MATH 135 Online (Spring 2020)
Practice Questions for the Midterm

Unit 1

1.1 Consider the following expression:

There exists a real number x such that for every $y \in \{-1, 1\}$ the equation $x^4 + y^4 = 1$ holds

Questions:

- Can you write this expression symbolically, without using any words or '¬' symbol?
- Can you write the negation of this expression symbolically, without using any words or '¬' symbol?
- Is this a *mathematical statement*, an *open sentence*, or *neither*?
- If this is a mathematical statement, is it *true* or *false*?
- If this is a true mathematical statement, can you prove it? If this is a false mathematical statement, can you disprove it?
- If this is an open sentence, can you explain what variables it depends on?
- What would happen if we change the order of quantifiers as follows:

  For every $y \in \{-1, 1\}$ there exists a real number x such that the equation $x^4 + y^4 = 1$ holds

1.2 Consider the following statement:

Every odd integer can be written either in the form $4k + 1$ or in the form $4k - 1$ for some integer k

A student attempted to write this expression symbolically as follows:

$$\forall n \in \mathbb{O}, \forall k \in \mathbb{Z}, (n = 4k + 1) \vee (n = 4k - 1)$$

- Is this symbolic expression correct? If the answer is "yes", explain why. If the answer is "no", explain what mistakes were made and how would you fix them.
- Can you find at least two different ways of expressing the given statement symbolically, without using any words or '¬' symbol?

Unit 2

2.1 Consider the logical expression $(A \wedge B) \Rightarrow (B \vee C)$.

Questions:

- What is the truth table for this logical expression?
- What is its negation?
- Is this logical expression logically equivalent to $A \Rightarrow C$? If the answer is "yes", can you prove it by a) using truth tables; and by b) using properties of boolean algebra? If the answer is "no", can you find A, B, C where the two statements are different?
- Can you write down some other variation of this expression that is still equivalent to the original one?
- Can you prove that the original expression is not logically equivalent to $(A \vee B) \Rightarrow (B \wedge C)$?

2.2 Consider the following statement:

Andrés can get his favourite candy by doing the dishes and by doing his homework

Questions:

- Can you write this statement in the form of a logical expression?
- Consider the statement

  If Andrés did the dishes but still did not receive his favourite candy, then he must have not done his homework

  Is this statement logically equivalent to the original statement?
- Can you write down some other variation of this expression that is still equivalent to the original one?
- What is the negation of this statement?
- What is the contrapositive of this statement?
- What is the converse of this statement?

Unit 3

3.1 Consider the following true mathematical statement:

For all integers a and b, if $8 \mid (a^2 + b^2 - 1)$ *then a is even or b is even*

- Can you write this statement symbolically?
- Can you provide examples of a and b that make the hypothesis of an implication true? Use this example to demonstrate that the conclusion is also true.
- Can you provide examples of a and b that make the hypothesis of an implication false?
- What are possible strategies that you can think of for proving this statement? Discuss their advantages and disadvantages.
- Can you prove this statement?
- Is the converse of this statement true or false? If it is "true", can you prove it? If it is "false", can you provide a counter example?
- Can this statement be turned into an if and only if statement?

3.2 Consider the following false statement:

For all positive odd integers a and b, if $a \neq 1$ *then* $a \nmid (2b - 4)$ *or* $a \nmid (3b - 9)$

- Can you find a counter example which disproves this statement?
- Can you find a condition on a that can be put in the statement below such that a) the hypothesis can be made true for at least one choice of a; and b) that would make the entire statement true?

For all positive odd integers a and b, if $a \neq 1$ *and* _____ *then* $a \nmid (2b - 4)$ *or* $a \nmid (3b - 9)$

- Can you find a condition on a that can be put in the statement below that would make the entire statement true?

For all positive odd integers a and b, $a \neq 1$ *and* _____ *if and only if* $a \nmid (2b - 4)$ *or* $a \nmid (3b - 9)$

Unit 4

4.1 Consider the sum

$$\sum_{n=-3}^{3} \left(2^{n+3} + 1\right)$$

- What is this sum equal to?
- How would you change the orders of summation so to make this sum equal to 19?
- Is the sum

$$\sum_{n=-1}^{8} \left(2^{n+2} + 1\right)$$

a result of reindexing the sum $\sum_{n=-3}^{3} (2^{n+3} + 1)$? Explain why or why not.

4.2 We say that a positive number n is *triangular* if there exists an integer k such that $n = \frac{k(k+1)}{2}$. The first five triangular numbers are $1, 3, 6, 10, 15$. Consider the following true statement:

For every $n \in \mathbb{N}$, the sum of the first n triangular numbers is equal to $\frac{n(n+1)(n+2)}{6}$

- How would you write this statement symbolically using the summation notation?
- When proving this statement by induction, what method would you choose? The Principle of Mathematical Induction or the Principle of Strong Induction?
- Do you need one base case or many base cases? How would you prove them?
- What statement has to be proved on the inductive step?
- Can you prove this statement?

Unit 5

5.1 An integer n is called a *perfect cube* if there exists an integer ℓ such that $n = \ell^3$. Let S denote the set of all odd integers that are also perfect cubes.

- Can you write this statement using the set builder notation in three different ways?
- Can you write it as an intersection of two sets?

5.2 Consider the following three sets:

$$A = \{n \in \mathbb{Z} \colon n \text{ is odd}\}, \quad B = \{4k + 3 \colon k \in \mathbb{Z}\}, \quad C = \{m \in \mathbb{Z} \colon 4 \mid (m - 1)\}$$

- Can you give examples of 3 elements in each of these sets?
- Can you write the following sets using set builder notation:

$$\begin{array}{lll} \overline{A} & \overline{B} & \overline{C} \\ \overline{A \cup B} & \overline{A \cup C} & \overline{B \cup C} \\ \overline{A \cap B} & \overline{A \cap C} & \overline{B \cap C} \\ \overline{A} \cup \overline{B} & \overline{A} \cup \overline{C} & \overline{B} \cup \overline{C} \\ \overline{A} \cap \overline{B} & \overline{A} \cap \overline{C} & \overline{B} \cap \overline{C} \end{array}$$

- What are the relations between A, B and C? Is $A \subseteq B$ or $B \subseteq A$? Is $A \subseteq C$ or $C \subseteq A$? Is $B \subseteq C$ or $C \subseteq B$? If one is a subset of the other, can you prove that it is a *proper* subset or that the two sets are equal?
- What should you do to sets B and C so to make them equal to A?