PURE MATHEMATICS DEPARTMENT

ANALYSIS AND TOPOLOGY COMPREHENSIVE EXAMINATION October 23, 1985

Answer all questions.

1. Find the Laurent expansion of $f(z) = \frac{1}{(z-1)(3-z)}$ in the annulus $1 < |z| < 3$.

2. Evaluate $\int_0^\infty \frac{dx}{4+x^4}$.

3. i) Prove that the open unit disc $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$ is homeomorphic to $\mathbb{R}^2$.

   ii) Is it true that any two simply connected open sets D_1 and D_2 in $\mathbb{R}^2$ are homeomorphic? Justffy pour answer.

4. Let θ_1 and θ_2 be non-zero complex numbers such that θ_1/θ_2 is not a rational number. Let f be an analytic function on $\mathbb{C}$ such that $f(z+\theta_1) = f(z) = f(z+\theta_2)$ for all z in $\mathbb{C}$. Prove that f is a constant function.

5. i) Give the definition of a filter and an ultrafilter.

   ii) Give an example of an ultrafilter on $\mathbb{Z}$.

   iii) Prove, by using Zorn's Lemma, that every filter is contained in some ultrafilter.

6. Let $C[0,1]$ denote the set of real valued continuous functions on the closed interval $[0,1]$. For any $f,g \in C[0,1]$ define δ_1 and δ_2 by

$$\delta_1(f,g) = \max_{0\le t\le 1} |f(t)-g(t)|$$

and

$$\delta_2(f,g) = \int_0^1 |f(t)-g(t)|dt.$$

   i) Show that δ_1 and δ_2 are metrics on $C[0,1]$.

   ii) Determine whether or not δ_1 and δ_2 are equivalent metrics.

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7. (i) State Banach's fixed point theorem (the contraction mapping theorem) for metric spaces.

(ii) Give a sketch of how this theorem may be used to prove the existence of a solution for certain integral or differential equations.

8. (i) Define compactness in a topological space;

(ii) Prove that a closed subset of a compact set is compact;

(iii) Prove that if K_1 and K_2 are disjoint compact sets in a Hausdorff space X then there exist disjoint neighbourhoods U_1 and U_2 of K_1 and K_2, respectively.

9. (i) Define completeness of a normed linear space;

(ii) Define the space $L^2[a,b]$ and show that it is complete under the norm

$$\|f\| = \left\{\int_a^b f^2(t)\,dt\right\}^{1/2} .$$

(iii) Define the Fourier series of an integrable function f;

(iv) Discuss various senses in which the Fourier series of f can be said to "represent" f. Consider, e.g., the cases when

(a) $f \in C[-\pi,\pi]$;

(b) $f \in L^1[-\pi,\pi]$;

(c) $f \in L^2[-\pi,\pi]$.

10. (i) If f is a bounded Lebesgue measurable function defined on a (Lebesgue) measurable set $E \subset [a,b]$, $-\infty < a < b < \infty$, define the Lebesgue integral of f over E.

(ii) If $\{f_n\}$ is a sequence of Lebesgue integrable functions such that $f_n \to f$ on $[a,b]$, under what further conditions is it true that $\int_a^b f_n(t)\,dt \to \int_a^b f(t)\,dt$? (No proof is required.)

(iii) If f is Lebesgue integrable on $[a,b]$ and

$$F(x) = \int_a^x f(t)\,dt, \qquad a \le x \le b,$$

what properties does F have? (No proof is required.)