Department of Pure Mathematics

Comprehensive Examination in Analysis and Topology—May 2004

In order to pass, it is important that you demonstrate sufficient knowledge in each of the areas: set theory/topology, real analysis, and complex analysis, covered by this exam.

Set theory and general topology

[3] 1. (a) State the Schroeder-Bernstein theorem, and then use it to show that the closed square $[-1,1]\times[-1,1]$ in $\mathbb{R}^2$ has the same cardinality as the open square $(-1,1)\times(-1,1)$.

[2] (b) Find an explicit bijection from the closed interval $[0,1]$ to the open interval $(0,1)$.

[4] (c) Let X, Y, Z be sets. If X and Y have equal cardinality, prove that the sets Z^X and Z^Y of functions from X to Z and from Y to Z, respectively, have equal cardinality. If Z^X and Z^Y have equal cardinality, must X and Y have equal cardinality? Explain.

[2] 2. (a) A subset A of a topological space X is called a *retract* of X provided there is a continuous mapping $f : X \to A$ whose restriction to A is the identity mapping on A. If X is Hausdorff and A is a retract of X, prove that A is closed in X.

[3] (b) Show that the disk $\{p \in \mathbb{R}^2 : \|p\| \leq 1\}$ is a retract of $\mathbb{R}^2$.

[2] (c) Is every closed set A in a metric space X a retract of X?

Real analysis

[4] 3. (a) Assuming the axiom of choice, prove that every Hilbert space $\mathcal{H}$ has a maximal orthonormal set.

[4] (b) If $\mathcal{H}$ is a Hilbert space with a countable orthonormal base, show that $\mathcal{H}$ is separable.

[3] (c) If $\mathcal{H}$ has an uncountable orthonormal base, show that $\mathcal{H}$ is not separable.

[4] 4. (a) If every open cover of a metric space X has a finite sub-cover, prove that every sequence in X has a convergent subsequence.

[3] (b) If $f : X \to Y$ is a continuous function between topological spaces and X is compact, show that the image $f(X)$ is compact.

[3] (c) If X is a closed and bounded subset of $\mathbb{R}^3$ and $f : X \to \mathbb{R}$ is continuous, prove that f attains a maximum value on X.

[3] (d) Let the space $C[0,1]$ of continuous real valued functions on the interval $[0,1]$ be given the norm

$$\|f\| = \max\{|f(x)| : x \in [0,1]\} \text{ for every } f \text{ in } C[0,1].$$

Is the set X of functions f in $C[0,1]$ such that $\|f\| \leq 1$ compact? Explain.

[7] 5. (a) Suppose that $\varphi : X \to X$ is a contraction on a complete metric space X. Prove Banach's Contraction Principle that φ has a unique fixed point.

[3] (b) Show that the integral equation

$$f(x) = x + \int_0^x tf(t)\,dt \text{ for all } x \text{ in } [0,1]$$

has a unique solution f in the space $C[0,1]$ of continuous real valued functions on the interval $[0,1]$.

[2] 6. (a) Give the definition of the outer measure of a set A of real numbers.

[3] (b) Using the definition of outer measure, demonstrate that a countable set of real numbers has outer measure 0.

[2] (c) If the outer measure of a set of real numbers is 0, must the set be countable? Explain.

[4] (d) If $f : \mathbb{R} \to \mathbb{R}$ has a positive and bounded derivative on $\mathbb{R}$, show that f maps sets of measure 0 in $\mathbb{R}$ to sets of measure 0.

[3] 7. (a) State the Stone-Weierstrass theorem for the algebra $C(X)$ of continuous, real valued functions on a compact space X.

[6] (b) If f is a continuous function on the interval $[0,1]$, prove that

$$\lim_{n\to\infty} n \int_0^1 x^n f(x)\,dx = f(1).$$

Complex analysis

[4] 8. Find a conformal mapping that maps the punctured disk $\{z : 0 < |z-1| < 1\}$ onto the unbounded region $\{z : |z| > 2\}$.

[5] 9. (a) State Morera's theorem. Suppose a sequence of analytic functions f_n, defined on a simply connected region Ω, converges uniformly to a function f on every compact subset of Ω. Using Morera's theorem prove that f is analytic as well on Ω.

[4] (b) Use the Weierstrass M-test in conjunction with part (a) to show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges to an analytic function on the open half-plane $\Omega = \{z = x + iy : x > 1\}$.

[8] 10. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2+1)^2}\,dx = \pi/2e.$

[4] 11. (a) Find the maximum of $|2z^2 - 1|$ over the closed unit disk $D = \{z \in \mathbb{C} : |z| \le 1\}$.

[5] (b) If f is an entire function and $\dfrac{f(z)}{z} \to 0$ as $|z| \to \infty$, prove that f is constant.