Analysis Comprehensive Exam January, 2005.

Do SIX problems including AT LEAST ONE from each section.

Set Theory and Topology

1. Show that any σ–algebra containing infinitely many distinct sets must be uncountable. Hint: first find countably many disjoint sets.

2. Prove that the "middle thirds" Cantor set is homeomorphic to $\prod_{i=1}^{\infty} P_i$ where each P_i is a 2 element set with the discrete topology, and the product is given the product topology.

Real Analysis

3. Let $\mathcal{P}_n$ denote the set of polynomials of degree at most n. Prove that if $f \in C_{\mathbb{R}}[0,1]$ (the continuous real valued functions on $[0,1]$), then there is a polynomial $p_0 \in \mathcal{P}_n$ so that $\|f - p_0\|_\infty \le \|f - p\|_\infty$ for all $p \in \mathcal{P}_n$, where $\|g\|_\infty = \sup_{0 \le x \le 1} |g(x)|$.

4. Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$. State clearly any convergence results which are used. Hint: begin with the Taylor series for $f(x) = \frac{1}{1+x^2}$ about $x = 0$ and integrate.

Measure Theory

5. Let μ be a finite regular Borel measure on $\mathbb{R}^n$; and let $V \subset \mathbb{R}^n$ be open. Define $f(x) = \mu(V + x)$.
   (a) Show by example that f need not be continuous.
   (b) Prove that f is upper semi-continuous; i.e. $f^{-1}(r, \infty)$ is open for all $r \in \mathbb{R}$.
   (c) Prove that if μ is Lebesgue measure on the unit ball, then f is continuous.

6. Let $f \in L^p(0, 2\pi)$ and $g \in L^1(0, 2\pi)$, where $1 \le p < \infty$. Considering f and g as 2π-periodic functions, one can define the convolution
$$f * g(t) = \frac{1}{2\pi} \int_0^{2\pi} f(t-x) g(x)\, dx.$$
Prove that $f * g$ belongs to $L^p(0, 2\pi)$ and that $\|f * g\|_p \le \|f\|_p \, \|g\|_1$. Be careful to point out where results from measure theory are used.

Complex Analysis

7. Show that $f(z) = z - e^{-z}$ takes the value π exactly once in the right half plane $H = \{z \in \mathbb{C} : \mathrm{Re}\, z \ge 0\}$ and that the solution is real.

8. Evaluate $\displaystyle\int_0^{\infty} \frac{x^2\, dx}{1 + x^6}$.