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Efficient Nested Simulation for CTE of Variable Annuities

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Outline

Variable Annuities (VAs)

Insurance policies that are exposed to market return (and risk)

- Payoffs depend on
  - death/survival of insured life (insurance risks)
  - equity market performance (additional market risks)
- Channel upside market potentials and provide downside protections
  - through policy riders, aka “minimum guarantees”

Common Contract Types

- Guaranteed Minimum Maturity Benefit (GMMB)
  - Guarantees the policyholder a specific amount at maturity
- Guaranteed Minimum Death Benefit (GMDB)
  - Guarantees the policyholder a specific amount upon death
- Guaranteed Minimum Accumulation Benefit (GMAB)
  - Policyholder has the option to renew the contract, at a new guarantee level

Guarantees ≈ “embedded options” that are nonlinear and path-dependent

Enterprise Risk Management (ERM) for VAs

Traditional Actuarial Techniques are Insufficient

- Insurance risks (death/survival) can be diversified
  - little uncertainty about total claim (CLT)
  - okay to use deterministic valuation (expected value suffices)
- Market risks have limited diversification
  - either all policies will generate claims or none will (common seg fund)
  - it is important to consider systemic risk and tail risk (tail risk measures)

Risk Management: Hedging and Valuation

- Dynamic hedging programs are popular
  - Perfect hedge is theoretically possible
- In practice, there is always hedging errors due to
  - Discrete hedging
  - Model error
- Key ERM task: estimate tail risk measures (CTE, VaR)

Hedging and Valuation

Example (Guaranteed Minimum Maturity Benefit (GMMB) – Simplified)

- $S_t :=$ fund value at time t
- $G_t :=$ guaranteed maturity value (may depend on t)
- $T :=$ maturity of the policy
- GMMB pays $\max\{S_T, G_T\}$ to the policyholder at time T
- The insurer liquidates S_T and pays $\max\{0, G_T - S_T\}$ at time T
- In this simple setting, the insurer has a short position on a put option

Dynamic Hedging

To hedge against market risk, a hedge (replicating) portfolio is set up

- Long risk-free bond
- Short the underlying asset

Re-balance periodically and wish to estimate the hedging error

Dynamic Hedging for VAs

Underlying asset prices $S_1, \dots, S_T$

At each time t, setup a hedge (replicating) portfolio consisting of

- $B_t :=$ amount invested in risk-free bond
- $\Delta_t S_t :=$ amount invested in underlying asset S_t
- $H_t := B_t + \Delta_t S_t =$ value of portfolio set up at time t
- $H_{t+1}^{BF} := B_t e^r + \Delta_t S_{t+1} =$ value of portfolio *brought forward* from time t
- The *hedging error* realized at time t is

$$HE_t = H_t - H_t^{BF}$$

  Essentially, "what you need" minus "what you have" at each t.

- The liability at time 0 of hedging errors for the VA policy is

$$L = \sum_{t=1}^{T} e^{-rt} HE_t$$

  The r.v. whose tail risk measure is estimated via nested simulation

Nested Simulation of Conditional Tail Expectation (CTE)

1. **Outer Sim**: asset sample paths (scenarios) $S_1^{(i)}, \ldots, S_T^{(i)}$ for $i = 1, \ldots, N$
2. Estimate the liability $\widehat{L}^{(i)}$ for each scenario i
   - **Inner Sim**: estimate $\widehat{B}_t^{(i)}$ and $\widehat{\Delta}_t^{(i)}$, $t = 0, \cdots, T-1$
3. Rank the estimated liabilities $\widehat{L}_{(1)} \geq \ldots \geq \widehat{L}_{(N)}$
4. Given confidence level α, the CTE_α is

$$CTE_\alpha = \frac{1}{(1-\alpha)N} \sum_{i=1}^{(1-\alpha)N} \widehat{L}_{(i)}$$

Features of CTE estimation

- The $(1-\alpha)N$ scenarios in the summation are called the **tail scenarios**
- Simulation efforts for non-tail scenarios are essentially "wasted"
  1. needed to rank & identify tail scen.
  2. does not affect accuracy of estimating CTE
- If somehow we can identify the tail efficiently...?

Computational Challenge in Nested Simulation

Computational Challenge

- Full nested simulation can be prohibitively difficult
- Total number of inner simulations required
  $=$ no. of inner sim $\times$ no. of outer sim $\times$ no. of policies
- Considerable professional and industry interest in solutions to the computational challenge.

Active Research to Address the Computational Challenge

- Representative policies: e.g. Gan et al., (2015)
- Proxy modeling in inner-loop via least-squares Monte Carlo (Broadie et al., 2015) and PDE (Feng, 2014)
- Strategic allocation of simulation budget: Broadie et al., (2011) and Gordy et al., (2010)

Importance-Allocated Nested Simulation (IANS)

Main Steps (fixed simulation budget)

1. Outer simulation of sample paths (the scenarios)
2. Proxy evaluation in every scenario (avoid inner sim)
3. Identify tail scenarios based on proxies (rank & select)
4. Concentrate total budget to tail scenarios (importance allocation)

Main Questions

1. Good proxy model?
   - Similar to the inner sim model, but much faster
2. Calibrate the proxy model?
   - Inner sim model param $\Rightarrow$ proxy model param
3. More tail scenarios as safety margin?
   - A proxy is a proxy
   - Tradeoff between tail coverage and budget concentration

IANS for GMMB (put option)

Example (GMMB, with additional details)

- S_t modeled by Regime-Switching (RS)
  - Switching between two Black-Scholes: "normal time" & "crisis time"
  - Incomplete market, no closed-form B_t & Δ_t, inner sim necessary
- 20yr maturity, monthly rebalancing

Main Questions Answered

1. Black-Scholes (BS) as the proxy model (closed-form B_t & Δ_t)
2. Match BS implied vol to expected RS vol in the same period
3. Safety margin:
   - $1 - \alpha = 5\% \Rightarrow \xi = 10\%$
   - $1 - \alpha = 20\% \Rightarrow \xi = 25\%$

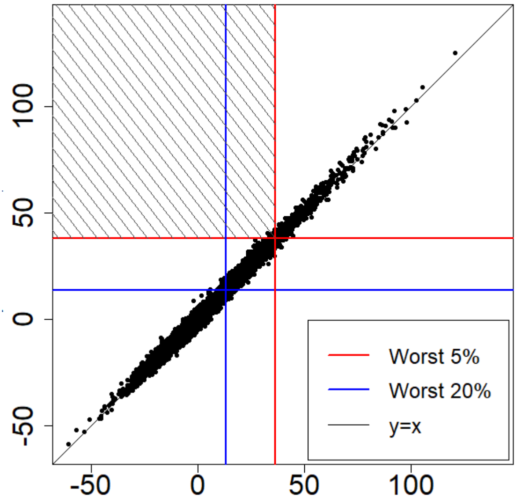
Numerical Experiment (GMMB) – Settings

Benchmarks for Comparisons

1. "true value": full nested sim with 10K outer sim & 10K inner sim
2. Standard Monte Carlo with the same total budget
   - SMC1. 2K outer sim & 500 inner sim
   - SMC2. 1K outer sim & 1K inner sim
   - SMC3. 200 outer sim & 5K inner sim
3. IANS with 2K outer sim
   - CTE95. $\xi = 10\%$, 5K inner sim for 200 tail scen.
   - CTE80. $\xi = 25\%$, 2K inner sim for 500 tail scen.

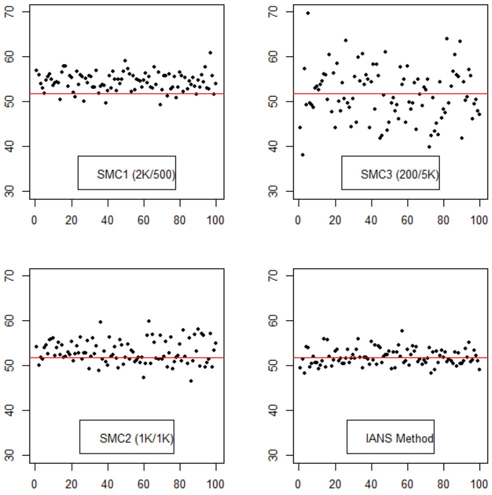
Repeat the experiment 100 times to assess accuracies

Numerical Experiment (GMMB) – Results



Q-Q plot of GMMB PV of HE (L) (proxy model vs. inner sim.) for 10K scenarios

Numerical Experiment (GMMB) – Results



Scatter plots of 100 CTE95 PV of HE (L). The "true value" is displayed in red.

Numerical Experiment (GMMB) – Results

	CTE95		CTE80	
	MSE	Normalized	MSE	Normalized
IANS	3.26	1	1.34	1
SMC1 (2K/500)	10.47	3.21	3.28	2.45
SMC2 (1K/1K)	8.13	2.49	2.39	1.78
SMC3 (200/5K)	33.94	10.41	11.60	8.66

Table: MSEs for different simulation procedures with the same simulation budget.

Findings

- IANS is superior than SMC
- IANS is better for more extreme CTE (higher budget concentration)

GMMB is such a simple VA, does IANS really work in more complex examples?

More Complex VA: GM-Accumulation-B

Example

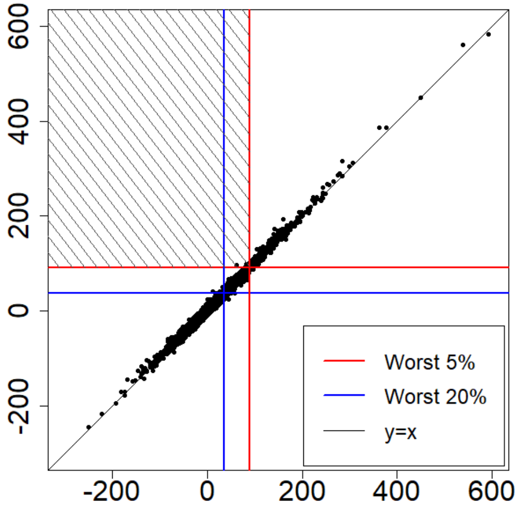
Guarantee a minimum fund value at both renewals and maturity.

- $R \in (0, T) :=$ renewal time
- $G_{R-} :=$ minimum guarantee prior to renewal
- $G_{R+}, S_{R+} := \max\{S_{R-}, G_{R-}\} :=$ renewed guarantee/fund value
- Prior to renewal: equivalent to a compound put option (put-on-put)
- After the last renewal: equivalent to a GMMB/put option

Additional Complexity

- BS proxy still have closed-form B_t & Δ_t, but more complicated
- Much harder to calibrate the "equivalent" volatilities

Numerical Experiment (GMAB) – Results



Q-Q plot of GMAB PV of HE (L) (proxy model vs. inner sim.) for 10K scenarios

Numerical Experiment (GMAB) – Results

Single Renewal	CTE95		CTE80	
	MSE	Normalized	MSE	Normalized
IANS	54.57	1	8.03	1
SMC1 (2K/500)	68.63	1.26	12.47	1.55
SMC2 (1K/1K)	87.36	1.60	18.26	2.27
SMC3 (200/5K)	343.45	6.29	70.39	8.77

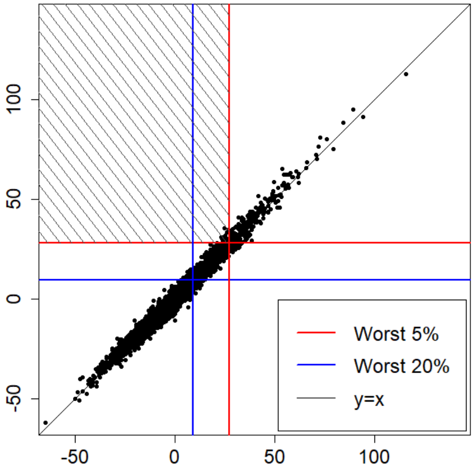
Table: MSEs for different simulation procedures with the same simulation budget.

Findings

- IANS is still superior than SMC
- Improvement is less significant

Numerical Experiment (GARCH) – Preliminary Results

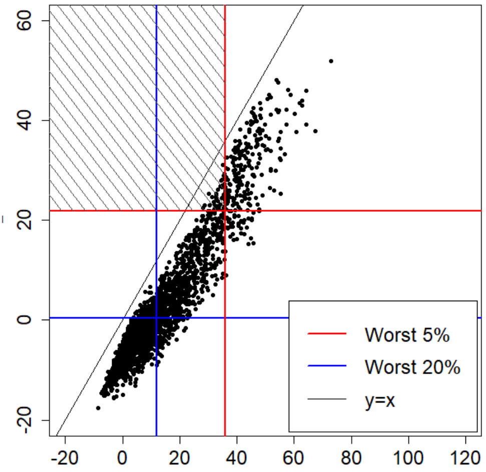
- GMMB under GARCH(1,1) Simulation Model



Q-Q plot of GMMB PV of HE (L) (proxy model vs. inner sim.) for 5K scenarios

Numerical Experiment (Dynamic Lapse) – Preliminary Results

- GMMB with dynamic lapse under Regime-Switching Model



Q-Q plot of GMMB PV of HE (L) (proxy model vs. inner sim.) for 5K scenarios

Concluding Remarks

What's new?

- Efficient nested simulation for tail risk estimation
- Concentrated simulation efforts on tail scen. identified via proxy
- Numerical demonstrations via improved accuracies in different VAs

What's next?

- Choose ξ based on α and contract complexity
- Fixed budget vs. Target accuracy
- Non-uniform budget allocation on tail scen.

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