Department of Pure Mathematics

Analysis Comprehensive Examination

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Attempt all questions below.

1. (a) Let $f : (0, 1) \to \mathbb{R}$ be a uniformly continuous function. Must f be bounded? Give a proof or a counter-example.

   (b) Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function, with the property that $\lim_{t\to\infty} f(t) = 1$. Must f be uniformly continuous? Give a proof or a counter-example.

2. Let $C(\ [0, 1], \mathbb{R}\ )$ denote the space of real-valued continuous functions on $[0, 1]$, and consider the subset $\mathcal{B} \subset C(\ [0, 1], \mathbb{R}\ )$ defined as follows:

$$\mathcal{B} \ = \ \left\{ p \ : \ \begin{array}{l} p \text{ polynomial function with real coefficients,} \\ \text{such that degree}(p) \text{ is not equal to 3,4, or 5} \end{array} \right\}.$$

   (a) Does the set $\mathcal{B}$ satisfy the hypotheses of the Stone-Weierstrass theorem? Justify your answer.

   (b) Is it true that every function in $C(\ [0, 1], \mathbb{R}\ )$ can be written as the limit of a uniformly convergent sequence of functions from $\mathcal{B}$? Justify your answer (proof or counter-example).

3. (a) State Zorn's Lemma.

   (b) In this part of the problem X is a fixed topological space. You are told that the space X was studied by the mathematician Tasty, who put into evidence a family $\mathcal{T}$ of subsets of X, called *Tasty sets*. It is known that:

      (i) Every Tasty set $T \in \mathcal{T}$ is a closed subset of X.

      (ii) The empty set $\emptyset$ is not a Tasty set.

      (iii) If $(T_i)_{i\in I}$ is a family of Tasty sets such that $\cap_{i\in I} T_i \neq \emptyset$, then $\cap_{i\in I} T_i$ is a Tasty set.

      (iv) If $K \subset X$ is a Tasty set which is compact and is not a single point, then K has a proper Tasty subset (i.e. $\exists\ H \in \mathcal{T}$ such that $H \subset K$, $H \neq K$).

   We say that $x \in X$ is a *Tasty element* if the one-element set $\{x\}$ is a Tasty set. Prove the following statement:

   "If K is a Tasty compact subset of X, then K contains a Tasty element (i.e. $\exists\ x \in K$ such that $\{x\} \in \mathcal{T}$".

4. Let $h : [0,\infty) \to \mathbb{R}$ be a continuous function. For every $n \geq 1$, define $f_n : [0,1] \to \mathbb{R}$ by the formula $f_n(t) = h(nt)$, $0 \leq t \leq 1$. Suppose that the functions $(f_n)_{n=1}^{\infty}$ form a uniformly equicontinuous family. What conclusion can you draw about h?

5. Calculate $\int_0^{\infty} \frac{\cos t}{t^2+1}\, dt$.

6. In this problem f is a holomorphic function defined on the entire complex plane $\mathbb{C}$. You are given that there exist a non-negative integer n and positive real numbers R, M_1, M_2 such that:
$$M_2|z|^n \ \leq\ |f(z)|\ \leq M_1|z|^n, \quad \forall\ z \in \mathbb{C} \text{ with } |z| \geq R.$$
Prove that $f(z)$ is a polynomial of degree exactly n.

7. Show that $\prod_{n=1}^{\infty}(1 + z^{n^2})$ converges for $|z| < 1$ to a function $f(z)$. Prove that the function f is analytic in the open unit disc.

8. For any complex number z let $\mathrm{Re}(z)$ denote the real part of z and let $\mathrm{Im}(z)$ denote the imaginary part of z. Put
$$T = \{z \in \mathbb{C} \ : \ |\mathrm{Re}(z)| + |\mathrm{Im}(z)| < 10\}, \quad U = \{z \in \mathbb{C} \ : \ |z| < 1\}.$$
   (a) Prove that there exists an analytic isomorphism f from T to U, such that $f(2 + 3i) = 0$.
   (b) Prove that there does not exist an analytic isomorphism from T to $\mathbb{C}$.

9. In this problem μ denotes the Lebesgue measure on the interval $[0,1]$, $(I_n)_{n=1}^{\infty}$ is a sequence of closed subintervals of $[0,1]$, and χ_n denotes the characteristic function of the interval I_n, for $n \geq 1$.
   (a) Suppose that $\mu(I_n) \leq 1/n^2$, $\forall\ n \geq 1$. Prove that $\chi_n \longrightarrow 0$ μ–almost everywhere when $n \to \infty$.
   (b) Suppose you are only given that $\mu(I_n) \leq 1/n$, $\forall\ n \geq 1$. Must it still be true that $\chi_n \longrightarrow 0$ μ–almost everywhere, when $n \to \infty$? Justify your answer (proof or counterexample).

10. Let X be a Banach space over $\mathbb{C}$; for every $a \in X$ and $r > 0$ we denote
$$\overline{B}(a;r) := \{x \in X \ : \ ||x - a|| \leq r\}.$$
Suppose that $(a_n)_{n=1}^{\infty}$ are elements of X, and $r_1 > r_2 > \cdots > r_n > \cdots$ are positive numbers such that:
$$\overline{B}(a_{n+1}; r_{n+1}) \subset \overline{B}(a_n; r_n), \quad \forall\ n \geq 1.$$
   (a) In addition to the above, suppose moreover that $\lim_{n\to\infty} r_n = 0$. Prove that $\cap_{n=1}^{\infty}\overline{B}(a_n;r_n) \neq \emptyset$.
   (b) If the hypothesis that $\lim_{n\to\infty} r_n = 0$ is removed, must it still be true that $\cap_{n=1}^{\infty}\overline{B}(a_n;r_n) \neq \emptyset$? Justify your answer (proof or counterexample).