University of Waterloo
Department of Pure Mathematics
Analysis and Topology Comprehensive Examination
February, 2009

Instructions:

1. There are four parts to the Exam, labeled Part A, Part B, Part C and Part D. You are required to answer a *minimum of one question for each part*, and *a total of 8 questions.*
2. No books, notes, calculators are permitted.
3. Good Luck!

Part A. Set theory and topology

Question A1.
Let X be a non-empty set, and denote by $\mathcal{P}(X)$ the *power set* of X. That is, $\mathcal{P}(X) = \{A : A \subseteq X\}$. The cardinality of X is denoted by $|X|$. Finally, $\aleph_0$ denotes the cardinality of the set $\mathbb{N}$ of natural numbers, while $\mathfrak{c}$ denotes the cardinality of the set $\mathbb{R}$ of real numbers.

(a) Prove that $|X| < |\mathcal{P}(X)|$.

(b) Find a cardinal number α so that $|\mathcal{C}([0,1],\mathbb{R})| = 2^\alpha$, where $\mathcal{C}([0,1],\mathbb{R}) = \{f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}$.

(c) A function $\varphi : \mathcal{P}(X) \to \mathcal{P}(X)$ is said to be *increasing* if $A, B \in \mathcal{P}(X)$ and $A \subseteq B$ implies $\varphi(A) \subseteq \varphi(B)$. Prove that if $\varphi : \mathcal{P}(X) \to \mathcal{P}(X)$ is increasing, then there exists a set $T \in \mathcal{P}(X)$ so that $\varphi(T) = T$.

Question A2.

(a) Let $\{X_\lambda\}_{\lambda\in\Lambda}$ denote a family of non-empty connected topological spaces. Let $X = \prod_{\lambda\in\Lambda} X_\lambda$, equipped with the product topology. Prove that X is connected.

(b) Recall that if $(X_n, \mathcal{T}_n)$ is a non-empty topological space for each $n \geq 1$, then the *box topology* on $X = \prod_{n\in\mathbb{N}} X_n$ has as a base all sets of the form $\prod_{n\in\mathbb{N}} U_n$ where $U_n \in \mathcal{T}_n$, $n \geq 1$. Show that the countable product $\mathbb{R}^\mathbb{N}$, equipped with the box topology, is not connected.

Part B. Measure theory

Question B1. Let $f : [0, 1] \to \mathbb{R}$ be a function.

(a) Define what it means to say that

(i) f is *absolutely continuous*, and

(ii) f is *of bounded variation.*

(b) Prove that if f is absolutely continuous, then f is of bounded variation.

(c) Prove that if $f : [0, 1] \to \mathbb{R}$ is absolutely continuous and $E \subseteq [0, 1]$ has Lebesgue measure zero, then $f(E) \subseteq \mathbb{R}$ has Lebesgue measure zero.

Question B2.

(a) Suppose that $f_n : [0, 1] \to \mathbb{R}$ and $f : [0, 1] \to \mathbb{R}$ are Lebesgue integrable and $\lim_{n\to\infty} ||f_n - f||_1 = 0$. Show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $A \subset [0, 1]$ with $m(A) < \delta$, $\int_A |f_n| < \varepsilon$ for all n.

Note: You *may assume without proof* that given a *single* Lebesgue integrable function $g : [0, 1] \to \mathbb{R}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $A \subset [0, 1]$ with $m(A) < \delta$, $\int_A |g| < \varepsilon$.

(b) Is the statement in (a) true when the assumption $\lim_{n\to\infty} ||f_n - f||_1 = 0$ is replaced by $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in [0, 1]$? Justify your answer.

Question B3. A function $f : \mathbb{R} \to \mathbb{R}$ is called additive if it satisfies

$$f(x + y) = f(x) + f(y) \quad (\forall x, y \in \mathbb{R}).$$

A function $f : \mathbb{R} \to \mathbb{R}$ is called locally Lebesgue integrable if it is Lebesgue integrable over every finite interval.

(a) Show that a locally Lebesgue integrable additive function f must be linear, i.e.,

$$f(x) = cx \quad (\forall x \in \mathbb{R})$$

for some real constant c.

(b) Assuming the fact stated in (a), prove that there are additive functions which are not locally Lebesgue integrable. [**Hint:** you may assume the Axiom of Choice.]

Part C. Complex analysis

Question C1. Find a conformal map taking the set

$$A := \{z \in \mathbb{C} : 0 < \arg z < \frac{\pi}{2},\ 0 < |z| < 1\}$$

onto the set $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

Question C2.

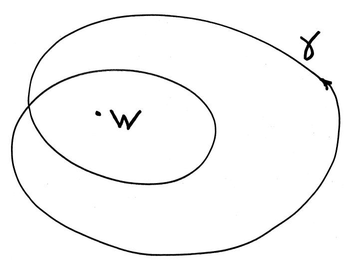
(a) State Rouché's Theorem.

(b) Show that the equation

$$ze^{a-z} = 1, \quad a > 1$$

has exactly one root in the open unit disc $|z| < 1$.

(c) Suppose that f is entire and that the image of the unit circle $e^{i\theta}$, $\theta \in [0, 2\pi]$, under f is the following curve γ - (it is assumed that the curve γ is traced only once):



For given w whose relative position to γ is as indicated in the above figure, determine the number of solutions to the equation

$$f(z) = w$$

for z in the open unit disc.

Question C3.

(a) Evaluate the residue of the function

$$\frac{\pi \cot(\pi z)}{(z - \frac{1}{2})^2}$$

at the point $z = \frac{1}{2}$.

(b) By using part (a) or by any other means, show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n - \frac{1}{2})^2} = \pi^2.$$

PART D. REAL ANALYSIS - FUNCTIONAL ANALYSIS

Question D1. Let $f : [0,1] \to \mathbb{R}$ be a function. For $z \in [0,1]$ and $r > 0$, we define the *oscillation* of f on $(z-r, z+r) \cap [0,1]$ to be

$$\operatorname{osc}[f, z, r] = \sup\{|f(x) - f(y)| : x, y \in (z-r, z+r) \cap [0,1]\}.$$

(a) Prove that f is continuous at $z \in [0,1]$ if and only if

$$\lim_{r \to 0} \operatorname{osc}[f, z, r] = 0.$$

(b) Let $g : [0,1] \to \mathbb{R}$ be an arbitrary function. Prove that the set of points at which g is continuous is a G_δ set.

(c) Prove that there is no function $h : [0,1] \to \mathbb{R}$ that is continuous *precisely* on the set of rational numbers in $[0,1]$.

Question D2. Let $\mathfrak{X}$ be a normed linear space and $\mathfrak{M}$ be a closed subspace of $\mathfrak{X}$.

(a) Prove that if $\mathfrak{X}$ and $\mathfrak{M}$ are complete, then so is $\mathfrak{X}/\mathfrak{M}$.

(b) Prove that if $\mathfrak{M}$ and $\mathfrak{X}/\mathfrak{M}$ are complete, then so is $\mathfrak{X}$.

Question D3. Recall that ℓ^2 denotes the Hilbert space of square summable sequences of real numbers. Recall also that a net $(\mathbf{x}_\lambda)_{\lambda \in \Lambda}$ of vectors in ℓ^2 is said to *converge in the weak topology* to the vector $\mathbf{x} \in \ell^2$ if for each $\mathbf{y} \in \ell^2$, $\lim_{\lambda \in \Lambda} \langle \mathbf{x}_\lambda, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$.

(a) Let $(\mathbf{x}_n)_{n=1}^\infty$ be a sequence in ℓ^2 converging in the weak topology to $\mathbf{x} \in \ell^2$. Prove that if $\|\mathbf{x}_n\|_2$ converges to $\|\mathbf{x}\|_2$ as n tends to ∞, then $\|\mathbf{x}_n - \mathbf{x}\|_2 \to 0$.

(b) Prove that if a sequence $(\mathbf{x}_n)_{n=1}^\infty$ of vectors in ℓ^2 converges in the weak topology to $\mathbf{x} \in \ell^2$, then $(\mathbf{x}_n)_{n=1}^\infty$ is bounded.

(c) Let $\{\mathbf{e}_n\}_{n=1}^\infty$ denote the standard basis of ℓ^2 and let

$$A = \{\mathbf{e}_m + m\,\mathbf{e}_n : 1 \leq m < n\}.$$

Prove that 0 is in the closure (in the weak topology) of A, but that no sequence in A converges in the weak topology to 0.