Department of Pure Mathematics

Analysis Comprehensive Examination

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DO EIGHT questions, at least ONE from each of the FOUR sections.

Real Analysis

1. Fix $p \in \mathbb{R}$. Let $f_n(x) = n^p x(1-x^2)^n$ for $0 \le x \le 1$.
   For which values of p does this converge (i) pointwise? (ii) uniformly?

2. Consider $\mathcal{F} = \left\{ F(x) = \int_0^x f(t)\,dt : f \in C[0,1],\ ||f||_\infty \le 1 \right\}$ as a subset of $C[0,1]$ with the sup norm.
   (a) Show that $\mathcal{F}$ is not closed. Describe the closure $\overline{\mathcal{F}}$.
   Hint: h has Lipschitz constant C if $|h(x) - h(y)| \le C|x-y|$ for all x, y.
   (b) Is $\overline{\mathcal{F}}$ compact?

3. (a) If $f \in C^n[a,b]$ has $n \ge 1$ continuous derivatives, prove that there is a sequence p_k of polynomials such that $p_k^{(i)}$ converges uniformly to $f^{(i)}$ for $0 \le i \le n$.
   (b) Let $\mathbb{P}_k$ denote the vector space of polynomials of degree at most k.
   If $f \in C[a,b]$, prove that there exists a polynomial $p \in \mathbb{P}_k$ which is closest in the sup norm on $[a,b]$ among all polynomials in $\mathbb{P}_k$.

Complex Variables

4. (a) Show that there is no function f which is analytic in a neighbourhood of z_0 such that $f^{(n)}(z_0) = (n!)^2$ for all $n \ge 0$.
   (b) Let f be an entire function such that $\lim_{z\to\infty} \dfrac{f(z)}{z} = 0$. Show that f is a constant.

5. (a) Evaluate $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2} dx$. Hint: $2\sin^2 x = 1 - \cos 2x$.
   (b) Define a polynomial $p(z) = z^{65} + 1234z^{55} + 987654321z^4 + 10001$.
   Determine the number of zeros of p in the disk $\{z \in \mathbb{C} : |z| \le 2\}$.

6. Find an explicit conformal map of $\mathbb{D}\backslash(-1,0] = \left\{ z \in \mathbb{C} : |z| < 1,\ z \notin (-1,0] \right\}$ onto the unit disk $\mathbb{D}$.

Topology and Set Theory

7. (a) Let A and B be subsets of a topological space X. If B is open, prove that $B \cap A = \emptyset$ if and only if $B \cap \overline{A} = \emptyset$.

   (b) Let X be a topological space. Show that $G \subseteq X$ is open if and only if $\overline{G \cap \overline{A}} = \overline{G \cap A}$ for every $A \subseteq X$.

   (c) Let A and B be dense subsets of X.
   Prove that if A and B are open, then $A \cap B$ is dense.
   Is this still true if neither A nor B is open? (Prove or disprove.)

8. Let ω_1 be the first uncountable ordinal. The *long line* is defined as $X = [0,1) \times \omega_1$ with the topology generated by the sets

$$[0,b) \times \{0\}, \quad (a,b) \times \{\alpha\} \qquad 0 \leq a < b \leq 1 \text{ and } \alpha \in \omega_1$$
$$(a,1)\times\{\alpha\} \cup [0,1)\times(\alpha,\beta) \cup [0,b)\times\{\beta\} \qquad a,b \in [0,1] \text{ and } \alpha < \beta \in \omega_1.$$

   (a) Prove that X is not compact.

   (b) If μ is a Borel probability measure supported on X, prove that $\overline{\mathrm{supp}(\mu)}$ is compact.

More Real Analysis

9. Let $f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \pi \\ -1 & \text{if } -\pi < x < 0 \end{cases}$

   (a) Find the Fourier series of f.

   (b) Explain as much as you know about the convergence of this Fourier series. Consider (i) pointwise convergence, (ii) uniform convergence on subsets, (iii) convergence in $L^2(-\pi,\pi)$ and (iv) convergence in $L^1(-\pi,\pi)$.

10. (a) Let $f_n, f \in L^1(0,1)$ such that f_n converges to f almost everywhere.
    Prove that f_n converges to f in $L^1(0,1)$ if and only if $\lim_{n\to\infty} ||f_n||_1 = ||f||_1$.

    (b) Conversely, if f_n converges to f in $L^1(0,1)$, does f_n converge to f almost everywhere? (Prove or disprove.)