October 25, 1988

Department of Pure Mathematics

Ph.D. Comprehensive exam in ANALYSIS and TOPOLOGY

ANSWER AS MANY QUESTIONS AS POSSIBLE.

1. Prove the theorem of Alexander: If $\mathcal{B}$ is a subase for the topology of a space X such that every cover of X by members of $\mathcal{B}$ has a finite subcover, then X is compact.

2.(a) Give the definition of the one point compactification of a topological space X.

(b) Give a compactification of the open interval $(0,1)$ (with usual topology) which is not topologically equivalent to the one point compactification.

(c) Must two topological spaces with homeomorphic one point compactifications be themselves homeomorphic?

3. Let $\mathcal{C}$ be a family of compact subsets of a Hausdorff space X such that the finite intersections of members of $\mathcal{C}$ are connected. Show that $\cap\mathcal{C}$ is connected.

4.(a) What is meant by a Lebesgue measurable function $f: \mathbb{R} \to \mathbb{R}$?

(b) Prove that the derivative f' of a differentiable function $f: \mathbb{R} \to \mathbb{R}$ is measurable, but not necessarily continuous.

(c) Is it true that if $f_n: [0,1] \to [0,\infty)$ are measurable and f_n decrease to 0 pointwise then $\int_0^1 f_n(x)\,dx \to 0$? Explain.

....question 4. continued on page 2...

4. con't.

(d) If $f: [0,1] \to [0,\infty)$ is integrable, prove that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\int_E f < \varepsilon$ whenever E is a measurable set with Lebesgue measure $m(E) < \delta$. Is this true when f is not integrable? Explain.

(e) What does the monotone convergence theorem say? Is the real valued function $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$, defined over the interval $[0,1)$, integrable over $[0,1)$, or not? Explain.

5. Give a brief answer to each of the following.

(a) What is the radius of convergence of the series $\sum_{n=1}^{\infty} (1 + \frac{1}{n})^n z^n$.

(b) If Ω is an open connected set in the plane and $f: \Omega \to \mathbb{C}$ is analytic then $f = g'$ for some analytic function g on Ω. Is this true? Explain.

(c) If a power series $\sum_{n=0}^{\infty} a_n z^n$ converges uniformly over all of $\mathbb{C}$, then all but finitely many a_n must vanish. Why is this so?

(d) If f is an entire function but not a polynomial then for some sequence z_n in $\mathbb{C}$, we have $|z_n| \to \infty$ but $f(z_n) \to 0$ as $n \to \infty$. Why is this so?

(e) If f is analytic and bounded on the punctured disk $\{z: 0 < |z| < 1\}$, then the singularity of f at 0 is removable. Explain why?

...question 5. continued on page 3...

5. con't.

(f) If f is analytic with a zero of order n at a point p, what is the order of the pole and the residue of $g = f'/f$ at p?

(g) Expand $f(z) = \frac{1}{2-z} + \frac{1}{1-z}$ in a Laurent series valid for the annulus $1 < |z| < 2$.

(h) Show that the punctured disk $\{z \in \mathbb{C}: 0 < |z| < 1\}$ is homeomorphic to the annulus $\{z: 1 < |z| < 2\}$, but not conformally equivalent to it.

6. Let $C[0,1]$ stand for the algebra of continuous real valued functions on the interval $[0,1]$. Let $\varphi: C[0,1] \to \mathbb{R}$ be a non-zero homomorphism of $\mathbb{R}$-algebras. Prove that there exists a point p in $[0,1]$ such that $\varphi(f) = f(p)$ for all $f \in C[0,1]$.

7.(a) Show that for all $c > 1$ the series $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges uniformly over the half plane $\{z: \mathrm{Re}(z) \geq c\}$. (Here $n^z = e^{z \log n}$.)

(b) Prove $f(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ represents an analytic function over the half plane $\{z: \mathrm{Re}(z) > 1\}$.

(c) Compute $f(2)$, and justify your answer.

8.(a) Let $C([0,1]^2)$ be the algebra of continuous real valued functions over the unit square $[0,1]^2$. Let A be the set of functions g in $C([0,1]^2)$ which can be written in the form $g(x,y) = \Sigma u_i(x)v_i(y)$ for some continuous functions $u_i, v_i \colon [0,1] \to \mathbb{R}$. Prove that A is dense in $C([0,1]^2)$, using the sup norm on $C([0,1]^2)$.

(b) Prove that for $f \in C([0,1]^2)$,

$$\int_0^1 \left(\int_0^1 f(x,y)\,dx\right) dy = \int_0^1 \left(\int_0^1 f(x,y)\,dy\right) dx.$$

9.(a) Let Ω be an open connected set in the plane, and let $u\colon \Omega \to \mathbb{R}$ be a real valued continuous function satisfying the mean value property. Namely, for all p in Ω and for each closed disk $D(p,r)$, centred at p and of radius r, inside Ω

$$u(p) = \frac{1}{2\pi}\int_0^{2\pi} u(p+re^{i\theta})\,d\theta.$$

Prove that such u cannot attain a maximum in Ω unless u is constant.

(b) Give an example of a non-constant function u satisfying the mean value property.