be the roots of $z^2-\sqrt{2}z+1$. Find the constants A, B such that $\frac{z}{z^2-\sqrt{2}z+1} = \frac{A}{z-\xi_1} + \frac{B}{z-\xi_2}$, and use this to get an explicit power series expansion in powers of z for the function $\frac{z}{z^2-\sqrt{2}z+1}$.

(d) Find an explicit formula for a_n in terms of the roots ξ_1, ξ_2 of $z^2-\sqrt{2}z+1$. How does the fact that $R > 0$ justify this formula?

(e) Calculate R.

(f) Compute $\int_\Gamma \frac{z}{z^2-\sqrt{2}z+1}\,dz$ when Γ is the contour $\{z = x+iy\colon |x|+|y| = 1\}$. Also compute this integral when Γ is the contour $\{z = x+iy\colon \max(|x|,|y|) = 1\}$.

3.(a) Define the outer Lebesgue measure of a set E of real numbers.

(b) Prove that the outer measure of the interval $[0,1]$ is 1 by showing that it is both ≤ 1 and ≥ 1.

(c) What is $L^1(\mathbb{R})$?

(d) If f is in $L^1(\mathbb{R})$, is it true that $\int_{\mathbb{R}} f(x)dx = \lim_{n\to\infty} \int_{-n}^{n} f(x)dx$? Explain.

(e) If f is a measurable function, not necessarily in $L^1(\mathbb{R})$, and $f \le 0$ ($\ge$), is it true $\int_{\mathbb{R}} f(x)dx = \lim_{n\to\infty} \int_{-n}^{n} f(x)dx$? Explain.

4.(a) Prove that any compact set A in a metric space X must be closed and bounded.

(b) Give an example of a closed and bounded set A in a metric space X such that A is not compact.

(c) Outline the proof that a continuous real-valued function f defined on a compact metric space X must achieve a maximum value on X.

4. (cont'd)

(d) Let X be the set of all functions $f: \mathbb{R} \to \mathbb{R}$ with the property that $|f(x)| \le |x|$ at every $x \in \mathbb{R}$. For given $g \in X$, $y \in \mathbb{R}$, $\varepsilon > 0$, consider the set of functions

$$S(g,y,\varepsilon) = \{f : f \in X \text{ and } |f(y)-g(y)| < \varepsilon\}.$$

Put a topology on X by taking, as a base, the family of all finite intersections of sets of type $S(g,y,\varepsilon)$.

(e) Explain why X is compact with this topology.

(f) In this topology does the sequence of functions $f_n = n\chi_{[n,\infty)}$ raise of χ converge? Explain. (χ_E is the characteristic function of E).

(g) Let Y be the set of functions in X which are continuous. Is Y closed in X? Explain.

5.(a) Compute the coefficients of the Fourier series for the function $f(x) = x^2$ over the interval $[-\pi,\pi]$.

(b) Using Parseval's equality or appropriate convergence theorems evaluate:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

6.(a) Let $g: [0,2\pi] \to \mathbb{R}$ be a continuous non-negative function. Let $M = \max\{g(x) : 0 \le x \le 2\pi\}$. Prove that either g is constant or $\frac{1}{2\pi}\int_0^{2\pi} g(x)\,dx < M$.

(b) Let E be an open connected set in $\mathbb{C}$. Let its closure $\bar{E}$ be compact. Let $f: \bar{E} \to \mathbb{C}$ be continuous with its restriction to E analytic. Let $M = \max\{|f(z)| : z \in \bar{E}\}$. Prove the maximum modulus principle that either f must be constant on $\bar{E}$ or $|f(z)| < M$ for all $z \in E$.

(c) Let $f(z) = 1/(z+1)^2$ define a function over the triangle $\bar{E}$ with vertices at $0, 2$ and i. Locate the points of $\bar{E}$ at which f attains its maximum modulus.

7.(a) What does it mean to say that a function $f: [0,1] \to \mathbb{R}$ is of bounded variation?

(b) Prove that every absolutely continuous $f: [0,1] \to \mathbb{R}$ must be of bounded variation.

(c) For any continuous function of two variables $f: [0,1]\times[0,1] \to \mathbb{R}$ and any $\varepsilon > 0$, there is a finite set of continuous functions of one variable $u_k, v_k: [0,1] \to \mathbb{R}$, $k = 1,\ldots,n$, such that $\left|f(x,y) - \sum_{k=1}^{n} u_k(x)v_k(y)\right| < \varepsilon$ for all $x,y \in [0,1]$. Explain why this is true.

(d) State and prove Fubini's theorem for continuous $f: [0,1]\times[0,1] \to \mathbb{R}$.