University of Waterloo
Department of Pure Mathematics
Analysis and Topology Comprehensive Examination
1pm–4pm, June 8, 2006

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Instructions: There are 3 sections in this exam, each section with 4 problems. Solve at least 2 and at most 3 problems from each section. Attempt at least 6 problems overall.

Set Theory and Topology

1. (a) Prove that the cardinality of $\mathbb{R}$ and $\mathbb{R}^2$ are the same.
(b) Using part (a), prove that the cardinality of $\mathbb{R}^2$ and $\mathbb{R}^3$ are the same.
(c) Prove that the unit sphere in $\mathbb{R}^3$

$$\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

has the same cardinality as $\mathbb{R}$.
(Hint: Use the Schroeder-Bernstein Theorem.)

2. (a) Prove that there is no surjective continuous map from the closed interval $[0,1]$ onto the open interval $(0,1)$.
(b) Give an example of a surjective continuous map from the open interval $(0,1)$ onto the closed interval $[0,1]$.
(c) Prove that no example in part (b) can be injective.

3. Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let $f : X_1 \to X_2$ be a continuous map such that

$$d_1(p,q) \ \leq \ d_2(f(p), f(q))$$

for every pair of points $p, q \in X_1$. Assume that f is *surjective*.
(a) Prove that f must be injective.
(b) If X_1 is complete, then must X_2 be complete? Give a proof or a counterexample.
(c) If X_2 is complete, then must X_1 be complete? Give a proof or a counterexample.

4. Let X be a metrizable topological space. Prove that the following are equivalent.
(i) X is bounded under every metric that induces the topology of X.
(ii) Every continuous function $f : X \to \mathbb{R}$ is bounded.
(iii) X is compact.
(Hints: Prove that (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(i). If f is not bounded, map X into $X \times \mathbb{R}$ by $x \mapsto (x, f(x))$. If $\{x_n\}$ is a sequence having no convergent subsequence, find a continuous function f with $f(x_n) = n$ using the Urysohn Lemma.)

5. Recall that a sequence $\{f_n\}$ in a Hilbert space H converges weakly to $f \in H$ if

$$\langle f_n, g \rangle \to \langle f, g \rangle \quad \text{for all } g \in H.$$

Also recall that $\{f_n\}$ converges strongly to $f \in H$ if

$$\|f_n - f\| \to 0.$$

Give an example of a sequence in $L^2(\mathbb{R})$ that converges weakly, but not strongly. Make sure to justify your assertions.

6. Let $\{f_n : [0,1] \to \mathbb{R}\}$ be a sequence of continuous functions satisfying

$$\int_0^1 (f_n(x))^2 \, dx \le 1$$

for all n. Define $g_n : [0,1] \to \mathbb{R}$ by

$$g_n(x) = \int_0^1 f_n(y) \sqrt{x+y} \, dy.$$

(a) Find a constant $K \ge 0$ such that $|g_n(x)| \le K$ for all n and $x \in [0,1]$.
(b) Prove that there exists a subsequence of $\{g_n\}$ that converges uniformly.
(Hint: Use the Arzelà-Ascoli Theorem.)

7. Let $\{\varphi_n : [0,1] \to \mathbb{R}\}$ be a sequence of nonnegative continuous functions such that the limit

$$\lim_{n\to\infty} \int_0^1 x^k \varphi_n(x) \, dx$$

exists for every nonnegative integer $k = 0, 1, 2, \ldots$. Prove that the sequence of real numbers

$$I_n(f) = \int_0^1 f(x) \, \varphi_n(x) \, dx$$

converges for every continuous function $f : [0,1] \to \mathbb{R}$.
(Hint: Use the Stone-Weierstrass Theorem to prove that the sequence $\{I_n(f)\}$ is Cauchy.)

Real Analysis Continued

8. Let u be harmonic in the unit disc $\mathbb{D}_1 \subset \mathbb{R}^2$ and continuous on the boundary $\partial\mathbb{D}_1$. Then u can be represented by the Poisson integral formula

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)u(1,\phi)}{1-2r\cos(\phi-\theta)+r^2}\,d\phi,$$

where $0 \le r < 1$ and $0 \le \theta \le 2\pi$.

(a) Fix some $0 < r < 1$ and let $\mathbb{D}_r$ be the disc of radius r around the origin. Prove that for all $\mathbf{x} \in \mathbb{D}_r$ we have the derivative estimate

$$|D^N u(\mathbf{x})| \le C_{N,r} \sup_{\mathbf{y}\in\mathbb{D}_1} |u(\mathbf{y})|,$$

where $D^N u$ is any N-th order partial derivative of u in cartesian coordinates, and $C_{N,r}$ is a constant depending only on N and r.

(b) Let Δ denote the Laplacian. Suppose u solves the equation $\Delta u = k$ in $\mathbb{D}_1$ for some constant k, and is continuous on the boundary $\partial\mathbb{D}_1$. Prove that u satisfies derivative estimates in part (a) provided that we allow the constants $C_{N,r}$ to depend also on k.

Complex Analysis

9. Use contour integration to evaluate the improper integral

$$\int_0^\infty \frac{1}{x^4+1}\,dx.$$

Make sure to justify your steps.

10. Suppose f is a bijection from the unit disc to itself such that f is analytic and fixes the origin. Prove that $|f'(0)| = 1$.

11. Let f and g be entire such that $|f(z)| < |g(z)|$ for all z satisfying $|z| \ge M$ for some real constant $M \ge 0$. Prove that f/g is rational.

12. Let U and V be open connected subsets of the complex plane. Let $f : U \to V$ be analytic. Assume that $f^{-1}(K)$ is compact whenever K is a compact subset of V.

(a) Prove that f is not a constant function.

(b) Prove that $f(U)$ is a closed subset of V.

(c) Prove that $f(U) = V$. (Hint: Use the Open Mapping Theorem.)