Analysis Comprehensive

January 2004

1) a) State Zorn's Lemma.

b) State the Axiom of Choice

c) Let $\mathfrak{A}$ and $\mathfrak{B}$ be nonempty sets. Let $A \subseteq \mathfrak{A}$ and let $B \subseteq \mathfrak{B}$. Define $\mathfrak{C}(A, B) = \{f : A \to B \mid f \text{ is } 1-1 \text{ and onto}\}$ and let

$$\mathfrak{F} = \{(A, B, f) \mid A \subseteq \mathfrak{A}, B \subseteq \mathfrak{B}, f \in \mathfrak{C}\}.$$

We write $(A, B, f) \preceq (A_1, B_1, g)$ if $A \subseteq A_1, B \subseteq B_1$ and g extends f. Show that $\preceq$ determines a partial order on $\mathfrak{F}$.

d) Show that either there exists an injective function f from $\mathfrak{A}$ into $\mathfrak{B}$ or there exists an injective function f from $\mathfrak{B}$ into $\mathfrak{A}$.

2) Recall that a metric space (X, d) is said to be separable if it has a countable dense subset.

a) Show that $\mathbb{R}^n$ is separable with respect to the usual metric.

b) Show that that (l_1, d_1) is separable, where

$$d_1(\{x_i\}, \{y_i\}) = \sum_{i=1}^{\infty} \mid x_i - y_i \mid .$$

c) Show (l_∞, d_∞) is not separable where

$$d_\infty(\{x_i\}, \{y_i\}) = sup \mid x_i - y_i \mid .$$

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3) a) Show that if $\{f_n\}$ is a sequence of continuous functions on the closed interval $[0, 1]$ that converges uniformly to f on $[0, 1]$, then f is continuous on $[0, 1]$.

b) Let $C[0, 1]$ be the space of continuous functions on $[0, 1]$. Define what it means for a family $\mathfrak{F}$ in $C[0, 1]$ to be equicontinuous.

c) Let $\{f_n\}$ be a sequence of continuous functions on the closed interval $[0, 1]$ that is converges uniformly to a function f on $[0, 1]$. Must $\{f_n\}$ be equicontinuous? Explain your answer.

4) A subset S of a metric space (X, d_X) is said to be G_δ if there exists a countable collection $\{U_i\}_{i=1}^{\infty}$ of open sets such

$$S = \bigcap_{i=1}^{\infty} U_i.$$

a) Show that every closed subset F of X is G_δ.

b) State the Baire Category Theorem for metric spaces.

c) Show that the rationals $\mathbb{Q}$ are not a G_δ set in $\mathbb{R}$.

5) Let $0 < \alpha < 1$. Let $P_\alpha\{0\} = [0, 1]$. Let $P_\alpha\{1\}$ be obtained from $P_\alpha\{0\}$ by removing the open interval of length $\frac{\alpha}{3}$ from the middle of $P_\alpha\{0\}$. Then construct $P_\alpha\{2\}$ from $P_\alpha\{1\}$ by removing open intervals of length $\frac{\alpha}{3^2}$ from the middle of the two closed subintervals of $P_\alpha\{1\}$. In general, $P_\alpha\{n+1\}$ is obtained from $P_\alpha\{n\}$ by removing the open interval of length $\frac{\alpha}{3^{n+1}}$ from the middle of each of the 2^n closed subintervals of $P_\alpha\{n\}$. Let

$$P_\alpha = \bigcap_{n=0}^{\infty} P_\alpha\{n\}.$$

P_α is called a generalized Cantor set.

a) Prove that P_α is closed and nowhere dense.

b) Find $m(P_\alpha)$ where m denotes the usual Lebesgue measure on $\mathbb{R}$.

c) Show that it is possible to decompose $\mathbb{R}$ into two sets A and B with A being of first category and $m(B) = 0$.

6) a) Define what it means for a complex analytic function to be entire and state Liouville's Theorem for entire functions.

b) Let f be an entire function such that $\mid f(z) \mid = \mid e^z \mid$ for all $z \in \mathbb{C}$. Prove that there exists some $\lambda \in \mathbb{C}$ such that $f(z) = \lambda e^z$ for all $z \in \mathbb{C}$.

c) Let f be an entire function such that $\mid f(z) \mid = \mid ze^z \mid$ for all $z \in \mathbb{C}$. Does it follow that there exists some $\lambda \in \mathbb{C}$ such that $f(z) = \lambda ze^z$ for all $z \in \mathbb{C}$? Explain your answer.

d) Find all entire functions h with the property that $\mid h(z) \mid \leq \sqrt{\mid z \mid}$ for all $z \in \mathbb{C}$.

7) a) Define the concept of a basis for a topology $\mathcal{T}$.

b) Let $\mathcal{X}$ be the set of all functions from $[0,1]$ to $\mathbb{R}$. For every $n \geq 1$, $t_1, \ldots, t_n \in [0,1]$ and $a_1 < b_1, \ldots, a_n < b_n$ in $\mathbb{R}$, we denote

$$\begin{aligned} \mathcal{V} &= \mathcal{V}(n, t_1, \ldots, t_n, a_1, \ldots, a_n, b_1, \ldots, b_n) \\ &= \{f \in \mathcal{X} \mid f(t_1) \in (a_1, b_1) \ldots, f(t_n) \in (a_n, b_n)\}. \end{aligned}$$

Show that the collection of all such $\mathcal{V}$ forms a basis for a topology $\mathcal{T}$ on $\mathcal{X}$.

c) Show that $(\mathcal{X}, \mathcal{T})$ is Hausdorff.

d) Let f and $f_1, f_2, \ldots, f_n, \ldots$ be functions in $\mathcal{X}$. Prove that the sequence $\{f_n\}$ converges to f in the topology $\mathcal{T}$ if and only if $\lim_{n\to\infty} f_n(t) = f(t)$ for all $t \in [0,1]$.

e) For every subset $A \subseteq [0,1]$, let $\chi_A \in \mathcal{X}$ denote the characteristic function of A. That is $\chi_A(t) = 1$ if $t \in A$ and $\chi_A = 0$ for $t \in [0,1] \setminus A$. Consider the set

$$\mathcal{H} = \{\chi_A \mid A \subseteq [0,1] \text{ is finite}\} \subset \mathcal{X}.$$

Prove that $\chi_{[0,1]} \in cl(\mathcal{H})$, where $cl(\mathcal{H})$ denotes the closure of $\mathcal{H}$ with respect to the topology $\mathcal{T}$.

f) In the notation of part e), prove or disprove the following statement: there exists a sequence $\{f_n\}$ in $\mathcal{H}$ that converges to $\chi_{[0,1]}$ in the topology $\mathcal{T}$.

8) a) State the Residue Theorem and the corresponding formula which allows one to compute Cauchy integrals by using residues.

b) Evaluate $\frac{1}{2\pi i}\int_\gamma \frac{e^z}{z^2(1-z^2)}dz$ where γ is the curve in the diagram below.

9) a) A map $\phi : (X, d_X) \to (Y, d_Y)$ is called an isometry if $d_Y(\phi(x_1), \phi(x_2)) = d_X(x_1, x_2)$. Determine all possible isometries $\phi : \mathbb{R} \to \mathbb{R}$.

b) Determine all possible isometries $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ and show that each such map is surjective.

c) Does there exist an isometry $\phi : \mathbb{R}^2 \to \mathbb{R}$? Explain.

10) a) Give the definition of a σ-algebra Ω. Give the precise definition of a measure μ defined on Ω .

b) Let $E \subset \mathbb{R}$ be such that $m(E) < \infty$ where m denotes the usual Lebesgue measure on $\mathbb{R}$. Show that the map $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(x) = m((x + E) \bigcap E)$$

is continuous.

c) Show that if $E \subset \mathbb{R}$ is such that $m(E) > 0$, then $E - E$ has nonempty interior.

d) Find all of the subgroups H of $\mathbb{R}$ with $m(H) > 0$.