UNIVERSITY OF WATERLOO

WATERLOO ONTARIO

Department of Pure Mathematics

Comprehensive Examination in Analysis

Duration: 3 hours November 3, 1981

Do <u>THREE</u> out of four questions from each section.

Topology and Set Theory

1. Let X be a set and let $\mathcal{P}(X)$ denote the set of all subsets of X. Prove that the cardinality of $\mathcal{P}(X)$ is greater than the cardinality of X.

2. A set S of real numbers is called rationally independent if an equality of the form $r_1x_1+\ldots+r_nx_n = 0$ with $x_1,\ldots,x_n$ a finite subset of S and $r_1,\ldots,r_n$ rational numbers implies $r_1 = \ldots = r_n = 0$. Prove that there exists a rationally independent set of real numbers T such that for each real number x there exist real numbers $x_1,\ldots,x_n$ from T and rational numbers $r_1,\ldots,r_n$ such that $x = r_1x_1+\ldots+r_nx_n$.

3. Prove that a subset A of $\mathbb{R}$ is open if and only if A is equal to the union of a countable number of pairwise disjoint open intervals.

4. Show that $(-1,1)$ and $\mathbb{R}$ are homeomorphic. Does there exist a homeomorphism from $[-1,1]$ to $\mathbb{R}$?

Real Analysis

5. Evaluate $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

6. Put $f(x) = \sum_{n=1}^{\infty} \frac{(\sin x + n \cos x)}{n^3}$ and put $S = \{f(x) \mid x \in \mathbb{R}\}$.

   Prove that S has a maximum.

7. a) If p is a prime number, show that $\frac{1-x^p-(1-x)^p}{p}$ has integer coefficients.

   b) Show that the polynomials with integer coefficients are dense in $C_R([1/4,3/4])$, real valued continuous functions on $[1/4,3/4]$.

   (Hint: use (a) to approximate constant functions.)

8. Let $f: \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function. Prove that the graph of f, $G(f) = \{(x,f(x) \mid x \in \mathbb{R}\}$ is a measurable set of planar Lebesgue measure zero.

Complex Analysis

9. Put $p_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}$ and let r_n be the minimum modulus of the roots of $p_n(z)$. Prove that $\lim_{n\to\infty} r_n = \infty$.

10. Define $f: \mathbb{C}-\{0\} \to \mathbb{C}$ by $f(z) = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$. Prove that there is a sequence of complex numbers, $(z_i)_0^{\infty}$ with $\lim_{i\to\infty} z_i = 0$ such that $f(z_i) = \pi$ for $i = 0,1,2,\ldots$.

11. Let D_1 and D_2 be connected open sets in $\mathbb{C}$ and let $f_i: D_i \to \mathbb{C}$ be analytic functions for $i = 1,2$. Consider the following statement: If f_1 and f_2 agree on a non-empty open subset of $D_1 \cap D_2$ then there is a unique analytic function $f: D_1 \cup D_2 \to \mathbb{C}$ such that f_i is the restriction of f to D_i. Show that this statement is false. What additional hypothesis will make it true and why?

12. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^6}$.