UNIVERSITY OF WATERLOO

Department of Pure Mathematics

Comprehensive Examination in Analysis   May 7, 1981 - 1:00-4:00 p.m.

MARKS

9   1. Let A be the set of all real valued sequences. Let B be the set of all functions on $\mathbb{R}$ which take only positive integer values. Prove that A has the same cardinality as $\mathbb{R}$, but that B has larger cardinality than $\mathbb{R}$. That is, prove that $\mathrm{card}(\mathbb{R}^{\mathbb{N}}) = \mathrm{card}(\mathbb{R}) < \mathrm{card}(\mathbb{N}^{\mathbb{R}})$.

NOTE: Give a proof of the strict inequality from scratch; without appeal to known theorems.

5   2. State Zorn's lemma and state the axiom of choice. Prove that Zorn's lemma implies the axiom of choice.

9   3. a) Prove that, if a metric space is compact, then it is sequentially compact.

Recall: A space is sequentially compact if every seqeunce has a convergent subsequence.

b) Give an example of a compact topological space, which is not sequentially compact.

9   4. a) Let $\mathcal{T}$ be a family of connected sets in a topological space X, such that for any $A,B \in \mathcal{T}$ the intersection $A \cap B$ is not empty. Prove that $\bigcup_{A\in\mathcal{T}} A$ is connected.

b) Give an example of a topological subspace of $\mathbb{R}^2$ which is connected but not locally connected.

Recall: A space is locally connected if every neighbourhood of a point contains a connected neighbourhood.

c) Show that in a normed linear space over R, an open set is connected if and only if it is pathwise connected.

MARKS

10 5. a) Prove that any uncountable subset A of $\mathbb{R}$ must have a limit point <u>in A</u>.

b) Let $f: \mathbb{R} \to \mathbb{R}$ be a function. Show that the set of points at which a jump discontinuity occurs is at most countable.

HINT: Show that for all positive integers n the sets $\{x: x \in \mathbb{R}, \frac{1}{n} \leq f(x+)-f(x-)\}$ are countable.

c) Show that any increasing function $f: \mathbb{R} \to \mathbb{R}$ has at most countably many discontinuities.

8 6. a) Suppose that $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n < \infty$. Prove that $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n} < \infty$.

b) Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n.$$

8 7. Let $f: \mathbb{R} \to \mathbb{C}$ be a continuous, 2π-periodic function and let α/π be an irrational number. Prove that $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x+n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt$, regardless of x.

Hint: First prove it for $f(x) = e^{ikx}$, $k = 0, \pm 1, \pm 2, \ldots$

8 8. a) Prove that if (X,μ) is a measure space and $f: X \to [0,\infty)$ is a measurable function such that $\int_X f d\mu = 0$, then $f(x) = 0$ almost everywhere.

b) With the same (X,μ) as above, let $g: X \to [0,\infty)$ be μ-measurable. For any μ-measurable set E let $\nu(E) = \int_E g d\mu$. Prove that ν is also a measure on the μ-measurable sets. Also prove that for any ν-integrable function $h: X \to R$ we have $\int_X h d\nu = \int_X hg d\mu$.

MARKS

16 9. a) Let D be an open region in the complex plane. Let $f_n: D \to \mathbb{C}$ be a sequence of analytic functions. Let $f: D \to \mathbb{C}$ be a function such that for each compact subset K inside D, $\sup\{|f_n(t)-f(t)|: t \in K\} \to 0$ as $n > \infty$. Prove that f is analytic on D.

b) Show that the series $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ defines an analytic function on the domain $D = \{z: \mathrm{Re}(z) > 1\}$.

c) Compute $\zeta(2)$ explicitly and justify your answer. <u>Hint</u>: Try Fourier series for $f(x) = x$.

10 10. a) Let $p(z)$, $q(z)$ be polynomials over $\mathbb{C}$ such that $\deg p(z)+2 \le \deg q(z)$ and such that $q(z)$ has no real roots. Prove that $\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx$ converges to the sum of the residues of $p(z)/q(z)$ in the upper half plane $\{z: \mathrm{Im}(z) > 0\}$.

b) Compute $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+x+1)(x^2+1)}$.

8 11. A complex valued function $f(z)$ of a complex variable is meromorphic with simple poles at 0 and 1. Outside the disk $\{z: z \in \mathbb{C}, |z| < 2\}$ $f(z)$ is bounded. Prove that $f(z)$ is a rational function of the form

$$f(z) = \frac{az^2+bz+c}{z(z-1)} .$$