Analysis Comprehensive Examination

Pure Mathematics, University of Waterloo
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Answer all of the first seven questions and one of the remaining three. In these questions $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{C}$ denotes the set of all complex numbers.

(18) 1. Let U be an open subset of $\mathbb{C}$, let $z_0 \in U$, let k be a natural number and let $f: U\backslash\{z_0\} \to \mathbb{C}$ be analytic.

a) What is meant by the Laurent expansion of f at z_0?

b) What does it mean to say that f has an essential singularity at z_0?

c) What does it mean to say that f has a pole of order k at z_0?

d) Define "the residue of f at z_0."

e) If f has a pole of order 3 at z_0 prove that the residue of f at z_0 is $\frac{1}{2}\lim\limits_{z\to z_0} \frac{d^2}{dz^2}((z-z_0)^3 f(z))$.

f) Use the Residue Theorem to find $\int_\gamma \frac{\sin \pi z}{z^4+z^2}\,dz$ where $\gamma(t) = 2e^{it}$ for $0 \le t \le 2\pi$.

(9) 2. Find the cardinality of the set of

a) all open subsets of $\mathbb{R}$,

b) all measurable subsets of $\mathbb{R}$,

c) all continuous $f: \mathbb{R} \to \mathbb{R}$,
and justify your answers.

(8) 3. For each natural number n let $X_n = \{0,1\}$ with the discrete topology and let $X = \prod_{n=1}^{\infty} X_n$ with the (Tychonoff) product topology. Show that X is homeomorphic to the Cantor set (with the topology inherited from $\mathbb{R}$).

(12) 4. Suppose that for each $n = 1,2,\ldots$, $f_n: [0,1] \to \mathbb{R}$ is Riemann integrable on $[0,1]$ and suppose $\{f_n\}_{n=1}^{\infty}$ converges pointwise on $[0,1]$ to $f: [0,1] \to \mathbb{R}$. Decide whether each of the following statements is true or false and prove your answers by either quoting appropriate theorems or giving counterexamples.

(a) f is Lebesgue integrable on $[0,1]$.

(b) if $\{f_n\}$ is uniformly bounded on $[0,1]$ then f is Riemann integrable on $[0,1]$.

(c) if $\{f_n\}$ is uniformly bounded then $\lim_{n\to\infty} \int_0^1 f_n(x)dx$ exists.

(d) if f is Lebesgue integrable on $[0,1]$ then $\lim_{n\to\infty} \int_0^1 f_n(x)dx$ exists and is equal to $\int_0^1 f(x)dx$.

(18) 5. Let $f: [0,\pi] \to \mathbb{R}$ be continuous and such that $f(0) = f(\pi) = 0$. Let g be the odd extension of f to $\mathbb{R}$ with period 2π.

(a) Show that the Fourier series of g can be written $\sum_{k\geq 1} b_k \sin kx$.

(b) State what you know concerning the convergence (pointwise, uniform and mean square) of the Fourier series of g.

(c) Assume $\sum_{k\geq 1} b_k \sin kx$ converges uniformly on $\mathbb{R}$ and let

$u(x,t) = \sum_{k=1}^{\infty} b_k e^{-k^2 t} \sin kx$ for $x \in \mathbb{R}$ and $t > 0$. Prove that

(i) u is of class C^{∞} on $\{(x,t) \mid x \in \mathbb{R} \text{ and } t > t_0\}$ for each $t_0 > 0$ and hence u is of class C^{∞} on $\mathbb{R}\times(0,+\infty)$,

5. (cont'd)

(ii) $\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$ for $x \in \mathbb{R}$, $t > 0$ and

(iii) $\lim_{t \downarrow 0} u(x,t) = f(x)$ uniformly for $x \in [0,\pi]$

(15) 6. Let $K: [0,1]\times[0,1] \in \mathbb{R}$ be continuous.

a) For $f \in L^2[0,1]$ show that $\int_0^1 K(x,t)f(t)dt \ (\stackrel{\text{def}}{=} Tf(x))$ exists for each $x \in [0,1]$ and prove that Tf is continuous on $[0,1]$.

b) Prove that if $\{f_n\}$ is a bounded sequence in $L^2[0,1]$ then $\{Tf_n\}$ has a uniformly convergent subsequence.

c) Assuming $\int_0^1 \int_0^1 K(x,t)^2 dxdt < 1$ and given $g \in L^2[0,1]$, prove that there exists $f \in L^2[0,1]$ such that $f(x) = g(x) + \int_0^1 K(x,t)f(t)dt$ for a.e. $x \in [0,1]$. Hint: Use the contraction mapping principle.

(10) 7. a) Define what is meant by a "simply connected open subset of $\mathbb{C}$" and state the Riemann mapping theorem.

b) Find a conformal mapping of $\{z \in \mathbb{C} \mid \text{Im} z > 0\}$ onto $\{z \in \mathbb{C} \mid |z| < 1\}$.

ANSWER ONE OF THE REMAINING THREE QUESTIONS

(10) 8. Let $f: [0,1] \to \mathbb{R}$. Prove that f is continuous if and only if its graph, $G = \{x,f(x)) \mid 0 \le x \le 1\}$, is a compact subset of $\mathbb{R}^2$.

(10) 9. Give an example of

a) a normed linear space which is not a Banach space,

b) a compact Hausdorff space which is not metrizable,

c) a sequence of real valued functions $f_1, f_2, \ldots$ on $[0,1]$ such that $\sum_{k\ge1} f_k(x)$ converges uniformly on $[0,1]$ but $\sum_{k\ge1} |f_k(x)|$ diverges for every $x \in [0,1]$.

(10) 10. Suppose f is a continuous bijection of a Hausdorff space X onto a Hausdorff space Y.

a) Prove that if X is compact then so is Y and f^{-1} is continuous.

b) If $X,Y \subseteq \mathbb{R}^n$ and X is open, state conditions on f which are sufficient to guarantee that Y is open and f^{-1} is differentiable.

c) If X is an open subset of $\mathbb{C}$ and $Y \subseteq \mathbb{C}$ state conditions on f which are sufficient to guarantee that Y is open and f^{-1} is analytic.