Transition Matrix Methods

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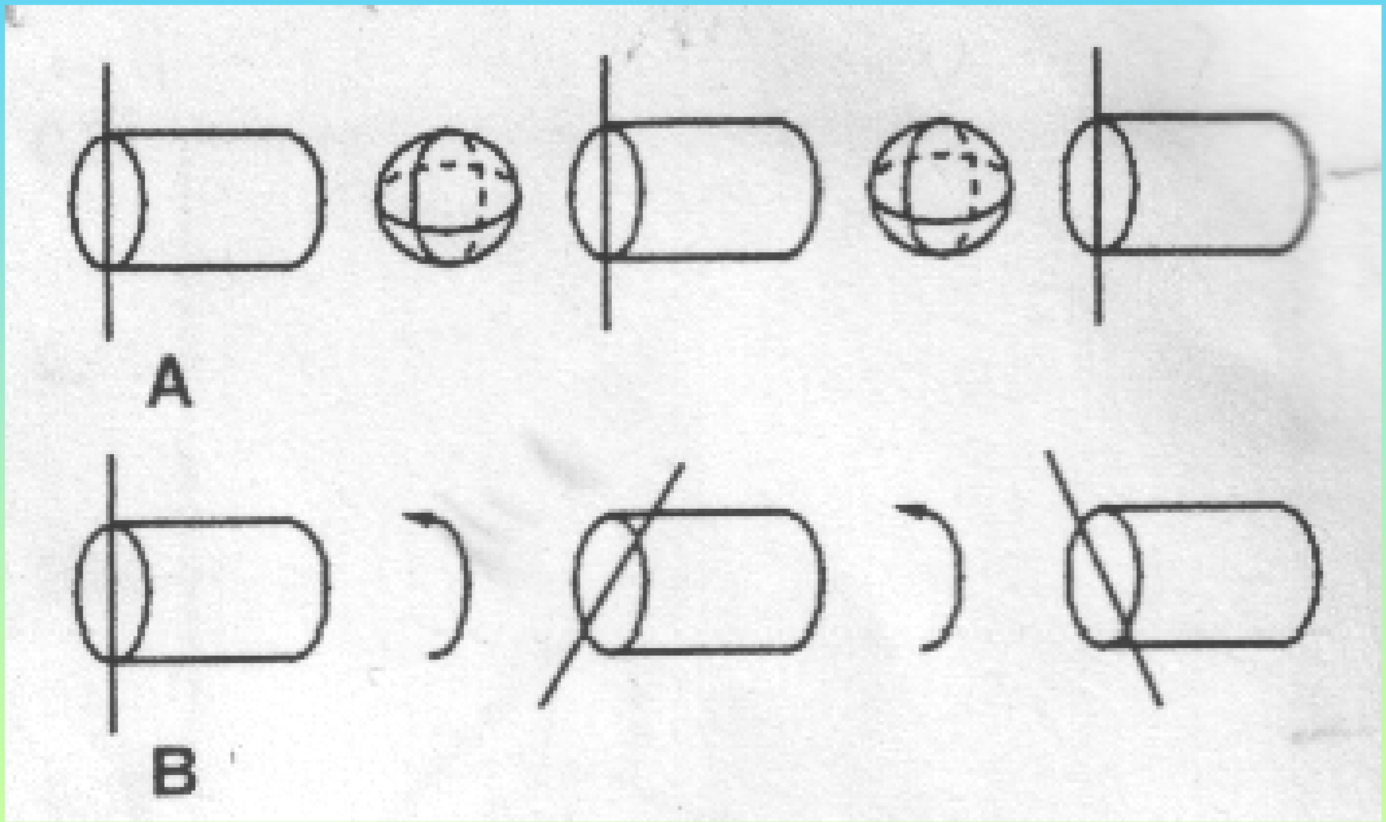
Outline

- Review of PMD
- Multicanonical Algorithm
- Transition Matrix Methods
- Numerical results
- Conclusions

Polarized Waveguide Modes

- For single mode waveguides with sufficient symmetry, light propagates in a superposition of two degenerate modes.
- In general the modal group velocities differ, yielding signal distortion in optical fibers (polarization mode dispersion).

PMD Emulator Types



PMD Emulators

- We describe a general communication system by a set of system parameters $\vec{\alpha}$.
- For PMD emulators, these are e.g. the polarization rotation angles associated with polarization scramblers separated by concatenated birefringent sections or the optical path lengths of successive mutually rotated birefringent segments.

System Simulation

- We generate random values of the local system parameters (rotation angles).

$$\vec{\alpha} = (\theta_1, \phi_1, \theta_2, \phi_2, \ldots, \theta_N, \phi_N)$$

- For each realization, we calculate one or more global system variables $\vec{O}$ (observables). In our case these are the PMD vectors τ, τ_ω …

Control Variables

- Random control variables

$$\vec{\alpha} = (\alpha_1, \alpha_2 \ldots, \alpha_N)$$

- Objective: Determine the probability distribution function (pdf) $p(\vec{O})$ of a set of observables

$$\vec{O} = \vec{O}(\vec{\alpha})$$

- Generate *N* sets of control variables $\vec{\alpha}$ according to the physical distribution $p(\vec{\alpha})$
- Compute or measure $\vec{O}$ for each set $\vec{\alpha}$

Monte Carlo Sampling

- If the function $I(\vec{O}_k, \vec{\alpha}^{(i)})$ is one inside the *k*:th histogram bin, then after *N* realizations,

$$p(\vec{O}_k) \approx \frac{1}{N} \sum_{i=1}^{N} I(\vec{O}_k, \vec{\alpha}^{(i)})$$

- Clearly few events arise in regions of small probability. Therefore, many samples are required to generate "worst case" events with small PMD statistics.

Markov Chains

- To insure that lower probability states are sampled more often, Markov chains are employed.
- Additing a small perturbation to the current srate yields a proposed transition
- A rule governs the acceptance of this transition.

Multicanonical Algorithm

- The transition rule should ensure:
  - Equal sampling of equal *O* regions
  - Rapid escape from local minima
  - (Detailed balance)
- These require that the acceptance is a particular function of the ratio of the pdf of the initial and final states.
- However, the pdf is unknown and must be determined iteratively.

Multicanonical Procedure

- A Markov chain leading to equal sampling of the system variable space, $\vec{O}$.
  - Starting from p_o = 1, accept all transitions that decrease p_0 and those that increase p_0 with a probability $p_0(\vec{O}^{(curr)})/p_0(\vec{O}^{(new)})$This gives the acceptance probability

$$\min\{1, p_0(\vec{O}^{(curr)})/p_0(\vec{O}^{(new)})\}$$

  - This yields a (Monte-Carlo) distribution H_1.
  - Iterate with $p_n = p_{n-1}H_{n-1}/c_n$

Result of Iterations

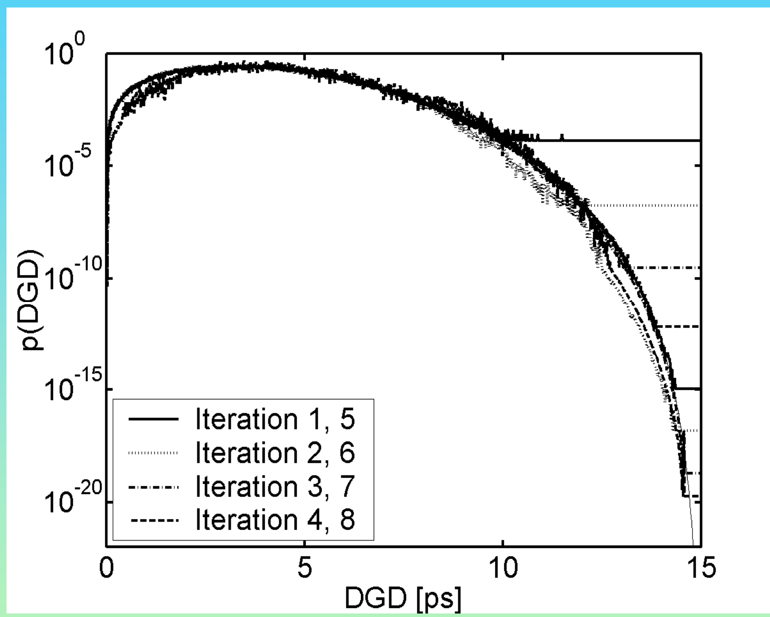
- In the tail regions regions, p is initially 1, yielding Monte-Carlo statistics and the number of events falls off as the PDF.
- As the iterations proceed, the states increasingly sample these low-probability regions.

MC Enhancements

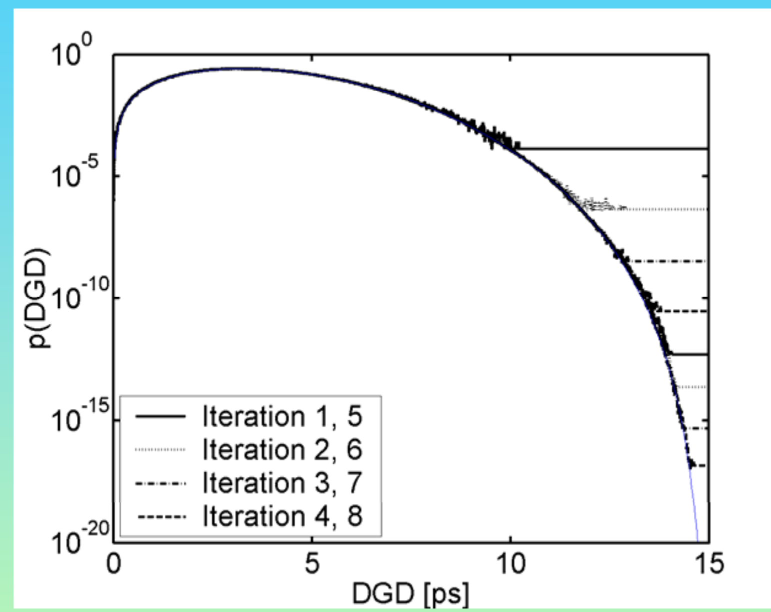
- Raise the intermediate result $p_n(E)$ to a power in the transition rule.
- Introduce a bias function $p_n(E) \to c_n p_n(E) H(E)$ into the transition rule and combine the results in overlapping regions.
- Interpolate the histogram in the transition rule.
- Employ different probability distributions

1^{st} Order PMD Results

Conventional



Modified

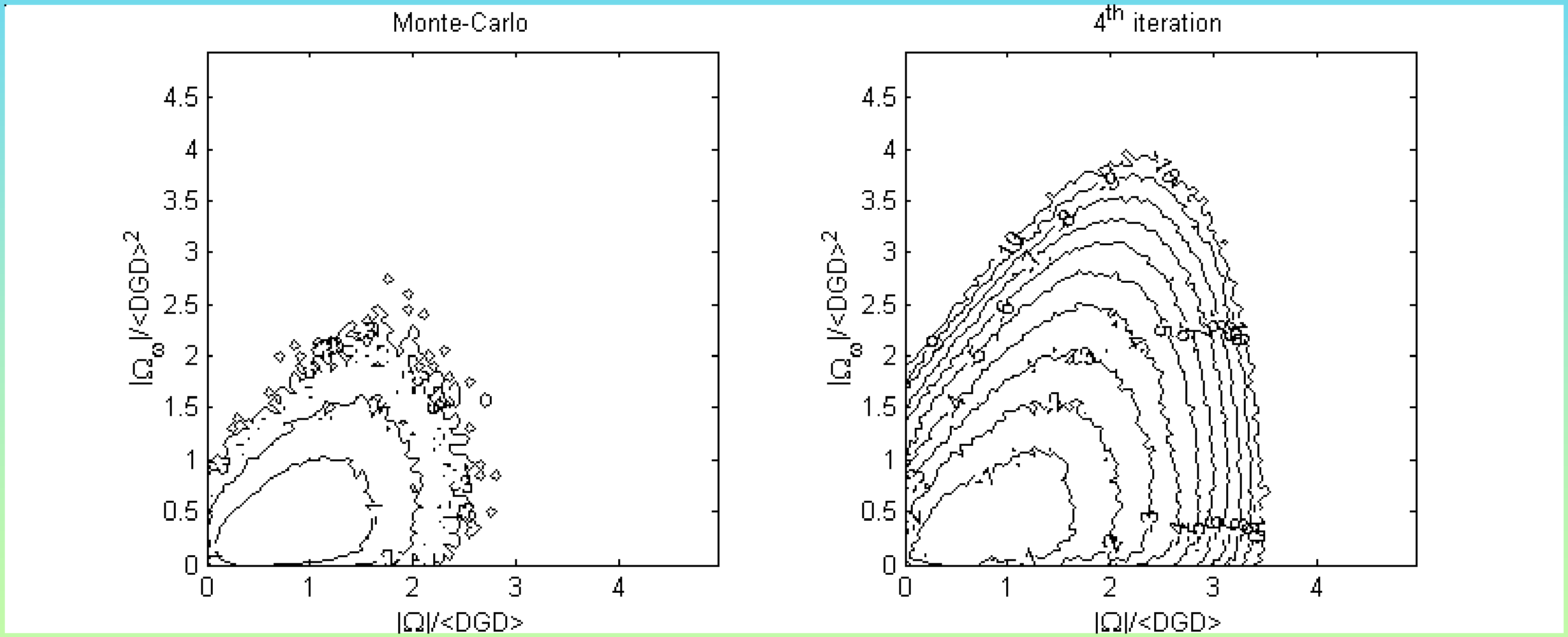


- 15 section PMD emulator
- Section DGD ($|\vec{\tau}_i| = 1ps$)
- $\vec{\tau}_i$ rotates randomly on Poincare sphere
- 500,000 samples/iteration
- 8 iterations

Joint PMD Calculation

- We employ a 15 section PMD emulator.
- 200,000 samples/iteration
- Number of histogram bins: 100x100
- We normalize the first order PMD to <PMD> and the second order PMD to <PMD>^{2}.

Joint pdf Results



Experimental Results

- Determined the DGD distribution of a microheater-based PMD emulator.
- * Measured the joint pdf of the first and second order PMD of a 8 section PMD emulator with General Photonics polarization controllers.
- Recorded the distribution of bit error rates in a recirculating fiber loop.

PMD Emulator

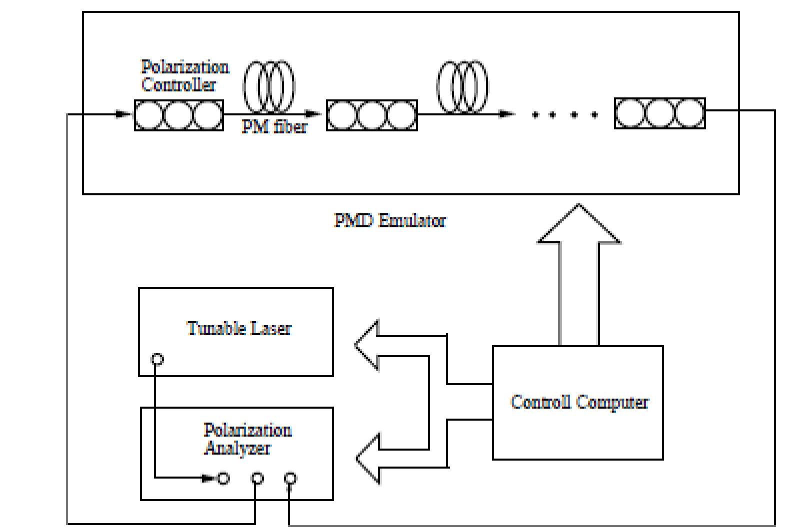


Fig. 1. Experimental setup

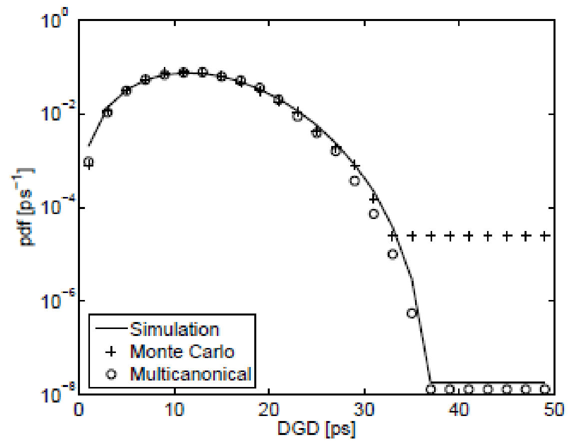


Fig. 2. The pdf of the DGD obtained with a 20,000 sample experimental Monte Carlo measurement (+), a multicanonical measurement with ten iterations of 2,000 samples (○) and a numerical multicanonical simulation employing twenty 50,000 sample iterations (solid line)

Biasing Method

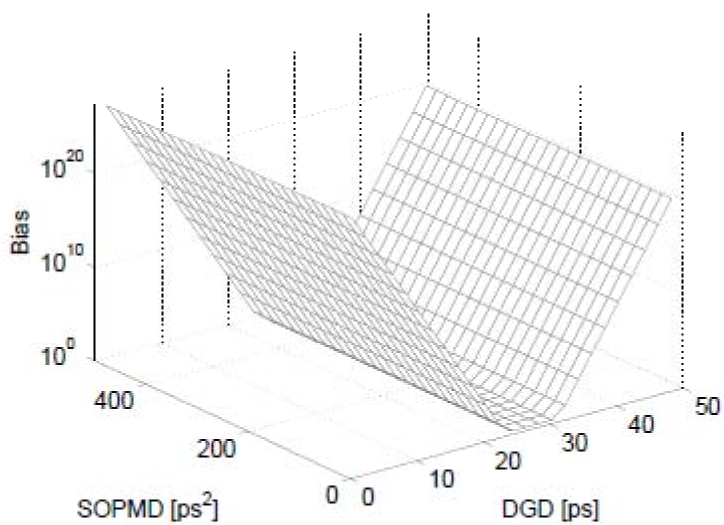


Fig. 4. The $\mathcal{R}_4 \equiv 24 < \tau < 32ps$ subregion bias function

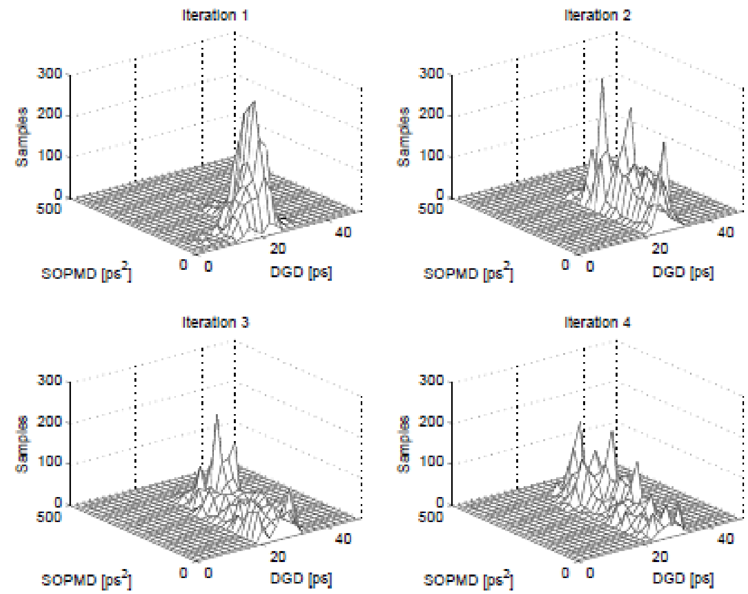


Fig. 5. The histograms obtained after each biased multicanonical iteration

Full pdf

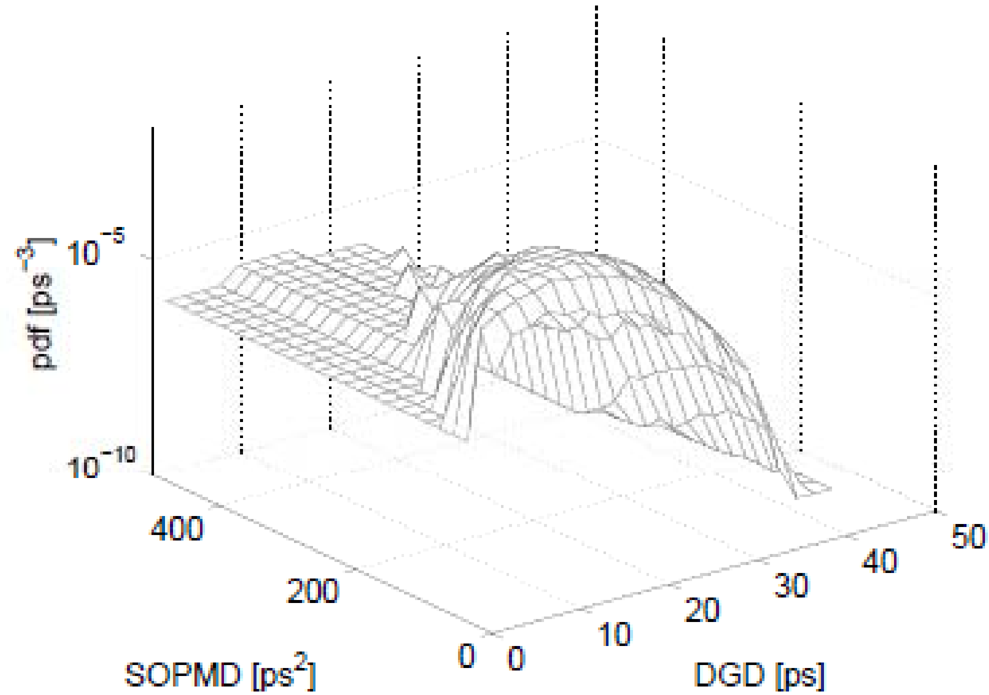


Fig. 6. The pdf obtained from the biased multicanonical method generated by combining results for all subregions $\mathcal{R}_1 \ldots \mathcal{R}_5$.

Transition Matrix Method

- For every accepted or rejected transition from the *n*:th to the *m*:th histogram bin in a biased calculation:
  - » Increment the corresponding element, M_{nm} , of an unbiased but unnormalized transition matrix, *M*, by unity.
  - » At the end normalize the rows of T_{nm} to unity, yielding the transition matrix *T*.

Rapid pdf updating

- By detailed balance, the pdf can be obtained from the recursion relation

$$p\left(E_{i+1}\right)=\frac{T_{i+1,i}}{T_{i,i+1}}\,p\left(E_i\right)$$

- This enables updates after every few steps.
- Alternatively transition only to final states visited less than the initial state.

Detailed Balance Violation

- The preceding method violates detailed balance
- More system realizations enter a histogram bin from the high probability side than the low probability side.
- If transitions out of the bin are discarded until the physical pdf is established this problem can be resolved.

Sampling Frequency

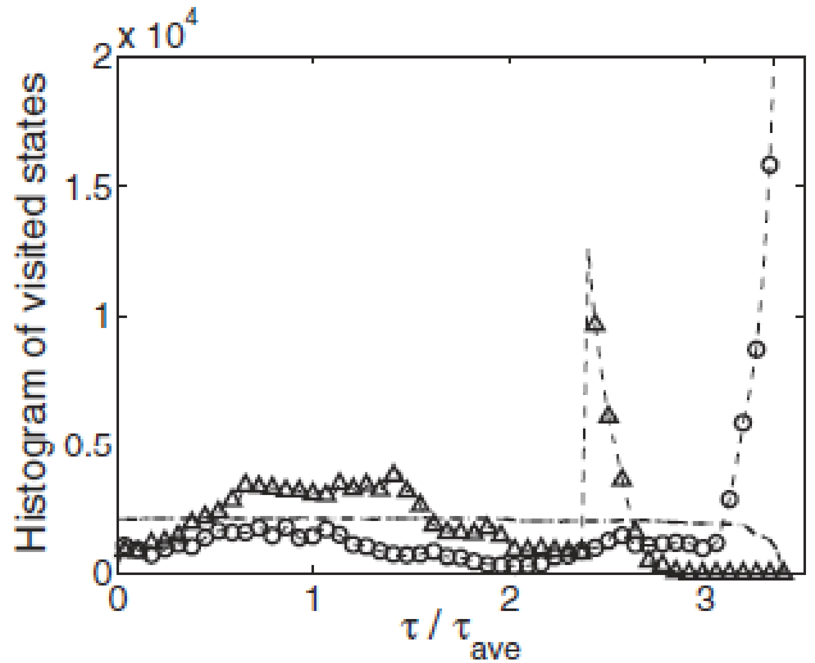


Fig. 2. Total number of times each histogram bin is visited for the standard multicanonical procedure (△), method 1 (○), and method 2 (dashed–dotted curve).

Method Comparison

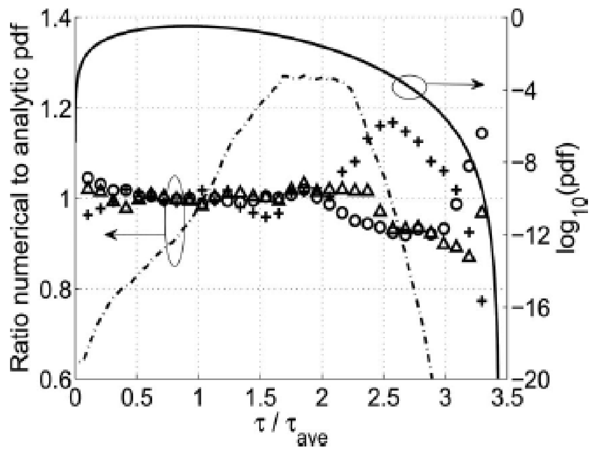


Fig. 1. Ratio between the numerical and analytic pdfs for the standard multicanonical procedure (△), our modified transition matrix procedure with a multicanonical acceptance rule (method 1, ○), an acceptance rule that rejects transitions to more visited histogram bins (method 2, dashed–dotted curve), and a procedure that restricts transitions out of a recently visited bin (method 3, crosses) as functions of the normalized DGD for a $N_{sec}=10$ segment fiber emulator.

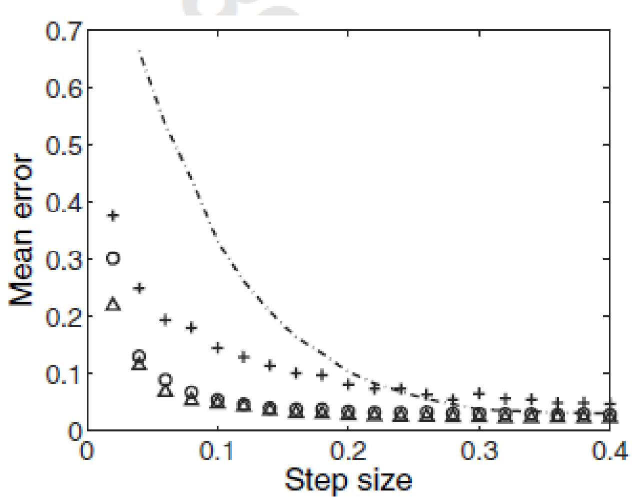


Fig. 3. Variation of the error, Eq. (2), weighted by the histogram bin probability as a function of the average DGD change over one Markov step for the standard multicanonical method (△), method 1 (○), method 2 (dashed–dotted curve), and method 3 (crosses).

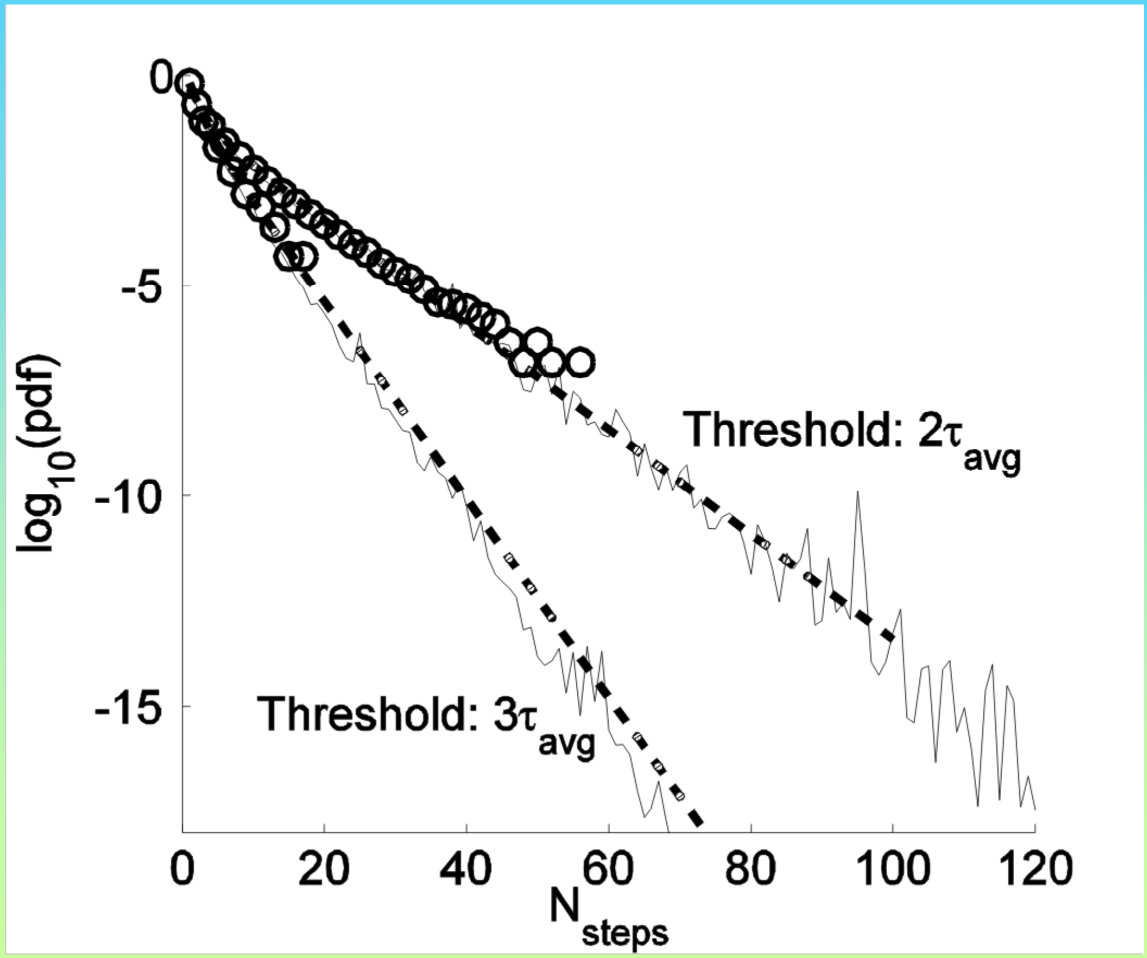
Outage Time Formulation

- Initial state: $S_n = P(E_n)$ for n in the non-outage region, zero elsewhere
- Repeatedly multiply by the transition matrix.
- At each step, sum and then set to zero the histogram values of non-outage states.
- Alternatively, employ the outage region submatrix eigenfunctions and eigenvalues.

Outage Time Calculation

- Outage conditions: (1) DGD > 2<DGD> and (2) DGD > 3<DGD>.
- Circles - 10^9 step unbiased Markov chain.
- Solid line - Biased TM calculation with three 5,000K sample MC iterations
- Dashed line - Multiplication with the unbiased TM from twenty 200K MC iterations or submatrix eigenvalue method.

Outage Time Results



Fading Channels

- Gain of Rayleigh fading channel

$$g = \mu_1(t) + \mu_2(t)$$

$$\mu_m(t) = \sum_{n=1}^{N_m} \cos\left(2\pi f_{\max} \sin\left(\frac{\pi(2n-1)}{4N_m}\right) t + \theta_{mn}\right)$$

- The phases θ_{mn} are the control variables.
- The durations of two successive fading events are employed as the observables.
- This yields relevant 1-dimensional pdf:s

PDF of Fading Events

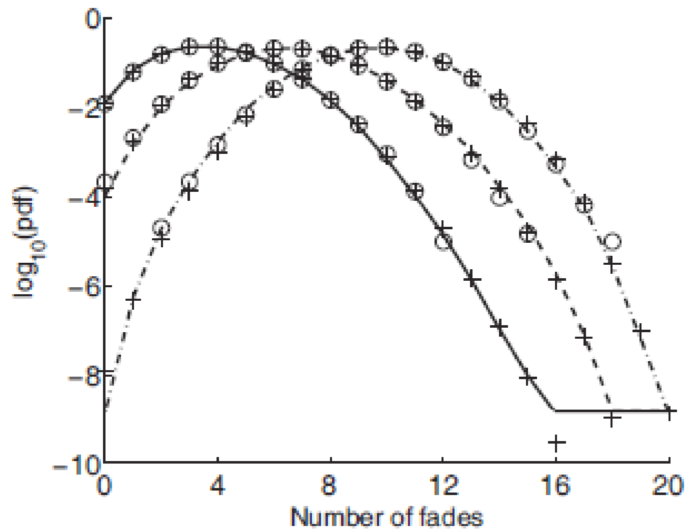


Fig. 1. The base 10 logarithm of the pdf of the number of fading events for $f_{\max} = 10$ Hz within $T_F = 1$ sec for outage threshold levels of $R = 5$ dB (dashed-dotted line), 10 dB (dashed line) and 15 dB (solid line) below the mean channel gain as evaluated with two 10^5-sample iterations of the multicanonical method as well as for our modified transition matrix procedure (+) with a 2×10^5 samples and a direct evaluation of Eq. (2) (○).

PDF of Fade Duration

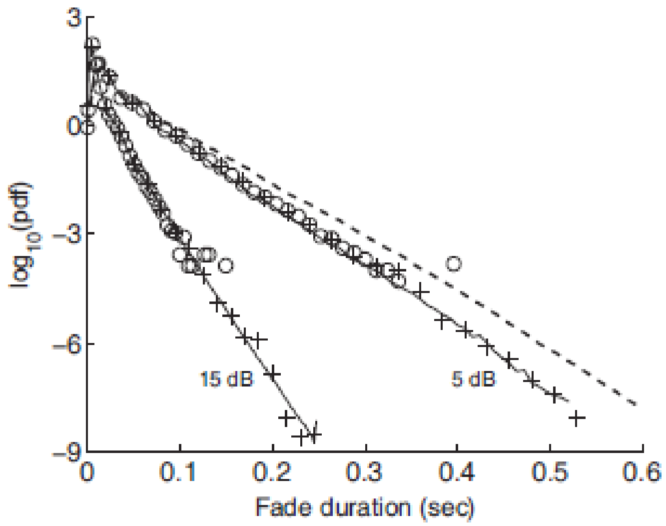


Fig. 2. The fade duration distribution for three 2×10^6-sample iterations of the multicanonical method (solid lines) with $N = 100$ bins for outage levels of $R = 5$ dB and 15 dB below the mean channel gain, respectively, and for 6×10^6 samples of the the modified transition matrix procedure $(+)$ and Eq. (2) $(\circ)$. The dashed line indicates the sampling bias introduced by the one-dimensional multicanonical calculation for $R = 5$ dB.

Conclusions

- Transition matrix methods enable rapid prediction of the dynamic behavior of general communication systems.
- However, standard procedures are more accurate since all correlations between states are present.
- The technique must be carefully applied when long-time correlations exist.