Analysis Comprehensive Exam

Do EIGHT problems including at least TWO from the Complex Analysis section.

Real analysis and topology.

1. Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and
$$\int_0^1 f(x)\phi'(x)dx = 0$$
for all differentiable functions $\phi : [0, 1] \to \mathbb{R}$ such that $\phi(0) = 0$. Prove $f = 0$.

2. Let (M, d) be a compact metric space and $\phi : M \to M$ be such that
$$d(\phi(x), \phi(y)) < d(x, y)$$
for all $x, y \in M$, $x \neq y$.
(a) Show that ϕ has a unique fixed point.
(b) Is the statement of (a) still true if M not compact? Justify your answer by giving a proof or a counterexample.

3. Let A be the subset of $C[0, 1]$ that consists of functions $f \in C[0, 1]$ such that $f(0) = 0$ and
$$|f(x) - f(y)| \leq \sqrt{|x - y|},$$
for all $x, y \in [0, 1]$. Show that A is a compact subset of $C[0, 1]$.

4. (a) Show that neither $L^1(\mathbb{R}) \subset L^2(\mathbb{R})$ nor $L^2(\mathbb{R}) \subset L^1(\mathbb{R})$.
(b) Assume $1 \leq r \leq p \leq s < \infty$ and $g \in L^r(\mathbb{R}) \cap L^s(\mathbb{R})$. Prove that $g \in L^p(\mathbb{R})$.

5. Prove the following special case of the Baire category theorem: Let X be a compact Hausdorff space. Show that if $\{A_n\}$ is a countable collection of closed sets in X, each of which has empty interior in X, then there is a point of X which is not in any set A_n.

6. (a) Prove that a totally bounded metric space is separable.

   (b) Prove that the unit ball in l^∞ is not compact.

7. Evaluate $\lim_{n\to\infty} \int_0^n (1+\frac{x}{n})^n e^{-2x} dx$.

8. Let $F : \mathbb{R} \to \mathbb{R}$. Prove that $\{x : \limsup_{y\to x} F(y) > \limsup_{y\to x^+} F(y)\}$ is countable.

Complex analysis.

9. Let f be analytic in an open set containing the closed unit disc. Suppose that $|f(z)| > 2$ for $|z| = 1$ and $|f(0)| < 2$. Prove that f has at least one zero in the open unit disc $|z| < 1$.

10. Suppose f is an entire function, and

$$|f(z)| \leq |z|^n$$

for all z. Prove that f must be a polynomial of degree at most n.

11. Use contour integration to evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{x^2+1} dx.$$