DEPARTMENT OF PURE MATHEMATICS
ANALYSIS & TOPOLOGY COMPREHENSIVE EXAM
MAY 2003

EXAMINERS: BRIAN FORREST AND DOUG PARK

N.B. The exam has 11 problems and is out of 100 points.

Set Theory and Topology

1. [**8 points**] Let $\Im_{\mathbb{R}}$ be the standard topology on $\mathbb{R}$.

a) Let $\Gamma = \{\emptyset\} \bigcup \{(a, b) \mid a \in \mathbb{R} \cup \{-\infty\} \text{ and } b \in \mathbb{R} \cup \{\infty\},\ a < b\}$ be the collection of open intervals in $\mathbb{R}$. Show that if $U \in \Im_{\mathbb{R}}$, then U is the union of at most countably many pairwise disjoint open intervals in Γ.

b) What is the cardinality $|\Im_{\mathbb{R}}|$?

2. [**10 points**] A map $f : X \to Y$ is said to be *proper* if for every compact subset $K \subset Y$, the inverse image $f^{-1}(K)$ is compact.

a) Suppose X is a compact space and Y is Hausdorff. Show that every continuous map $f : X \to Y$ is proper.

b) Give an example of a continuous map which is not proper.

c) Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is a proper continuous map. Show that f is a *closed* map, i.e. $f(C)$ is closed in $\mathbb{R}^n$ whenever C is a closed subset of $\mathbb{R}^m$.

3. [**5 points**] Show that if X is infinite, then you can find two disjoint subsets S and T such that $S \cup T = X$ and $|S| = |T| = |X|$.

4. [**5 points**] An accurate map of Ontario is laid out flat on a table in a classroom at the University of Waterloo. Prove that there is exactly one point on the map lying directly over the point which it represents.

Complex Analysis

5. [8 points] a) Evaluate

$$I = \int_{|z|=1} \frac{\cos^3 z}{z^3}\, dz\,,$$

where the direction of integration is counterclockwise.

b) Find the terms of order ≤ 3 in the power series expansion of the function $f(z) = z^2/(z-2)$ at $z = 1$.

6. [10 points] a) Find a fractional linear transformation that maps the upper half plane into the unit circle in such a way that $z = i$ is mapped to 0 and the point at ∞ is mapped to -1.

b) Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Prove that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

7. [10 points] a) Let

$$\sum_{n=0}^{\infty} a_n z^n$$

be a complex series with the radius of convergence $0 < R \leq \infty$. Let $0 < r < R$. Show that the series converges uniformly to some function $f(z)$ on the disk $D(0,r) = \{z \in \mathbb{C} : |z| \leq r\}$.

b) Let $A \subset \mathbb{C}$ be open. Let $\{f_n\}$ be a sequence of analytic functions defined on A. Assume also that f_n converges uniformly to f on every closed disk D contained in A. Prove that f is analytic on A, and that $f_n' \to f'$ pointwise on A and uniformly on each closed disk in A.

Real Analysis

8. [10 points] Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be normed linear spaces. Let $T : X \to Y$ be linear. Let

$$\|T\| = \sup\{ \|Tx\|_Y \ : \ \|x\|_X \leq 1 \}.$$

We say that T is bounded if $\|T\| < \infty$.

a) Prove that the following are equivalent:

i) T is bounded.
ii) T is continuous.

b) Let $1 \leq p \leq \infty$ and let $\frac{1}{p} + \frac{1}{q} = 1$ (if $p = 1$ then $q = \infty$ and if $p = \infty$, then $q = 1$). For each $f \in L^q[0,1]$, define $\Phi_f : L^p[0,1] \to \mathbb{R}$ by

$$\Phi_f(g) = \int_{[0,1]} f(x)g(x)dx$$

Show that $\|\Phi_f\| \leq \|f\|_q$.

c) Show in fact that $\|\Phi_f\| = \|f\|_q$ for $1 < p < \infty$.

9. [12 points] Let $f(x)$ be a function defined on $[0,1]$. Let $E_n = \{x \in [0,1] \mid \text{for every } \delta > 0 \text{ there exist } y, z \in [0,1] \text{ with } |x - y| < \delta \text{ and } |x - z| < \delta \text{ but } |f(y) - f(z)| \geq \frac{1}{n} \}$.

Let

$$E = \bigcup_{n=1}^{\infty} E_n.$$

a) Show that E is the set of points of discontinuity of $f(x)$.

b) Show that each E_n is closed and hence measurable.

c) Show that if f is Riemann integrable over $[0,1]$, then $m(E_n) = 0$ for every $n \in \mathbb{N}$. In particular, show that $f(x)$ is continuous except on a set of measure 0.

Real Analysis Continued

10. [10 points] **a)** Let (X, d) be a metric space. Let $\{f_n(x)\}$ be a sequence of continuous real valued functions on X that converges uniformly to $f(x)$ on X. Show that $f(x)$ is also continuous.

b) Dini's Theorem: Let (X, d) be a compact metric space. Let $\{f_n(x)\}$ be a sequence of continuous functions on X such that $f_n(x) \leq f_{n+1}(x)$ for each $n \in \mathbb{N}$ and $f(x) = \lim_{n\to\infty} f_n(x)$. Show that $f(x)$ is continuous on X if and only if the sequence converges uniformly.

c) Show that Dini's Theorem fails on $[0, \infty)$ by giving a sequence $\{f_n(x)\}$ of continuous functions on $[0, \infty)$ such that $f_n(x) \leq f_{n+1}(x)$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} f_n(x) = 1$ for each x but for which the convergence is not uniform.

11. [12 points] **a)** Show that $\text{Span}\{1, x^7, x^8, x^9, \cdots\}$ is dense in $(C[0,1], \|\cdot\|_\infty)$, where $C[0,1] = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous on } [0,1]\}$, and $\|f\|_\infty = \max\{|f(x)| : x \in [0,1]\}$.

b) Let I be a closed ideal in $(C[0,1], \|\cdot\|_\infty)$ such that for each $x \in [0,1]$ there exists $f \in I$ with $f(x) \neq 0$. Show that $I = C[0,1]$.

c) Let $f \in C[0,1]$ be such that $f(0) = 0$ and

$$\int_0^1 f(x) x^k dx = 0 \quad \text{for each } k \geq 7.$$

Prove or disprove that $f = 0$.