Course Notes for AMATH 732

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Contents

Preface

These notes are based in part on lectures notes developed by P. Tenti.

Useful references

1. Course notes.

2. Bender, C. M., and Orzag, S. A. *Advanced Mathematical Methods for Scientist and Engineers*, McGraw-Hill (1978). QA371.B43 1978.

   The book I learned from. A wealth of topics. Lots and lots of problems. Some very difficult.

3. Lin, C. C., and L. A. Segel, L. A. *Mathematics Applied to Deterministic Problems in the Natural Sciences*, MacMillan (1974). QA37.2.L55 1974

   This is one of the best applied mathematics texts available. It was reprinted by SIAM in 1988. Parts of it can be read on google books.

4. Murdock, J. A. *Perturbations: Theory and Methods*. QA871.M87 1999

5. Nayfeh, A. H. *Introduction to Perturbation Techniques*. QA371.N32 1981

6. Bleistein, N., and Handelsman, R. A. *Asymptotic Expansions of Integrals*, Holt, Rinehart and Winston, New York (1975). QA311.B58 1975

   Title says it all. Dover republished an unabridged corrected version in 1986.

7. Ablowitz, M. J., and Fokas, A. S. *Complex Variables: Introduction and Applications*, Cambridge University Press (2003). QA331.7.A25

   Useful treatment of asymptotic evaluation of integrals (e.g., method of steepest descent).

Chapter 1

Introduction

Before the 18^{th} century, Applied Mathematics and its methods received the close attention of the best mathematicians who were driven by a desire to explain the physical universe. Applied Mathematics can be thought of as a three step process:

$$
\begin{array}{ccc}
\text{Physical} & 1 & \text{Mathematical} \\
\text{Situation} & \Longrightarrow & \text{Formulation} \\
 & & \Big\Downarrow 2 \\
\text{Physical} & 3 & \text{Solution by Purely} \\
\text{Interpretation} & \Longleftarrow & \text{Formal Operations} \\
\text{of the Solution} & & \text{of the Math Problem}
\end{array}
$$

Over the centuries step 2 took on a life of its own. Mathematics was studied on its own, devoid of contact with a physical problem. This is pure mathematics. Applied mathematics deals with all three steps.

The goal of asymptotic and perturbation methods is to find useful, approximate solutions to difficult problems that arise from the desire to understand a physical process. Exact solutions are usually either impossible to obtain or too complicated to be useful. Approximate, useful solutions are often tested by comparison with experiments or observations rather than by 'rigourous' mathematical methods. Hence we will not be concerned with 'rigorous' proofs in this course. The derivation of approximate solutions can be done in two different ways. First, one can find an approximate set of equations that can be solved, or, one can find an approximate solution of a set of equations. Usually one must do both.

A key turning point in the history of mathematics was the brilliant discovery of the theory of limits of Gauss (1777–1855) and Cauchy (1789–1857). In the limit process, usually characterized by an infinite expansion, we do not attempt to obtain the exact solution but merely to approach it with arbitrary precision. Thus, the desire for absolute accuracy (zero error) was replaced by one for arbitrarily great accuracy (arbitrarily small error):

$$
\begin{array}{c} \text{absolute} \\ \text{accuracy} \end{array} \Longrightarrow \begin{array}{c} \text{arbitrarily great} \\ \text{accuracy} \end{array}
$$

and

$$
\begin{array}{c} \text{zero} \\ \text{error} \end{array} \Longrightarrow \begin{array}{c} \text{arbitrarily small} \\ \text{error} \end{array} .
$$

We are no longer interested in what happens after a finite number of steps but wish to know what happens eventually if the number of steps is increased indefinitely. The obvious difficulty with this is that in most real applications you can only sum up a finite number of terms. In fact, for many problems that we will tackle we will obtain only the first two or three terms in a series. We are then not particularly interested in what happens as the number of terms goes to zero but rather in how accurate, or useful, an approximation using a few terms is. Since observations have limited accuracy, there is no need to make the error arbitrarily small.

This gives rise to a different limiting process and different questions: What error occurs after a finite number of steps? How can we minimize the error for a given number of steps? This is a branch of applied analysis.

1.0.1 The Role of Numerical Analysis

An obvious question, particularly in this day and age, is 'If the problem is so difficult why not solve it on a computer'. Ultimately you may end up doing this, but using asymptotic and perturbation techniques to find useful, approximate answers is an extremely important first step. It should always be done whenever possible. Approximate solutions have many benefits. They provide necessary checks, and aid in the understanding and interpretation, of numerical solutions. They illuminate potential problems, e.g., regions in parameter space where singularities exist and where special numerical approaches may be required. They can give tremendous insight into how the solution depends on the parameters of the problem and help determine what the important parameters are.

Example 1.0.1 *Newtonian, constant density, steady state flow past a finite object* $\Omega \subset \mathbb{R}^3$.

$$\left\{\begin{array}{rcl} \mathbf{u}\cdot\vec{\nabla}\cdot\mathbf{u} & = & -\frac{1}{\rho_0}\vec{\nabla}p+\nu\nabla^2\mathbf{u} \\ \mathbf{u}|_{\partial\Omega} & = & 0 \\ \mathbf{u}\to\mathbf{u}_\infty & \text{as} & |\mathbf{x}|\to\infty \end{array}\right. \tag{1.1}$$

where

$$\begin{array}{rcl} \mathbf{u}(x,y,z) & = & \text{fluid velocity} \\ \rho_0 & = & \text{mass density} \\ p & = & \text{hydrodynamic pressure} \\ \mathbf{u}_\infty & = & \text{constant for field flow} \\ \nu & = & \text{kinematic viscosity} \end{array}$$

Much work needs to be done on (1.1), e.g., prove existence and uniqueness. Such a proof may not be constructive, i.e. it may not be helpful in finding the solution. No knowledge of fluid mechanics is required: the problem of proving existence and uniqueness is part of step 2 in our three step process and is part of pure analysis. To obtain an actual solution is another matter. In general, it is impossible. The biggest source of difficulty lies in the nonlinear term $\mathbf{u}\cdot\vec{\nabla}\mathbf{u}$ and in the viscous term $\nu\nabla^2\mathbf{u}$. One way of tackling this problem is to assume that the viscous term $\nu\nabla^2\mathbf{u}$ term is negligible. This would appear to be very reasonable for, e.g., an airplane in air, since the viscosity of air is very small ($\approx 10^{-5}$ m^2 s^{-1}). Dropping this term requires abandonment of the boundary condition $\mathbf{u}|_{\partial\Omega} = 0$ (this condition, which says that the fluid velocity is zero on the solid boundary, is a consequence of viscosity). This results in a linear potential problem

$$\begin{aligned} \mathbf{u} &= \vec{\nabla}\phi \\ \nabla^2\phi &= 0 \\ \nabla\phi\cdot\widehat{n} &= 0 \text{ on } \partial\Omega \end{aligned} \tag{1.2}$$

This approximate linear problem can be solved for some geometries and many general results can be proved as much is known about solutions of Laplace's equation. This makes it very tempting to use the simplified problem (1.2). In fact researchers in the late 1800's and early 1900's used this model and proved that airplanes can't fly!

Today there is a strong tendency to solve problems like (1.1) on a computer. This can be a lot of work and if the mathematical model does not correctly describe the physics then the numerical solution is garbage no matter how accurately you solve the model equations. In fact (1.1) is useful only for laminar flows (e.g., flow over a streamlined body like an airplane wing) because the model is very inaccurate for turbulent flows.

Computers, while very useful and often necessary, should be used in the last stage of a scientific investigation. Analytic work on a mathematical problem is necessary to provide a rough understanding of possible solutions. Phases 1 and 3 must be considered even in cases where we think we already have a good mathematical model at our disposal. It is here that perturbation theory has proved invaluable.

1.0.2 Numerical Noise vs. Physical Noise

Example 1.0.2 (C. Lanczos) *Solving the* 2×2 *linear system*

$$\left.\begin{aligned} x + y &= 2.00001 \\ x + 1.00001y &= 2.00002 \end{aligned}\right\} \tag{1.3}$$

we obtain the solution

$$\begin{aligned} x &= 1.00001 \\ y &= 1. \end{aligned}$$

Suppose that the values on the R.H.S. were obtained from measurements which have limited accuracy. Suppose they are accurate to $\pm 10^{-3}$.

Someone else takes the measurement and gets:

$$x + y = 2.001 \tag{1.4}$$

$$x + 1.00001y = 2.002. \tag{1.5}$$

Solving yields the solution $(x, y) = (-97.999, 100)$. A very different solution! The difficulty here is that in this system of equations $x + y$ is well represented but $x - y$ is poorly represented. Setting

$$\xi = \frac{1}{2}(x+y) \quad \text{and} \quad \eta = \frac{1}{2}(x-y), \tag{1.6}$$

gives the system

$$2\xi = 2.00001 \tag{1.7}$$

$$2.00001\xi - 0.00001\eta = 2.00002. \tag{1.8}$$

The first equation immediately gives ξ. Changing the right-hand side by a tiny amount will change the solution by a tiny amount. In this sense $x + y = 2\xi$ is well represented. To get η we will need to divide by 10^{-5}, resulting in

$$\eta = 10^5\Big(2.00001\xi - 2.00002\Big). \tag{1.9}$$

Thus, the value of η is very sensitive to small changes in the measured values.

Here the problem is very simple to understand, but suppose we had a large system and went to the computer to find the solution. Roundoff error would play havoc giving completely erroneous results. The 'exact' numerical solution of a mathematical problem may have no physical significance.

Exercise: *Write (1.3) in matrix form as*

$$A\vec{x} = \vec{s}, \tag{1.10}$$

where $\vec{x} = (x, y)^T$. What are the eigenvalues of the matrix A and how do they imply sensitivity of the solution to the source term $\vec{s}$?

1.0.3 Perturbation Theory and Asymptotic Analysis in Applied Mathematics

Most mathematical problems facing applied mathematicians, scientists, and engineers have features which preclude exact solutions. Some of these features include:

- nonlinear terms in the equations
- variable coefficients
- nonlinear boundary conditions at known boundaries
- linear or nonlinear boundary conditions at unknown boundaries

Perturbation Theory (PT) is the collective name for a group of techniques developed for the purpose of deriving *approximate* solutions, valid in certain limiting cases which are helpful in understanding the essential processes in simple terms. These often serve as benchmarks for fully numerical solutions. They often have highly accurate predictive capability even when applied outside the range of conditions for which the method is justified. Approximate solutions obtained by perturbation theory usually consist of the first two or three terms of a certain series expansion in the neighbourhood of a point at which the solution has an essential singularity. Asymptotic and perturbations methods can be helpful in several ways. First, they can help by directly finding an approximate solution to your problems. Secondly, these methods can be used to find approximations to exact solutions which are difficult to understand (e.g., solutions written in terms of Bessel functions of large or complicated arguements, or in terms of elliptic function). A third approach is to use asymptotic methods to derive simpler problems which can then be solved exactly (or approximately using perturbation and asymptotic methods again!).

The series obtained by perturbation and asymptotic methods is usually divergent and ordinary results from calculus do not apply. Asymptotic Analysis is the new branch of analysis developed to study such series.

Perturbation Theory has its origin in celestial mechanics. From Newtonian Mechanics it is known that the motion of a celestial body, (e.g. the Earth) is specified by

$$M\ddot{x}_i \quad = \quad F_i^{(0)} + \mu F_i^{(1)} + \mu^2 F_i^{(2)} + \cdots\,, \tag{1.11}$$

for $i = 1, 2, 3$, where the $\mathbf{F}^{(j)}(x_1, x_2, x_3, t)$ represent the gravitational forces emanating from other bodies. $\mathbf{F}^{(0)}$ is the largest force, due to the sun.

The other terms, $\mu\mathbf{F}^{(1)}, \mu^2\mathbf{F}^{(2)}, \ldots$ are successively smaller forces due to the moon and other planets. These other forces are *perturbations* of the main force due to the sun. In particular, $\mu \ll 1$ is a small parameter.

In about 1830 Poisson suggested looking for a solution of (1.11) in a series of powers of μ:

$$x_i(t) = x_i^{(0)}(t) + \mu x_i^{(1)}(t) + \mu^2 x_i^{(2)}(t) + \cdots\,, \tag{1.12}$$

the reasoning behind this being that the solution is a function of μ as well as time t: $x_i = x_i(t, \mu)$. Substituting this expansion into (1.11), expanding the $F_i^{(j)}$s in power series of μ,

$$\begin{aligned} &F_i^{(0)}(\mathbf{x}^{(0)} + \mu\mathbf{x}^{(1)} + \mu^2\mathbf{x}^{(2)} + \cdots, t) \\ &\quad = F_i^{(0)}(\mathbf{x}^{(0)}, t) \\ &\qquad + \vec{\nabla} F_i^{(0)}(\mathbf{x}^{(0)}, t) \cdot [\mu\mathbf{x}^{(1)} + \mu^2\mathbf{x}^{(2)} + \cdots] \\ &\qquad\quad + \cdots\,, \end{aligned} \tag{1.13}$$

and equating like powers of μ gives a series of ODEs to solve.

The first, obtained from the coefficients of μ^0, is

$$M\ddot{x}_i^{(0)} \quad = \quad F_i^{(0)}(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, t) \quad i = 1, 2, 3.$$

This is called the *reduced equation* or the *reduced problem*. It is obtained by setting $\mu = 0$. One *must* be able to solve the reduced problem in order to proceed.

Before Poincaré (1859–1912) the mathematical status of perturbation series of the form (1.12) was rarely considered. One could rarely find more than a few terms, let alone determine if the series converged or not. Indeed, it was often not known whether a solution existed or not.

Poincaré shifted the attention from the convergence of a power series, such as $\sum_{n=1}^{\infty} \mu^n x^{(n)}(t)$ where the emphasis is on the limiting behaviour of $\sum_{n=1}^{N} \mu^n x^{(n)}(t)$ as $N \to \infty$ for fixed μ and t, to the new concept of *asymptotic analysis* of finding the limiting behaviour of $\sum_{n=1}^{N} \mu^n x^{(n)}(t)$ as $\mu \to 0$ or $t \to \infty$ for fixed N.

Chapter 2

Simple linear systems and roots of polynomials

2.1 Introduction and simple linear systems

Reference: Lin & Segel

The general idea behind perturbation theory is the following:

(A) Non-dimensionalize the problem to introduce a small parameter, traditionally called ϵ or μ.

(B) Estimate the size of the terms in your model and drop small ones obtaining a **reduced problem**.

(C) Solve the reduced problem.

(D) Compute perturbative corrections.

Basic Simplification Procedure (BSP): Set $\epsilon = 0$ to get the reduced problem. Solve.

Example 2.1.1 *(From Lin & Segel, page 186): Solve approximately*

$$\begin{aligned} \epsilon x + 10y &= 21, \\ 5x + y &= 7, \end{aligned} \tag{2.1}$$

for $\epsilon = 0.01$.

Solution:

(A) Step (A) is already done: equation nondimensionalized and a small parameter ϵ has been introduced.

(B) The Basic Simplification Procedure assumes that the presence of a small parameter in the coefficient of a term indicates that that term is small. Using the BSP, we set $\epsilon = 0$ to get the reduced problem, giving

$$\begin{aligned} 10y_0 &= 21, \\ 5x_0 + y_0 &= 7, \end{aligned} \tag{2.2}$$

where we have introduced x_0 and y_0 to denote the approximate solution.

(C) *The reduced problem is easily solved giving*

$$(x_0, y_0) = (0.98, 2.1). \tag{2.3}$$

(D) *We next find perturbative corrections. The most common approach in perturbation theory is the following. The solution of the system (2.16) depends on ϵ. Denote the solution by $(x, y) = (x(\epsilon), y(\epsilon))$ and assume a Taylor Series for $x(\epsilon)$ and $y(\epsilon)$ exists:*

$$\begin{aligned} x(\epsilon) &= x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots, \\ y(\epsilon) &= y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots. \end{aligned} \tag{2.4}$$

Substituting these expansions into (2.16) gives

$$\begin{aligned} \epsilon^0(10y_0 - 21) + \epsilon(x_0 + 10y_1) + \epsilon^2(x_1 + 10y_2) + \cdots &= 0, \\ \epsilon^0(5x_0 + y_0) + \epsilon(5x_1 + y_1) + \epsilon^2(5x_2 + y_2) + \cdots &= 0. \end{aligned} \tag{2.5}$$

Since these equations should be satisfied for all ϵ in a neighbourhood of 0, the coefficient of each power of ϵ must be zero. Thus we get a sequence of problems:

(a) *The $\mathcal{O}(1)$ terms (those with coefficient ϵ^0) give*

$$\begin{aligned} 10y_0 - 21 &= 0, \\ 5x_0 + y_0 &= 0. \end{aligned} \tag{2.6}$$

This is the reduced problem we have already solved.

(b) *The $\mathcal{O}(\epsilon)$ terms give*

$$\begin{aligned} x_0 + 10y_1 &= 0, \\ 5x_1 + y_1 &= 0. \end{aligned} \tag{2.7}$$

From this we find

$$y_1 = -\frac{x_0}{10} = -0.098, \tag{2.8}$$

and

$$x_1 = -\frac{y_1}{5} = 0.0196. \tag{2.9}$$

(c) *The $\mathcal{O}(\epsilon^2)$ terms give*

$$\begin{aligned} x_1 + 10y_2 &= 0, \\ 5x_2 + y_2 &= 0. \end{aligned} \tag{2.10}$$

giving

$$(x_2, y_2) = (0.000392, -0.00196). \tag{2.11}$$

Thus, to order ϵ^2, we have

$$\begin{aligned} x &= 0.98 + 0.0196\epsilon + 0.000392\epsilon^2 + \cdots, \\ x &= 2.1 - 0.098\epsilon - 0.00196\epsilon^2 + \cdots. \end{aligned} \tag{2.12}$$

For $\epsilon = 0.01$ the first three terms give

$$(x, y) \approx (0.9801960392, 2.099019804). \tag{2.13}$$

The exact solution is

$$(x, y) = \Big(\frac{49}{50-\epsilon}, \frac{105-7\epsilon}{50-\epsilon}\Big), \tag{2.14}$$

which, for $\epsilon = 0.01$ *gives*

$$(x, y) = \Big(\frac{49}{49.99}, \frac{104.93}{49.99}\Big) = (0.9801960392\ldots, 2.09901980396\ldots). \tag{2.15}$$

The first three terms in the perturbation expansion gives the solution to the accuracy of my calculator!

Some important points:

(i) We had to solve the $\mathcal{O}(1)$ problem (i.e., the reduced problem) first. All the subsequent problems depended on it. One always needs to have a reduced problem that can be solved. Trivial in this case, but not always.

(ii) The solution of the reduced problem $(x_0, y_0) = (0.98, 2.1)$ is very close to the exact solution. This is indication that the terms neglected to obtain the reduced problem were indeed small. For the exact solution $\epsilon x = 0.0098\cdots \ll 10y = 20.99\ldots$, so approximating the first equation by dropping ϵx was OK.

The next example shows one way things can go wrong. It is a simple example which allows us to understand why perturbation theory fails in this case.

Example 2.1.2 *(From Lin & Segel): Find an approximate solution of the system*

$$\begin{aligned} \epsilon x + y &= 0.1, \\ x + 101y &= 11, \end{aligned} \tag{2.16}$$

for $\epsilon = 0.01$.

The reduced problem is

$$\begin{aligned} y_0 &= 0.1, \\ x_x + 101y_0 &= 11, \end{aligned} \tag{2.17}$$

which has the solution $(x_0, y_0) = (0.9, 0.1)$. *Solving the system exactly, we have*

$$\begin{aligned} (1-101\epsilon)x &= 11 - 10.1 = 0.9, \\ (101\epsilon - 1)y &= 0.11 - 0.1 = 0.01 \end{aligned} \tag{2.18}$$

so

$$(x, y) = (-90, 1). \tag{2.19}$$

The solution of the reduced problem is way off. What went wrong?

For the exact solution $\epsilon x = -0.9$ *is comparable to the other two terms in the first equation. In obtaining the reduced problem by dropping the* ϵx *term we assumed that it was small compared with the other terms. In this example this assumption is incorrect and it leads to a poor reduced problem.*

In real problems we won't know the exact solution (otherwise we wouldn't be using perturbation methods!), so how can we realize our perturbation solution is wrong? In this example, assuming we haven't noticed the problem we proceed to find perturbative corrections. This leads to

$$\begin{aligned} x &= 0.9 + 90.9\epsilon + 9180.9\epsilon^2 + \cdots, \\ y &= 0.1 - 0.9\epsilon - 90.9\epsilon^2 + \cdots, \end{aligned} \tag{2.20}$$

which, for $\epsilon = 0.01$, gives

$$\begin{aligned} x &= 0.9 + .909 + 0.91809 + \cdots\,, \\ y &= 0.1 - 0.009 - 0.00909 + \cdots\,. \end{aligned} \tag{2.21}$$

It looks like the series will not converge (of course we can't really tell with only three terms). In general the $\mathcal{O}(\epsilon)$ correction should be small compared with the leading-order ($\mathcal{O}(1)$ terms, and the $\mathcal{O}(\epsilon^2)$ terms should be small compared to the $\mathcal{O}(\epsilon)$ terms. This is clearly not the case here.

The exact solution for x is

$$x = \frac{0.9}{1 - 101\epsilon}. \tag{2.22}$$

Thus, $x(\epsilon)$ has a singularity at epsilon $= 1/101 = 0.009901\ldots$ and the Taylor Series expansion for $x(\epsilon)$ cannot converge for $\epsilon = 0.01$.

For $\epsilon = 0.002$, say, the first three terms of the expansion gives a very good approximation ($x \approx 1.1185236$ vs the exact solution $x = 1.12782\ldots$).

Perturbative methods often work only if the small parameter(s), ϵ in this case, is small enough. How small 'small enough' is may be difficult to determine.

- Dropping terms uncritically can be dangerous!
- Learning how to simplify a problem consistently is difficult and a **very** important part of this course.

In most problems you will have to introduce a small parameter, or perhaps several small parameters. Where does ϵ come from? Two possibilities:

- Introduce ϵ artificially.
- Obtain ϵ from scaling and non-dimensionalization.

The latter is the most important when dealing with physical problems.

2.2 Roots of polynomials

References: Murdoch or Bender & Orzag.

Example 2.2.1 *(From Bender & Orzag): Artificial introduction of ϵ.*

Find approximate solutions of

$$x^3 - 4.001x + 0.002 = 0. \tag{2.23}$$

Tricky, but

$$x^3 - 4x = x(x-2)(x+2) = 0, \tag{2.24}$$

is easy. Consider (2.23) a perturbation of (2.24). There are many ways to do this, one is to consider the problem

$$x^3 - (4+\epsilon)x + 2\epsilon = 0. \tag{2.25}$$

where we are interested in the solution when $\epsilon = 0.001$. As above, assume the solutions $x(\epsilon)$ have a Taylor series expansion

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots\,. \tag{2.26}$$

Substituting into (2.25) and collecting like powers of ϵ gives

$$(x_0^3 - 4x_0) + (3x_0^2x_1 - 4x_1 - x_0 + 2)\epsilon + (3x_0^2x_2 + 3x_1^2x_0 - 4x_2 - x_1)\epsilon^2 + \mathcal{O}(\epsilon^3) = 0. \tag{2.27}$$

The coefficient of each power of ϵ must be zero, giving a sequence of problems to be solved.

1. $\mathcal{O}(1)$ problem:
$$x_0^3 - 4x_0 = 0 \tag{2.28}$$
giving the three roots $x_0 = -2, 0, 2$. Note that we chose ϵ so that at $\epsilon = 0$ our problem reduced to this simple problem that we already noticed we could easily solve.

2. $\mathcal{O}(\epsilon)$ problem:
$$(3x_0^2 - 4)x_1 = x_0 - 2. \tag{2.29}$$
This is easily solved:
$$x_1 = \frac{x_0 - 1}{3x_0^2 - 4}. \tag{2.30}$$
Each value of x_0 gives a different value for x_1. Note that the denominator $3x_0^2 - 4$ is non-zero for each of our values for x_0.

3. $\mathcal{O}(\epsilon^2)$ problem:
$$(3x_0^2 - 4)x_2 = x_1 - 3x_1^2x_0, \tag{2.31}$$
so
$$x_2 = \frac{x_1 - 3x_1^2x_0}{3x_0^2 - 4}. \tag{2.32}$$
Note that the denominator is the same as in the $\mathcal{O}(\epsilon)$ problem. This is no coincidence. More on this later.

Taking $x_0 = -2$, one root is
$$x^{(1)} = -2 - \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 + \mathcal{O}(\epsilon^3), \tag{2.33}$$
which gives $x^{(1)} \approx -2.000499875$ for $\epsilon = 0.001$.

Comment:

- There may be many ways to introduce a small parameter. Some good, some bad.
- The $\mathcal{O}(1)$ problem (the reduced problem) must be solvable. In the preceding example this problem was a cubic polynomial that we could easily solve, as opposed to the original cubic problem. The higher-order problems were all simple linear problems. Once the leading-order problem was solved the higher-order corrections were simple. This is common to all problems involving finding roots of polynomials, but it is not always the case for other types of problems. Sometimes the higher-order problems get more difficult to solve.

2.2.1 Order of the error

If we truncate our solution at $\mathcal{O}(\epsilon^n)$ then how can we estimate the error? We know that the error is due to the terms $\mathcal{O}(\epsilon^n)$ and higher but that does not mean the error is bounded by $C\epsilon^n$ for some constant $C > 0$. The coefficients of the ϵ^m terms for $m > n$ may grow very rapidly. The series may not converge and in fact many useful asymptotic series do not.

Definition 2.2.1 *We will call $\mathcal{O}_F(\epsilon^n)$ the* **formal order of truncation** *and by this mean that terms of $\mathcal{O}(\epsilon^n)$ and higher are neglected. It says nothing about the error.*

From now on we will use the notation $\mathcal{O}_F(\epsilon^n)$ unless we know the error is bounded by $C\epsilon^n$ in which case the error is $\mathcal{O}(\epsilon^n)$. For our root problem we can say something more.

Let

$$f(x,\epsilon) = x^3 - (4+\epsilon)x + 2\epsilon. \tag{2.34}$$

Then $f(x,\epsilon) = 0$ implicitely defines $x(\epsilon)$ — actually three different $x(\epsilon)$, one for each root. The Implicit Function Theorem guarantees that a unique function is defined by

$$f(x(\epsilon),\epsilon) = 0; \quad x(0) = x_0, \tag{2.35}$$

where x_0 is one of the roots of $f(x,0) = 0$, i.e., $x_0 = -2$, 0, or 2, for a non-zero interval containing $\epsilon = 0$.

Theorem 2.2.1 Implicit Function Theorem: *Let $f(x,\epsilon)$ be a function having continuous partial derivatives (including mixed derivatives) up to order r. Let x_0 satisfy $f(x_0,0) = 0$ and $f_x(x_0,0) \neq 0$. Then there is an $\epsilon_0 > 0$ and a unique C^r function $x = x(\epsilon)$ defined for all $0 \leq |\epsilon| \leq \epsilon_0$ such that*

$$f(x(\epsilon),\epsilon) = 0 \quad \textit{and} \quad x(0) = x_0. \tag{2.36}$$

You can read about the Implicit Function Theorem in, for example, Murdoch 'Perturbations: Theory and Methods', Marsden & Hoffman 'Elementary Classical Analysis' or Apostol 'Calculus: Volume II'.

The function $f(x,\epsilon)$ need not be a polynomial. If it is then it is C^∞ (only a finite number of derivatives being non-zero) and, provided $f_x(x_0,0) \neq 0$, $x(\epsilon)$ exists and is C^∞. For the previous example

$$f(x,\epsilon) = 3x^3 - (4+\epsilon)x + 2\epsilon, \tag{2.37}$$

and

$$f_x(x,\epsilon) = 3x^2 - 4 \tag{2.38}$$

which is nonzero for all three roots. Thus, by the Implicit Function Theorem, the solution $x(\epsilon)$ exists for all $0 \leq |\epsilon| \leq \epsilon_0$ for some $\epsilon_0 > 0$. The theorem does not help us determine the size of ϵ_0. Taylor's Theorem (see below) can be used to show that, using a third-order approximation for example,

$$\Big|x(\epsilon) - (x_0 + x_1\epsilon + x_2\epsilon^2)\Big| \leq M\frac{\epsilon^3}{6}, \tag{2.39}$$

where

$$M = \max\Big\{\Big|\frac{\partial^3 x}{\partial \xi^3}(\xi)\Big| : \xi \in [0,\epsilon_0]\Big\}, \tag{2.40}$$

which gives us some information about the error.

Theorem 2.2.2 Taylor's Theorem: *Let $x(\epsilon)$ be a C^r function on $|\epsilon| < \epsilon_0$. For $k \le r-1$ let $p_k(\epsilon)$ be the Taylor polynomial*

$$p_k(\epsilon) = \sum_0^k \frac{x^{(n)}}{n!} \tag{2.41}$$

where $x^{(n)}$ denotes the n^{th} derivative of x. Then if $x(\epsilon)$ is approximated by $p_k(\epsilon)$ the error is

$$R_k(\epsilon) = x(\epsilon) - p_k(\epsilon) = \int_0^\epsilon x^{(k+1)}(\eta)\frac{(\epsilon-\eta)^k}{k!}\,d\eta \tag{2.42}$$

and for each $\epsilon_1 \in (0, \epsilon_0)$

$$|R_k(\epsilon)| \le \frac{M_k(\epsilon_1)}{(k+1)!}|\epsilon|^{k+1} \quad \text{for } |\epsilon| \le \epsilon_1 \tag{2.43}$$

where

$$M_k(\epsilon_1) = \max\Big\{|f^{(k+1)}(\epsilon)| \text{ for } |\epsilon| \le \epsilon_1\Big\} \tag{2.44}$$

For a proof, which is based on the fundamental theorem of calculus and integration by parts, see any first year Calculus book. It is also discussed in the text by Murdoch.

2.2.2 Sometimes you don't expand in powers of ϵ

The presence of a small parameter ϵ in your problem does not necessarily imply that the perturbation series solution is in integer powers of ϵ. Consider the following.

Example 2.2.2 *Find approximate roots $x(\epsilon)$ of*

$$f(x, \epsilon) = x^3 - x^2 + \epsilon = 0. \tag{2.45}$$

Solution: Proceeding as before substitute

$$x = x_0 + x_1\epsilon + x_2\epsilon^2 + \cdots \tag{2.46}$$

into the equation giving

$$x_0^3 - x_0^2 + \Big((3x_0^2 - 2x_0)x_1 + 1\Big)\epsilon + \Big((3x_0^2 - 2x_0)x_2 + 3x_0x_1^2 - x_1^2\Big)\epsilon^2 + \cdots = 0. \tag{2.47}$$

This leads to the following sequence of problems:

1. *$\mathcal{O}(1)$ problem:*
$$x_0^3 - x_0^2 = 0. \tag{2.48}$$
which has two roots: $x_0 = 1$ and $x_0 = 0$. The latter is a double root.

2. *$\mathcal{O}(\epsilon)$ problem:*
$$(3x_0^2 - 2x_0)x_1 + 1 = 0, \tag{2.49}$$
giving
$$x_1 = -\frac{1}{3x_0^2 - 2x_0}. \tag{2.50}$$

3. *$\mathcal{O}(\epsilon^2)$ problem: The solution is*
$$x_2 = -\frac{3x_0x_1^2 - x_1^2}{3x_0^2 - 2x_0}. \tag{2.51}$$

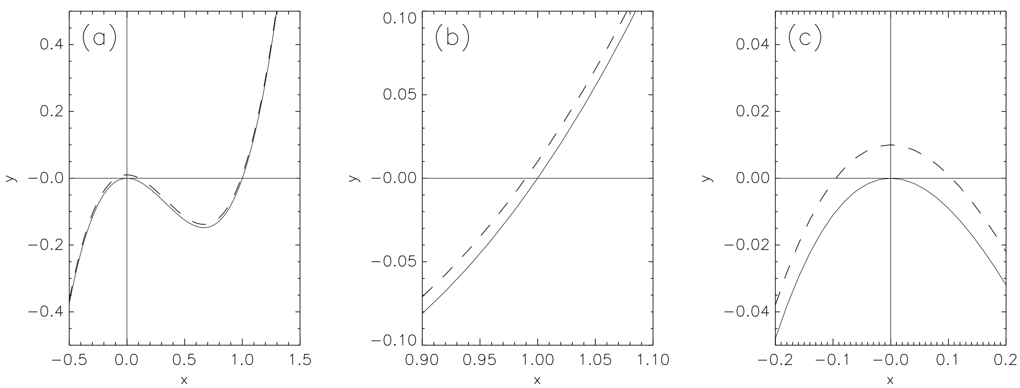

Figure 2.1: (a) Plots of the functions y = f(x,0) (solid curve) and $y = f(x, 0.01)$ (dashed curve) where $f(x,\epsilon) = x^3 - x^2 + \epsilon$. (b) Neighbourhood of $x = 1$. (c) Neighbourhood of $x = 0$.

For the single root $x_0 = 1$ we find $x_1 = -1$ and $x_2 = -2$, so an approximation to one root is

$$x^{(1)} = 1 - \epsilon - 2\epsilon^2 + \mathcal{O}(\epsilon^3), \tag{2.52}$$

(why can we use $\mathcal{O}$ instead of $\mathcal{O}_F$?). For the double root $x_0 = 0$ both x_1 and x_2 are undefined since the denominator $3x_0^2 - 2x_0 = 0$!

What went wrong and how can we resolve the problem? Note that $f_x(x_0, 0) = 3x_0^2 - 2x_0$ is equal to zero at $x_0 = 0$ so at the double root the conditions of the Implicit Function Theory are not satisfied.

The curves $f(x,\epsilon)$ for $\epsilon = 0$ and 0.01 are illustrated in Figure 2.1. Consider the simple root near $x = 1$. Let $g(x) = f(x, 0)$. For $\epsilon = 0$ the polynomial $y = g(x)$ can be approximated by the tangent line $y = g'(1)(x-1) = x - 1$ in a neighbourhood of $x = 1$. In this example the function $f(x,\epsilon)$ is obtained by adding ϵ to $f(x,0)$ which simply shifts the curve up a distance ϵ. The tangent line is shifted up to $y = g'(1)(x-1) + \epsilon = x - 1 + \epsilon$. This curve intersects the x-axis at $x = 1 - \epsilon/g'(1) = 1 - \epsilon$ which approximates the root of $f(x,\epsilon) = 0$ which is near $x = 1$. Adding ϵ to $g(x)$ shifts the root by $\Delta x = -\epsilon/g'(1)$. In other words, the first correction to the approximate root $x_0 = 1$ is *linear* in ϵ. This is illustrated in Figure 2.2(a).

For the double roots the problem is different. The tangent line to $y = f(x,0)$ at $x = 0$ is the line $y = 0$. Adding ϵ to $f(x,0)$ shifts the tangent line up to $y = \epsilon$ which never crosses the x-axis. We need a higher-order approximation to $f(x,0)$ in this case if we want to estimate the roots $f(x,\epsilon) = 0$ that are close to the origin. In the neighbourhood of $x = 0$ we need to approximate the polynomial with a quadratic. The quadratic passing through $(x,y) = (0,0)$ with the same slope and curvature as $y = f(x,0)$ is $y_q(x) = -x^2$ (this simplest way to see this is that as $x \to 0$, the term x^3 becomes much smaller than $-x^2$ so for sufficiently small x, $x^3 - x^2 \approx -x^2$).

In a small neighbourhood of $x = 0$, $f(x,\epsilon)$ can be approximated by $y = -x^2 + \epsilon$. Its roots are $x = \pm\epsilon^{1/2}$, which may be imaginary if $\epsilon < 0$. Hence

> **For a double root we must expand $x(\epsilon)$ in powers of $\epsilon^{1/2}$. Similarly for roots of order n we must expand $x(\epsilon)$ in powers of $\epsilon^{1/n}$.**

To find perturbative corrections to the double root at $x = 0$ we need to set

$$x(\epsilon) = x_0 + \epsilon^{1/2}x_1 + \epsilon x_2 + \epsilon^{3/2}x_3 + \cdots . \tag{2.53}$$

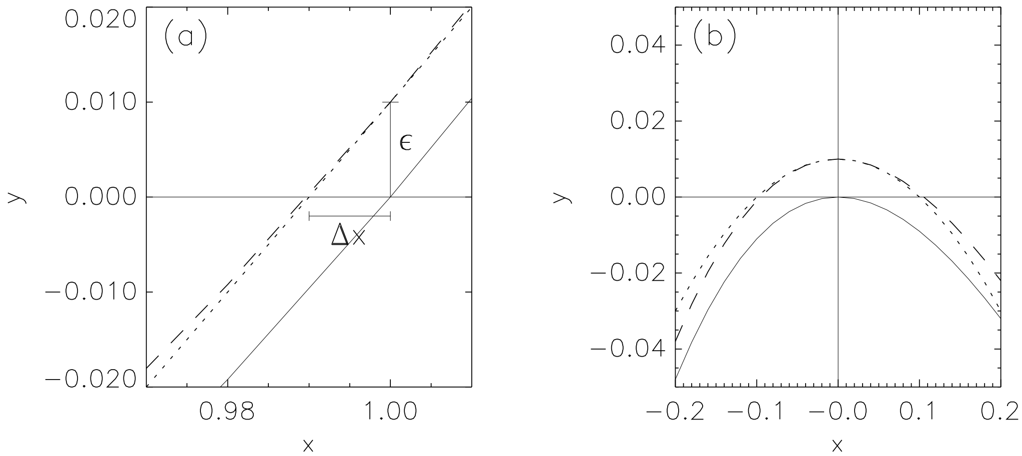

Figure 2.2: As in figure 2.1. (a) Neighbourhood of root at $x = 1$. Dotted curve is linear fit (tangent line) to $y = f(x, 0.01)$ at $x = 1$. (b) Neighbourhood of roots near $x = 0$. Dotted curve is quadratic fit to $y = f(x, 0.01)$ at $x = 0$.

Substituting into $f(x, \epsilon) = 0$ gives

$$\Big(x_0 + \epsilon^{1/2}x_1 + \epsilon x_2 + \epsilon^{3/2}x_3 + \cdots\Big)^3 - \Big(x_0 + \epsilon^{1/2}x_1 + \epsilon x_2 + \epsilon^{3/2}x_3 + \cdots\Big)^2 + \epsilon = 0. \tag{2.54}$$

Expanding and collecting like powers of ϵ leads to

$$\begin{aligned} x_0^3 - x_0^2 + (3x_0^2 - 2x_0)x_1\epsilon^{1/2} + \Big((3x_0^2 - 2x_0)x_2 + 3x_0x_1^2 - 2x_0x_1 - x_1^2 + 1\Big)\epsilon \\ + \Big((3x_0^2 - 2x_0)x_3 + 6x_0x_1x_2 + x_1^3 - 2x_1x_2\Big)\epsilon^{3/2} + \cdots = 0. \end{aligned} \tag{2.55}$$

For the double roots $x_0 = 0$ this simplifies to

$$\Big(-x_1^2 + 1\Big)\epsilon + \Big(x_1^3 - 2x_1x_2\Big)\epsilon^{3/2} + \cdots = 0, \tag{2.56}$$

hence the two roots near zero are

$$x^{2,3} = \pm\epsilon^{1/2} + \frac{1}{2}\epsilon + \mathcal{O}_F(\epsilon^{3/2}). \tag{2.57}$$

[Note: instead of subsituting (2.53) it is easier in this case to use $x_0 = 0$ and substitute $x(\epsilon) = \epsilon^{1/2}x_1 + \cdots$. This simplifies the algebra, particularly if you are finding the solution by hand].

2.2.3 Solving by rescaling: a singular perturbation problem

By an appropriate rescaling we can replace $\mathcal{O}_F$ in the previous solution with $\mathcal{O}$. Let $\mu = \epsilon^{1/2}$ and $x = \mu y$ so that the two roots near $x = 0$, $x^{(2,3)}$, become $y \approx \pm 1$. The polynomial become

$$\mu y^3 - y^2 + 1 = 0. \tag{2.58}$$

Expanding as

$$y = y_0 + \epsilon y_1 + \mu^2 y_2 + \cdots, \tag{2.59}$$

leads to

$$\mu\Big(y_0 + y_1\mu + y_2\mu^2 + y_3\mu^3 + \cdots\Big)^3 - \Big(y_0 + y_1\mu + y_2\mu^2 + y_3\mu^3 + \cdots\Big)^2 + 1 = 0. \tag{2.60}$$

Expanding and collecting like powers of μ leads to

$$-y_0^2 + 1 + (y_0^3 - 2y_0y_1)\mu + (3y_0^2y_1 - 2y_0y_2 - y_1^2)\mu^2 + \cdots = 0. \tag{2.61}$$

Solving this leads to

$$y = \pm 1 + \frac{1}{2}\mu \pm \frac{5}{8}\mu^2 + \mathcal{O}(\mu^3), \tag{2.62}$$

where we can say $\mathcal{O}(\mu^3)$ because the conditions of the implicit function theorem are satisfied. Using $\mu = \epsilon^{1/2}$ and $y = x/\epsilon^{1/2}$ recovers (2.57).

We now have a different problem. The cubic polynomial (2.58) has three roots. Our perturbation solution has only found two of them! What happen to the other one?

We already know that the missing root is $x^{(1)} = 1 - \epsilon - 2\epsilon^2 + \mathcal{O}(\epsilon^3)$. In terms of y and μ this becomes

$$y^{(1)} = \frac{1}{\mu} - \mu - 2\mu^3 + \mathcal{O}(\mu^5). \tag{2.63}$$

This has a singularity at $\mu = 0$. The rescaling $x = \mu y$ is only valid if $\mu \neq 0$.

2.2.4 Finding the singular root: Introduction to the method of dominant balance

In the examples we have considered thus far we have always used the Basic Simplification Procedure (set the small parameter to zero) to obtain the reduced problem. This is not always appropriate, and indeed often is not in singular perturbation problems.

Consider again the problem

$$\mu y^3 - y^2 + 1 = 0, \tag{2.64}$$

where $\mu \ll 1$.

The equation has three terms in it. We wish to simplify the problem and that can only be done by dropping one of the three terms. The idea here is that two of the three terms are much larger than the third so to a first approximation they are equal. This gives the reduced problem. There are three possible cases:

Case 1: μy^3 is much smaller than $-y^2$ and 1. This leads to the reduced problem $y_0^2 = 1$ from which we have already seen two roots are obtained. For two of the three roots μy^3 is indeed small compared with $-y^2$ and 1.

Case 2: y^2 is much smaller than μy^3 and 1. If this is true then $\mu y^3 \approx -1$ which means $y \approx 1^{1/3}/\mu^{1/3}$. Note there are three roots corresponding to each of the cubic roots of 1: 1, $e^{i2\pi/3}$ and $e^{i4\pi/3}$. Since $\mu \ll 1$, y is very large. But that means $y^2 \gg 1$ contradicting our assumption that $y^2 \ll 1$. Thus this case is not consistent and must be discarded.

Case 3: 1 is much smaller than μy^3 and y^2. Solving $\mu y_0^3 = y_0^2$ gives $y_0 = 0$, which violates our assumption that $y^2 \gg 1$, or $y_0 = 1/\mu$. If $y \approx 1/\mu$ then $\mu y^3 \approx y^2 \approx 1/\mu^2 \gg 1$ so this solution is consistent with our assumption that 1 is small compared with the other terms. The full solution is now obtained by expanding $y(\mu)$ as

$$y = \frac{1}{\mu} + y_0 + y_1\mu + y_2\mu^2 + \cdots . \tag{2.65}$$

Proceeding we would obtain (2.63).

2.3 Problems

1. Find approximate solutions of the following problems by finding the first three terms in a perturbation series solution (in an appropriate power of ϵ) using perturbation methods. For problem (a) explain whether the missing terms are $O_F(\epsilon^?)$ or $O(\epsilon^?)$. You should find all of the roots, including complex roots.

   (a) $x^2 + (5 + \epsilon)x - 6 + 3\epsilon = 0.$

   (b) $x^2 + (4 + \epsilon)x + 4 - \epsilon = 0.$

   (c) $(x - 1)^2(x + 2) + \epsilon = 0.$

   (d) $x^3 + \epsilon + 1 = 0.$

   (e) $\epsilon x^3 + x^2 + 2x + 1 = 0.$

   (f) $\epsilon x^5 + (x - 2)^2(x + 1) = 0.$

   (g) $\epsilon x^4 + \epsilon x^3 + x^2 - 3x + 2 = 0.$

Chapter 3

Nondimensionalization and scaling

The chapter is based on material from Lin and Segel (1974). It is strongly recommended that you read the relevent sections of this book.

3.1 Nondimensionalizing to get ϵ

Example 3.1.1 (The Projectile Problem) *Consider a vertically launched projectile of mass m leaving the surface of the Earth with speed v. Find the height of the projectile as a function of time.*

Ignore:

- the Earth's rotation;
- the presence of air (i.e., friction);
- relativistic effects;
- the fact that the Earth is not a perfect sphere;
- etc., etc., etc.

Assume:

- Earth is a perfect sphere;
- Newtonian mechanics apply.

Include:

- Fact that the gravitational force varies with height.

Solution

Let the x-axis extend radially from the centre of the Earth through the projectile. Let $x = 0$ at the Earth's surface. Let M_E and R be the mass and radius of the Earth.

Let $x(t)$ be the height of the projectile at time t. The initial conditions are

$$x(0) = 0 \quad \text{and} \quad \dot{x}(0) = v > 0, \tag{3.1}$$

where the dot denotes differentiation.

From Newtonian mechanics

$$\ddot{x}(t) = -\frac{GM_E}{(x+R)^2} = -\frac{gR^2}{(x+R)^2} \tag{3.2}$$

where $g = GM_E/R^2 \approx 9.8$ m s^{-2} is the gravitational acceleration at $x = 0$.

Summary of the problem:

$$\begin{aligned} \ddot{x} &= -\frac{gR^2}{(x+R)^2}, \\ x(0) &= 0, \\ \dot{x}(0) &= v. \end{aligned} \tag{3.3}$$

We can separate the solution procedure into three steps: (1) dimensional analysis; (2) use the ODE to deduce some useful facts; and (3) nondimensionalize (rescale) the problem to obtain a good reduced problem and find an approximate solution.

1. *Dimensional analysis.*

Physical Quantity	*Dimension*
t, *time*	T
x, *height*	L
R, *radius of Earth*	L
V, *initial speed*	LT^{-1}
g, *acceleration at* $x = 0$	LT^{-2}

There are two dimensions involved: time and length. We need to scale both by introducing nondimensional time and space variables via,

$$t = T_c\tilde{t} \quad \text{and} \quad x = L_c\tilde{x}. \tag{3.4}$$

where T_c and L_c are characteristic time and length scales. They hold the dimensions while $\tilde{t}$ and $\tilde{x}$ are dimensionless. *There are many choices for T_c and L_c.*

Typical values of v, R and g are

$$\begin{aligned} v &\approx 100 \text{ m s}^{-1}, \\ R &\approx 6.4 \times 10^6 \text{ m} \\ g &\approx 10 \text{ m s}^{-2}. \end{aligned}$$

While the values of R and g are fixed the value of v is a choice. This choice is such that the projectile rises high enough for the height variation of the gravitational force has an effect (it will be small).

2. *Use the ODE to say something useful about the solution.*

1. Existence - Uniqueness Theorems for 2^{nd} order ODEs ensures that there is a unique solution up to some time $t_0 > 0$.

2. Multiplying the ODE by $\dot{x}$ and integrating from 0 to $t_{\max}$, where $t_{\max}$ is the time the projectile reaches its maximum height $x_{\max}$ gives

$$x_{\max} = \frac{v^2 R}{2gR - v^2} = \frac{v^2}{2g}\left(\frac{1}{1 - \frac{v^2}{2gR}}\right) \tag{3.5}$$

Note that

1. $x_{\max} \to \infty$ as $v \to \sqrt{2gR} \approx 10^4$ m s^{-1}.

2. For $v \approx 100$ m s^{-1}, $g \approx 10$ m s^{-2}, $R \approx 6.4 \times 10^6$ m,

$$\frac{v^2}{2gR} \approx \frac{10^4}{2 \times 10 \times 6 \times 10^6} \approx 10^{-4} \tag{3.6}$$

$$\Rightarrow x_{\max} \approx \frac{v^2}{2g}. \tag{3.7}$$

3. *Nondimensionalization*

We now consider three possible choices for the time and length scales T_c and L_c. The first two will turn out to be bad choices but they serve to illustrate some of the things that can go wrong and also illustrate the point that you need to put some thought into your choice of scales.

Procedure A:

Take $L_c = R$ and $T_c = R/v$, which is the time needed to travel a distance R at speed v. Then

$$\frac{dx}{dt} = \frac{d\tilde{t}}{dt}\frac{d}{d\tilde{t}}(L_c\tilde{x}) = \frac{L_c}{T_c}\frac{d\tilde{x}}{d\tilde{t}} = v\frac{d\tilde{x}}{d\tilde{t}} \tag{3.8}$$

which makes sense as $L_c/T_c = v$ is the velocity scale. Next

$$\frac{d^2x}{dt^2} = \frac{L_c}{T_c^2}\frac{d^2\tilde{x}}{d\tilde{t}^2} = \frac{v^2}{R}\frac{d^2\tilde{x}}{d\tilde{t}^2} \tag{3.9}$$

Therefore the ODE becomes:

$$\frac{v^2}{R}\frac{d^2\tilde{x}}{d\tilde{t}^2} = -\frac{gR^2}{(R\tilde{x} + R)^2} = -\frac{g}{(\tilde{x} + 1)^2}, \tag{3.10}$$

or

$$\frac{v^2}{gR}\frac{d^2\tilde{x}}{d\tilde{t}^2} = -\frac{1}{(1 + \tilde{x})^2}. \tag{3.11}$$

Recall that $v^2/2gR \approx 10^{-4}$which is very small. Hence

$$\epsilon = \frac{v^2}{gR} \tag{3.12}$$

is a small dimensionless parameter.

Scaling the initial conditions we have

$$x(0) = 0 \quad \rightarrow \quad \tilde{x}(0) = 0 \tag{3.13}$$

$$\dot{x}(0) = v \quad \rightarrow \quad v\frac{d\tilde{x}}{d\tilde{t}}(0) = v \Rightarrow \frac{d\tilde{x}}{d\tilde{t}}(0) = 1, \tag{3.14}$$

hence the final scaled, nondimensional problem is

$$\begin{aligned}
\epsilon\frac{d^2\tilde{x}}{d\tilde{t}^2} &= \frac{-1}{(1+\tilde{x})^2}, \\
\tilde{x}(0) &= 0, \\
\frac{d\tilde{x}}{d\tilde{t}}(0) &= 1.
\end{aligned} \tag{3.15}$$

Because we have only scaled the variables and have not dropped any terms we have not introduced any errors. *No approximation has been made yet and the solution of this scaled problem is the correct solution.* The difficulty lies with the reduced problem. The reduced problem, obtained by setting $\epsilon = 0$, is

$$\begin{aligned}
0 &= -\frac{1}{(1+\tilde{x}_0)^2}, \\
\tilde{x}_0(0) &= 0, \\
\frac{d\tilde{x}_0}{d\tilde{t}}(0) &= 1,
\end{aligned} \tag{3.16}$$

which has no solution! This is a **bad reduced problem**. The small parameter ϵ multiplying the second derivative of $\tilde{x}$ *incorrectly suggests that this term is small.* In fact, at $t = 0$ the r.h.s. is exactly equal to -1. Thus, if $\epsilon = 10^{-4}$, at $t = 0$ $d^2\tilde{x}/d\tilde{t}^2$ must be equal to 10^4, which is very large compared with 1. We need to scale the dimensional variables so the presence of the small parameter ϵ correctly identifies negligible terms. **This is very important.**

Procedure B:

The quantity $\sqrt{\frac{R}{g}}$ has units of time, so let's try $T_c = \sqrt{\frac{R}{g}}$ and take $L_c = R$ as before. This gives

$$\frac{d^2\tilde{x}}{d\tilde{t}^2} = -\frac{1}{(1+\tilde{x})^2}, \tag{3.17}$$

$$\tilde{x}(0) = 0, \tag{3.18}$$

$$\frac{d\tilde{x}}{d\tilde{t}}(0) = \sqrt{\frac{v^2}{Rg}} = \sqrt{\epsilon}, \tag{3.19}$$

where, as before, $\epsilon = v^2/gR \approx 10^{-4} \ll 1$.

As in the previous case, no approximations have been made yet so the solution of this problem is the correct solution. There are, however, two problems with this scale.

1. The ODE has not been simplified!

2. The solution of the reduced problem has $\tilde{x}$ becoming negative for $\tilde{t} > 0$ (since the initial velocity is zero and the initial acceleration is negative). Hence, the solution of the reduced problem has the projectile going the wrong way!

These are both indications of a **bad reduced problem**!

Procedure C:

To get a good reduced problem we must properly scale the variables. You must *think about how you nondimensionalize the problem!*

In procedure A we obtained

$$\epsilon\frac{d^2\tilde{x}}{d\tilde{t}^2} = -\frac{1}{(1+\tilde{x})^2} \tag{3.20}$$

As already pointed out, the problem here is that $\frac{d^2\tilde{x}}{d\tilde{t}^2}$ must be very large so that $\epsilon\frac{d^2\tilde{x}}{d\tilde{t}^2}$ balances the r.h.s. since both sides are equal to negative one at $t = 0$. The nondimensionalization should be done so that the coefficients reflect the size of the whole term.

We'll now do the scaling properly. We have already shown that the maximum height reached by the projectile is

$$x_{\max} = \frac{v^2}{2g}\left(\frac{1}{1-\frac{v^2}{2gR}}\right) \approx \frac{v^2}{2g}, \tag{3.21}$$

since $v^2/(2gR) \approx 10^{-4}$. Thus

$$\frac{x_{\max}}{R} \approx \frac{v^2}{2gR} \approx 10^{-4} \Rightarrow x_{\max} \ll R, \tag{3.22}$$

showing that R is not a good choice for the length scale:

- If we set $x = R\tilde{x}$ then

$$\begin{aligned} 0 &\leq x \leq \frac{V^2}{2g}, \\ \Rightarrow 0 &\leq \tilde{x} \leq \frac{V^2}{2gR} \approx 10^{-4}. \end{aligned} \tag{3.23}$$

  This scaling is not a good choice because $\tilde{x}$ is very tiny, i.e., much smaller than one.

- If we set $x = \frac{V^2}{g}\tilde{x}$ then

$$0 \leq \tilde{x} \leq \frac{1}{2}, \tag{3.24}$$

  i.e. $\tilde{x}$ is an O(1) number. Thus $L_c = v^2/g$ is a much better choice for the length scale. It is in fact the only choice because this scaling reflects the maximum value of $x(t)$.

- v is the obvious velocity scale since the velocity of the projectile must vary between v and $-v$ as the projectile rises and returns to the Earth's surface. If $v = L_c/T_c$ then $T_c = L_c/v = v/g$, is the only logical time scale, since it ensures $\tilde{t}$ is O(1).

- Suppose the time scale is not obvious. Then leave it undetermined for a while. Have:

$$\begin{aligned} \frac{L_c}{T_c^2}\frac{d^2\tilde{x}}{d\tilde{t}^2} &= -\frac{gR^2}{(R+L_c\tilde{x})^2} = \frac{-g}{(1+\frac{L_c}{R}\tilde{x})^2} \\ \Rightarrow \frac{v^2/g}{T_c^2}\frac{d^2\tilde{x}}{d\tilde{t}^2} &= \frac{-g}{(1+\frac{v^2}{gR}\tilde{x})^2} \\ \Rightarrow \left(\frac{v/g}{T_c}\right)^2\frac{d^2\tilde{x}}{d\tilde{t}^2} &= -\frac{1}{(1+\epsilon\tilde{x})^2} \end{aligned} \tag{3.25}$$

where $\epsilon = v^2/(gR) \ll 1$ as before. Since the r.h.s. ≈ -1 , the l.h.s. ≈ -1. To have $\frac{d^2\tilde{x}}{d\tilde{t}^2}$ close to one (in magnitude) means $\frac{v/g}{T_c}$ should be close to 1. Therefor one should choose $T_c = v/g$.

The problem is now

$$\begin{aligned} \frac{d^2\tilde{x}}{d\tilde{t}^2} &= -\frac{1}{(1+\epsilon\tilde{x})^2}, \\ \tilde{x}(0) &= 0, \\ \frac{d\tilde{x}}{d\tilde{t}}(0) &= 1. \end{aligned} \tag{3.26}$$

Setting $\epsilon = 0$ gives the reduced problem

$$\begin{aligned} \Rightarrow \frac{d^2\tilde{x}_0}{d\tilde{t}^2} &= -1, \\ \tilde{x}_0(0) &= 0, \\ \frac{d\tilde{x}_0}{d\tilde{t}} &= 1, \end{aligned} \tag{3.27}$$

which has the solution

$$\tilde{x}_0(t) = \tilde{t} - \frac{\tilde{t}^2}{2}. \tag{3.28}$$

Note that $\max\{\tilde{x}_0\}$ is 1/2 as expected. Note also that $\tilde{x}_o(\tilde{t})$ attains its maximum value at $\tilde{t} = 1$, hence the time scale $T_c = v/g$ can also be interpreted as the characteristic flight time.

3.2 More on Scaling

The goal of scaling is to introduce non-dimensional variables that have order of magnitude equal to 1.

Definition 3.2.1 *A number A has order of magnitude 10^n, n an integer, if*

$$3 \cdot 10^{n-1} < |A| \leq 3 \cdot 10^n \tag{3.29}$$

or if

$$n - \frac{1}{2} < \log_{10}|A| \leq n + \frac{1}{2} \tag{3.30}$$

($\log_{10} 3 \approx \frac{1}{2}$).

By order of magnitude of a function, we mean the order of magnitude of the maximum, or the least upper bound of the function.

Suppose we have a model of the form:

$$f\left(u, \frac{du}{dx}\right) = 0, \; x \in [a, b] \tag{3.31}$$

To properly scale u and x we choose

$$U = \max\{|u| : \; x \in [a, b]\} \tag{3.32}$$

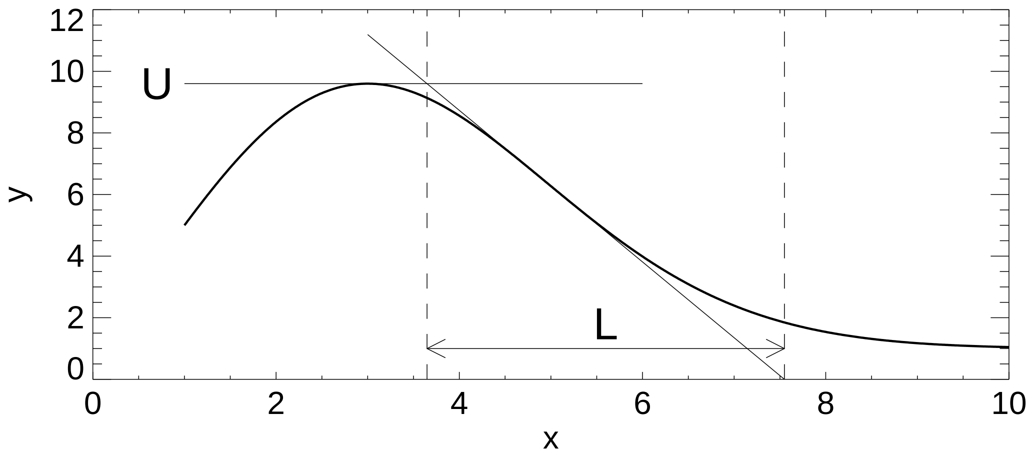

Figure 3.1: Scaling illustration.

so that in setting

$$u = U\tilde{u} \tag{3.33}$$

the function $\tilde{u}$ has order of magnitude 1. We next need to scale x via

$$x = L\tilde{x} \tag{3.34}$$

so that

$$\frac{du}{dx} = \frac{U}{L}\frac{d\tilde{u}}{d\tilde{x}} \tag{3.35}$$

results in $\frac{d\tilde{u}}{d\tilde{x}}$ having order of magnitude 1.

This means we should have

$$\begin{aligned} \frac{U}{L} &= \max\left\{\left|\frac{du}{dx}\right| : \ x \in [a,b]\right\} \\ \Rightarrow L &= \frac{\max|u|}{\max\left|\frac{du}{dx}\right|} \end{aligned} \tag{3.36}$$

Note: If u is known this is easy. If u is unknown this can be difficult.

Example 3.2.1 *Consider the function*

$$u = a\sin(\lambda x), \ a > 0 \text{ on } [0, 2\pi]. \tag{3.37}$$

Solution: Obviously $U = a$ and

$$L = \frac{\max|u|}{\max\left|\frac{du}{dx}\right|} = \frac{a}{a\lambda} = \frac{1}{\lambda}, \tag{3.38}$$

giving

$$\tilde{u} = \sin\tilde{x} \tag{3.39}$$

In general, a model will be of the type

$$f(u, u', u'', \dots, u^{(n)}) = 0 \tag{3.40}$$

One could take L so that

$$\frac{U}{L} = \max|u'| \quad \text{or} \quad \frac{U}{L^2} = \max|u''| \quad \text{or} \quad \dots \quad \text{or} \quad \frac{U}{L^n} = \max|u^{(n)}|. \tag{3.41}$$

You should choose L so that the largest of the non-dimensional derivatives has order of magnitude $1 \Rightarrow L$ is smallest of above choices. Thus, take

$$L = \min\left\{\frac{\max|u|}{\max|u'|}, \left(\frac{\max|u|}{\max|u''|}\right)^{1/2}, \dots, \left(\frac{\max|u|}{\max|u^{(n)}|}\right)^{1/n}\right\}. \tag{3.42}$$

Example 3.2.2 *Consider the function*

$$u = a\sin\lambda x. \tag{3.43}$$

Solution: Have

$$\left(\frac{\max|u|}{\max|u^{(n)}|}\right)^{1/n} = \left(\frac{a}{a\lambda^n}\right)^{1/n} = \frac{1}{\lambda} \tag{3.44}$$

so $L = 1/\lambda$.

Example 3.2.3 *Consider the function*

$$u = a\sin\lambda x + 0.0001a\sin 10\lambda x. \tag{3.45}$$

Solution: Have $\max|u| \approx a$ so take $U = a$. Next,

$$\begin{aligned}
\max|u^{(n)}| &= \max\left|a\lambda^n\begin{pmatrix}\cos(\lambda x)\\ \text{or}\\ \sin(\lambda x)\end{pmatrix} + 10^{n-3}a\lambda^n\begin{pmatrix}\cos(10\lambda x)\\ \text{or}\\ \sin(10\lambda x)\end{pmatrix}\right|\\
&= a\lambda^n\max\left|\begin{pmatrix}\cos(\lambda x)\\ \text{or}\\ \sin(\lambda x)\end{pmatrix} + 10^{n-3}\begin{pmatrix}\cos(10\lambda x)\\ \text{or}\\ \sin(10\lambda x)\end{pmatrix}\right|\\
&\approx \begin{cases} a\lambda^n & \text{for} \quad n \le 3\\ a\lambda^n 10^{n-3} & \text{for} \quad n \gg 1\end{cases}
\end{aligned} \tag{3.46}$$

Thus, for $n \le 3$ one should take $L = 1/\lambda$ while for $n \ge 3$ one should take $L = 1/(10^{1-3/n}\lambda)$ which is approximately $1/10\lambda$. Figure 3.2 shows plots of u and some of its derivatives, clearly illustrating that for large derivatives the fast oscillations dominate and determine the appropriate length scale.

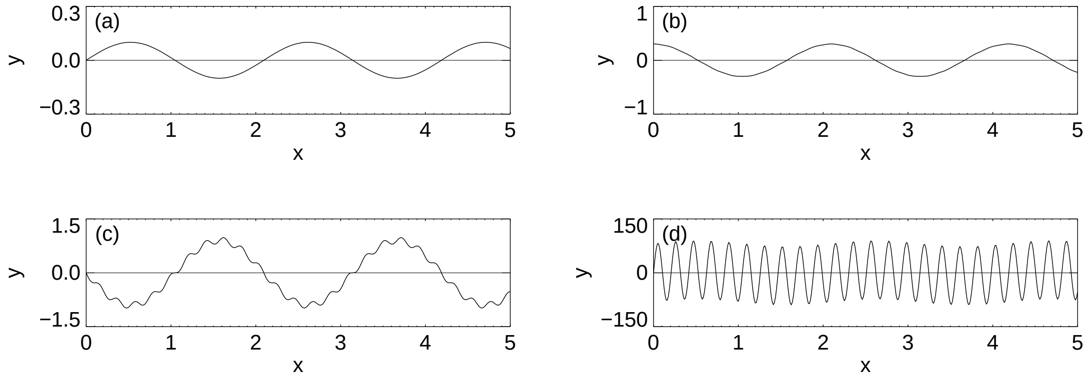

Figure 3.2: Plots of $u(x)$ and some of its derivatives where $u(x) = a\sin(\lambda x) + 0.001a\sin(10\lambda x)$ with $a = 0.1$ and $\lambda = 3$. (a) $u(x)$. (b) $u'(x)$. (c) $u''(x)$. (d) $u^{(4)}(x)$.

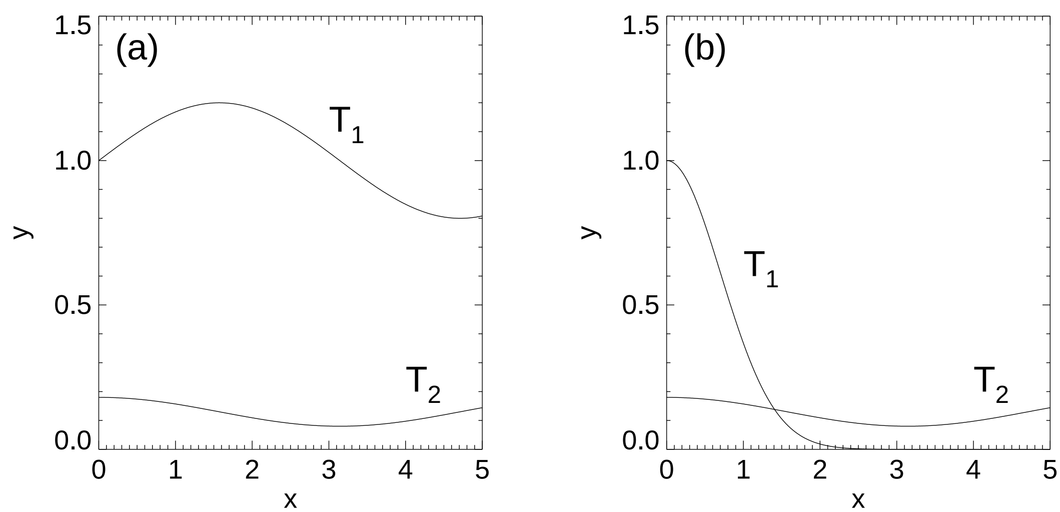

Figure 3.3: (a) Orthodoxy satisfied on $[0, 5]$. (b) Orthodoxy not satisfied on $[0, 5]$.

3.3 Orthodoxy

Suppose we are comparing two terms in a model, $T_1(x)$ and $T_2(x)$, for $x \in [a, b]$, which have been appropriately scaled . We now wish to compare the sizes of each and neglect one if it is small compared to the other.

Problem: The scaling may show that $\max|T_2| \ll \max|T_1|$, but this does not mean that $|T_2| \ll |T_1|$ on all of $[a, b]$.

Definition 3.3.1 *Orthodoxy is said to be satisfied if one term is much smaller than the other on the whole interval.*

If orthodoxy is not satisfied then the intervals on which orthodoxy is not satisfied may be so small that the effects are negligible, e.g., $T_1(x) = \sin x$ and $T_2 = 0.01\cos x$, or multiple scales are needed.

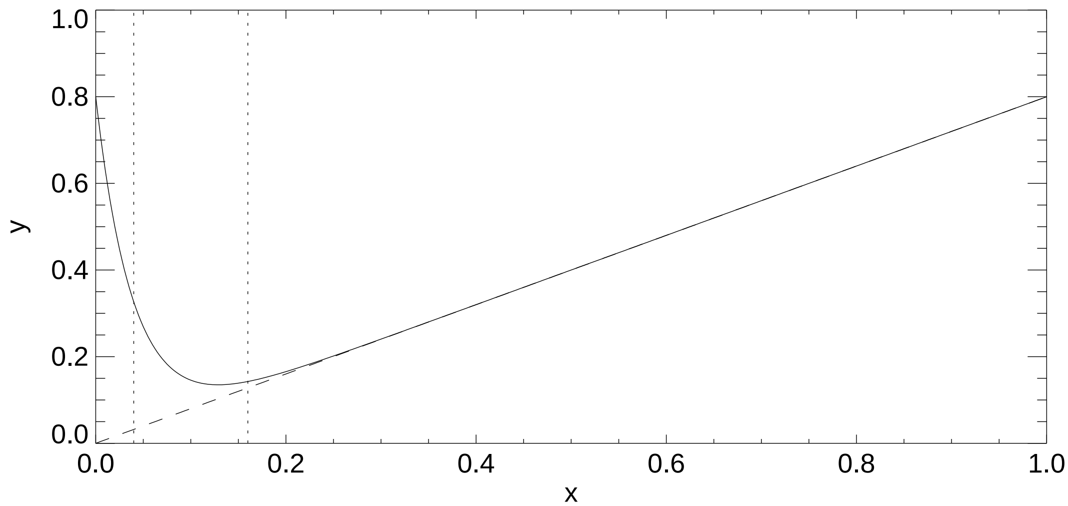

Figure 3.4: Solid: $y = a(x - \exp(-x/\epsilon)$ for $a = 0.8$ and $\epsilon = 0.04$. Dashed: $y = ax$. Vertical dotted lines are $x = \epsilon$ and $x = 4\epsilon$.

Example 3.3.1 *Consider the function* $u(x) = a(x + e^{-x/\epsilon})$ *for* $x \in [0, 1]$, $a > 0$ *and* $0 < \epsilon \ll 1$ *(see Figure 3.4). What scales for* x *should be used?*

The derivative of $u(x)$ is

$$u'(x) = a\left(1 - \frac{1}{\epsilon}e^{-x/\epsilon}\right) = \begin{cases} a\left(1 - \frac{1}{\epsilon}\right) \approx -a/\epsilon \text{ at } x = 0; \\ \\ a\left(1 - \frac{1}{\epsilon}e^{-\frac{1}{\epsilon}}\right) \approx a \text{ at } x = 1; \end{cases} \tag{3.47}$$

for $0 < \epsilon \ll 1$. Taking $L = \frac{\max|u|}{\max|u'|} = \frac{a}{a/\epsilon}$ gives $L = \epsilon$ when $\epsilon \ll 1$. This is a good length scale near the origin (see figure) but not in the region far away from the origin. Away from the origin, say on $[4\epsilon, 1]$

$$\max|u'| = u'(1) \approx a. \tag{3.48}$$

Using $U = a$ and $L = \epsilon$ gives $\tilde{u} = \epsilon\tilde{x} + \exp(-\tilde{x})$ and $\tilde{u}'(\tilde{x}) = \epsilon - \exp(-\tilde{x})$. The interval of interest is now very large, namely $\tilde{x} \in [0, \epsilon^{-1}]$. For $\tilde{x} \gg 1$, which is most of the interval since $\epsilon \ll 1$, $\tilde{u}'(\tilde{x})$ is very tiny. For most of the domain of interest the correct length scale is $L = 1$

Functions such as this one need to be treated differently in different parts of the domain. There is an inner region, near the origin, in which $u(x)$ varies rapidly, and an outer region, away from the origin, where u varies much more slowly.

Inner Region: Within a few multiples of ϵ of $x = 0$

- $\max|u| \approx a$
- $\max|u'| \approx \frac{a}{\epsilon} \Rightarrow U = a, L = \epsilon$

Therefor we should set $u(x) = a\tilde{u}_i$ and $x = \epsilon\tilde{x}_i$ where subscript i denotes inner region. With this scaling

$$u(x) = a(x + e^{-x/\epsilon}) \Rightarrow \tilde{u}_i(\tilde{x}_i) = \epsilon\tilde{x}_i + e^{-\tilde{x}_i} \tag{3.49}$$

The leading order behaviour of $\tilde{u}$ in the inner region is $e^{-\tilde{x}_i}$. We say $\tilde{u}_i(\tilde{x}_i) \sim e^{-\tilde{x}_i}$ as $\epsilon \to 0$ with $\tilde{x}_i$ fixed, where "$\sim$" denotes "is asymptotic to". More on this shortly.

Outer Region: Many multiples of ϵ away from the origin.

In the outer region

$$u' = a\left(1 - \frac{1}{\epsilon}e^{-x/\epsilon}\right) \approx a. \tag{3.50}$$

Both $\max|u|$ and $\max|u'|$ are close to a, hence we should take $U = a$ and $L = 1$. Setting $u = a\tilde{u}_0$ and $x = 1 \cdot \tilde{x}_0$, where the 1 carries the dimensions (if problem hasn't been nondimensionalized yet) we have $\tilde{u} = \tilde{x}_0 + e^{-\tilde{x}_0/\epsilon} \sim \tilde{x}_0$ as $\epsilon \to 0$ for any fixed, nonzero $\tilde{x}_0$ (i.e., for any $\tilde{x}_0$, no matter how small, ϵ can be made sufficiently small, e.g., $\tilde{x}_0/4$ such that the second term is negligible.

Inner and outer regions arise naturally in many problems as illustrated in the above examples. The inner region is often called a boundary layer.

Example 3.3.2 *Consider the problem*

$$\begin{aligned} \epsilon g'' + g' &= 0 \text{ on } [0,1],\ 0 < \epsilon \ll 1, \\ g(0) &= a, \\ g(1) &= b, \end{aligned} \tag{3.51}$$

where $0 < \epsilon \ll 1$.

Solution: The exact solution is

$$\begin{aligned} g &= \left(\frac{b - ae^{-1/\epsilon}}{1 - e^{-1/\epsilon}}\right) + \left(\frac{a-b}{1 - e^{-1/\epsilon}}\right)e^{-x/\epsilon}, \\ &\approx b + (a-b)e^{-x/\epsilon}. \end{aligned} \tag{3.52}$$

Example 3.3.3 *Consider the problem*

$$\begin{aligned} \epsilon f'' - f' &= 0 \text{ on } [0,1],\ 0 < \epsilon \ll 1 \\ f(0) &= a \\ f(1) &= b \end{aligned} \tag{3.53}$$

Solution:

$$\begin{aligned} f &= \left(\frac{b - ae^{1/\epsilon}}{1 - e^{1/\epsilon}}\right) + \left(\frac{a-b}{1 - e^{1/\epsilon}}\right)e^{x/\epsilon}, \\ &\approx b + (a-b)e^{(x-1)/\epsilon}. \end{aligned} \tag{3.54}$$

These two problems only differ by a change in sign of the second term in the differential equation. The solutions are qualitatively very different. The first has a term $e^{-x/\epsilon}$ which decays rapidly near the origin (left side of the domain). The second has a term $e^{(x-1)/\epsilon}$ which decays rapidly as one moves into the domain from the right boundary at $x = 1$. The solutions are shown in Figure 3.5.

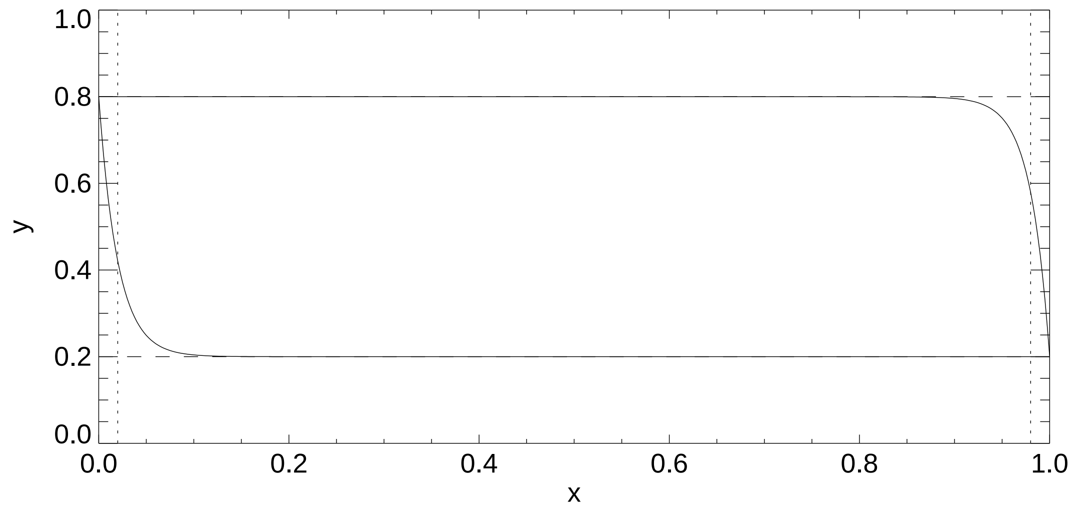

Figure 3.5: Solid curves: solutions of examples 4.2 and 4.3 for $a = 0.8$, $b = 0.2$, and $\epsilon = 0.02$. Dashed lines indicate values of a and b while the vertical dotted lines are $x = \epsilon$ and $x = 1 - \epsilon$.

Question: Attempting to solve $\epsilon y'' + y' = 0$ via regular perturbation methods gives the reduced problem

$$\begin{aligned} y' &= 0, \\ y(0) &= a, \\ y(1) &= b. \end{aligned} \tag{3.55}$$

This is a first-order ODE with two boundary conditions! We can only use one of them. Which one? The solution above shows that we must pick $y(1) = b$ which yields the outer solution. For the second problem, $\epsilon y'' - y' = 0$, the reduced problem is identical but we must now use the boundary condition $y(0) = a$. How can we determine which boundary condition to use without knowing the solution? What happens if ϵ is negative? We will return to questions of this type later when we study boundary layers and matched asymptotics.

Example 3.3.4 *Consider the IVP*

$$\begin{aligned} \ddot{x}(t) + \pi^2 x(t) &= \sin(t) + \epsilon,\ \ t \in \mathbb{R} \\ x(0) &= 1 \\ x'(0) &= 0. \end{aligned} \tag{3.56}$$

1. Find the exact solution.
2. Find $x(t, 0)$ and $x(t, \epsilon)$ and make a sketch. Is orthodoxy satisfied?
3. Is lack of orthodoxy important?

Solution:

1. The general solution of the DE is

$$x(t) = A \sin \pi t + B \cos \pi t + \frac{1}{\pi^2 - 1} \sin t + \frac{\epsilon}{\pi^2}. \tag{3.57}$$

Applying the boundary conditions gives

$$x(t) = -\frac{1}{\pi(\pi^2 - 1)} \sin \pi t + \frac{1}{\pi^2 - 1} \sin t + \frac{\epsilon}{\pi^2}\Big(1 - \cos \pi t\Big). \tag{3.58}$$

2. Near the zeros of $\ddot{x}$, x and $\sin t$ the term ϵ in the ODE will not be much smaller than these terms so orthodoxy is not satisfied

3. It does not matter that orthodoxy is not satisfied in this case.

$$|x(t,0) - x(t,\epsilon)| = \frac{\epsilon}{\pi^2}|1 - \cos \pi t| \leq \frac{2\epsilon}{\pi^2} \ll 1, \tag{3.59}$$

where 1 gives the order of magnitude of the solution (and hence is the appropriate quantity to compare to).

3.4 Example: Inviscid, compressible irrotational flow past a cylinder

Background: (not examinable)

- Inviscid flow means neglect viscosity and heat conduction, (i.e. adiabatic flow).

This type of flow is a good approximation for cases where a fast moving object (i.e. a plane) moves through the air on a time scale much smaller than that required for significant diffusion. It is valid only outside the boundary layer.

Thermodynamics tells us that for isentropic flow the pressure p and density ρ are related by an equation of state $p = p(\rho)$ or $\rho = \rho(p)$. Two important cases are

- For a perfect gas at constant temperature

$$\frac{p}{\rho} = C; \tag{3.60}$$

- For a Perfect Gas at constant entropy

$$\frac{p}{\rho^\gamma} = C, \tag{3.61}$$

where C is a constant and $\gamma = \frac{C_P}{C_V} \approx 1.4$. We will assume isentropic flow (constant entropy).

Let $\mathbf{v}(x,y,z,t)$ be the fluid velocity. The motion of the fluid is governed by the following conservation laws:

1. **Conservation of mass:**

$$\rho_t + \vec{\nabla} \cdot (\rho \mathbf{v}) = 0 \tag{3.62}$$

2. **Conservation of linear momentum:**

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v} \cdot \vec{\nabla}\right)\mathbf{v}\right) = -\vec{\nabla} p \tag{3.63}$$

Definition 3.4.1 Irrotationality: *If fluid particles have no angular momentum then* $\vec{\nabla} \times \mathbf{v} = 0$*.*

Definition 3.4.2 *The* **sound speed** *is defined by*

$$a = \sqrt{\frac{dp}{d\rho}} = \sqrt{\gamma \frac{p}{\rho}}. \tag{3.64}$$

Theorem 3.4.1 (Kelvin, 1868) *For inviscid flow with* $p = p(\rho)$*, if the fluid is initially irrotational and the speed* U *of the flow is less that speed of sound then the flow remains irrotational for all time.*

In this theorem U is the maximum deviation from the flow speed at 'infinity', or far from the cylinder. That is, U should be found in a reference frame fixed with the fluid at infinity.

If $\vec{\nabla} \times \mathbf{v} = 0$ at $t = 0$ then, assuming the conditions of Kelvin's Theorem are satisfied, $\vec{\nabla} \times \mathbf{v} = 0$ for all time $\Rightarrow \mathbf{v} = \vec{\nabla}\phi$ for some velocity potential ϕ. The introduction of a velocity potential greatly simplifies things because the three components of the velocity vector are replaced by a single scalar field.

Using

$$\frac{1}{\rho}\vec{\nabla} p = -\frac{\vec{\nabla} p}{p^{1/\gamma}} C^{1/\gamma} = -\frac{\gamma}{\gamma - 1}\vec{\nabla}(p^{1-1/\gamma})C^{1/\gamma}, \tag{3.65}$$

the momentum equation can be written as

$$\vec{\nabla}\Big(\frac{\partial \phi}{\partial t} + \frac{1}{2}|\vec{\nabla}\phi|^2 + \frac{\gamma}{\gamma - 1} p^{1-1/\gamma} C^{1/\gamma}\Big) = 0. \tag{3.66}$$

Thus,

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\vec{\nabla}\phi|^2 + \frac{a^2}{\gamma - 1} = g(t), \tag{3.67}$$

where $g(t)$ is an undetermined function of time. Assuming a steady uniform far-field flow $\mathbf{v} = (U_\infty, 0, 0)$ with sound speed a_∞^2 gives

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\vec{\nabla}\phi|^2 + \frac{a^2}{\gamma - 1} = \frac{1}{2}U_\infty^2 + \frac{a_\infty^2}{\gamma - 1}. \tag{3.68}$$

The continuity equation can be written as

$$\Big(\frac{\partial}{\partial t} + \vec{\nabla}\phi \cdot \vec{\nabla}\Big) a^2 = -(\gamma - 1) a^2 \nabla^2 \phi, \tag{3.69}$$

Applying the operator $(\partial/\partial t + \vec{\nabla}\phi \cdot \vec{\nabla}$ to (3.68) then yields a single PDE for the velocity potential:

$$a^2 \nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = \frac{\partial}{\partial t}|\vec{\nabla}\phi|^2 + \vec{\nabla}\phi \cdot [(\vec{\nabla}\phi \cdot \vec{\nabla})\vec{\nabla}\phi]. \tag{3.70}$$

We now simplify to 2 dimensions and use

Theorem 3.4.2 (Conformal Mapping Theorem:) *Any simply connected region* $A \subset \mathbb{C}$ *can be transformed (bijectively and analytically) to a disk.*

Using this theorem, for the 2-D case we can assume the object is a disk of radius R. Assuming steady state the model equations give

$$\left(1-\frac{u^2}{a^2}\right)\phi_{xx}-\frac{2uv}{a^2}\phi_{xy}+\left(1-\frac{v^2}{a^2}\right)\phi_{yy}=0, \tag{3.71}$$

where $\mathbf{v}=(u,v)=\vec{\nabla}\phi$.

The discriminant of the PDE is

$$\Delta=\left(\frac{uv}{a^2}\right)^2-\left(1-\frac{u^2}{a^2}\right)\left(1-\frac{v^2}{a^2}\right)=M^2-1 \tag{3.72}$$

where $M=\frac{|\mathbf{v}|}{a}$ is the Mach number:

$M<1$	subsonic flow	equation (3.71) is elliptic $\rightarrow$ static situations
$M=1$	sonic flow	equation (3.71) is parabolic $\rightarrow$ diffusive situations
$M>1$	supersonic flow	equation (3.71) is hyperbolic $\rightarrow$ wave situations

Next we nondimensionalize. Let

$$\begin{aligned}(x,y)&=R(\tilde{x},\tilde{y}),\\(u,v)&=U_\infty(\tilde{u},\tilde{v}).\end{aligned} \tag{3.73}$$

Recall that R is the radius of the cylinder and U_∞ is the far field flow. Then

$$(u,v)=\vec{\nabla}\phi\rightarrow U_\infty(\tilde{u},\tilde{v})=\frac{1}{R}\tilde{\vec{\nabla}}\phi \tag{3.74}$$

So we should set $\phi=RU_\infty\tilde{\phi}$. Putting the terms linear in ϕ on the left and the terms cubic in ϕ on the right gives

$$\frac{U_\infty}{R}\left[\tilde{\phi}_{\tilde{x}\tilde{x}}+\tilde{\phi}_{\tilde{y}\tilde{y}}\right]=\frac{U_\infty^2}{a^2}\left(\tilde{u}^2\frac{U_\infty}{R}\tilde{\phi}_{\tilde{x}\tilde{x}}+2\tilde{u}\tilde{v}\frac{U_\infty}{R}\tilde{\phi}_{\tilde{x}\tilde{y}}+\tilde{v}^2\frac{U_\infty}{R}\tilde{\phi}_{\tilde{y}\tilde{y}}\right), \tag{3.75}$$

where a is a function of x and y. We need to express it in terms of a_∞, the sound speed at infinity. Using (3.68) to eliminate a and dropping the tildes gives

The nondimensional governing equation:

$$\nabla^2\phi=M_\infty^2\left(\phi_x^2\phi_{xx}+2\phi_x\phi_y\phi_{xy}+\phi_y^2\phi_{yy}-\frac{\gamma-1}{2}\nabla^2\phi(1-\phi_x^2-\phi_y^2)\right), \tag{3.76}$$

where $M_\infty=\frac{U_\infty}{a_\infty}$ is the free stream Mach number.

For air at $\approx 20°$C and atmospheric pressure and for $U_\infty\approx 100$ km hr^{-1}, $M_\infty^2\approx 0.1$, so M_∞^2 is a small parameter.

The boundary conditions: No flow through solid boundary and fluid velocity goes to far-field velocity $(1,0)$ at infinity:

$$\begin{aligned}\vec{\nabla}\phi\cdot\widehat{n}&=0 \text{ on } x^2+y^2=1,\\(\phi_x,\phi_y)&\rightarrow(1,0)\quad \text{as } |\mathbf{x}|\rightarrow\infty.\end{aligned} \tag{3.77}$$

The solution will depend on the circulation around the disk. We will assume zero circulation which implies that the flow is symmetric above and below the disk.

Regular Perturbation Theory Solution:

Assume M_∞^2 is small and set

$$\phi = \phi_0(x,y) + M_\infty^2 \phi_1(x,y) + M_\infty^4 \phi_2(x,y) + \cdots \tag{3.78}$$

$\mathcal{O}(1)$ **problem:** At leading order we have

$$\begin{aligned} \nabla^2 \phi_0 &= 0, \\ \vec{\nabla}\phi_0 &\to (U_\infty, 0) \text{ as } |\mathbf{x}| \to \infty, \\ \vec{\nabla}\phi_0 \cdot \widehat{n} &= 0 \text{ on } x^2 + y^2 = 1. \end{aligned} \tag{3.79}$$

In addition ϕ_0 is symmetric about $y = 0$. This Neumann problem for ϕ_0 has the solution

$$\phi_0(r,\theta) = \left(r + \frac{1}{r}\right)\cos\theta. \tag{3.80}$$

Without symmetry condition we get an additional term $A\theta$ for arbitrary A.

$\mathcal{O}(M_\infty^2)$ **problem**: In polar coordinates at the next order we have

$$\begin{aligned} \frac{\partial^2}{\partial r^2}\phi_1 + \frac{1}{r}\frac{\partial}{\partial r}\phi_1 + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\phi_1 &= (\gamma - 1)\left[\left(\frac{1}{r^7} - \frac{1}{r^5}\right)\cos\theta + \frac{1}{r^3}\cos 3\theta\right], \\ \phi_1 &\to 0 \text{ as } r \to \infty, \\ \frac{\partial \phi_1}{\partial r} &= 0 \text{ on } x^2 + y^2 = 1, \\ \phi_1(r,\theta) &= \phi_1(r,-\theta) \quad \text{(symmetry)} \end{aligned} \tag{3.81}$$

which can be solved to yield the total solution

$$\begin{aligned} \phi = \left(r + \frac{1}{r}\right)\cos\theta + \frac{\gamma-1}{2}M_\infty^2 \Bigg(&\left(\frac{13}{12r} - \frac{1}{2r^3} + \frac{1}{12r^5}\right)\cos\theta \\ &+ \left(\frac{1}{12r^3} - \frac{1}{4r}\right)\cos 3\theta \Bigg) + \mathcal{O}_F(M_\infty^4). \end{aligned} \tag{3.82}$$

Remarks:

1. Real life problems can be difficult.
2. Getting the first two terms in a Perturbation Theory expansion can be a lot of work.
3. Problem: What is the error? It is believed that the series is uniformly valid (definition below) but this has not been proven (as of mid-90's. I may be out of date). Hence, this is an example of RPT.

Definition 3.4.3 *A series expansion $\sum \epsilon^2 \xi^2(\cdot,\cdot)$ is said to be uniformly valid if it converges uniformly over all parts of the domain as $\epsilon \to 0$. The series is said to be uniformly ordered if all ξ_n are bounded, in which case the series may not converge.*

More on this later.

Chapter 4

Resonant Forcing and Method of Strained Coordinates: Another example from Singular Perturbation Theory

4.1 The simple pendulum

Consider a mass m suspended from a fixed frictionless pivot via an inextensible, massless string. Let θ be the angle of the string from the vertical. The only force acting on the mass is gravity and the tension in the string (i.e., ignore presence of air). The governing equations for a mass initially at rest at an angle a are

$$\begin{aligned} \frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0, \\ \theta(0) = a, \\ \frac{d\theta}{dt}(0) = 0. \end{aligned} \tag{4.1}$$

The solution of the linear problem, obtained by assuming θ is small and approximating $\sin\theta$ by θ is

$$\theta = a\cos\Big(\sqrt{\frac{g}{\ell}}t\Big). \tag{4.2}$$

According to this solution the mass oscillates with frequency $\sqrt{g/\ell}$ and period $T_\ell = 2\pi\sqrt{\ell/g}$. The full nonlinear problem can be solved exactly in terms of Jacobian elliptic functions. Since these can only be expressed in terms of power series we might as well seek a Perturbation Theory solution which will give a power series solution directly. As a first step we need to scale the variables.

To begin with consider the energy of the system. The governing nonlinear ODE has the energy conservation law

$$\frac{d}{dt}\Big(\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \frac{g}{\ell}\cos\theta\Big) = 0, \tag{4.3}$$

which, after using the initial conditions, gives

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 + \frac{g}{\ell}\cos a = \frac{g}{\ell}\cos\theta. \tag{4.4}$$

From this we can deduce that $|\theta| \leq a$ and that θ oscillates periodically between $\pm a$. Therefore scale θ by a:

$$\theta = a\tilde{\theta}. \tag{4.5}$$

For the time scale take the inverse of the linear frequency, thus set

$$t = \sqrt{\frac{\ell}{g}}\tau. \tag{4.6}$$

The scaled problem is

$$\begin{aligned} \frac{d^2\tilde{\theta}}{d\tau^2} + \frac{\sin a\tilde{\theta}}{a} &= 0, \\ \tilde{\theta}(0) &= 1, \\ \frac{d\tilde{\theta}(0)}{d\tau} &= 0. \end{aligned} \tag{4.7}$$

We will assume that a is small. Note that for small a $\sin(a\tilde{\theta})/a$ is $\mathcal{O}(1)$ hence so is the scaled acceleration $d^2\tilde{\theta}/d\tau^2$. This suggests we have appropriately scaled t.

The Taylor series expansion of $\sin a\tilde{\theta}$ converges for all $a\tilde{\theta}$, so we can write the governing DE in (4.7) as

$$\frac{d^2\tilde{\theta}}{d\tau^2} + \tilde{\theta} - \frac{a^2}{3!}\tilde{\theta}^3 + \frac{a^4}{5!}\tilde{\theta}^5 + \cdots = 0. \tag{4.8}$$

The small parameter a appears only in even powers, hence we seek a Perturbation Theory solution of the form

$$\tilde{\theta} = \theta_0(\tau) + a^2\theta_1(\tau) + a^4\theta_2(\tau) + \cdots . \tag{4.9}$$

$\mathcal{O}(1)$ problem: At leading order we have

$$\begin{aligned} \frac{d^2\theta_0}{d\tau^2} + \theta_0 &= 0, \\ \theta_0(0) &= 1, \\ \frac{d\theta_0}{d\tau}(0) &= 0, \end{aligned} \tag{4.10}$$

which has solution

$$\theta_0 = \cos\tau. \tag{4.11}$$

$\mathcal{O}(a^2)$ problem: At the next order we have

$$\begin{aligned} \frac{d^2\theta_1}{d\tau^2} + \theta_1 &= \frac{1}{3!}\cos^3\tau = \frac{1}{24}\cos 3\tau + \frac{1}{8}\cos\tau, \\ \theta_1(0) &= \frac{d\theta_1}{d\tau}(0) = 0. \end{aligned} \tag{4.12}$$

The general solution of (4.12) is:

$$\theta_1(\tau) = -\frac{1}{192}\cos 3\tau + \frac{1}{16}\tau\sin\tau + A\cos\tau + B\sin\tau. \tag{4.13}$$

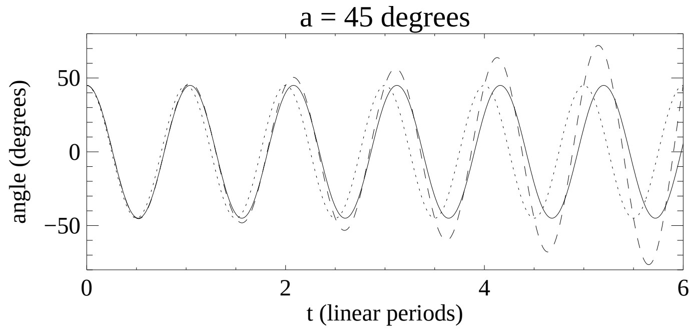

Figure 4.1: Comparison of regular perturbation theory solution with linear and nonlinear solutions for initial angle of 45°. Dotted curve: linear solution. Solid curves: nonlinear solution. Dashed curves: regular perturbation theory solution.

Applying the boundary conditions gives

$$\theta_1 = \frac{1}{192}[\cos\tau - \cos 3\tau] + \frac{\tau}{16}\sin\tau \tag{4.14}$$

so that the total solution is

$$\tilde{\theta} = \cos\tau + a^2\left(\frac{1}{192}(\cos\tau - \cos 3\tau) + \frac{\tau}{16}\sin\tau\right) + \mathcal{O}_F(a^4). \tag{4.15}$$

Problem: The amplitude of the $(a^2/16)\tau\sin\tau$ term grows in time. It is as important as the leading order term, $\cos\tau$, when $a^2\tau/16$ is order 1. Thus, *the perturbation series breaks down by a time of* $O(1/a^2)$*, at which point* $a^2\theta_1$ *is no longer much smaller than* θ_0*.* The breakdown is illustrated in Figure 4.1 for $a = \pi/4$. Note that while the perturbation solution becomes very bad after three or four periods it is better than the linear solution for times up to close to 2 linear periods. At this time the linear solution has drifted away from the nonlinear solution whereas the phase of perturbation solution is much better.

Physically the perturbation solution goes awry because the linear (i.e., the leading-order) and nonlinear solutions drift apart in time. The $O(a^2)$ error made in linearizing the problem to get the leading-order problem for θ_o are cumulative and eventually destroy the approximation. The regular perturbation solution tries to correct for this but does not do so correctly — the phase is improved at the cost of a growing amplitude.

The secular term $(a^2/16)\tau\sin\tau$ appears in the $O(a^2)$ solution because of the appearance of the **resonant forcing term** $\cos\tau$ in the DE for θ_1 (resonant forcing because the forcing term has the same frequency as the homogeneous solution, or more generally because the forcing term is a solution of the homogeneous solution):

$$\frac{d^2\theta_1}{d\tau^2} + \theta_1 = \frac{1}{24}\cos 3\tau + \underbrace{\frac{1}{8}\cos\tau.}_{\substack{\text{resonant}\\ \text{forcing}\\ \text{term}}}$$

The appearance of a resonant forcing term means this is another example of a Singular Perturbation Theory problem.

How can we fix this problem? From energy considerations we know that the amplitude is given by the initial condition. The nonlinearity does not change this. We also know that the solution is periodic. Nonlinearity modifies the shape and period of the oscillations. It increases the period because the true restoring force, $(g/l)\sin(\theta)$ is less than the linearized restoring force $(g/l)\theta$. The properties of the linear and nonlinear solutions are compared in table 4.1.

property	linear solution	nonlinear solution
amplitude	a	a
shape	sinusoidal	non-sinusoidal shape
period	$2\pi\sqrt{l/g}$	increases with amplitude

Table 4.1: Properties of linear and nonlinear solutions.

Because the periods of the linear and nonlinear solutions are different they slowly drift out of phase. Eventually they will be completely out of phase.

The Fix: We must allow the period, or equivalently the frequency, to be a function of a.

Recall the original unscaled problem was

$$\begin{aligned}\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta &= 0,\\ \theta(0) &= a,\\ \frac{d\theta}{dt}(0) &= 0.\end{aligned} \tag{4.16}$$

As before, set $\theta = a\tilde{\theta}$, since this is the amplitude of the nonlinear solution. In our previous attempt we set

$$t = \sqrt{\frac{\ell}{g}}\tau,$$

i.e. we used a time scale $T_c = \sqrt{\ell/g}$, which was independent of a, and proportional to the period of the linearized solution. We need a time scale which is relevant to the nonlinear solution, one which depends on a. Since we do not know how the period depends on a we are forced to introduce an unknown function $\sigma(a)$ via

$$t = \sqrt{\frac{\ell}{g}\frac{\tau}{\sigma(a)}}. \tag{4.17}$$

This is known as the **method of strained coordinates (MSC)** (we have 'strained' time by an unknown function $\sigma(a)$). We will return to this method later.

Since in the limit $a \to 0$ the period does go to $\sqrt{\ell/g}$ we can take $\sigma(0) = 1$. With this time scaling the nondimensionalized problem is

$$\begin{aligned}\sigma^2(a)\frac{d^2\tilde{\theta}}{d\tau^2} + \frac{\sin a\tilde{\theta}}{a} &= 0,\\ \tilde{\theta}(0) &= 1,\\ \frac{d\tilde{\theta}}{d\tau}(0) &= 0.\end{aligned} \tag{4.18}$$

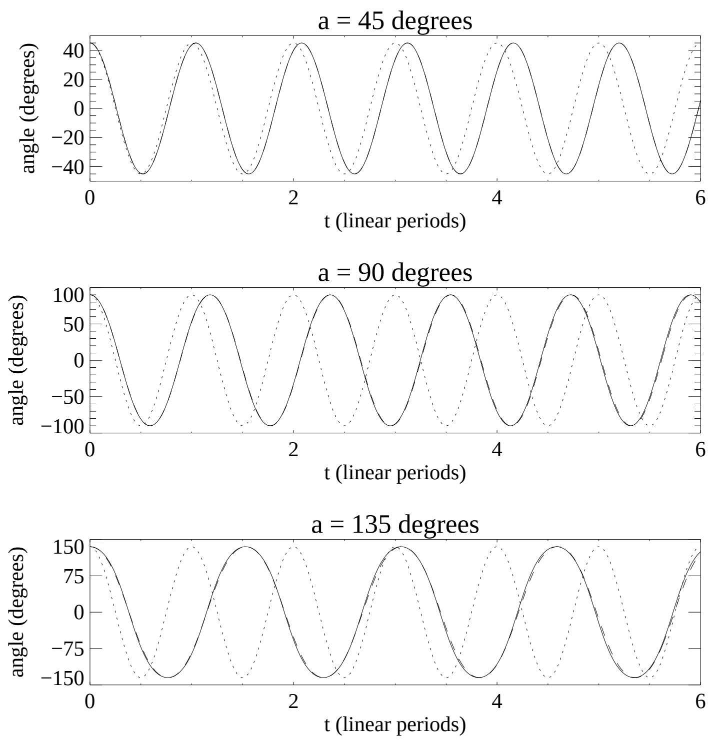

Figure 4.2: Comparison of singular perturbation theory solution with linear and nonlinear solutions for different initial angles. Dotted curves: linear solution. Solid curves: nonlinear solution. Dashed curves: singular perturbation theory solution. The dashed curves are almost identical to the nonlinear solution.

We now expand both $\tilde{\theta}$ and σ in powers of a^2, via

$$\begin{aligned} \tilde{\theta} &= \theta_0(\tau) + a^2\theta_1(\tau) + a^4\theta_2(\tau) + \cdots , \\ \sigma(a) &= 1 + a^2\sigma_1 + a^4\sigma_2 + \cdots . \end{aligned} \tag{4.19}$$

Substituting the series into the differential equation gives

$$\begin{aligned} &\Big(1 + 2\sigma_1 a^2 + (2\sigma_2 + \sigma_1^2)a^4 + \cdots\Big)\Big(\frac{d^2\theta_0}{d\tau^2} + a^2\frac{d^2\theta_1}{d\tau^2} + \cdots\Big) \\ &\quad + \big(\theta_0 + a^2\theta_1 + a^4\theta_2 + \cdots\big) - \frac{a^2}{6}\big(\theta_0 + a^2\theta_1 + a^4\theta_2\big)^3 + \mathcal{O}(a^4) = 0. \end{aligned} \tag{4.20}$$

$\mathcal{O}(1)$ **Problem:** The leading-order problem is unchanged

$$\left.\begin{aligned} &\tfrac{d^2\theta_0}{d\tau^2} + \theta_0 = 0, \\ &\theta_0(0) = 1, \\ &\tfrac{d\theta_0}{d\tau}(0) = 0. \end{aligned}\right\} \quad \Rightarrow \quad \theta_0 = \cos\tau$$

$\mathcal{O}(a^2)$ **Problem:** At $\mathcal{O}(a^2)$ we have

$$\begin{aligned} 2\sigma_1\frac{d^2\theta_0}{d\tau^2} + \frac{d^2\theta_1}{d\tau^2} + \theta_1 - \frac{1}{6}\theta_0^3 &= 0, \\ \theta_1(0) &= 0, \\ \frac{d\theta_1}{d\tau}(0) &= 0. \end{aligned} \tag{4.21}$$

$$\Rightarrow \frac{d^2\theta_1}{d\tau^2} + \theta_1 \quad = \quad \underbrace{\frac{1}{24}\cos 3\tau + \frac{1}{8}\cos\tau.}_{\text{We had this before}} + 2\sigma_1 \cos\tau$$

There is a new resonant forcing term, namely $2\sigma_1 \cos\tau$. By *choosing* $\sigma_1 = -1/16$ the resonant forcing terms are eliminated. There is in fact no choice about this. The only way to eliminate the secular growth in the $O(\epsilon)$ solution is be eliminating the resonant forcing term. This reduces the problem to

$$\frac{d^2\theta_1}{d\tau^2} + \theta_1 = \frac{1}{24}\cos 3\tau, \tag{4.22}$$

which, with the initial conditions, gives

$$\theta_1 = -\frac{1}{192}\left(\cos\tau - \cos 3\tau\right). \tag{4.23}$$

The total solution, so far, is

$$\begin{aligned} \tilde{\theta} &= \cos\tau + \frac{a^2}{192}\left(\cos\tau - \cos 3\tau\right) + \mathcal{O}_F(a^4), \\ \sigma(a) &= 1 - \frac{a^2}{16} + \mathcal{O}_F(a^4), \end{aligned} \tag{4.24}$$

where

$$\tau = \sqrt{\frac{g}{\ell}}\sigma(a)t. \tag{4.25}$$

The dimensional solution is

$$\theta(t) = a\tilde{\theta}(\tau) = a\tilde{\theta}\left(\sqrt{\frac{g}{\ell}}\sigma(a)t\right), \tag{4.26}$$

or

$$\begin{aligned} \theta(t) = {} & a\cos\left(\sqrt{\frac{g}{\ell}}\left(1 - \frac{a^2}{16} + \cdots\right)t\right) \\ & + \frac{a^3}{192}\left[\cos\left(\sqrt{\frac{g}{\ell}}\left(1 - \frac{a^2}{16} + \cdots\right)t\right) - \cos\left(3\sqrt{\frac{g}{\ell}}\left(1 - \frac{a^2}{16} + \cdots\right)t\right)\right] \\ & + \mathcal{O}_F(a^5). \end{aligned} \tag{4.27}$$

The nonlinear solution frequency is $\sigma(a)\sqrt{g/\ell} = \left(1 - \frac{a^2}{16} + \cdots\right)\sqrt{g/\ell} < \sqrt{g/\ell}$ which makes sense because we know that the period of the nonlinear solution must be larger than the period of the linear solution because the forcing in the nonlinear problem, $(g/l)\sin\theta$, is smaller that the forcing in the linear problem, $(g/l)\theta$ (i.e., the acceleration of the nonlinear pendulum is smaller than for the linear pendulum). The SPT solution (4.27) is shown in figure 4.2 showing excellent agreement with the full nonlinear solution over six linear periods for very large initial angles.

Chapter 5

Asymptotic Series

5.1 Asymptotics: large and small terms

Notation

For order of magnitude of a number of function we will use the symbol O_M:

$$\begin{aligned} 90 &= \mathcal{O}_M(100) \\ 0.0072\sin x &= \mathcal{O}_M(10^{-2}) \end{aligned}$$

Definition 5.1.1 (The $\mathcal{O}$ "big-oh" Symbol) *Let f and g be two functions defined on a region $\mathcal{D}$ in $\mathbb{R}^n$ or $\mathbb{C}^n$. Then*

$$f(x) = \mathcal{O}(g(x)) \quad \textit{on } \mathcal{D} \tag{5.1}$$

means that

$$|f(x)| \leq k|g(x)| \quad \forall x \in \mathcal{D} \tag{5.2}$$

for some constant k.

We will usually be interested in the relative behaviour of two functions in the neighbourhood of a point x_0. In that case when we write

$$f(x) \;=\; \mathcal{O}(g(x)) \text{ as } x \to x_0$$

we mean there exists a constant k and a neighbourhood of x_0, $\mathcal{U}$, such that

$$|f(x)| \leq k|g(x)| \text{ for } x \in \mathcal{U}$$

Remarks

1. If $g(x) \neq 0$ then $f(x) = \mathcal{O}(g(x))$ in $\mathcal{D}$ or $f(x) = \mathcal{O}(g(x))$ as $x \to x_0$ can be written as $\frac{f(x)}{g(x)} < \infty$ in $\mathcal{D}$, or $\frac{f(x)}{g(x)}$ is bounded as $x \to x_0$.

2. $\mathcal{O}(g(x))$ on its own has no meaning. The equals sign in "$f(x) = \mathcal{O}(g(x))$" is an abuse of notation.

$$f(x) = \mathcal{O}(g(x)) \Rightarrow 2f(x) = \mathcal{O}(g(x)) \tag{5.3}$$

   but this does not mean that $2f(x) = f(x)$.

3. $f(x) = \mathcal{O}(g(x))$ does not imply that $g(x) = \mathcal{O}(f(x))$. For example, $x^2 = \mathcal{O}(x)$ as $x \to 0$ since $|x^2| < 5|x|$ for $|x| < 5$, but $x \neq \mathcal{O}(x^2)$ as $x \to 0$ because it is not true that $|x| < k|x^2|$ for some constant k in a neighbourhood of 0.

4. An expression containing $\mathcal{O}$ is to be considered a class of functions. For example, $\mathcal{O}(1)+\mathcal{O}(x^2)$ in $0 < x < \infty$ denotes the class of all functions of the type $f+g$ where $f = \mathcal{O}(1)$ and $g = \mathcal{O}(x^2)$.

5. If $f(x) = c$ is a constant, $f = \mathcal{O}(1)$ no matter what the value of c is.

$$\begin{aligned} 10^{-9} &= \mathcal{O}(1), \\ 1 &= \mathcal{O}(1), \\ 10^{9} &= \mathcal{O}(1). \end{aligned}$$

Example 5.1.1

- $x^2 = \mathcal{O}(x)$ on [-2,2] since $x^2 < 5|x|$ on $[-2, 2]$.
- $x^2 \neq \mathcal{O}(x)$ on $[1, \infty]$ since $\frac{|x^2|}{|x|} = |x|$ is unbounded on $[1, \infty]$.
- $\sin(x) = \mathcal{O}(1)$ on $\mathbb{R}$.
- $x^2 = \mathcal{O}(x)$ as $x \to 0$ since $\frac{x^2}{x} = x$ is bounded as $x \to 0$.
- $e^x - 1 = \mathcal{O}(x)$ as $x \to 0$ since $\frac{|e^x-1|}{|x|}$ is bounded as $x \to 0$.

Definition 5.1.2 (The o "little-oh" symbol) *Let f and g be functions defined on a region $\mathcal{D}$ and let x_0 be a limit point of $\mathcal{D}$. Then*

$$f(x) = \mathrm{o}(g(x)) \text{ as } x \to x_0,$$

means that

$$\frac{f(x)}{g(x)} \to 0 \text{ as } x \to x_0.$$

Example 5.1.2

- $x^3 = \mathrm{o}(x^2)$ as $x \to 0$.
- $x^3 = \mathrm{o}(x^4)$ as $x \to \infty$.
- $x^n = \mathrm{o}(e^x)$ as $x \to \infty$.

Note that $f(x) \ll g(x)$ as $x \to x_0$ is the same as $f = \mathrm{o}(g(x))$ as $x \to x_0$.

Definition 5.1.3 (Asymptotic Equivalence) *Let f and g be defined in a region $\mathcal{D}$ with limit point x_0. We write*

$$f \sim g \text{ as } x \to x_0 \tag{5.4}$$

to mean that

$$\frac{f(x)}{g(x)} \to 1 \text{ as } x \to x_0 \tag{5.5}$$

Note:

1. x_0 could be $\pm\infty$.
2. $f \sim g$ as $x \to x_0$ implies that $f = \mathcal{O}(g(x))$ and $g = \mathcal{O}(f(x))$. The converse is not true. For example, $f(x) = x$, $g(x) = 5x$.

Example 5.1.3

- $$x + \frac{1}{x} \sim \frac{1}{x} \quad \text{as } x \to 0,$$
  since
  $$\frac{x + \frac{1}{x}}{\frac{1}{x}} = x^2 + 1 \to 1 \quad \text{as } x \to 0.$$
- $$x + \frac{1}{x} \sim x \quad \text{as } x \to \infty.$$
- $$x^3 + 9x^4 - \frac{3}{2}x^5 \sim \begin{cases} x^3 & \text{as } x \to 0; \\ -\frac{3}{2}x^5 & \text{as } x \to 0. \end{cases}$$
- $$e^{x-9/x} \sim \begin{cases} e^{-9/x} & \text{as } x \to 0; \\ e^x & \text{as } x \to \infty. \end{cases}$$

Note: $f \sim g$ as $x \to x_0 \Rightarrow g \sim f$ as $x \to x_0$.

Note: $f \sim g$ means that $f - g \ll g$.

Example 5.1.4 *The functions $f = e^x + x$ and $g = e^x$ are asymptotic to one another as $x \to \infty$ as*

$$\frac{f-g}{g} = \frac{x}{e^x} \to 0 \text{ as } x \to \infty$$

Note that the difference $f - g$ *does not* go to 0! The difference goes to infinity as $x \to \infty$. Saying $f \sim g$ as $x \to x_0$ does not mean that f and g get close in an absolute sense, it only means that they get close in a relative sense: $f - g$ can blow up but $f - g$ gets small relative to f or g (i.e., gets small in the sense that $(f-g)/g \to 0$). Saying something is large or small can only be done in comparison with something else. You shouldn't say 0.0000001 is small. It is small compared to 1 (which, if someone says 0.0000001 is small, is what they mean implicitely), but it is large compared with 10^{-20}.

Definition 5.1.4 (Asymptotic Series) *To say that*

$$g(x) \sim x^4 - 3x^2 - 2x + \cdots \text{ as } x \to \infty,$$

means the following:

1. *$g \sim x^4$, i.e. $\frac{g}{x^4} \to 1$ as $x \to \infty$,*

2. $g - x^4 \sim -3x^2$, *i.e.* $\frac{g-x^4}{-3x^2} \to 1$ *as* $x \to \infty$,

3. $g - x^4 + 3x^2 \sim -2x$, *i.e.* $\frac{g-x^4+3x^2}{-2x} \to 1$ *as* $x \to \infty$,

etc. The series on the right hand side is an example of an asymptotic series. In the series the fastest growing term comes first. Each successive term must grow more slowly than the preceding term.

Asymptotic series are very useful for finding approximate values of integrals and functions, which we consider next.

5.2 Asymptotic Expansions

We begin by finding an asymptotic expression for an integral.

5.2.1 The Exponential Integral

The exponential integral function $\mathrm{Ei}(x)$ is defined by:

$$\mathrm{Ei}(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt \quad \text{for } x > 0. \tag{5.6}$$

This is not very useful as it stands – can we find a useful approximation? Successively integrating by parts gives

$$\mathrm{Ei}(x) = \underbrace{e^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots + \frac{(-1)^{n-1}(n-1)!}{x^n}\right)}_{S_n(x)} + \underbrace{(-1)^n n! \int_x^\infty \frac{e^{-t}}{x^{n+1}} dt}_{R_n(x)}. \tag{5.7}$$

As $n \to \infty$, $S_n(x)$ gives a divergent series as is easily seen from the ratio test. The ratio of the (m+1)st and mth terms is

$$\frac{\frac{(-1)^m m!}{x^{m+1}}}{\frac{(-1)^{m-1}(m-1)!}{x^m}} = \frac{m}{x} \to \infty \text{ as } m \to \infty \tag{5.8}$$

for fixed x. Suppose we change the question from "What is the limit of $S_n(x)$ as $n \to \infty$ for fixed x?", to "What is the limit as $x \to \infty$ for fixed n".

Have

$$\begin{aligned} |\mathrm{Ei}(x) - S_n(x)| &= |R_n(x)| \\ &\le n! \int_x^\infty \frac{e^{-t}}{t^{n+1}}\,dt \\ &\le \frac{n!}{x^{n+1}} \int_x^\infty e^{-t}\,dt \end{aligned}$$

so

$$|\mathrm{Ei}(x) - S_n(x)| \le \frac{n!}{x^{n+1}} e^{-x} \to 0 \text{ as } x \to \infty. \tag{5.9}$$

Hence for fixed n, $S_n(x)$ gives a good approximation to $\mathrm{Ei}(x)$ if x is sufficiently large. An alternative derivation of this result is the following. Because the error term R_n alternates in sign $S_{2n-1} <$

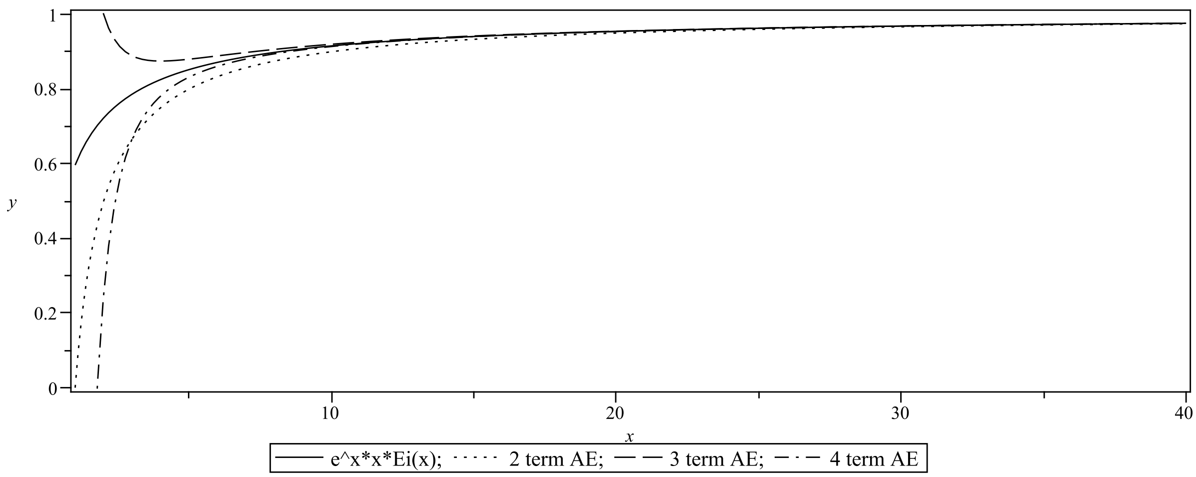

Figure 5.1: Comparison of $xe^x Ei(x)$ and asymptotic approximations using two (dots), three (dashes) and four (dash-dot) terms of the Asymptotic Expansion.

$Ei(x) < S_{2n}$ so the magnitude of the error is less than the magnitude of the first omitted term, namely $e^{-x}n!/x^{n+1}$, as above.

Because of this result we can write

$$\mathrm{Ei}(x) \sim e^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots\right) \text{ as } x \to \infty. \tag{5.10}$$

This is an asymptotic expansion of $\mathrm{Ei}(x)$. Figure 5.1 compares $xe^x\mathrm{Ei}(x)$ with $xe^xS_i(x)$ for $i = 1, 2, 3$. The first two terms of the Asymptotic Expansion, $1 - 1/x$, is within 1% of the exact value for x larger than about 13.3. Using the first four terms the error is less than 1% for x larger than about 6.3.

Now we can ask the question "For a given value of x for what value of n, call it $N(x)$, does $S_n(x)$ give the best approximation to $\mathrm{Ei}(x)$?". The answer to this question is difficult to determine precisely as we only have an upper bound on the magnitude of the error which is easy to use. We can approximate the answer by minimizing our bound on the error. This means choosing n so the first neglected term in the alternating series is minimized. As shown above the ratio of the magnitudes of the (n+1)st and nth terms is

$$\frac{n}{x} < 1 \quad \text{if } n \leq x. \tag{5.11}$$

The terms decrease until $n > x$ thus the minimum is at $nN(x) = \lfloor x \rfloor$, the greatest integer less than x. This implies that as a function of n, $|\mathrm{Ei}(x) - S_n(x)|$ initially decreases monotonically until n exceeds x after which it increases monotonically. This is illustrated in Figure 5.2 which compares $S_n(x)$ with $Ei(x)$ as a function of n for $x = 5$ and 10. Alternatively,

$$|R_n(x)| \leq \frac{n!e^{-x}}{x^{n+1}} = \frac{e^{-x}}{x} \cdot \frac{1}{x} \cdot \frac{2}{x} \cdot \frac{3}{x} \cdot \cdots \cdot \frac{n}{x}. \tag{5.12}$$

The factors $1/x$, $2/x$, $\ldots$ are less than 1, hence decrease R_n until n becomes larger than x.

In summary, for the exponential integral, for fixed x our upper bound on the error is minimized when $n = \lfloor x \rfloor$. Hence, $S_{\lfloor x \rfloor}(x)$ is an estimate $\mathrm{Ei}(x)$ with error $R_{\lfloor x \rfloor}(x) < \frac{e^{-x}\lfloor x \rfloor!}{x^{\lfloor x \rfloor + 1}}$.

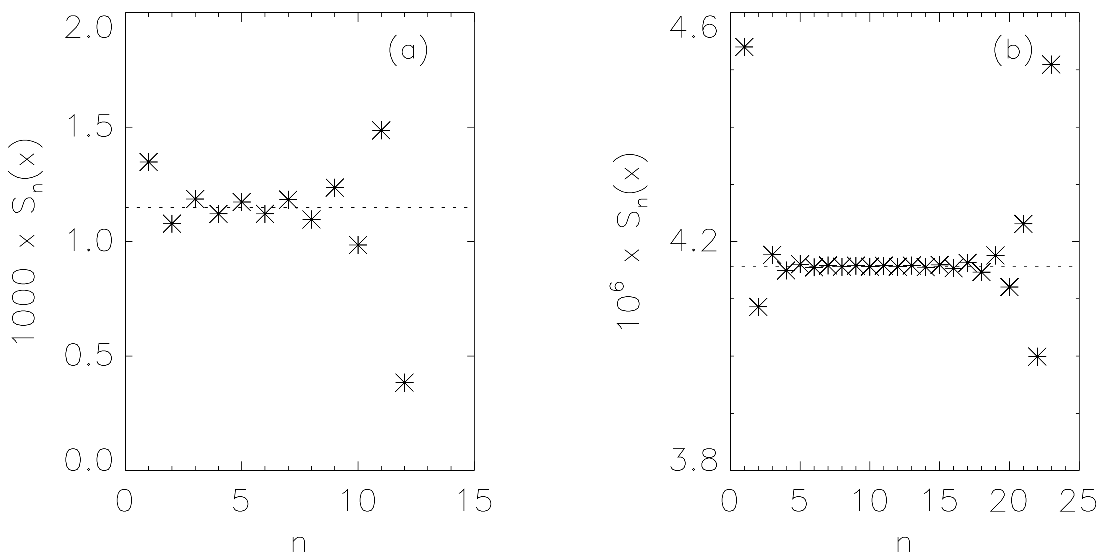

Figure 5.2: Comparison of $\mathrm{Ei}(x)$ (dotted line) with values of partial sums S_n as a function of n. (a) $x = 5.0$. (b) $x = 10.0$.

Rule of Thumb: For an alternating divergent series use $S_N(x)$ where the $(N+1)^{\text{st}}$ term in the asymptotic series is the smallest.

The rule of thumb is a rough guide. In practice we can often take n much less than $n_{\text{opt}} = \lfloor x \rfloor$, depending on the level of accuracy required. This is particularly true for large x as shown in Figure 5.2. Here it can be seen that the $S_n(x)$ are very close to $\mathrm{Ei}(x)$ over a much broader range of values of n when $x = 10$ than when $x = 5$.

Example 5.2.1 *For $x = 10$, $R_4(10) \le \frac{4! e^{-10}}{10^5} \approx 1.1 \times 10^{-8}$. The error bound gives an approximate error of*

$$\left|\frac{R_4(10)}{S_4(10)}\right| \times 100\% \quad = \quad 0.26\%,$$

whereas using the optimal value of n the approximate error is

$$\left|\frac{R_{10}(10)}{S_{10}(10)}\right| \times 100\% \quad = \quad 0.04\%.$$

The actual error using S_4 is

$$\left|\frac{\mathrm{Ei}(10) - S_4(10)}{\mathrm{Ei}(10)}\right| \times 100\% \quad = \quad 0.18\%$$

and it is 0.0193% using S_{10} and -0.0202% using S_{11}.

Important point: For a given x there is a *minimum* error (which is less than the error bound, in this case $|R_{\lfloor x \rfloor}| \le \lfloor x \rfloor! e^{-x}/x^{\lfloor x \rfloor + 1}$) that can be made. In contrast, for a convergent power series the error can be made arbitrarily small if we are prepared to sum enough terms. In this example the minimum error decreases as x increases.

5.2.2 Asymptotic Sequences and Asymptotic Expansions (Poincaré 1886)

Definition 5.2.1 *A set of functions* $\{\varphi_n(x)\}, n = 1, 2, 3, \ldots$ *for* $x \in \mathcal{D}(= \mathbb{R}, \mathbb{R}^n, \mathbb{C})$ *is an asymptotic sequence (AS) as* $x \to x_0$ *if for each* n*,* $\varphi_n(x)$ *is defined on* $\mathcal{D}$ *and* $\varphi_{n+1}(x) = \mathrm{o}(\varphi_n(x))$ *as* $x \to x_0$.

Example 5.2.2

- $\{(x - x_0)^n\}$ is an asymptotic sequence as $x \to x_0$, but is **not** an asymptotic sequence as $x \to \infty$.
- $\{e^{-x}x^{-a_n}\}$ is an asymptotic sequence as $x \to \infty$ where $a_n \in \mathbb{R}$ with $a_n + 1 > a_n$.
- $\{\ln(x)^{-n}\}$ is an asymptotic sequence as $x \to \infty$.

Definition 5.2.2 *Let* x, x_0 *and* $\mathcal{D}$ *be defined as above and let* $f(x)$ *be a function on* $\mathcal{D}$. *Let* $\{\varphi_n(x)\}$ *be an asymptotic series as* $x \to x_0$. *The '***formal***' series*

$$f = \sum_{n=1}^{N} a_n \varphi_n(x) \tag{5.13}$$

is said to be an asymptotic expansion of f *as* $x \to x_0$ *to* N *terms provided*

$$f(x) - \sum_{n=1}^{N} a_n \varphi_n(x) = \left\{ \begin{array}{c} \mathrm{o}(\varphi_N(x)) \\ \text{or} \\ \mathcal{O}(\varphi_{N+1}(x)) \end{array} \right\} \text{ as } x \to x_0. \tag{5.14}$$

Note that (5.14) gives some information about the error, i.e.

$$\text{error} = f(x) - \sum_{n=1}^{N} a_n \varphi_n(x) \to 0$$

faster than $\varphi_N(x) \to 0$ as $x \to x_0$ or it blows up more slowly. This means that the error is small compared to $\varphi_N(x)$. Of course this may only be useful if $\varphi_N(x) \to 0$ as $x \to x_0$ and only for x sufficiently close to x_0.

Important Point: The accuracy of an asymptotic approximation is limited. It has nothing to do with ordinary convergence. In the case of a function $f(x)$ expressed as a convergent power series we can make the error arbitrarily small if we are prepared to sum enough terms. In an asymptotic expansion the potential accuracy is limited.

Example 5.2.3

For $\mathrm{Ei}(x)$ the smallest we can guarantee the error to be less than

$$\frac{n! e^{-x}}{x^{n+1}},$$

with $n = \lfloor x \rfloor$ for any given x. This is an upper bound on the error, so the actual error might be a lot smaller but without further analysis we can't say any more about the error. Thus, there is nothing we can do to reduce the error using this asymptotic expansion (a function has many asymptotic expansions, a different one may give a better error estimate).

Example 5.2.4 *The Bessel function $J_o(x)$ has the power series expansion*

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots . \tag{5.15}$$

which converges to $J_0(x)$ for all x. The power series is completely useless unless x is small. For example,

$$J_0(4) = 1 - 4 + 4 - \frac{16}{9} + \cdots , \tag{5.16}$$

and 8 terms are need to get three digits of accuracy. An asymptotic expansion for $J_0(x)$ is

$$\begin{aligned} J_0(x) \sim \sqrt{\frac{2}{\pi x}} \bigg\{ & \left(1 - \frac{9}{128x^2} + \cdots\right) \cos\left(x - \frac{\pi}{4}\right) \\ & + \left(\frac{1}{8x} - \frac{75}{1024x^2} + \cdots\right) \sin\left(x - \frac{\pi}{4}\right) \bigg\} \quad \text{as } x \to \infty. \end{aligned} \tag{5.17}$$

This series is divergent for all x. This non-divergent asymptotic series is, however, extremely useful. The leading order term

$$\sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right) \tag{5.18}$$

gives $J_0(x)$ to three digit accuracy for all $x \geq 4$! Example approximations are shown in Figure 5.3. There it can be seen that the leading-order asymptotic approximation is very good for $x \geq 1$ whereas the 4, 10 and 20-term power series approximations are useful for $x <\approx 3$, 7.5 and 15 respectively. Finding $J_0(99)$ using the power-series would clearly be difficult but easy using the asymptotic approximation! We will discuss finding the asymptotic expansion for the Bessel function in the next chapter.

Claim 5.2.1 *If $f(x)$ and $\{\varphi_n(x)\}$ are known where $\{\varphi_n\}$ is an asymptotic series, then the asymptotic expansion for f in terms of the φ_n is unique.*

Proof: Need to find a_n's such that

$$f \sim a_1\varphi_1 + a_2\varphi_2 + \cdots \quad \text{as } x \to x_0. \tag{5.19}$$

This means that

$$\begin{aligned} f - a_1\varphi_1 = \mathrm{o}(\varphi_1(x)) & \text{ as } x \to x_0, \\ \Rightarrow \frac{f - a_1\varphi_1}{\varphi_1} &= \frac{f}{\varphi_1} - a_1 \to 0 \text{ as } x \to x_0. \end{aligned}$$

Thus, take

$$a_1 = \lim_{x \to x_0} \frac{f}{\varphi_1}. \tag{5.20}$$

Next

$$\begin{aligned} f - a_1\varphi_1 - a_2\varphi_2 &= \mathrm{o}(\varphi_2), \\ \Rightarrow \frac{f - a_1\varphi_1 - a_2\varphi_2}{\varphi_2} &= \frac{f - a_1\varphi_1}{\varphi_2} - a_2 \to 0 \text{ as } x \to x_0. \end{aligned}$$

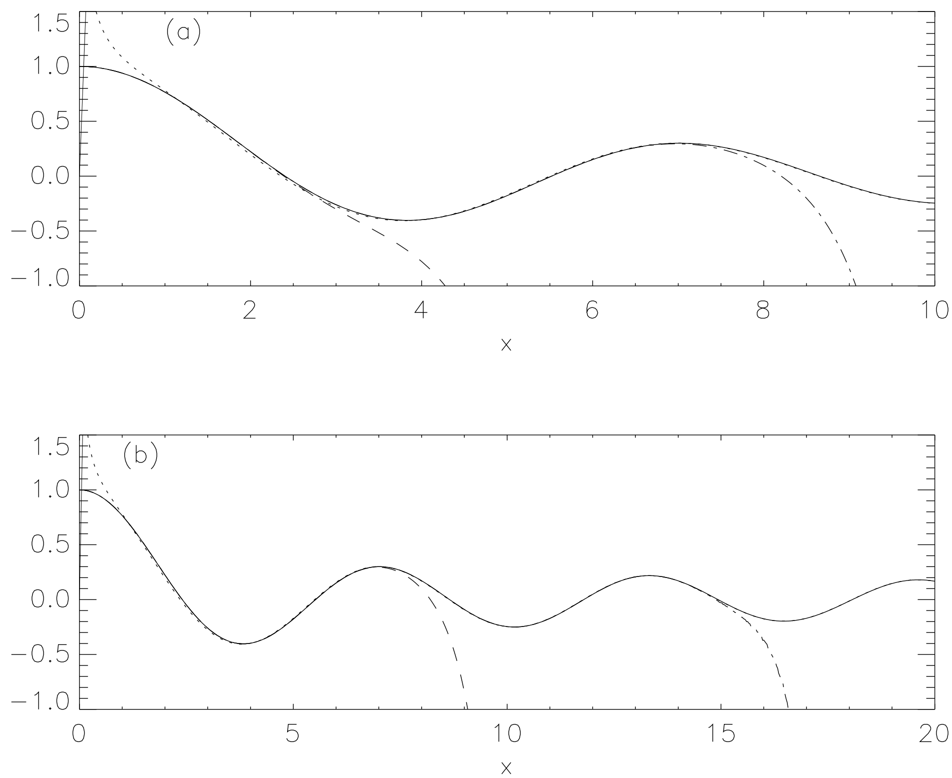

Figure 5.3: Comparison of approximation of $J_0(x)$ with power series or asymptotic series. In both panels the solid curve is $J_0(x)$ and the dotted curve is the leading-order term of the asymptotic expansion. (a) Dashed: 4 term power series approximation. Dash-dot: 10 term power series approximation. (b) Dashed: 10 term power series approximation. Dash-dot: 20 term power series approximation.

Therefore take

$$a_2 = \lim_{x \to x_0} \frac{f - a_1\varphi_1}{\varphi_2}. \tag{5.21}$$

The pattern is clear.

Note:

1. This might give something useless, such as all a_n's are zero, as would happen, for example, if $f = e^{-x}$ and $\varphi_n(x) = \frac{1}{x^n}$ as $x \to \infty$.

2. If the asymptotic series is not known, there will be many possible asymptotic expansions for f. For example,

$$\begin{aligned}
\sin 2\epsilon &\sim 2\epsilon - \frac{4}{3}\epsilon^3 + \frac{4}{15}\epsilon^5 + \cdots \text{ as } \epsilon \to 0, \\
\sin 2\epsilon &\sim 2\tan\epsilon - 2\tan^3\epsilon + 2\tan^5\epsilon + \cdots \text{ as } \epsilon \to 0, \\
\sin 2\epsilon &\sim 2\left(\frac{3\epsilon}{3+2\epsilon^2}\right) - \frac{7}{12}\left(\frac{3\epsilon}{3+2\epsilon^2}\right)^5 + \cdots \text{ as } \epsilon \to 0.
\end{aligned}$$

5.2.3 The Incomplete Gamma Function

Example 5.2.5 *The incomplete Gamma function is defined as*

$$\gamma(a,x) \quad = \quad \int_0^x e^{-t}t^{a-1}\,dt \tag{5.22}$$

for $a, x > 0$.

1. *Derive a power series expansions which converges for all* x. *Show it is useless for large* x.

2. *Find an asymptotic expansion for* γ *by writing (5.22) as*

$$\begin{aligned}\gamma(a,x) &= \int_0^\infty e^{-t}t^{a-1}dt - \int_x^\infty e^{-t}t^{a-1}\,dt \\ &= \Gamma(a) - \mathrm{Ei}_{a-1}(x).\end{aligned}$$

Solution:

1. Using the convergent power series expansion of e^{-t} write

$$\begin{aligned}e^{-t}t^{a-1} &= t^{a-1}\sum_{n=0}^{\infty}\frac{(-1)^n t^n}{n!} \\ &= \sum_{n=0\infty}\frac{(-1)^n t^{n+a-1}}{n!}.\end{aligned} \tag{5.23}$$

The partial sums converge uniformly on any interval $[0, x]$ so we can integrate term by term to get

$$\gamma(a,x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{x^{n+a}}{n+a} = \sum_{n=0}^{\infty} a_n, \tag{5.24}$$

where

$$a_n = \frac{(-1)^n}{n!}\frac{x^{n+a}}{n+a}. \tag{5.25}$$

Applying the ratio test,

$$\frac{a_{n+1}}{a_n} = \frac{x}{(a+n+1)(n+1)} \to 0 \text{ as } n \to \infty, \tag{5.26}$$

showing that the series converges for all x. For any fixed N the partial sum

$$S_N(x) = x^a\sum_{n=0}^{N}\frac{(-1)^n x^n}{(a+n)n!} \to \infty \text{ as } x \to \infty. \tag{5.27}$$

Thus, for large x a large number of terms from the power series are needed to obtain a reasonably accurate approximation. This makes the power series useless for large x.

2. Proceeding as for $\mathrm{Ei}(x)$, several integration by parts yields

$$\begin{aligned}\mathrm{Ei}_{a-1}(x) = x^a e^{-x}&\left(\frac{1}{x} + \frac{(a-1)}{x^2} + \cdots + \frac{(a-1)^{[n-1]}}{x^n}\right) \\ &+ (a-1)^{[n]}\int_x^\infty e^{-t}t^{a-(n+1)}dt,\end{aligned} \tag{5.28}$$

where $k^{[n]} = k(k-1)(k-2)\cdots(k-n+1)$.
Set

$$\begin{aligned} S_n(x,a) &= x^a e^{-x}\left(\frac{1}{x} + \frac{(a-1)}{x^2} + \cdots + \frac{(a-1)^{[n-1]}}{x^n}\right), \\ R_n(x,a) &= (a-1)^{[n]} \int_x^\infty e^{-t} t^{a-n+1}\, dt. \end{aligned} \tag{5.29}$$

As before $S_n(x,a)$ is divergent as $n \to \infty$. For fixed x the integral in R_n converges for all $a > 0$ and $\lim_{x\to\infty} R_n(x,a) = 0$. Have

$$\mathrm{Ei}_{a-1}(x) \sim x^a e^{-x}\left(\frac{1}{x} + \frac{a-1}{x^2} + \cdots + \frac{(a-1)^{[n-1]}}{x^n} + \cdots\right), \tag{5.30}$$

as $x \to \infty$.

Appendix A: USEFULL FORMULAE

Trigonometric Identities:

$$
\begin{aligned}
\sin^3(t) &= \frac{3}{4}\sin(t) - \frac{1}{4}\sin(3t), \\
\cos^3(t) &= \frac{3}{4}\cos(t) + \frac{1}{4}\cos(3t), \\
\sin^5(t) &= \frac{5}{8}\sin(t) - \frac{5}{16}\sin(3t) + \frac{1}{16}\sin(5t), \\
\cos^5(t) &= \frac{5}{8}\cos(t) + \frac{5}{16}\cos(3t) + \frac{1}{16}\cos(5t), \\
(A\cos t + B\sin t)^3 &= \frac{3}{4}A(A^2+B^2)\cos t + \frac{3}{4}B(A^2+B^2)\sin t \\
&\quad + \frac{1}{4}A(A^2-3B^2)\cos 3t - \frac{1}{4}B(B^2-3A^2)\sin 3t
\end{aligned}
$$

$$
\begin{aligned}
\sin(nt)\cos(mt) &= \frac{\sin((n+m)t) + \sin((n-m)t)}{2}, \\
\sin(nt)\sin(mt) &= \frac{\cos((n-m)t) - \cos((n+m)t)}{2}, \\
\cos(nt)\cos(mt) &= \frac{\cos((n+m)t) + \cos((n-m)t)}{2},
\end{aligned}
$$

Solutions of homogeneous ODEs for $y(x)$:

$$
\begin{aligned}
y'' + \frac{a}{x}y' + \frac{b}{x^2}y = 0 \qquad &\rightarrow \qquad \text{try } y \propto x^n, \\
y' = \frac{1}{4}y(4-y) \qquad &\rightarrow \qquad \frac{dx}{dy} = (\frac{1}{y} + \frac{1}{4-y})
\end{aligned}
$$

Particular solutions of common forced ODEs:

$$\begin{aligned}
y'' + \lambda^2 y = \sin\lambda t \qquad & y_p = -\frac{1}{2\lambda} t\cos\lambda t, \\
y'' + \lambda^2 y = \cos\lambda t \qquad & y_p = \frac{1}{2\lambda} t\sin\lambda t, \\
y'' + \lambda^2 y = \sin\alpha t \qquad & y_p = \frac{1}{\lambda^2-\alpha^2}\sin\alpha t \quad \text{for } \lambda\neq\alpha, \\
y'' + \lambda^2 y = \cos\alpha t \qquad & y_p = \frac{1}{\lambda^2-\alpha^2}\cos\alpha t \quad \text{for } \lambda\neq\alpha, \\
y' - \lambda y = e^{\lambda t} \qquad & y_p = te^{\lambda t}, \\
y'' - \lambda y' = 1 \qquad & y_p = -\frac{t}{\lambda}, \\
y'' - \lambda y' = e^{\lambda t} \qquad & y_p = \frac{t}{\lambda}e^{\lambda t} - \frac{1}{\lambda^2}e^{\lambda t}
\end{aligned}$$

Taylor Series:

$$\tanh(x) = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots\,, \tag{9.60}$$

Expansions:

$$\begin{aligned}
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3, \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4, \\
(a+b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5, \\
(a_o + a_1\mu + a_2\mu^2 + \cdots)^2 &= a_o^2 + 2a_0a_1\mu + (2a_oa_2 + a_1^2)\mu^2 + \cdots\,, \\
(a_o + a_1\mu + a_2\mu^2 + \cdots)^3 &= a_o^3 + 3a_0^2a_1\mu + (3a_o^2a_2 + 3a_oa_1^2)\mu^2 + \cdots\,, \\
(a_o + a_1\mu + a_2\mu^2 + \cdots)^4 &= a_o^4 + 4a_0^3a_1\mu + (4a_o^3a_2 + 6a_o^2a_1^2)\mu^2 + \cdots\,, \\
(a_o + a_1\mu + a_2\mu^2 + \cdots)^5 &= a_o^5 + 5a_0^4a_1\mu + (5a_o^4a_2 + 10a_o^3a_1^2)\mu^2 + \cdots\,.
\end{aligned}$$

Methods:

- Lighthill: $y = Y(X)$ is replaced with

$$x = X + \epsilon x_1(X) + \cdots\,.$$

- Pritulo: $y = y_o(x) + \epsilon y_1(x) + \cdots$ is replaced by

$$Y_o(X) + \epsilon Y_1(X) + \cdots \quad \text{with} \quad x = X + \epsilon x_1(X) + \cdots\,.$$

- MSC and Poincaré-Linstedt: $\tau = \omega(\epsilon)t$

Solutions to Selected Problems

Problems from chapter 2

1(a). Have a second order polynomial, hence two roots to find. Setting $\epsilon = 0$ gives two distinct roots -6 and 1 hence expand in powers of ϵ. Get

$$x^{(1)} = -6 - \frac{3}{7}\epsilon - \frac{12}{7^3}\epsilon^2 + O(\epsilon^3),$$
$$x^{(2)} = -6 - \frac{4}{7}\epsilon + \frac{12}{7^3}\epsilon^2 + O(\epsilon^3).$$

1(c). Polynomial of degree three, hence need to find three roots. Setting $\epsilon = 0$ gives a double root at $x_0 = 1$ and a single root $x_0 = -2$. Near the single root expand in powers of ϵ to find $x^{(1)} = -2 + \epsilon/9 + (2/243)\epsilon^2 + O(\epsilon^3)$. Near the double root expand in powers of $\epsilon^{1/2}$ to get $x^{(2,3}) = 1 \pm i\epsilon^{1/2}/\sqrt{3} + \epsilon/18 + O(\epsilon^{3/2})$.

1(e). Need to find three roots. Setting $\epsilon = 0$ gives $x_0 = -1$ as a double root. To find the two roots near $x_0 = -1$ expand in powers of $\epsilon^{1/2}$. Find $x^{1/2} = -1 \pm \epsilon^{1/2} - 3\epsilon/2 + O(\epsilon^{3/2})$. For the third root dominant balance is between ϵx^3 and x^2 so $\epsilon x^3 \approx -x^2$ or $x \approx -1/\epsilon$. Thus set $x = -1/\epsilon + x_1 + x_2\epsilon + x_3\epsilon^2 + \cdots$. Fine $x^{(3)} = -1/\epsilon + 2 + 3\epsilon + O(\epsilon^2)$.

1(g). Need to find four roots. Setting $\epsilon = 0$ give a quadratic equation with two distinct roots. For these expand in powers of ϵ giving $x^(1) = 1+2\epsilon+18\epsilon^2+O(\epsilon^3)$ and $x^(2) = 2-24\epsilon+488\epsilon^2+O(\epsilon^3)$. For the other two roots the dominant balance is between ϵx^4 and x^2 which gives $x \approx \pm i\epsilon^{-1/2}$. Let $\mu = \epsilon^{1/2}$ and $y = \mu x = y_0 + y_1\mu + y_2\mu^2 + \cdots$. Get $y^4 + \mu y^3 + y^2 - 3\mu y + 2\mu^2 = 0$. The leading order problem gives $y_0 = \pm i$ and $y_0 = 0$ as a double root. Only first two of interest. Since $\pm i$ are distinct single roots expand in powers of μ. Find $y = \pm i - 2\mu \pm 3i\mu^2 + O(\mu^3)$ or $x^{3,4} = \pm i/\epsilon^{1/2} - 2 \pm 3i\epsilon^{1/2} + O(\epsilon)$.