Department of Pure Mathematics

Analysis and Topology Comprehensive Examination

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Read the instructions carefully.

Part A Answer question 1 and **any** 3 of questions 2 to 6.

1. The Cantor ternary set consists of the real numbers $\sum_{j=1}^{\infty} \epsilon_j 3^{-j}$ where $\epsilon_j \in \{0, 2\}$, $j \geq 1$.

   (a) Show that the Cantor set is closed, nowhere dense and that every point is an accumulation point.

   (b) Find the cardinality of the Cantor set.

   (c) Show that the Lebesgue measure of the Cantor set is zero.

   (d) Show that the Cantor set is totally disconnected.

   (e) Prove that the Cantor set is homeomorphic to the space $\prod_1^{\infty}\{0, 1\}$ with the product topology.

2. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous real-valued functions on a complete metric space X which converge pointwise. Prove there exists a constant M and a non–empty open set $U \subseteq X$ such that $\sup_n |f_n(x)| \leq M$ for all $x \in U$.

3. For $f : \mathbb{R} \to \mathbb{R}$ and $t \in \mathbb{R}$, define $f_t : \mathbb{R} \to \mathbb{R}$ via $f_t(x) = f(x + t)$.

   (a) Suppose $f \in L^1(\mathbb{R}, dx)$, where dx denotes Lebesgue measure on $\mathbb{R}$. Prove that

   $$\lim_{t \to 0} \|f_t - f\|_1 = 0\,.$$

   (b) Suppose $g \in L^{\infty}(\mathbb{R}, dx)$. Is it true that $\lim_{t \to 0} \|g_t - g\|_{\infty} = 0$?

4. (a) Define what it means for a function $f : [0, 1] \to \mathbb{R}$ to be *absolutely continuous.*

   (b) Define what it means for a function $f : [0, 1] \to \mathbb{R}$ to be of *bounded variation.*

   (c) Prove that if $f : [0, 1] \to \mathbb{R}$ is absolutely continuous and $E \subseteq [0, 1]$ has Lebesgue measure 0, then $f(E) := \{f(x) : x \in E\}$ also has Lebesgue measure 0.

   (d) Give an example of a function of bounded variation which is not absolutely continuous.

5. Suppose $f : [0, 2\pi] \to \mathbb{C}$ satisfies $f(0) = f(2\pi)$ and $\|f\|_1 \equiv \frac{1}{2\pi}\int_0^{2\pi} |f(x)|\,dx < \infty$.

   The Fourier coefficients of f are given by $\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} f(x)\,dx$, $n \in \mathbb{Z}$.

   (a) If f is differentiable and $\|f'\|_1 \equiv \frac{1}{2\pi}\int_0^{2\pi} |f'(x)|dx < \infty$ prove that $|\hat{f}(n)| \leq \frac{\|f'\|_1}{|n|}$ for all $n \neq 0$.

   (b) If $\sum_{n=-\infty}^{\infty} |\hat{f}(n)n| < \infty$ prove that f is continuously differentiable.

6. Fix $1 < p < \infty$. Let $\{\mathbf{x}_n\}_{n=1}^{\infty}$ denote a sequence in $\ell^p(\mathbb{N})$ (the space of real-valued p–summable sequences), where for each $n \geq 1$, $\mathbf{x}_n = (x_n(i))_{i=1}^{\infty}$. We say that $\{\mathbf{x}_n\}_{n=1}^{\infty}$ *converges weakly* to $\mathbf{x} = (x(i))_{i=1}^{\infty} \in \ell^p(\mathbb{N})$ if for every $\varphi \in (\ell^p(N))^*$, the dual of $\ell^p(\mathbb{N})$, we have

$$\lim_{n\to\infty} \varphi(\mathbf{x}_n) = \varphi(\mathbf{x})\,.$$

   Prove that the following are equivalent:

   (a) $\{\mathbf{x}_n\}_{n=1}^{\infty}$ converges weakly to $\mathbf{x}$;

   (b) $\sup_{n\geq 1} \|\mathbf{x}_n\|_p < \infty$ and for each $i \geq 1, \lim_{n\to\infty} x_n(i) = x(i)$.

Part B Answer **any** 3 of questions 7 to 10.

7. Evaluate the integral $\int_0^{\infty} \frac{\sin x}{x}\,dx$, and justify your methods.

8. (a) How many zeros does the function defined by $p(z) = z^4 - 5z + 1$ have in the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$?

   (b) Suppose that $\mathcal{C}$ denotes a family of closed rectifiable curves, and that for each $C \in \mathcal{C}, C$ does not pass through any of the points $0, -1$ and 1. Suppose, furthermore, that if $C \in \mathcal{C}$ and $\lambda \in \{-1, 0, 1\}$, then the winding number of C around λ is either 0 or 1.

   Calculate all possible values of the integral $\int_C \frac{dz}{z(z^2-1)}$ as C ranges over $\mathcal{C}$.

9. Prove Schwarz's Lemma: Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and suppose that $f : \mathbb{D} \to \mathbb{C}$ is analytic, with $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. Then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.

10. (a) Let D be an open connected subset of $\mathbb{C}$ and let $(f_n)_{n=1}^{\infty}$ be a sequence of functions which are analytic and never vanishing on D. If the sequence $\{f_n\}$ converges to a function f uniformly on compact subsets of D, prove that either f has no zeros in D or f is identically zero on D.

    (b) Give an example where the limit function f is identically zero.

Part C Answer **any** 2 of questions 11 to 13.

11. (a) Prove that if $\kappa \geq \aleph_0$ is a cardinal number, then $\kappa \cdot \kappa = \kappa$.

    (b) Find the cardinality of the set $\mathcal{C}([0,1],\mathbb{R})$ of continuous, real-valued functions on the interval $[0,1]$.

    (c) Is there a topological version of the Schröder–Bernstein Theorem, namely: suppose that X and Y are topological spaces and that $f : X \to Y$ and $g : Y \to X$ are continuous injections. Does there exist a homeomorphism $h : X \to Y$?

12. Let $X = \{(x,0) : x \in [-1,1]\} \cup \{(0,y) : y \in [-1,1]\}$ and let $Y = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

    (a) Exhibit an explicit bijection f between X and Y.

    (b) Is it possible for a bijection between X and Y to be continuous?

13. Suppose that $\{X_\alpha\}_{\alpha\in\Lambda}$ is a family of Hausdorff topological spaces, and that each X_α contains at least two points. Suppose also that $\prod\limits_{\alpha\in\Lambda} X_\alpha$ carries the product topology.

    Prove that the following are equivalent:

    (a) $\Pi_{\alpha\in\Lambda} X_\alpha$ is separable - i.e. it contains a countable dense subset.

    (b) Each X_α is separable and there are at most $\mathfrak{c}$ factors, where $\mathfrak{c}$ denotes the cardinality of the set of real numbers.