University of Waterloo
Department of Pure Mathematics

Analysis and Topology Comprehensive Examination

1:00 – 4:00 p.m., Wednesday, January 30, 2008

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Do all 10 questions.

1. a) Prove that if A is a compact subset of a metric space X and $f : X \to X$ is continuous then f is uniformly continuous on A.

   b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ and suppose that $f(x, y_0)$ is continuous as a function of x for each y_0 and $f(x_0, y)$ is continuous as a function of y for each x_0. Is f continuous? Prove or give a counterexample.

2. Let (X, d) be a complete metric space. Suppose that F_n is a non-empty closed subset of X for $n = 1, 2, \ldots$ and that $F_n \supseteq F_{n+1}$ for $n = 1, 2, \ldots$. Put $\gamma_n = \sup\{d(x, y) : x \text{ and } y \text{ in } F_n\}$ and suppose that $\lim_{n\to\infty} \gamma_n = 0$.

   a) Prove that $\bigcap_{n=1}^{\infty} F_n$ consists of a single point.

   b) Show that if we remove the condition that F_n be closed then $\bigcap_{n=1}^{\infty} F_n$ need not be a single point.

3. a) Give the definition of a partially ordered set S.

   b) Give the definition of a maximal element of a partially ordered set.

   c) State Zorn's Lemma.

   d) A set T of real numbers is said to be rationally linearly independent if whenever $r_1x_1 + \cdots + r_nx_n = 0$ with distinct $x_1, \ldots, x_n$ in T and $r_1, \ldots, r_n$ rational numbers then $r_1 = \cdots = r_n = 0$. Prove that there is a rationally linearly independent set S of real numbers with the property that each real number x can be expressed in the form $x = r_1x_1 + \cdots + r_mx_m$ with $x_1, \ldots, x_m$ from S and $r_1, \ldots, r_m$ rational numbers.

4. Let f be a continuous function on $[-1,1]$ which satisfies

$$\int_{-1}^{1} f(x)x^{2k}dx = 0$$

for $k = 0, 1, 2, \ldots$. Show that f is an odd function, in other words, show that $f(-x) = -f(x)$ for x in $[-1,1]$.

5. Construct, with a justification, a meromorphic function whose only poles are poles of order 1 at the points $\{m + ni \mid m, n \in \mathbb{Z}\}$.

6. Let $f \in L^1[0,1]$ and suppose that

$$\int_0^1 f(x)g(x)dx = 0,$$

for all measurable functions g on $[0,1]$. Show that $f = 0$ almost everywhere with respect to Lebesgue measure.

7. a) Let V be the vector space of continuous complex functions on $[0,1]$ with inner product

$$(f,g) = \int_0^1 f(t)\overline{g}(t)dt.$$

   Prove that V, $(\ ,\ )$ is not a Hilbert space.

   b) Let $X_1, X_2, \ldots$ be topological spaces. Define the Cartesian product topology on $\prod_{n=1}^{\infty} X_n$.

   c) State Tychonoff's product theorem.

8. a) For any complex number z let $\mathrm{Im}(z)$ denote the imaginary part of z. Find a holomorphic function $f : A \to D$ where $A = \{z \in \mathbb{C} \mid 0 < Im(z) < 1\}$ and $D = \{z \in \mathbb{C} \mid |z| < 1\}$.

   b) Prove that D and $\mathbb{C}$ are homeomorphic but that there is no holomorphic function f from C onto D.

9. Let $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y, z) = x + y + z - \sin xyz$.

   a) Show that there exists an open ball B in $\mathbb{R}^2$ around $(0, 0)$ and a continuous function $g : B \to \mathbb{R}$ such that $g(0, 0) = 0$ and $f(x, y, g(x, y)) = 0$.

   b) Determine the Jacobian of g at $(0, 0)$.

10. Let $f \in L^1(\mathbb{R})$, and continuous. Recall the Poisson summation formula which states that, under certain conditions,

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n),$$

where $\hat{f}(n) = \int_{-\infty}^{\infty} f(t) \exp(-2\pi i t n) dt$. Sufficient conditions are that the sum on the r.h.s. converge absolutely and that the sum

$$\sum_{n=-\infty}^{\infty} f(n + t)$$

converge uniformly for $0 \leq t \leq 1$. Use the Poisson summation formula to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1} = \frac{\pi}{2}\left(1 + \frac{2}{e^{2\pi} - 1}\right) - \frac{1}{2}.$$