Course Notes for AMATH 732

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Contents

Preface

These notes are based to a large degree on lectures notes developed by P. Tenti. Duncan Mowbray LaTeX'd a first draft of a large chunk of these notes.

Useful references

1. Course notes.

2. Bender, C. M., and Orzag, S. A. *Advanced Mathematical Methods for Scientist and Engineers*, McGraw-Hill (1978). QA371.B43 1978.

   The book I learned from. A wealth of topics. Lots and lots of problems. Some very difficult.

3. Lin, C. C., and L. A. Segel, L. A. *Mathematics Applied to Deterministic Problems in the Natural Sciences*, MacMillan (1974). QA37.2.L55 1974

   This is one of the best applied mathematics texts available. It was reprinted by SIAM in 1988. Parts of it can be read on google books.

4. Murdock, J. A. *Perturbations: Theory and Methods*. QA871.M87 1999

5. Nayfeh, A. H. *Introduction to Perturbation Techniques*. QA371.N32 1981

6. Bleistein, N., and Handelsman, R. A. *Asymptotic Expansions of Integrals*, Holt, Rinehart and Winston, New York (1975). QA311.B58 1975

   Title says it all. Dover republished an unabridged corrected version in 1986.

7. Ablowitz, M. J., and Fokas, A. S. *Complex Variables: Introduction and Applications*, Cambridge University Press (2003). QA331.7.A25

   Useful treatment of asymptotic evaluation of integrals (e.g., method of steepest descent).

Chapter 1

Introduction

Before the 18^{th} century, Applied Mathematics and its methods received the close attention of the best mathematicians who were driven by a desire to explain the physical universe. Applied Mathematics can be thought of as a three step process:

$$\begin{array}{ccc}
\text{Physical} & 1 & \text{Mathematical} \\
\text{Situation} & \Longrightarrow & \text{Formulation} \\
 & & \Big\Downarrow 2 \\
\text{Physical} & 3 & \text{Solution by Purely} \\
\text{Interpretation} & \Longleftarrow & \text{Formal Operations} \\
\text{of the Solution} & & \text{of the Math Problem}
\end{array}$$

Over the centuries step 2 took on a life of its own. Mathematics was studied on its own, devoid of contact with a physical problem. This is pure mathematics. Applied mathematics deals with all three steps.

The goal of asymptotic and perturbation methods is to find useful, approximate solutions to difficult problems that arise from the desire to understand a physical process. Exact solutions are usually either impossible to obtain or too complicated to be useful. Approximate, useful solutions are often tested by comparison with experiments or observations rather than by 'rigourous' mathematical methods. Hence we will not be concerned with 'rigorous' proofs in this course. The derivation of approximate solutions can be done in two different ways. First, one can find an approximate set of equations that can be solved, or, one can find an approximate solution of a set of equations. Usually one must do both.

A key turning point in the history of mathematics was the brilliant discovery of the theory of limits of Gauss (1777–1855) and Cauchy (1789–1857). In the limit process, usually characterized by an infinite expansion, we do not attempt to obtain the exact solution but merely to approach it with arbitrary precision. Thus, the desire for absolute accuracy (zero error) was replaced by one for arbitrarily great accuracy (arbitrarily small error):

$$\begin{array}{c}\text{absolute} \\ \text{accuracy}\end{array} \Longrightarrow \begin{array}{c}\text{arbitrarily great} \\ \text{accuracy}\end{array}$$

and

$$\begin{array}{c}\text{zero} \\ \text{error}\end{array} \Longrightarrow \begin{array}{c}\text{arbitrarily small} \\ \text{error}\end{array}.$$

We are no longer interested in what happens after a finite number of steps but wish to know what happens eventually if the number of steps is increased indefinitely. The obvious difficulty with this is that in most real applications you can only sum up a finite number of terms. In fact, for many problems that we will tackle we will obtain only the first two or three terms in a series. We are then not particularly interested in what happens as the number of terms goes to zero but rather in how accurate, or useful, an approximation using a few terms is. Since observations have limited accuracy, there is no need to make the error arbitrarily small.

This gives rise to a different limiting process and different questions: What error occurs after a finite number of steps? How can we minimize the error for a given number of steps? This is a branch of applied analysis.

1.0.1 The Role of Numerical Analysis

An obvious question, particularly in this day and age, is 'If the problem is so difficult why not solve it on a computer'. Ultimately you may end up doing this, but using asymptotic and perturbation techniques to find useful, approximate answers is an extremely important first step. It should always be done whenever possible. Approximate solutions have many benefits. They provide necessary checks, and aid in the understanding and interpretation, of numerical solutions. They illuminate potential problems, e.g., regions in parameter space where singularities exist and where special numerical approaches may be required. They can give tremendous insight into how the solution depends on the parameters of the problem and help determine what the important parameters are.

Example 1.0.1 *Newtonian, constant density, steady state flow past a finite object* $\Omega \subset \mathbb{R}^3$.

$$\left\{\begin{array}{rcl} \mathbf{u}\cdot\vec{\nabla}\cdot\mathbf{u} &=& -\frac{1}{\rho_0}\vec{\nabla}p+\nu\nabla^2\mathbf{u} \\ \mathbf{u}|_{\partial\Omega} &=& 0 \\ \mathbf{u}\to\mathbf{u}_\infty & \text{as} & |\mathbf{x}|\to\infty \end{array}\right. \tag{1.1}$$

where

$$\begin{aligned} \mathbf{u}(x,y,z) &= \text{fluid velocity} \\ \rho_0 &= \text{mass density} \\ p &= \text{hydrodynamic pressure} \\ \mathbf{u}_\infty &= \text{constant for field flow} \\ \nu &= \text{kinematic viscosity} \end{aligned}$$

Much work needs to be done on (1.1), e.g., prove existence and uniqueness. Such a proof may not be constructive, i.e. it may not be helpful in finding the solution. No knowledge of fluid mechanics is required: the problem of proving existence and uniqueness is part of step 2 in our three step process and is part of pure analysis. To obtain an actual solution is another matter. In general, it is impossible. The biggest source of difficulty lies in the nonlinear term $\mathbf{u} \cdot \vec{\nabla}\mathbf{u}$ and in the viscous term $\nu\nabla^2\mathbf{u}$. One way of tackling this problem is to assume that the viscous term $\nu\nabla^2\mathbf{u}$ term is negligible. This would appear to be very reasonable for, e.g., an airplane in air, since the viscosity of air is very small ($\approx 10^{-5}$ m^2 s^{-1}). Dropping this term requires abandonment of the boundary condition $\mathbf{u}|_{\partial\Omega} = 0$ (this condition, which says that the fluid velocity is zero on the solid boundary, is a consequence of viscosity). This results in a linear potential problem

$$\begin{aligned} \mathbf{u} &= \vec{\nabla}\phi \\ \nabla^2\phi &= 0 \\ \nabla\phi \cdot \widehat{n} &= 0 \text{ on } \partial\Omega \end{aligned} \tag{1.2}$$

This approximate linear problem can be solved for some geometries and many general results can be proved as much is known about solutions of Laplace's equation. This makes it very tempting to use the simplified problem (1.2). In fact researchers in the late 1800's and early 1900's used this model and proved that airplanes can't fly!

Today there is a strong tendency to solve problems like (1.1) on a computer. This can be a lot of work and if the mathematical model does not correctly describe the physics then the numerical solution is garbage no matter how accurately you solve the model equations. In fact (1.1) is useful only for laminar flows (e.g., flow over a streamlined body like an airplane wing) because the model is very inaccurate for turbulent flows.

Computers, while very useful and often necessary, should be used in the last stage of a scientific investigation. Analytic work on a mathematical problem is necessary to provide a rough understanding of possible solutions. Phases 1 and 3 must be considered even in cases where we think we already have a good mathematical model at our disposal. It is here that perturbation theory has proved invaluable.

1.0.2 Numerical Noise vs. Physical Noise

Example 1.0.2 (C. Lanczos) *Solving the* 2×2 *linear system*

$$\left.\begin{aligned} x + y &= 2.00001 \\ x + 1.00001y &= 2.00002 \end{aligned}\right\} \tag{1.3}$$

we obtain the solution

$$\begin{aligned} x &= 1.00001 \\ y &= 1. \end{aligned}$$

Suppose that the values on the R.H.S. were obtained from measurements which have limited accuracy. Suppose they are accurate to $\pm 10^{-3}$.

Someone else takes the measurement and gets:

$$x + y = 2.001 \tag{1.4}$$

$$x + 1.00001y = 2.002. \tag{1.5}$$

Solving yields the solution $(x, y) = (-97.999, 100)$. A very different solution! The difficulty here is that in this system of equations $x + y$ is well represented but $x - y$ is poorly represented. Setting

$$\xi = \frac{1}{2}(x + y) \quad \text{and} \quad \eta = \frac{1}{2}(x - y), \tag{1.6}$$

gives the system

$$2\xi = 2.00001 \tag{1.7}$$
$$2.00001\xi - 0.00001\eta = 2.00002. \tag{1.8}$$

The first equation immediately gives ξ. Changing the right-hand side by a tiny amount will change the solution by a tiny amount. In this sense $x + y = 2\xi$ is well represented. To get η we will need to divide by 10^{-5}, resulting in

$$\eta = 10^5 \Big(2.00001\xi - 2.00002\Big). \tag{1.9}$$

Thus, the value of η is very sensitive to small changes in the measured values.

Here the problem is very simple to understand, but suppose we had a large system and went to the computer to find the solution. Roundoff error would play havoc giving completely erroneous results. The 'exact' numerical solution of a mathematical problem may have no physical significance.

Exercise: *Write (1.3) in matrix form as*

$$A\vec{x} = \vec{s}, \tag{1.10}$$

where $\vec{x} = (x, y)^T$. What are the eigenvalues of the matrix A and how do they imply sensitivity of the solution to the source term $\vec{s}$?

1.0.3 Perturbation Theory and Asymptotic Analysis in Applied Mathematics

Most mathematical problems facing applied mathematicians, scientists, and engineers have features which preclude exact solutions. Some of these features include:

- nonlinear terms in the equations
- variable coefficients
- nonlinear boundary conditions at known boundaries
- linear or nonlinear boundary conditions at unknown boundaries

Perturbation Theory (PT) is the collective name for a group of techniques developed for the purpose of deriving *approximate* solutions, valid in certain limiting cases which are helpful in understanding the essential processes in simple terms. These often serve as benchmarks for fully numerical solutions. They often have highly accurate predictive capability even when applied outside the range of conditions for which the method is justified. Approximate solutions obtained by perturbation theory usually consist of the first two or three terms of a certain series expansion in the neighbourhood of a point at which the solution has an essential singularity. Asymptotic and perturbations methods can be helpful in several ways. First, they can help by directly finding an approximate solution to your problems. Secondly, these methods can be used to find approximations to exact solutions which are difficult to understand (e.g., solutions written in terms of Bessel functions of large or complicated arguements, or in terms of elliptic function). A third approach is to use asymptotic methods to derive simpler problems which can then be solved exactly (or approximately using perturbation and asymptotic methods again!).

The series obtained by perturbation and asymptotic methods is usually divergent and ordinary results from calculus do not apply. Asymptotic Analysis is the new branch of analysis developed to study such series.

Perturbation Theory has its origin in celestial mechanics. From Newtonian Mechanics it is known that the motion of a celestial body, (e.g. the Earth) is specified by

$$M\ddot{x}_i = F_i^{(0)} + \mu F_i^{(1)} + \mu^2 F_i^{(2)} + \cdots , \tag{1.11}$$

for $i = 1, 2, 3$, where the $\mathbf{F}^{(j)}(x_1, x_2, x_3, t)$ represent the gravitational forces emanating from other bodies. $\mathbf{F}^{(0)}$ is the largest force, due to the sun.

The other terms, $\mu\mathbf{F}^{(1)}, \mu^2\mathbf{F}^{(2)}, \ldots$ are successively smaller forces due to the moon and other planets. These other forces are *perturbations* of the main force due to the sun. In particular, $\mu \ll 1$ is a small parameter.

In about 1830 Poisson suggested looking for a solution of (1.11) in a series of powers of μ:

$$x_i(t) = x_i^{(0)}(t) + \mu x_i^{(1)}(t) + \mu^2 x_i^{(2)}(t) + \cdots , \tag{1.12}$$

the reasoning behind this being that the solution is a function of μ as well as time t: $x_i = x_i(t, \mu)$. Substituting this expansion into (1.11), expanding the $F_i^{(j)}$s in power series of μ,

$$\begin{aligned} &F_i^{(0)}(\mathbf{x}^{(0)} + \mu\mathbf{x}^{(1)} + \mu^2\mathbf{x}^{(2)} + \cdots , t) \\ &\quad = F_i^{(0)}(\mathbf{x}^{(0)}, t) \\ &\qquad + \vec{\nabla} F_i^{(0)}(\mathbf{x}^{(0)}, t) \cdot [\mu\mathbf{x}^{(1)} + \mu^2\mathbf{x}^{(2)} + \cdots] \\ &\qquad\quad + \cdots , \end{aligned} \tag{1.13}$$

and equating like powers of μ gives a series of ODEs to solve.

The first, obtained from the coefficients of μ^0, is

$$M\ddot{x}_i^{(0)} = F_i^{(0)}(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, t) \quad i = 1, 2, 3.$$

This is called the *reduced equation* or the *reduced problem*. It is obtained by setting $\mu = 0$. One *must* be able to solve the reduced problem in order to proceed.

Before Poincaré (1859–1912) the mathematical status of perturbation series of the form (1.12) was rarely considered. One could rarely find more than a few terms, let alone determine if the series converged or not. Indeed, it was often not known whether a solution existed or not.

Poincaré shifted the attention from the convergence of a power series, such as $\sum_{n=1}^{\infty} \mu^n x^{(n)}(t)$ where the emphasis is on the limiting behaviour of $\sum_{n=1}^{N} \mu^n x^{(n)}(t)$ as $N \to \infty$ for fixed μ and t, to the new concept of *asymptotic analysis* of finding the limiting behaviour of $\sum_{n=1}^{N} \mu^n x^{(n)}(t)$ as $\mu \to 0$ or $t \to \infty$ for fixed N.

Chapter 2

Simple linear systems and roots of polynomials

2.1 Introduction and simple linear systems

Reference: Lin & Segel

The general idea behind perturbation theory is the following:

(A) Non-dimensionalize the problem to introduce a small parameter, traditionally called ϵ or μ.

(B) Estimate the size of the terms in your model and drop small ones obtaining a **reduced problem**.

(C) Solve the reduced problem.

(D) Compute perturbative corrections.

Basic Simplification Procedure (BSP): Set $\epsilon = 0$ to get the reduced problem. Solve.

Example 2.1.1 *(From Lin & Segel, page 186): Solve approximately*

$$\begin{aligned} \epsilon x + 10y &= 21, \\ 5x + y &= 7, \end{aligned} \tag{2.1}$$

for $\epsilon = 0.01$.

Solution:

(A) Step (A) is already done: equation nondimensionalized and a small parameter ϵ has been introduced.

(B) The Basic Simplification Procedure assumes that the presence of a small parameter in the coefficient of a term indicates that that term is small. Using the BSP, we set $\epsilon = 0$ to get the reduced problem, giving

$$\begin{aligned} 10y_0 &= 21, \\ 5x_0 + y_0 &= 7, \end{aligned} \tag{2.2}$$

where we have introduced x_0 and y_0 to denote the approximate solution.

(C) *The reduced problem is easily solved giving*

$$(x_0, y_0) = (0.98, 2.1). \tag{2.3}$$

(D) *We next find perturbative corrections. The most common approach in perturbation theory is the following. The solution of the system (2.16) depends on* ϵ. *Denote the solution by* $(x, y) = (x(\epsilon), y(\epsilon))$ *and assume a Taylor Series for* $x(\epsilon)$ *and* $y(\epsilon)$ *exists:*

$$\begin{aligned} x(\epsilon) &= x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots, \\ y(\epsilon) &= y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots. \end{aligned} \tag{2.4}$$

Substituting these expansions into (2.16) gives

$$\begin{aligned} \epsilon^0(10y_0 - 21) + \epsilon(x_0 + 10y_1) + \epsilon^2(x_1 + 10y_2) + \cdots &= 0, \\ \epsilon^0(5x_0 + y_0) + \epsilon(5x_1 + y_1) + \epsilon^2(5x_2 + y_2) + \cdots &= 0. \end{aligned} \tag{2.5}$$

Since these equations should be satisfied for all ϵ *in a neighbourhood of* 0, *the coefficient of each power of* ϵ *must be zero. Thus we get a sequence of problems:*

(a) *The* $\mathcal{O}(1)$ *terms (those with coefficient* ϵ^0*) give*

$$\begin{aligned} 10y_0 - 21 &= 0, \\ 5x_0 + y_0 &= 0. \end{aligned} \tag{2.6}$$

This is the reduced problem we have already solved.

(b) *The* $\mathcal{O}(\epsilon)$ *terms give*

$$\begin{aligned} x_0 + 10y_1 &= 0, \\ 5x_1 + y_1 &= 0. \end{aligned} \tag{2.7}$$

From this we find

$$y_1 = -\frac{x_0}{10} = -0.098, \tag{2.8}$$

and

$$x_1 = -\frac{y_1}{5} = 0.0196. \tag{2.9}$$

(c) *The* $\mathcal{O}(\epsilon^2)$ *terms give*

$$\begin{aligned} x_1 + 10y_2 &= 0, \\ 5x_2 + y_2 &= 0. \end{aligned} \tag{2.10}$$

giving

$$(x_2, y_2) = (0.000392, -0.00196). \tag{2.11}$$

Thus, to order ϵ^2, *we have*

$$\begin{aligned} x &= 0.98 + 0.0196\epsilon + 0.000392\epsilon^2 + \cdots, \\ x &= 2.1 - 0.098\epsilon - 0.00196\epsilon^2 + \cdots. \end{aligned} \tag{2.12}$$

For $\epsilon = 0.01$ *the first three terms give*

$$(x, y) \approx (0.9801960392, 2.099019804). \tag{2.13}$$

The exact solution is

$$(x,y) = \Big(\frac{49}{50-\epsilon}, \frac{105-7\epsilon}{50-\epsilon}\Big), \tag{2.14}$$

which, for $\epsilon = 0.01$ gives

$$(x,y) = \Big(\frac{49}{49.99}, \frac{104.93}{49.99}\Big) = (0.9801960392\ldots, 2.09901980396\ldots). \tag{2.15}$$

The first three terms in the perturbation expansion gives the solution to the accuracy of my calculator!

Some important points:

(i) We had to solve the $\mathcal{O}(1)$ problem (i.e., the reduced problem) first. All the subsequent problems depended on it. One always needs to have a reduced problem that can be solved. Trivial in this case, but not always.

(ii) The solution of the reduced problem $(x_0, y_0) = (0.98, 2.1)$ is very close to the exact solution. This is indication that the terms neglected to obtain the reduced problem were indeed small. For the exact solution $\epsilon x = 0.0098\cdots \ll 10y = 20.99\ldots$, so approximating the first equation by dropping ϵx was OK.

The next example shows one way things can go wrong. It is a simple example which allows us to understand why perturbation theory fails in this case.

Example 2.1.2 *(From Lin & Segel): Find an approximate solution of the system*

$$\begin{aligned} \epsilon x + y &= 0.1, \\ x + 101y &= 11, \end{aligned} \tag{2.16}$$

for $\epsilon = 0.01$.

The reduced problem is

$$\begin{aligned} y_0 &= 0.1, \\ x_x + 101y_0 &= 11, \end{aligned} \tag{2.17}$$

which has the solution $(x_0, y_0) = (0.9, 0.1)$. Solving the system exactly, we have

$$\begin{aligned} (1-101\epsilon)x &= 11 - 10.1 = 0.9, \\ (101\epsilon - 1)y &= 0.11 - 0.1 = 0.01 \end{aligned} \tag{2.18}$$

so

$$(x,y) = (-90, 1). \tag{2.19}$$

The solution of the reduced problem is way off. What went wrong?

For the exact solution $\epsilon x = -0.9$ is comparable to the other two terms in the first equation. In obtaining the reduced problem by dropping the ϵx term we assumed that it was small compared with the other terms. In this example this assumption is incorrect and it leads to a poor reduced problem.

In real problems we won't know the exact solution (otherwise we wouldn't be using perturbation methods!), so how can we realize our perturbation solution is wrong? In this example, assuming we haven't noticed the problem we proceed to find perturbative corrections. This leads to

$$\begin{aligned} x &= 0.9 + 90.9\epsilon + 9180.9\epsilon^2 + \cdots, \\ y &= 0.1 - 0.9\epsilon - 90.9\epsilon^2 + \cdots, \end{aligned} \tag{2.20}$$

which, for $\epsilon = 0.01$*, gives*

$$x = 0.9 + .909 + 0.91809 + \cdots\,, \qquad y = 0.1 - 0.009 - 0.00909 + \cdots\,. \tag{2.21}$$

It looks like the series will not converge (of course we can't really tell with only three terms). In general the $\mathcal{O}(\epsilon)$ *correction should be small compared with the leading-order (*$\mathcal{O}(1)$ *terms, and the* $\mathcal{O}(\epsilon^2)$ *terms should be small compared to the* $\mathcal{O}(\epsilon)$ *terms. This is clearly not the case here.*

The exact solution for x *is*

$$x = \frac{0.9}{1 - 101\epsilon}. \tag{2.22}$$

Thus, $x(\epsilon)$ *has a singularity at epsilon* $= 1/101 = 0.009901\ldots$ *and the Taylor Series expansion for* $x(\epsilon)$ *cannot converge for* $\epsilon = 0.01$*.*

For $\epsilon = 0.002$*, say, the first three terms of the expansion gives a very good approximation (*$x \approx 1.1185236$ *vs the exact solution* $x = 1.12782\ldots$ *).*

Perturbative methods often work only if the small parameter(s), ϵ in this case, is small enough. How small 'small enough' is may be difficult to determine.

- Dropping terms uncritically can be dangerous!
- Learning how to simplify a problem consistently is difficult and a **very** important part of this course.

In most problems you will have to introduce a small parameter, or perhaps several small parameters. Where does ϵ come from? Two possibilities:

- Introduce ϵ artificially.
- Obtain ϵ from scaling and non-dimensionalization.

The latter is the most important when dealing with physical problems.

2.2 Roots of polynomials

References: Murdoch or Bender & Orzag.

Example 2.2.1 *(From Bender & Orzag): Artificial introduction of* ϵ*.*

Find approximate solutions of

$$x^3 - 4.001x + 0.002 = 0. \tag{2.23}$$

Tricky, but

$$x^3 - 4x = x(x-2)(x+2) = 0, \tag{2.24}$$

is easy. Consider (2.23) a perturbation of (2.24). There are many ways to do this, one is to consider the problem

$$x^3 - (4+\epsilon)x + 2\epsilon = 0. \tag{2.25}$$

where we are interested in the solution when $\epsilon = 0.001$. As above, assume the solutions $x(\epsilon)$ have a Taylor series expansion

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots\,. \tag{2.26}$$

Substituting into (2.25) and collecting like powers of ϵ gives

$$(x_0^3 - 4x_0) + (3x_0^2 x_1 - 4x_1 - x_0 + 2)\epsilon + (3x_0^2 x_2 + 3x_1^2 x_0 - 4x_2 - x_1)\epsilon^2 + \mathcal{O}(\epsilon^3) = 0. \tag{2.27}$$

The coefficient of each power of ϵ must be zero, giving a sequence of problems to be solved.

1. $\mathcal{O}(1)$ problem:

$$x_0^3 - 4x_0 = 0 \tag{2.28}$$

giving the three roots $x_0 = -2, 0, 2$. Note that we chose ϵ so that at $\epsilon = 0$ our problem reduced to this simple problem that we already noticed we could easily solve.

2. $\mathcal{O}(\epsilon)$ problem:

$$(3x_0^2 - 4)x_1 = x_0 - 2. \tag{2.29}$$

This is easily solved:

$$x_1 = \frac{x_0 - 1}{3x_0^2 - 4}. \tag{2.30}$$

Each value of x_0 gives a different value for x_1. Note that the denominator $3x_0^2 - 4$ is non-zero for each of our values for x_0.

3. $\mathcal{O}(\epsilon^2)$ problem:

$$(3x_0^2 - 4)x_2 = x_1 - 3x_1^2 x_0, \tag{2.31}$$

so

$$x_2 = \frac{x_1 - 3x_1^2 x_0}{3x_0^2 - 4}. \tag{2.32}$$

Note that the denominator is the same as in the $\mathcal{O}(\epsilon)$ problem. This is no coincidence. More on this later.

Taking $x_0 = -2$, one root is

$$x^{(1)} = -2 - \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 + \mathcal{O}(\epsilon^3), \tag{2.33}$$

which gives $x^{(1)} \approx -2.000499875$ for $\epsilon = 0.001$.

Comment:

- There may be many ways to introduce a small parameter. Some good, some bad.
- The $\mathcal{O}(1)$ problem (the reduced problem) must be solvable. In the preceding example this problem was a cubic polynomial that we could easily solve, as opposed to the original cubic problem. The higher-order problems were all simple linear problems. Once the leading-order problem was solved the higher-order corrections were simple. This is common to all problems involving finding roots of polynomials, but it is not always the case for other types of problems. Sometimes the higher-order problems get more difficult to solve.

2.2.1 Order of the error

If we truncate our solution at $\mathcal{O}(\epsilon^n)$ then how can we estimate the error? We know that the error is due to the terms $\mathcal{O}(\epsilon^n)$ and higher but that does not mean the error is bounded by $C\epsilon^n$ for some constant $C > 0$. The coefficients of the ϵ^m terms for $m > n$ may grow very rapidly. The series may not converge and in fact many useful asymptotic series do not.

Definition 2.2.1 *We will call* $\mathcal{O}_F(\epsilon^n)$ *the* **formal order of truncation** *and by this mean that terms of* $\mathcal{O}(\epsilon^n)$ *and higher are neglected. It says nothing about the error.*

From now on we will use the notation $\mathcal{O}_F(\epsilon^n)$ unless we know the error is bounded by $C\epsilon^n$ in which case the error is $\mathcal{O}(\epsilon^n)$. For our root problem we can say something more.

Let

$$f(x, \epsilon) = x^3 - (4 + \epsilon)x + 2\epsilon. \tag{2.34}$$

Then $f(x, \epsilon) = 0$ implicitely defines $x(\epsilon)$ — actually three different $x(\epsilon)$, one for each root. The Implicit Function Theorem guarantees that a unique function is defined by

$$f(x(\epsilon), \epsilon) = 0; \quad x(0) = x_0, \tag{2.35}$$

where x_0 is one of the roots of $f(x, 0) = 0$, i.e., $x_0 = -2$, 0, or 2, for a non-zero interval containing $\epsilon = 0$.

Theorem 2.2.1 Implicit Function Theorem: *Let* $f(x, \epsilon)$ *be a function having continuous partial derivatives (including mixed derivatives) up to order* r. *Let* x_0 *satisfy* $f(x_0, 0) = 0$ *and* $f_x(x_0, 0) \neq 0$. *Then there is an* $\epsilon_0 > 0$ *and a unique* C^r *function* $x = x(\epsilon)$ *defined for all* $0 \leq |\epsilon| \leq \epsilon_0$ *such that*

$$f(x(\epsilon), \epsilon) = 0 \quad \text{and} \quad x(0) = x_0. \tag{2.36}$$

You can read about the Implicit Function Theorem in, for example, Murdoch 'Perturbations: Theory and Methods', Marsden & Hoffman 'Elementary Classical Analysis' or Apostol 'Calculus: Volume II'.

The function $f(x, \epsilon)$ need not be a polynomial. If it is then it is C^∞ (only a finite number of derivatives being non-zero) and, provided $f_x(x_0, 0) \neq 0$, $x(\epsilon)$ exists and is C^∞. For the previous example

$$f(x, \epsilon) = 3x^3 - (4 + \epsilon)x + 2\epsilon, \tag{2.37}$$

and

$$f_x(x, \epsilon) = 3x^2 - 4 \tag{2.38}$$

which is nonzero for all three roots. Thus, by the Implicit Function Theorem, the solution $x(\epsilon)$ exists for all $0 \leq |\epsilon| \leq \epsilon_0$ for some $\epsilon_0 > 0$. The theorem does not help us determine the size of ϵ_0. Taylor's Theorem (see below) can be used to show that, using a third-order approximation for example,

$$\Big| x(\epsilon) - (x_0 + x_1\epsilon + x_2\epsilon^2) \Big| \leq M\frac{\epsilon^3}{6}, \tag{2.39}$$

where

$$M = \max\Big\{ \Big| \frac{\partial^3 x}{\partial \xi^3}(\xi) \Big| : \xi \in [0, \epsilon_0] \Big\}, \tag{2.40}$$

which gives us some information about the error.

Theorem 2.2.2 Taylor's Theorem: *Let $x(\epsilon)$ be a C^r function on $|\epsilon| < \epsilon_0$. For $k \le r-1$ let $p_k(\epsilon)$ be the Taylor polynomial*

$$p_k(\epsilon) = \sum_0^k \frac{x^{(n)}}{n!} \tag{2.41}$$

where $x^{(n)}$ denotes the n^{th} derivative of x. Then if $x(\epsilon)$ is approximated by $p_k(\epsilon)$ the error is

$$R_k(\epsilon) = x(\epsilon) - p_k(\epsilon) = \int_0^\epsilon x^{(k+1)}(\eta)\frac{(\epsilon-\eta)^k}{k!}\, d\eta \tag{2.42}$$

and for each $\epsilon_1 \in (0, \epsilon_0)$

$$|R_k(\epsilon)| \le \frac{M_k(\epsilon_1)}{(k+1)!}|\epsilon|^{k+1} \quad \text{for } |\epsilon| \le \epsilon_1 \tag{2.43}$$

where

$$M_k(\epsilon_1) = \max\Big\{|f^{(k+1)}(\epsilon)| \text{ for } |\epsilon| \le \epsilon_1\Big\} \tag{2.44}$$

For a proof, which is based on the fundamental theorem of calculus and integration by parts, see any first year Calculus book. It is also discussed in the text by Murdoch.

2.2.2 Sometimes you don't expand in powers of ϵ

The presence of a small parameter ϵ in your problem does not necessarily imply that the perturbation series solution is in integer powers of ϵ. Consider the following.

Example 2.2.2 *Find approximate roots $x(\epsilon)$ of*

$$f(x,\epsilon) = x^3 - x^2 + \epsilon = 0. \tag{2.45}$$

Solution: Proceeding as before substitute

$$x = x_0 + x_1\epsilon + x_2\epsilon^2 + \cdots \tag{2.46}$$

into the equation giving

$$x_0^3 - x_0^2 + \Big((3x_0^2 - 2x_0)x_1 + 1\Big)\epsilon + \Big((3x_0^2 - 2x_0)x_2 + 3x_0x_1^2 - x_1^2\Big)\epsilon^2 + \cdots = 0. \tag{2.47}$$

This leads to the following sequence of problems:

1. *$\mathcal{O}(1)$ problem:*

   $$x_0^3 - x_0^2 = 0. \tag{2.48}$$

   which has two roots: $x_0 = 1$ and $x_0 = 0$. The latter is a double root.

2. *$\mathcal{O}(\epsilon)$ problem:*

   $$(3x_0^2 - 2x_0)x_1 + 1 = 0, \tag{2.49}$$

   giving

   $$x_1 = -\frac{1}{3x_0^2 - 2x_0}. \tag{2.50}$$

3. *$\mathcal{O}(\epsilon^2)$ problem: The solution is*

   $$x_2 = -\frac{3x_0x_1^2 - x_1^2}{3x_0^2 - 2x_0}. \tag{2.51}$$

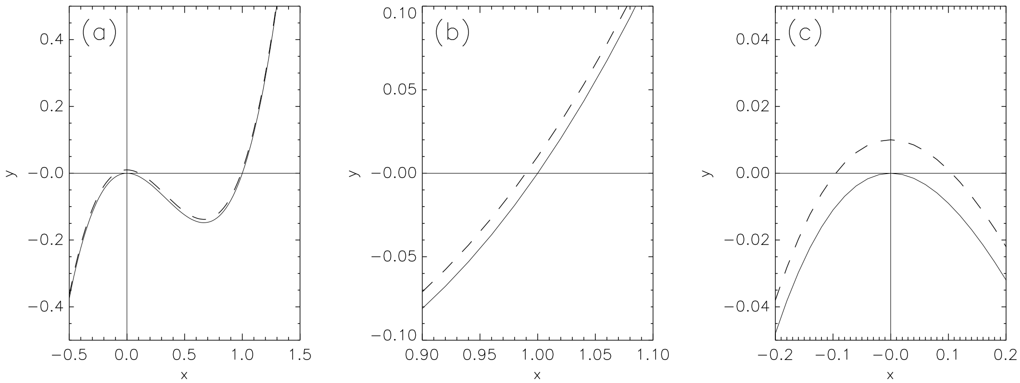

Figure 2.1: (a) Plots of the functions y = f(x,0) (solid curve) and $y = f(x, 0.01)$ (dashed curve) where $f(x,\epsilon) = x^3 - x^2 + \epsilon$. (b) Neighbourhood of $x = 1$. (c) Neighbourhood of $x = 0$.

For the single root $x_0 = 1$ we find $x_1 = -1$ and $x_2 = -2$, so an approximation to one root is

$$x^{(1)} = 1 - \epsilon - 2\epsilon^2 + \mathcal{O}(\epsilon^3), \tag{2.52}$$

(why can we use $\mathcal{O}$ instead of $\mathcal{O}_F$?). For the double root $x_0 = 0$ both x_1 and x_2 are undefined since the denominator $3x_0^2 - 2x_0 = 0$!

What went wrong and how can we resolve the problem? Note that $f_x(x_0, 0) = 3x_0^2 - 2x_0$ is equal to zero at $x_0 = 0$ so at the double root the conditions of the Implicit Function Theory are not satisfied.

The curves $f(x,\epsilon)$ for $\epsilon = 0$ and 0.01 are illustrated in Figure 2.1. Consider the simple root near $x = 1$. Let $g(x) = f(x, 0)$. For $\epsilon = 0$ the polynomial $y = g(x)$ can be approximated by the tangent line $y = g'(1)(x - 1) = x - 1$ in a neighbourhood of $x = 1$. In this example the function $f(x,\epsilon)$ is obtained by adding ϵ to $f(x, 0)$ which simply shifts the curve up a distance ϵ. The tangent line is shifted up to $y = g'(1)(x - 1) + \epsilon = x - 1 + \epsilon$. This curve intersects the x-axis at $x = 1 - \epsilon/g'(1) = 1 - \epsilon$ which approximates the root of $f(x,\epsilon) = 0$ which is near $x = 1$. Adding ϵ to $g(x)$ shifts the root by $\Delta x = -\epsilon/g'(1)$. In other words, the first correction to the approximate root $x_0 = 1$ is *linear* in ϵ. This is illustrated in Figure 2.2(a).

For the double roots the problem is different. The tangent line to $y = f(x, 0)$ at $x = 0$ is the line $y = 0$. Adding ϵ to $f(x, 0)$ shifts the tangent line up to $y = \epsilon$ which never crosses the x-axis. We need a higher-order approximation to $f(x, 0)$ in this case if we want to estimate the roots $f(x,\epsilon) = 0$ that are close to the origin. In the neighbourhood of $x = 0$ we need to approximate the polynomial with a quadratic. The quadratic passing through $(x, y) = (0, 0)$ with the same slope and curvature as $y = f(x, 0)$ is $y_q(x) = -x^2$ (this simplest way to see this is that as $x \to 0$, the term x^3 becomes much smaller than $-x^2$ so for sufficiently small x, $x^3 - x^2 \approx -x^2$).

In a small neighbourhood of $x = 0$, $f(x,\epsilon)$ can be approximated by $y = -x^2 + \epsilon$. Its roots are $x = \pm\epsilon^{1/2}$, which may be imaginary if $\epsilon < 0$. Hence

> **For a double root we must expand $x(\epsilon)$ in powers of $\epsilon^{1/2}$. Similarly for roots of order n we must expand $x(\epsilon)$ in powers of $\epsilon^{1/n}$.**

To find perturbative corrections to the double root at $x = 0$ we need to set

$$x(\epsilon) = x_0 + \epsilon^{1/2}x_1 + \epsilon x_2 + \epsilon^{3/2}x_3 + \cdots . \tag{2.53}$$

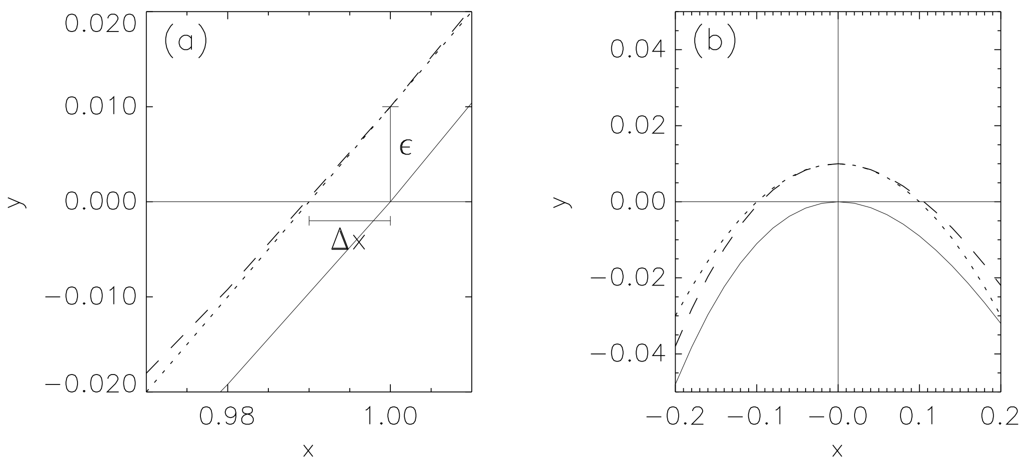

Figure 2.2: As in figure 2.1. (a) Neighbourhood of root at $x = 1$. Dotted curve is linear fit (tangent line) to $y = f(x, 0.01)$ at $x = 1$. (b) Neighbourhood of roots near $x = 0$. Dotted curve is quadratic fit to $y = f(x, 0.01)$ at $x = 0$.

Substituting into $f(x, \epsilon) = 0$ gives

$$\Big(x_0 + \epsilon^{1/2}x_1 + \epsilon x_2 + \epsilon^{3/2}x_3 + \cdots\Big)^3 - \Big(x_0 + \epsilon^{1/2}x_1 + \epsilon x_2 + \epsilon^{3/2}x_3 + \cdots\Big)^2 + \epsilon = 0. \tag{2.54}$$

Expanding and collecting like powers of ϵ leads to

$$\begin{aligned} x_0^3 - x_0^2 + (3x_0^2 - 2x_0)x_1\epsilon^{1/2} + \Big((3x_0^2 - 2x_0)x_2 + 3x_0x_1^2 - 2x_0x_1 - x_1^2 + 1\Big)\epsilon \\ + \Big((3x_0^2 - 2x_0)x_3 + 6x_0x_1x_2 + x_1^3 - 2x_1x_2\Big)\epsilon^{3/2} + \cdots = 0. \end{aligned} \tag{2.55}$$

For the double roots $x_0 = 0$ this simplifies to

$$\Big(-x_1^2 + 1\Big)\epsilon + \Big(x_1^3 - 2x_1x_2\Big)\epsilon^{3/2} + \cdots = 0, \tag{2.56}$$

hence the two roots near zero are

$$x^{2,3} = \pm\epsilon^{1/2} + \frac{1}{2}\epsilon + \mathcal{O}_F(\epsilon^{3/2}). \tag{2.57}$$

[Note: instead of subsituting (2.53) it is easier in this case to use $x_0 = 0$ and substitute $x(\epsilon) = \epsilon^{1/2}x_1 + \cdots$. This simplifies the algebra, particularly if you are finding the solution by hand].

2.2.3 Solving by rescaling: a singular perturbation problem

By an appropriate rescaling we can replace $\mathcal{O}_F$ in the previous solution with $\mathcal{O}$. Let $\mu = \epsilon^{1/2}$ and $x = \mu y$ so that the two roots near $x = 0$, $x^{(2,3)}$, become $y \approx \pm 1$. The polynomial become

$$\mu y^3 - y^2 + 1 = 0. \tag{2.58}$$

Expanding as

$$y = y_0 + \epsilon y_1 + \mu^2 y_2 + \cdots\ , \tag{2.59}$$

leads to

$$\mu\Big(y_0 + y_1\mu + y_2\mu^2 + y_3\mu^3 + \cdots\Big)^3 - \Big(y_0 + y_1\mu + y_2\mu^2 + y_3\mu^3 + \cdots\Big)^2 + 1 = 0. \tag{2.60}$$

Expanding and collecting like powers of μ leads to

$$-y_0^2 + 1 + (y_0^3 - 2y_0y_1)\mu + (3y_0^2y_1 - 2y_0y_2 - y_1^2)\mu^2 + \cdots = 0. \tag{2.61}$$

Solving this leads to

$$y = \pm 1 + \frac{1}{2}\mu \pm \frac{5}{8}\mu^2 + \mathcal{O}(\mu^3), \tag{2.62}$$

where we can say $\mathcal{O}(\mu^3)$ because the conditions of the implicit function theorem are satisfied. Using $\mu = \epsilon^{1/2}$ and $y = x/\epsilon^{1/2}$ recovers (2.57).

We now have a different problem. The cubic polynomial (2.58) has three roots. Our perturbation solution has only found two of them! What happen to the other one?

We already know that the missing root is $x^{(1)} = 1 - \epsilon - 2\epsilon^2 + \mathcal{O}(\epsilon^3)$. In terms of y and μ this becomes

$$y^{(1)} = \frac{1}{\mu} - \mu - 2\mu^3 + \mathcal{O}(\mu^5). \tag{2.63}$$

This has a singularity at $\mu = 0$. The rescaling $x = \mu y$ is only valid if $\mu \neq 0$.

2.2.4 Finding the singular root: Introduction to the method of dominant balance

In the examples we have considered thus far we have always used the Basic Simplification Procedure (set the small parameter to zero) to obtain the reduced problem. This is not always appropriate, and indeed often is not in singular perturbation problems.

Consider again the problem

$$\mu y^3 - y^2 + 1 = 0, \tag{2.64}$$

where $\mu \ll 1$.

The equation has three terms in it. We wish to simplify the problem and that can only be done by dropping one of the three terms. The idea here is that two of the three terms are much larger than the third so to a first approximation they are equal. This gives the reduced problem. There are three possible cases:

Case 1: μy^3 is much smaller than $-y^2$ and 1. This leads to the reduced problem $y_0^2 = 1$ from which we have already seen two roots are obtained. For two of the three roots μy^3 is indeed small compared with $-y^2$ and 1.

Case 2: y^2 is much smaller than μy^3 and 1. If this is true then $\mu y^3 \approx -1$ which means $y \approx 1^{1/3}/\mu^{1/3}$. Note there are three roots corresponding to each of the cubic roots of 1: 1, $e^{i2\pi/3}$ and $e^{i4\pi/3}$. Since $\mu \ll 1$, y is very large. But that means $y^2 \gg 1$ contradicting our assumption that $y^2 \ll 1$. Thus this case is not consistent and must be discarded.

Case 3: 1 is much smaller than μy^3 and y^2. Solving $\mu y_0^3 = y_0^2$ gives $y_0 = 0$, which violates our assumption that $y^2 \gg 1$, or $y_0 = 1/\mu$. If $y \approx 1/\mu$ then $\mu y^3 \approx y^2 \approx 1/\mu^2 \gg 1$ so this solution is consistent with our assumption that 1 is small compared with the other terms. The full solution is now obtained by expanding $y(\mu)$ as

$$y = \frac{1}{\mu} + y_0 + y_1\mu + y_2\mu^2 + \cdots . \tag{2.65}$$

Proceeding we would obtain (2.63).

2.3 Problems

1. Find approximate solutions of the following problems by finding the first three terms in a perturbation series solution (in an appropriate power of ϵ) using perturbation methods. For problem (a) explain whether the missing terms are $O_F(\epsilon^?)$ or $O(\epsilon^?)$. You should find all of the roots, including complex roots.

   (a) $x^2 + (5+\epsilon)x - 6 + 3\epsilon = 0$.

   (b) $x^2 + (4+\epsilon)x + 4 - \epsilon = 0$.

   (c) $(x-1)^2(x+2) + \epsilon = 0$.

   (d) $x^3 + \epsilon + 1 = 0$.

   (e) $\epsilon x^3 + x^2 + 2x + 1 = 0$.

   (f) $\epsilon x^5 + (x-2)^2(x+1) = 0$.

   (g) $\epsilon x^4 + \epsilon x^3 + x^2 - 3x + 2 = 0$.

Chapter 3

Nondimensionalization and scaling

The chapter is based on material from Lin and Segel (1974). It is strongly recommended that you read the relevent sections of this book.

3.1 Nondimensionalizing to get ϵ

Example 3.1.1 (The Projectile Problem) *Consider a vertically launched projectile of mass m leaving the surface of the Earth with speed v. Find the height of the projectile as a function of time.*

Ignore:

- the Earth's rotation;
- the presence of air (i.e., friction);
- relativistic effects;
- the fact that the Earth is not a perfect sphere;
- etc., etc., etc.

Assume:

- Earth is a perfect sphere;
- Newtonian mechanics apply.

Include:

- Fact that the gravitational force varies with height.

Solution

Let the x-axis extend radially from the centre of the Earth through the projectile. Let $x = 0$ at the Earth's surface. Let M_E and R be the mass and radius of the Earth.

Let $x(t)$ be the height of the projectile at time t. The initial conditions are

$$x(0) = 0 \quad \text{and} \quad \dot{x}(0) = v > 0, \tag{3.1}$$

where the dot denotes differentiation.

From Newtonian mechanics

$$\ddot{x}(t) = -\frac{GM_E}{(x+R)^2} = -\frac{gR^2}{(x+R)^2} \tag{3.2}$$

where $g = GM_E/R^2 \approx 9.8$ m s^{-2} is the gravitational acceleration at $x = 0$.

Summary of the problem:

$$\begin{aligned} \ddot{x} &= -\frac{gR^2}{(x+R)^2}, \\ x(0) &= 0, \\ \dot{x}(0) &= v. \end{aligned} \tag{3.3}$$

We can separate the solution procedure into three steps: (1) dimensional analysis; (2) use the ODE to deduce some useful facts; and (3) nondimensionalize (rescale) the problem to obtain a good reduced problem and find an approximate solution.

1. *Dimensional analysis.*

Physical Quantity	*Dimension*
t, time	T
x, height	L
R, radius of Earth	L
V, initial speed	LT^{-1}
g, acceleration at $x = 0$	LT^{-2}

There are two dimensions involved: time and length. We need to scale both by introducing nondimensional time and space variables via,

$$t = T_c\tilde{t} \quad \text{and} \quad x = L_c\tilde{x}. \tag{3.4}$$

where T_c and L_c are characteristic time and length scales. They hold the dimensions while $\tilde{t}$ and $\tilde{x}$ are dimensionless. *There are many choices for T_c and L_c.*

Typical values of v, R and g are

$$\begin{aligned} v &\approx 100 \text{ m s}^{-1}, \\ R &\approx 6.4 \times 10^6 \text{ m} \\ g &\approx 10 \text{ m s}^{-2}. \end{aligned}$$

While the values of R and g are fixed the value of v is a choice. This choice is such that the projectile rises high enough for the height variation of the gravitational force has an effect (it will be small).

2. *Use the ODE to say something useful about the solution.*

1. Existence - Uniqueness Theorems for 2nd order ODEs ensures that there is a unique solution up to some time $t_0 > 0$.

2. Multiplying the ODE by $\dot{x}$ and integrating from 0 to $t_{\max}$, where $t_{\max}$ is the time the projectile reaches its maximum height $x_{\max}$ gives

$$x_{\max} = \frac{v^2 R}{2gR - v^2} = \frac{v^2}{2g}\left(\frac{1}{1 - \frac{v^2}{2gR}}\right) \tag{3.5}$$

Note that

1. $x_{\max} \to \infty$ as $v \to \sqrt{2gR} \approx 10^4$ m s^{-1}.

2. For $v \approx 100$ m s^{-1}, $g \approx 10$ m s^{-2}, $R \approx 6.4 \times 10^6$ m,

$$\frac{v^2}{2gR} \approx \frac{10^4}{2 \times 10 \times 6 \times 10^6} \approx 10^{-4} \tag{3.6}$$

$$\Rightarrow x_{\max} \approx \frac{v^2}{2g}. \tag{3.7}$$

3. *Nondimensionalization*

We now consider three possible choices for the time and length scales T_c and L_c. The first two will turn out to be bad choices but they serve to illustrate some of the things that can go wrong and also illustrate the point that you need to put some thought into your choice of scales.

Procedure A:

Take $L_c = R$ and $T_c = R/v$, which is the time needed to travel a distance R at speed v. Then

$$\frac{dx}{dt} = \frac{d\tilde{t}}{dt}\frac{d}{d\tilde{t}}(L_c\tilde{x}) = \frac{L_c}{T_c}\frac{d\tilde{x}}{d\tilde{t}} = v\frac{d\tilde{x}}{d\tilde{t}} \tag{3.8}$$

which makes sense as $L_c/T_c = v$ is the velocity scale. Next

$$\frac{d^2x}{dt^2} = \frac{L_c}{T_c^2}\frac{d^2\tilde{x}}{d\tilde{t}^2} = \frac{v^2}{R}\frac{d^2\tilde{x}}{d\tilde{t}^2} \tag{3.9}$$

Therefore the ODE becomes:

$$\frac{v^2}{R}\frac{d^2\tilde{x}}{d\tilde{t}^2} = -\frac{gR^2}{(R\tilde{x} + R)^2} = -\frac{g}{(\tilde{x}+1)^2}, \tag{3.10}$$

or

$$\frac{v^2}{gR}\frac{d^2\tilde{x}}{d\tilde{t}^2} = -\frac{1}{(1+\tilde{x})^2}. \tag{3.11}$$

Recall that $v^2/2gR \approx 10^{-4}$which is very small. Hence

$$\epsilon = \frac{v^2}{gR} \tag{3.12}$$

is a small dimensionless parameter.

Scaling the initial conditions we have

$$x(0) = 0 \quad \to \quad \tilde{x}(0) = 0 \tag{3.13}$$

$$\dot{x}(0) = v \quad \to \quad v\frac{d\tilde{x}}{d\tilde{t}}(0) = v \Rightarrow \frac{d\tilde{x}}{d\tilde{t}}(0) = 1, \tag{3.14}$$

hence the final scaled, nondimensional problem is

$$\begin{aligned} \epsilon\frac{d^2\tilde{x}}{d\tilde{t}^2} &= \frac{-1}{(1+\tilde{x})^2}, \\ \tilde{x}(0) &= 0, \\ \frac{d\tilde{x}}{d\tilde{t}}(0) &= 1. \end{aligned} \tag{3.15}$$

Because we have only scaled the variables and have not dropped any terms we have not introduced any errors. *No approximation has been made yet and the solution of this scaled problem is the correct solution.* The difficulty lies with the reduced problem. The reduced problem, obtained by setting $\epsilon = 0$, is

$$\begin{aligned} 0 &= -\frac{1}{(1+\tilde{x}_0)^2}, \\ \tilde{x}_0(0) &= 0, \\ \frac{d\tilde{x}_0}{d\tilde{t}}(0) &= 1, \end{aligned} \tag{3.16}$$

which has no solution! This is a **bad reduced problem**. The small parameter ϵ multiplying the second derivative of $\tilde{x}$ *incorrectly suggests that this term is small.* In fact, at $t = 0$ the r.h.s. is exactly equal to -1. Thus, if $\epsilon = 10^{-4}$, at $t = 0$ $d^2\tilde{x}/d\tilde{t}^2$ must be equal to 10^4, which is very large compared with 1. We need to scale the dimensional variables so the presence of the small parameter ϵ correctly identifies negligible terms. **This is very important.**

Procedure B:

The quantity $\sqrt{\frac{R}{g}}$ has units of time, so let's try $T_c = \sqrt{\frac{R}{g}}$ and take $L_c = R$ as before. This gives

$$\frac{d^2\tilde{x}}{d\tilde{t}^2} = -\frac{1}{(1+\tilde{x})^2}, \tag{3.17}$$

$$\tilde{x}(0) = 0, \tag{3.18}$$

$$\frac{d\tilde{x}}{d\tilde{t}}(0) = \sqrt{\frac{v^2}{Rg}} = \sqrt{\epsilon}, \tag{3.19}$$

where, as before, $\epsilon = v^2/gR \approx 10^{-4} \ll 1$.

As in the previous case, no approximations have been made yet so the solution of this problem is the correct solution. There are, however, two problems with this scale.

1. The ODE has not been simplified!

2. The solution of the reduced problem has $\tilde{x}$ becoming negative for $\tilde{t} > 0$ (since the initial velocity is zero and the initial acceleration is negative). Hence, the solution of the reduced problem has the projectile going the wrong way!

These are both indications of a **bad reduced problem**!

Procedure C:

To get a good reduced problem we must properly scale the variables. You must *think about how you nondimensionalize the problem!*

In procedure A we obtained

$$\epsilon\frac{d^2\tilde{x}}{d\tilde{t}^2} = -\frac{1}{(1+\tilde{x})^2} \tag{3.20}$$

As already pointed out, the problem here is that $\frac{d^2\tilde{x}}{d\tilde{t}^2}$ must be very large so that $\epsilon\frac{d^2\tilde{x}}{d\tilde{t}^2}$ balances the r.h.s. since both sides are equal to negative one at $t = 0$. The nondimensionalization should be done so that the coefficients reflect the size of the whole term.

We'll now do the scaling properly. We have already shown that the maximum height reached by the projectile is

$$x_{\max} = \frac{v^2}{2g}\left(\frac{1}{1-\frac{v^2}{2gR}}\right) \approx \frac{v^2}{2g}, \tag{3.21}$$

since $v^2/(2gR) \approx 10^{-4}$. Thus

$$\frac{x_{\max}}{R} \approx \frac{v^2}{2gR} \approx 10^{-4} \Rightarrow x_{\max} \ll R, \tag{3.22}$$

showing that R is not a good choice for the length scale:

- If we set $x = R\tilde{x}$ then

$$\begin{aligned} 0 \le x &\le \frac{V^2}{2g}, \\ \Rightarrow 0 \le \tilde{x} &\le \frac{V^2}{2gR} \approx 10^{-4}. \end{aligned} \tag{3.23}$$

  This scaling is not a good choice because $\tilde{x}$ is very tiny, i.e., much smaller than one.

- If we set $x = \frac{V^2}{g}\tilde{x}$ then

$$0 \le \tilde{x} \le \frac{1}{2}, \tag{3.24}$$

  i.e. $\tilde{x}$ is an O(1) number. Thus $L_c = v^2/g$ is a much better choice for the length scale. It is in fact the only choice because this scaling reflects the maximum value of $x(t)$.

- v is the obvious velocity scale since the velocity of the projectile must vary between v and $-v$ as the projectile rises and returns to the Earth's surface. If $v = L_c/T_c$ then $T_c = L_c/v = v/g$, is the only logical time scale, since it ensures $\tilde{t}$ is O(1).

- Suppose the time scale is not obvious. Then leave it undetermined for a while. Have:

$$\begin{aligned} \frac{L_c}{T_c^2}\frac{d^2\tilde{x}}{d\tilde{t}^2} &= -\frac{gR^2}{(R+L_c\tilde{x})^2} = \frac{-g}{(1+\frac{L_c}{R}\tilde{x})^2} \\ \Rightarrow \frac{v^2/g}{T_c^2}\frac{d^2\tilde{x}}{d\tilde{t}^2} &= \frac{-g}{(1+\frac{v^2}{gR}\tilde{x})^2} \\ \Rightarrow \left(\frac{v/g}{T_c}\right)^2\frac{d^2\tilde{x}}{d\tilde{t}^2} &= -\frac{1}{(1+\epsilon\tilde{x})^2} \end{aligned} \tag{3.25}$$

where $\epsilon = v^2/(gR) \ll 1$ as before. Since the r.h.s. ≈ -1 , the l.h.s. ≈ -1. To have $\frac{d^2\tilde{x}}{d\tilde{t}^2}$ close to one (in magnitude) means $\frac{v/g}{T_c}$ should be close to 1. Therefor one should choose $T_c = v/g$.

The problem is now

$$\begin{aligned} \frac{d^2\tilde{x}}{d\tilde{t}^2} &= -\frac{1}{(1+\epsilon\tilde{x})^2}, \\ \tilde{x}(0) &= 0, \\ \frac{d\tilde{x}}{d\tilde{t}}(0) &= 1. \end{aligned} \tag{3.26}$$

Setting $\epsilon = 0$ gives the reduced problem

$$\begin{aligned} \Rightarrow \frac{d^2\tilde{x}_0}{d\tilde{t}^2} &= -1, \\ \tilde{x}_0(0) &= 0, \\ \frac{d\tilde{x}_0}{d\tilde{t}} &= 1, \end{aligned} \tag{3.27}$$

which has the solution

$$\tilde{x}_0(t) = \tilde{t} - \frac{\tilde{t}^2}{2}. \tag{3.28}$$

Note that $\max\{\tilde{x}_0\}$ is 1/2 as expected. Note also that $\tilde{x}_o(\tilde{t})$ attains its maximum value at $\tilde{t} = 1$, hence the time scale $T_c = v/g$ can also be interpreted as the characteristic flight time.

3.2 More on Scaling

The goal of scaling is to introduce non-dimensional variables that have order of magnitude equal to 1.

Definition 3.2.1 *A number A has order of magnitude 10^n, n an integer, if*

$$3 \cdot 10^{n-1} < |A| \leq 3 \cdot 10^n \tag{3.29}$$

or if

$$n - \frac{1}{2} < \log_{10}|A| \leq n + \frac{1}{2} \tag{3.30}$$

$(\log_{10} 3 \approx \frac{1}{2})$.

By order of magnitude of a function, we mean the order of magnitude of the maximum, or the least upper bound of the function.

Suppose we have a model of the form:

$$f\left(u, \frac{du}{dx}\right) = 0, \ x \in [a, b] \tag{3.31}$$

To properly scale u and x we choose

$$U = \max\{|u| : \ x \in [a, b]\} \tag{3.32}$$

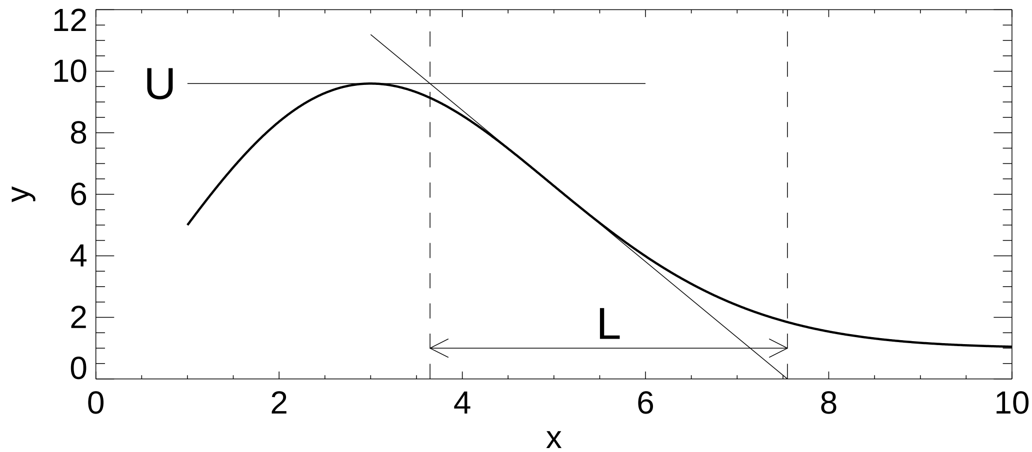

Figure 3.1: Scaling illustration.

so that in setting

$$u = U\tilde{u} \tag{3.33}$$

the function $\tilde{u}$ has order of magnitude 1. We next need to scale x via

$$x = L\tilde{x} \tag{3.34}$$

so that

$$\frac{du}{dx} = \frac{U}{L}\frac{d\tilde{u}}{d\tilde{x}} \tag{3.35}$$

results in $\frac{d\tilde{u}}{d\tilde{x}}$ having order of magnitude 1.

This means we should have

$$\begin{aligned} \frac{U}{L} &= \max\left\{\left|\frac{du}{dx}\right| : \ x \in [a,b]\right\} \\ \Rightarrow L &= \frac{\max|u|}{\max\left|\frac{du}{dx}\right|} \end{aligned} \tag{3.36}$$

Note: If u is known this is easy. If u is unknown this can be difficult.

Example 3.2.1 *Consider the function*

$$u = a\sin(\lambda x), \ a > 0 \text{ on } [0, 2\pi]. \tag{3.37}$$

Solution: Obviously $U = a$ and

$$L = \frac{\max|u|}{\max\left|\frac{du}{dx}\right|} = \frac{a}{a\lambda} = \frac{1}{\lambda}, \tag{3.38}$$

giving

$$\tilde{u} = \sin\tilde{x} \tag{3.39}$$

In general, a model will be of the type

$$f(u, u', u'', \ldots, u^{(n)}) = 0 \tag{3.40}$$

One could take L so that

$$\frac{U}{L} = \max|u'| \quad \text{or} \quad \frac{U}{L^2} = \max|u''| \quad \text{or} \quad \ldots \quad \text{or} \quad \frac{U}{L^n} = \max|u^{(n)}|. \tag{3.41}$$

You should choose L so that the largest of the non-dimensional derivatives has order of magnitude $1 \Rightarrow L$ is smallest of above choices. Thus, take

$$L = \min\left\{\frac{\max|u|}{\max|u'|}, \left(\frac{\max|u|}{\max|u''|}\right)^{1/2}, \cdots, \left(\frac{\max|u|}{\max|u^{(n)}|}\right)^{1/n}\right\}. \tag{3.42}$$

Example 3.2.2 *Consider the function*

$$u = a\sin\lambda x. \tag{3.43}$$

Solution: Have

$$\left(\frac{\max|u|}{\max|u^{(n)}|}\right)^{1/n} = \left(\frac{a}{a\lambda^n}\right)^{1/n} = \frac{1}{\lambda} \tag{3.44}$$

so $L = 1/\lambda$.

Example 3.2.3 *Consider the function*

$$u = a\sin\lambda x + 0.0001a\sin 10\lambda x. \tag{3.45}$$

Solution: Have $\max|u| \approx a$ so take $U = a$. Next,

$$\begin{aligned}
\max|u^{(n)}| &= \max\left|a\lambda^n \begin{pmatrix}\cos(\lambda x)\\ \text{or}\\ \sin(\lambda x)\end{pmatrix} + 10^{n-3}a\lambda^n \begin{pmatrix}\cos(10\lambda x)\\ \text{or}\\ \sin(10\lambda x)\end{pmatrix}\right| \\
&= a\lambda^n \max\left|\begin{pmatrix}\cos(\lambda x)\\ \text{or}\\ \sin(\lambda x)\end{pmatrix} + 10^{n-3}\begin{pmatrix}\cos(10\lambda x)\\ \text{or}\\ \sin(10\lambda x)\end{pmatrix}\right| \\
&\approx \begin{cases} a\lambda^n & \text{for} \quad n \le 3 \\ a\lambda^n 10^{n-3} & \text{for} \quad n \gg 1 \end{cases}
\end{aligned} \tag{3.46}$$

Thus, for $n \le 3$ one should take $L = 1/\lambda$ while for $n \ge 3$ one should take $L = 1/(10^{1-3/n}\lambda)$ which is approximately $1/10\lambda$. Figure 3.2 shows plots of u and some of its derivatives, clearly illustrating that for large derivatives the fast oscillations dominate and determine the appropriate length scale.

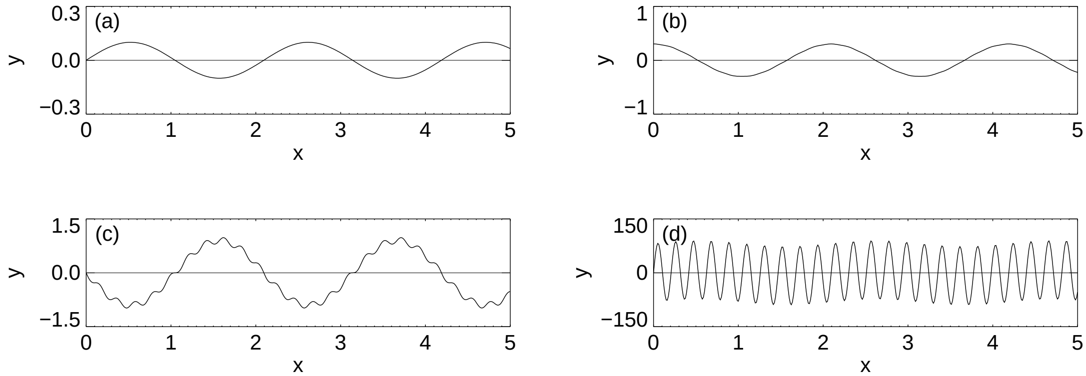

Figure 3.2: Plots of $u(x)$ and some of its derivatives where $u(x) = a\sin(\lambda x) + 0.001a\sin(10\lambda x)$ with $a = 0.1$ and $\lambda = 3$. (a) $u(x)$. (b) $u'(x)$. (c) $u''(x)$. (d) $u^{(4)}(x)$.

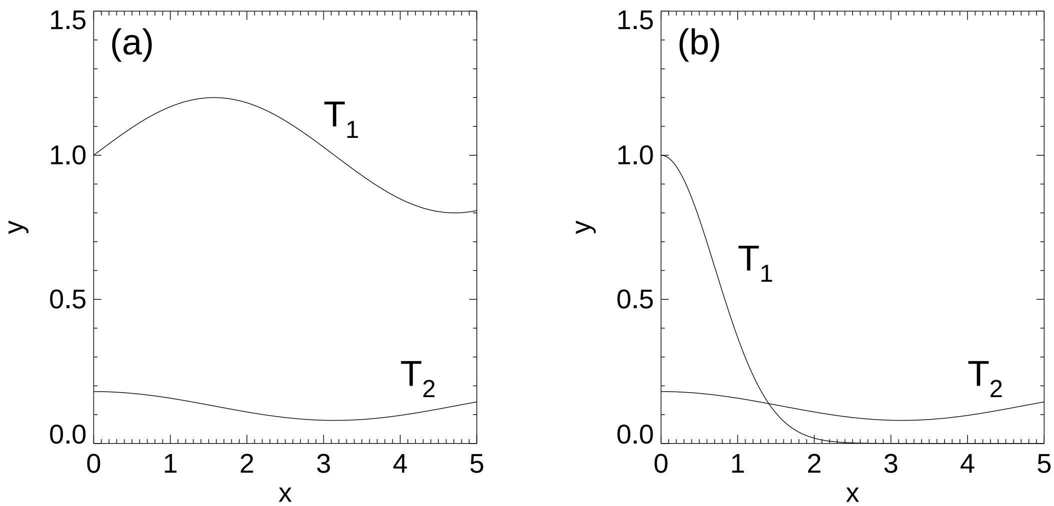

Figure 3.3: (a) Orthodoxy satisfied on $[0, 5]$. (b) Orthodoxy not satisfied on $[0, 5]$.

3.3 Orthodoxy

Suppose we are comparing two terms in a model, $T_1(x)$ and $T_2(x)$, for $x \in [a, b]$, which have been appropriately scaled . We now wish to compare the sizes of each and neglect one if it is small compared to the other.

Problem: The scaling may show that $\max|T_2| \ll \max|T_1|$, but this does not mean that $|T_2| \ll |T_1|$ on all of $[a, b]$.

Definition 3.3.1 *Orthodoxy is said to be satisfied if one term is much smaller than the other on the whole interval.*

If orthodoxy is not satisfied then the intervals on which orthodoxy is not satisfied may be so small that the effects are negligible, e.g., $T_1(x) = \sin x$ and $T_2 = 0.01\cos x$, or multiple scales are needed.

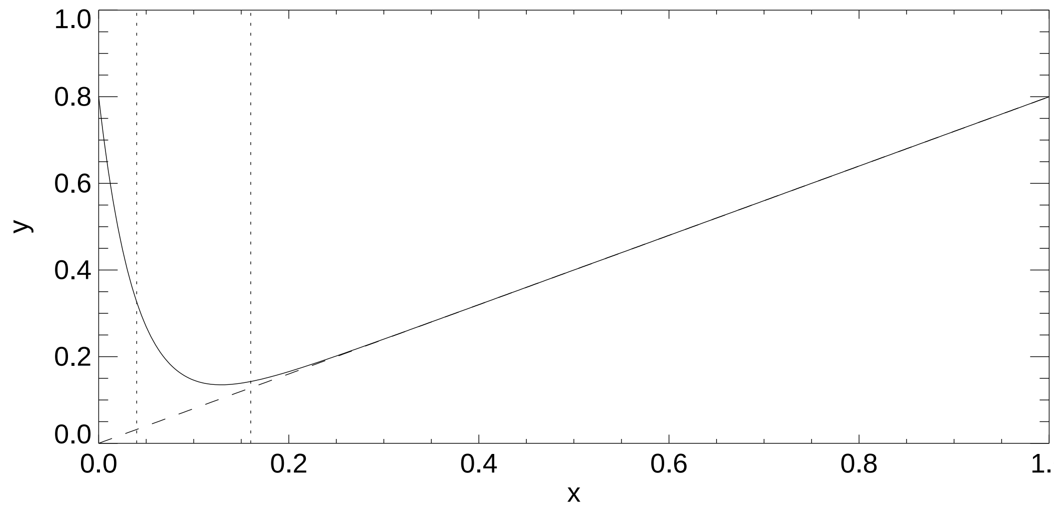

Figure 3.4: Solid: $y = a(x - \exp(-x/\epsilon)$ for $a = 0.8$ and $\epsilon = 0.04$. Dashed: $y = ax$. Vertical dotted lines are $x = \epsilon$ and $x = 4\epsilon$.

Example 3.3.1 *Consider the function $u(x) = a(x + e^{-x/\epsilon})$ for $x \in [0, 1]$, $a > 0$ and $0 < \epsilon \ll 1$ (see Figure 3.4). What scales for x should be used?*

The derivative of $u(x)$ is

$$u'(x) = a\left(1 - \frac{1}{\epsilon}e^{-x/\epsilon}\right) = \begin{cases} a\left(1 - \frac{1}{\epsilon}\right) \approx -a/\epsilon \text{ at } x = 0; \\ \\ a\left(1 - \frac{1}{\epsilon}e^{-\frac{1}{\epsilon}}\right) \approx a \text{ at } x = 1; \end{cases} \tag{3.47}$$

for $0 < \epsilon \ll 1$. Taking $L = \frac{\max|u|}{\max|u'|} = \frac{a}{a/\epsilon}$ gives $L = \epsilon$ when $\epsilon \ll 1$. This is a good length scale near the origin (see figure) but not in the region far away from the origin. Away from the origin, say on $[4\epsilon, 1]$

$$\max|u'| = u'(1) \approx a. \tag{3.48}$$

Using $U = a$ and $L = \epsilon$ gives $\tilde{u} = \epsilon\tilde{x} + \exp(-\tilde{x})$ and $\tilde{u}'(\tilde{x}) = \epsilon - \exp(-\tilde{x})$. The interval of interest is now very large, namely $\tilde{x} \in [0, \epsilon^{-1}]$. For $\tilde{x} \gg 1$, which is most of the interval since $\epsilon \ll 1$, $\tilde{u}'(\tilde{x})$ is very tiny. For most of the domain of interest the correct length scale is $L = 1$

Functions such as this one need to be treated differently in different parts of the domain. There is an inner region, near the origin, in which $u(x)$ varies rapidly, and an outer region, away from the origin, where u varies much more slowly.

Inner Region: Within a few multiples of ϵ of $x = 0$

- $\max|u| \approx a$
- $\max|u'| \approx \frac{a}{\epsilon} \Rightarrow U = a, L = \epsilon$

Therefor we should set $u(x) = a\tilde{u}_i$ and $x = \epsilon\tilde{x}_i$ where subscript i denotes inner region. With this scaling

$$u(x) = a(x + e^{-x/\epsilon}) \Rightarrow \tilde{u}_i(\tilde{x}_i) = \epsilon\tilde{x}_i + e^{-\tilde{x}_i} \tag{3.49}$$

The leading order behaviour of $\tilde{u}$ in the inner region is $e^{-\tilde{x}_i}$. We say $\tilde{u}_i(\tilde{x}_i) \sim e^{-\tilde{x}_i}$ as $\epsilon \to 0$ with $\tilde{x}_i$ fixed, where "$\sim$" denotes "is asymptotic to". More on this shortly.

Outer Region: Many multiples of ϵ away from the origin.

In the outer region

$$u' = a\left(1 - \frac{1}{\epsilon}e^{-x/\epsilon}\right) \approx a. \tag{3.50}$$

Both $\max|u|$ and $\max|u'|$ are close to a, hence we should take $U = a$ and $L = 1$. Setting $u = a\tilde{u}_0$ and $x = 1 \cdot \tilde{x}_0$, where the 1 carries the dimensions (if problem hasn't been nondimensionalized yet) we have $\tilde{u} = \tilde{x}_0 + e^{-\tilde{x}_0/\epsilon} \sim \tilde{x}_0$ as $\epsilon \to 0$ for any fixed, nonzero $\tilde{x}_0$ (i.e., for any $\tilde{x}_0$, no matter how small, ϵ can be made sufficiently small, e.g., $\tilde{x}_0/4$ such that the second term is negligible.

Inner and outer regions arise naturally in many problems as illustrated in the above examples. The inner region is often called a boundary layer.

Example 3.3.2 *Consider the problem*

$$\begin{aligned}\epsilon g'' + g' = 0 \text{ on } [0,1],\ 0 < \epsilon \ll 1, \\ g(0) = a, \\ g(1) = b,\end{aligned} \tag{3.51}$$

where $0 < \epsilon \ll 1$.

Solution: The exact solution is

$$\begin{aligned}g &= \left(\frac{b - ae^{-1/\epsilon}}{1 - e^{-1/\epsilon}}\right) + \left(\frac{a-b}{1 - e^{-1/\epsilon}}\right)e^{-x/\epsilon}, \\ &\approx b + (a-b)e^{-x/\epsilon}.\end{aligned} \tag{3.52}$$

Example 3.3.3 *Consider the problem*

$$\begin{aligned}\epsilon f'' - f' = 0 \text{ on } [0,1],\ 0 < \epsilon \ll 1 \\ f(0) = a \\ f(1) = b\end{aligned} \tag{3.53}$$

Solution:

$$\begin{aligned}f &= \left(\frac{b - ae^{1/\epsilon}}{1 - e^{1/\epsilon}}\right) + \left(\frac{a-b}{1 - e^{1/\epsilon}}\right)e^{x/\epsilon}, \\ &\approx b + (a-b)e^{(x-1)/\epsilon}.\end{aligned} \tag{3.54}$$

These two problems only differ by a change in sign of the second term in the differential equation. The solutions are qualitatively very different. The first has a term $e^{-x/\epsilon}$ which decays rapidly near the origin (left side of the domain). The second has a term $e^{(x-1)/\epsilon}$ which decays rapidly as one moves into the domain from the right boundary at $x = 1$. The solutions are shown in Figure 3.5.

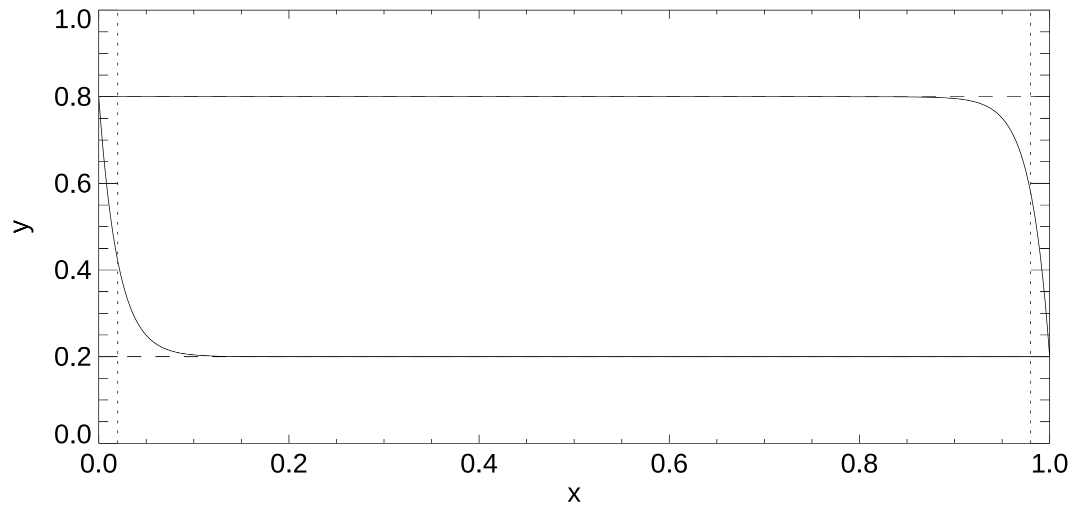

Figure 3.5: Solid curves: solutions of examples 4.2 and 4.3 for $a = 0.8$, $b = 0.2$, and $\epsilon = 0.02$. Dashed lines indicate values of a and b while the vertical dotted lines are $x = \epsilon$ and $x = 1 - \epsilon$.

Question: Attempting to solve $\epsilon y'' + y' = 0$ via regular perturbation methods gives the reduced problem

$$\begin{aligned} y' &= 0, \\ y(0) &= a, \\ y(1) &= b. \end{aligned} \tag{3.55}$$

This is a first-order ODE with two boundary conditions! We can only use one of them. Which one? The solution above shows that we must pick $y(1) = b$ which yields the outer solution. For the second problem, $\epsilon y'' - y' = 0$, the reduced problem is identical but we must now use the boundary condition $y(0) = a$. How can we determine which boundary condition to use without knowing the solution? What happens if ϵ is negative? We will return to questions of this type later when we study boundary layers and matched asymptotics.

Example 3.3.4 *Consider the IVP*

$$\begin{aligned} \ddot{x}(t) + \pi^2 x(t) &= \sin(t) + \epsilon, \;\; t \in \mathbb{R} \\ x(0) &= 1 \\ x'(0) &= 0. \end{aligned} \tag{3.56}$$

1. Find the exact solution.
2. Find $x(t, 0)$ and $x(t, \epsilon)$ and make a sketch. Is orthodoxy satisfied?
3. Is lack of orthodoxy important?

Solution:

1. The general solution of the DE is

$$x(t) = A \sin \pi t + B \cos \pi t + \frac{1}{\pi^2 - 1} \sin t + \frac{\epsilon}{\pi^2}. \tag{3.57}$$

Applying the boundary conditions gives

$$x(t) = -\frac{1}{\pi(\pi^2-1)}\sin \pi t + \frac{1}{\pi^2-1}\sin t + \frac{\epsilon}{\pi^2}\Big(1-\cos \pi t\Big). \tag{3.58}$$

2. Near the zeros of $\ddot{x}$, x and $\sin t$ the term ϵ in the ODE will not be much smaller than these terms so orthodoxy is not satisfied

3. It does not matter that orthodoxy is not satisfied in this case.

$$|x(t,0) - x(t,\epsilon)| = \frac{\epsilon}{\pi^2}|1-\cos \pi t| \leq \frac{2\epsilon}{\pi^2} \ll 1, \tag{3.59}$$

where 1 gives the order of magnitude of the solution (and hence is the appropriate quantity to compare to).

3.4 Example: Inviscid, compressible irrotational flow past a cylinder

Background: (not examinable)

- Inviscid flow means neglect viscosity and heat conduction, (i.e. adiabatic flow).

This type of flow is a good approximation for cases where a fast moving object (i.e. a plane) moves through the air on a time scale much smaller than that required for significant diffusion. It is valid only outside the boundary layer.

Thermodynamics tells us that for isentropic flow the pressure p and density ρ are related by an equation of state $p = p(\rho)$ or $\rho = \rho(p)$. Two important cases are

- For a perfect gas at constant temperature

$$\frac{p}{\rho} = C; \tag{3.60}$$

- For a Perfect Gas at constant entropy

$$\frac{p}{\rho^\gamma} = C, \tag{3.61}$$

where C is a constant and $\gamma = \frac{C_P}{C_V} \approx 1.4$. We will assume isentropic flow (constant entropy).

Let $\mathbf{v}(x, y, z, t)$ be the fluid velocity. The motion of the fluid is governed by the following conservation laws:

1. **Conservation of mass:**

$$\rho_t + \vec{\nabla}\cdot(\rho\mathbf{v}) = 0 \tag{3.62}$$

2. **Conservation of linear momentum:**

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v}\cdot\vec{\nabla}\right)\mathbf{v}\right) = -\vec{\nabla}p \tag{3.63}$$

Definition 3.4.1 Irrotationality: *If fluid particles have no angular momentum then* $\vec{\nabla}\times\mathbf{v}=0$.

Definition 3.4.2 *The* **sound speed** *is defined by*

$$a=\sqrt{\frac{dp}{d\rho}}=\sqrt{\gamma\frac{p}{\rho}}. \tag{3.64}$$

Theorem 3.4.1 (Kelvin, 1868) *For inviscid flow with* $p=p(\rho)$*, if the fluid is initially irrotational and the speed* U *of the flow is less that speed of sound then the flow remains irrotational for all time.*

In this theorem U is the maximum deviation from the flow speed at 'infinity', or far from the cylinder. That is, U should be found in a reference frame fixed with the fluid at infinity.

If $\vec{\nabla}\times\mathbf{v}=0$ at $t=0$ then, assuming the conditions of Kelvin's Theorem are satisfied, $\vec{\nabla}\times\mathbf{v}=0$ for all time $\Rightarrow \mathbf{v}=\vec{\nabla}\phi$ for some velocity potential ϕ. The introduction of a velocity potential greatly simplifies things because the three components of the velocity vector are replaced by a single scalar field.

Using

$$\frac{1}{\rho}\vec{\nabla}p=-\frac{\vec{\nabla}p}{p^{1/\gamma}}C^{1/\gamma}=-\frac{\gamma}{\gamma-1}\vec{\nabla}(p^{1-1/\gamma})C^{1/\gamma}, \tag{3.65}$$

the momentum equation can be written as

$$\vec{\nabla}\Big(\frac{\partial\phi}{\partial t}+\frac{1}{2}|\vec{\nabla}\phi|^2+\frac{\gamma}{\gamma-1}p^{1-1/\gamma}C^{1/\gamma}\Big)=0. \tag{3.66}$$

Thus,

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}|\vec{\nabla}\phi|^2+\frac{a^2}{\gamma-1}=g(t), \tag{3.67}$$

where $g(t)$ is an undetermined function of time. Assuming a steady uniform far-field flow $\mathbf{v}=(U_\infty,0,0)$ with sound speed a_∞^2 gives

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}|\vec{\nabla}\phi|^2+\frac{a^2}{\gamma-1}=\frac{1}{2}U_\infty^2+\frac{a_\infty^2}{\gamma-1}. \tag{3.68}$$

The continuity equation can be written as

$$\Big(\frac{\partial}{\partial t}+\vec{\nabla}\phi\cdot\vec{\nabla}\Big)a^2=-(\gamma-1)a^2\nabla^2\phi, \tag{3.69}$$

Applying the operator $(\partial/\partial t+\vec{\nabla}\phi\cdot\vec{\nabla}$ to (3.68) then yields a single PDE for the velocity potential:

$$a^2\nabla^2\phi-\frac{\partial^2\phi}{\partial t^2}=\frac{\partial}{\partial t}|\vec{\nabla}\phi|^2+\vec{\nabla}\phi\cdot[(\vec{\nabla}\phi\cdot\vec{\nabla})\vec{\nabla}\phi]. \tag{3.70}$$

We now simplify to 2 dimensions and use

Theorem 3.4.2 (Conformal Mapping Theorem:) *Any simply connected region* $A\subset\mathbb{C}$ *can be transformed (bijectively and analytically) to a disk.*

Using this theorem, for the 2-D case we can assume the object is a disk of radius R. Assuming steady state the model equations give

$$\left(1-\frac{u^2}{a^2}\right)\phi_{xx}-\frac{2uv}{a^2}\phi_{xy}+\left(1-\frac{v^2}{a^2}\right)\phi_{yy}=0, \tag{3.71}$$

where $\mathbf{v}=(u,v)=\vec{\nabla}\phi$.

The discriminant of the PDE is

$$\Delta=\left(\frac{uv}{a^2}\right)^2-\left(1-\frac{u^2}{a^2}\right)\left(1-\frac{v^2}{a^2}\right)=M^2-1 \tag{3.72}$$

where $M=\frac{|\mathbf{v}|}{a}$ is the Mach number:

$M<1$	subsonic flow	equation (3.71) is elliptic $\rightarrow$ static situations
$M=1$	sonic flow	equation (3.71) is parabolic $\rightarrow$ diffusive situations
$M>1$	supersonic flow	equation (3.71) is hyperbolic $\rightarrow$ wave situations

Next we nondimensionalize. Let

$$\begin{aligned}(x,y)&=R(\tilde{x},\tilde{y}),\\(u,v)&=U_\infty(\tilde{u},\tilde{v}).\end{aligned} \tag{3.73}$$

Recall that R is the radius of the cylinder and U_∞ is the far field flow. Then

$$(u,v)=\vec{\nabla}\phi\rightarrow U_\infty(\tilde{u},\tilde{v})=\frac{1}{R}\tilde{\vec{\nabla}}\phi \tag{3.74}$$

So we should set $\phi=RU_\infty\tilde{\phi}$. Putting the terms linear in ϕ on the left and the terms cubic in ϕ on the right gives

$$\frac{U_\infty}{R}\left[\tilde{\phi}_{\tilde{x}\tilde{x}}+\tilde{\phi}_{\tilde{y}\tilde{y}}\right]=\frac{U_\infty^2}{a^2}\Big(\tilde{u}^2\frac{U_\infty}{R}\tilde{\phi}_{\tilde{x}\tilde{x}}+2\tilde{u}\tilde{v}\frac{U_\infty}{R}\tilde{\phi}_{\tilde{x}\tilde{y}}+\tilde{v}^2\frac{U_\infty}{R}\tilde{\phi}_{\tilde{y}\tilde{y}}\Big), \tag{3.75}$$

where a is a function of x and y. We need to express it in terms of a_∞, the sound speed at infinity. Using (3.68) to eliminate a and dropping the tildes gives

The nondimensional governing equation:

$$\nabla^2\phi=M_\infty^2\Big(\phi_x^2\phi_{xx}+2\phi_x\phi_y\phi_{xy}+\phi_y^2\phi_{yy}-\frac{\gamma-1}{2}\nabla^2\phi(1-\phi_x^2-\phi_y^2)\Big), \tag{3.76}$$

where $M_\infty=\frac{U_\infty}{a_\infty}$ is the free stream Mach number.

For air at $\approx 20°$C and atmospheric pressure and for $U_\infty\approx 100$ km hr^{-1}, $M_\infty^2\approx 0.1$, so M_∞^2 is a small parameter.

The boundary conditions: No flow through solid boundary and fluid velocity goes to far-field velocity $(1,0)$ at infinity:

$$\begin{aligned}\vec{\nabla}\phi\cdot\hat{n}=0 \text{ on } x^2+y^2=1,\\(\phi_x,\phi_y)\rightarrow(1,0)\quad\text{ as } |\mathbf{x}|\rightarrow\infty.\end{aligned} \tag{3.77}$$

The solution will depend on the circulation around the disk. We will assume zero circulation which implies that the flow is symmetric above and below the disk.

Regular Perturbation Theory Solution:

Assume M_∞^2 is small and set

$$\phi = \phi_0(x,y) + M_\infty^2 \phi_1(x,y) + M_\infty^4 \phi_2(x,y) + \cdots \tag{3.78}$$

$\mathcal{O}(1)$ **problem:** At leading order we have

$$\begin{aligned} \nabla^2 \phi_0 &= 0, \\ \vec{\nabla}\phi_0 &\to (U_\infty, 0) \text{ as } |\mathbf{x}| \to \infty, \\ \vec{\nabla}\phi_0 \cdot \widehat{n} &= 0 \text{ on } x^2+y^2=1. \end{aligned} \tag{3.79}$$

In addition ϕ_0 is symmetric about $y=0$. This Neumann problem for ϕ_0 has the solution

$$\phi_0(r,\theta) = \left(r + \frac{1}{r}\right)\cos\theta. \tag{3.80}$$

Without symmetry condition we get an additional term $A\theta$ for arbitrary A.

$\mathcal{O}(M_\infty^2)$ **problem**: In polar coordinates at the next order we have

$$\begin{aligned} \frac{\partial^2}{\partial r^2}\phi_1 + \frac{1}{r}\frac{\partial}{\partial r}\phi_1 + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\phi_1 &= (\gamma-1)\left[\left(\frac{1}{r^7} - \frac{1}{r^5}\right)\cos\theta + \frac{1}{r^3}\cos 3\theta\right], \\ \phi_1 &\to 0 \text{ as } r \to \infty, \\ \frac{\partial\phi_1}{\partial r} &= 0 \text{ on } x^2+y^2=1, \\ \phi_1(r,\theta) &= \phi_1(r,-\theta) \quad \text{(symmetry)} \end{aligned} \tag{3.81}$$

which can be solved to yield the total solution

$$\begin{aligned} \phi = \left(r+\frac{1}{r}\right)\cos\theta + \frac{\gamma-1}{2}M_\infty^2 \Bigg(&\left(\frac{13}{12r} - \frac{1}{2r^3} + \frac{1}{12r^5}\right)\cos\theta \\ &+ \left(\frac{1}{12r^3} - \frac{1}{4r}\right)\cos 3\theta \Bigg) + \mathcal{O}_F(M_\infty^4). \end{aligned} \tag{3.82}$$

Remarks:

1. Real life problems can be difficult.

2. Getting the first two terms in a Perturbation Theory expansion can be a lot of work.

3. Problem: What is the error? It is believed that the series is uniformly valid (definition below) but this has not been proven (as of mid-90's. I may be out of date). Hence, this is an example of RPT.

Definition 3.4.3 *A series expansion* $\sum \epsilon^2 \xi^2(\cdot,\cdot)$ *is said to be uniformly valid if it converges uniformly over all parts of the domain as* $\epsilon \to 0$. *The series is said to be uniformly ordered if all* ξ_n *are bounded, in which case the series may not converge.*

More on this later.

Chapter 4

Resonant Forcing and Method of Strained Coordinates: Another example from Singular Perturbation Theory

4.1 The simple pendulum

Consider a mass m suspended from a fixed frictionless pivot via an inextensible, massless string. Let θ be the angle of the string from the vertical. The only force acting on the mass is gravity and the tension in the string (i.e., ignore presence of air). The governing equations for a mass initially at rest at an angle a are

$$\begin{aligned} \frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta &= 0, \\ \theta(0) &= a, \\ \frac{d\theta}{dt}(0) &= 0. \end{aligned} \tag{4.1}$$

The solution of the linear problem, obtained by assuming θ is small and approximating $\sin\theta$ by θ is

$$\theta = a\cos\Big(\sqrt{\frac{g}{\ell}}t\Big). \tag{4.2}$$

According to this solution the mass oscillates with frequency $\sqrt{g/\ell}$ and period $T_\ell = 2\pi\sqrt{\ell/g}$. The full nonlinear problem can be solved exactly in terms of Jacobian elliptic functions. Since these can only be expressed in terms of power series we might as well seek a Perturbation Theory solution which will give a power series solution directly. As a first step we need to scale the variables.

To begin with consider the energy of the system. The governing nonlinear ODE has the energy conservation law

$$\frac{d}{dt}\Big(\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \frac{g}{\ell}\cos\theta\Big) = 0, \tag{4.3}$$

which, after using the initial conditions, gives

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 + \frac{g}{\ell}\cos a = \frac{g}{\ell}\cos\theta. \tag{4.4}$$

From this we can deduce that $|\theta| \leq a$ and that θ oscillates periodically between $\pm a$. Therefore scale θ by a:

$$\theta = a\tilde{\theta}. \tag{4.5}$$

For the time scale take the inverse of the linear frequency, thus set

$$t = \sqrt{\frac{\ell}{g}}\tau. \tag{4.6}$$

The scaled problem is

$$\begin{aligned} \frac{d^2\tilde{\theta}}{d\tau^2} + \frac{\sin a\tilde{\theta}}{a} &= 0, \\ \tilde{\theta}(0) &= 1, \\ \frac{d\tilde{\theta}(0)}{d\tau} &= 0. \end{aligned} \tag{4.7}$$

We will assume that a is small. Note that for small a $\sin(a\tilde{\theta})/a$ is $\mathcal{O}(1)$ hence so is the scaled acceleration $d^2\tilde{\theta}/d\tau^2$. This suggests we have appropriately scaled t.

The Taylor series expansion of $\sin a\tilde{\theta}$ converges for all $a\tilde{\theta}$, so we can write the governing DE in (4.7) as

$$\frac{d^2\tilde{\theta}}{d\tau^2} + \tilde{\theta} - \frac{a^2}{3!}\tilde{\theta}^3 + \frac{a^4}{5!}\tilde{\theta}^5 + \cdots = 0. \tag{4.8}$$

The small parameter a appears only in even powers, hence we seek a Perturbation Theory solution of the form

$$\tilde{\theta} = \theta_0(\tau) + a^2\theta_1(\tau) + a^4\theta_2(\tau) + \cdots . \tag{4.9}$$

$\mathcal{O}(1)$ problem: At leading order we have

$$\begin{aligned} \frac{d^2\theta_0}{d\tau^2} + \theta_0 &= 0, \\ \theta_0(0) &= 1, \\ \frac{d\theta_0}{d\tau}(0) &= 0, \end{aligned} \tag{4.10}$$

which has solution

$$\theta_0 = \cos\tau. \tag{4.11}$$

$\mathcal{O}(a^2)$ problem: At the next order we have

$$\begin{aligned} \frac{d^2\theta_1}{d\tau^2} + \theta_1 &= \frac{1}{3!}\cos^3\tau = \frac{1}{24}\cos 3\tau + \frac{1}{8}\cos\tau, \\ \theta_1(0) &= \frac{d\theta_1}{d\tau}(0) = 0. \end{aligned} \tag{4.12}$$

The general solution of (4.12) is:

$$\theta_1(\tau) = -\frac{1}{192}\cos 3\tau + \frac{1}{16}\tau\sin\tau + A\cos\tau + B\sin\tau. \tag{4.13}$$

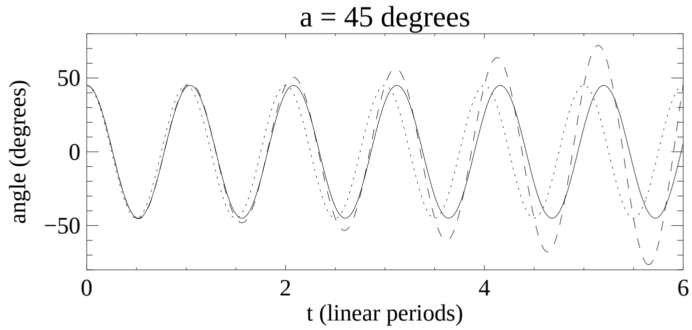

Figure 4.1: Comparison of regular perturbation theory solution with linear and nonlinear solutions for initial angle of 45°. Dotted curve: linear solution. Solid curves: nonlinear solution. Dashed curves: regular perturbation theory solution.

Applying the boundary conditions gives

$$\theta_1 = \frac{1}{192}[\cos\tau - \cos 3\tau] + \frac{\tau}{16}\sin\tau \tag{4.14}$$

so that the total solution is

$$\tilde{\theta} = \cos\tau + a^2\left(\frac{1}{192}(\cos\tau - \cos 3\tau) + \frac{\tau}{16}\sin\tau\right) + \mathcal{O}_F(a^4). \tag{4.15}$$

Problem: The amplitude of the $(a^2/16)\tau\sin\tau$ term grows in time. It is as important as the leading order term, $\cos\tau$, when $a^2\tau/16$ is order 1. Thus, *the perturbation series breaks down by a time of $O(1/a^2)$, at which point $a^2\theta_1$ is no longer much smaller than θ_0.* The breakdown is illustrated in Figure 4.1 for $a = \pi/4$. Note that while the perturbation solution becomes very bad after three or four periods it is better than the linear solution for times up to close to 2 linear periods. At this time the linear solution has drifted away from the nonlinear solution whereas the phase of perturbation solution is much better.

Physically the perturbation solution goes awry because the linear (i.e., the leading-order) and nonlinear solutions drift apart in time. The $O(a^2)$ error made in linearizing the problem to get the leading-order problem for θ_o are cumulative and eventually destroy the approximation. The regular perturbation solution tries to correct for this but does not do so correctly — the phase is improved at the cost of a growing amplitude.

The secular term $(a^2/16)\tau\sin\tau$ appears in the $O(a^2)$ solution because of the appearance of the **resonant forcing term** $\cos\tau$ in the DE for θ_1 (resonant forcing because the forcing term has the same frequency as the homogeneous solution, or more generally because the forcing term is a solution of the homogeneous solution):

$$\frac{d^2\theta_1}{d\tau^2} + \theta_1 = \frac{1}{24}\cos 3\tau + \underbrace{\frac{1}{8}\cos\tau.}_{\substack{\text{resonant}\\\text{forcing}\\\text{term}}}$$

The appearance of a resonant forcing term means this is another example of a Singular Perturbation Theory problem.

How can we fix this problem? From energy considerations we know that the amplitude is given by the initial condition. The nonlinearity does not change this. We also know that the solution is periodic. Nonlinearity modifies the shape and period of the oscillations. It increases the period because the true restoring force, $(g/l)\sin(\theta)$ is less than the linearized restoring force $(g/l)\theta$. The properties of the linear and nonlinear solutions are compared in table 4.1.

property	linear solution	nonlinear solution
amplitude	a	a
shape	sinusoidal	non-sinusoidal shape
period	$2\pi\sqrt{l/g}$	increases with amplitude

Table 4.1: Properties of linear and nonlinear solutions.

Because the periods of the linear and nonlinear solutions are different they slowly drift out of phase. Eventually they will be completely out of phase.

The Fix: We must allow the period, or equivalently the frequency, to be a function of a.

Recall the original unscaled problem was

$$\begin{aligned} \frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta &= 0, \\ \theta(0) &= a, \\ \frac{d\theta}{dt}(0) &= 0. \end{aligned} \tag{4.16}$$

As before, set $\theta = a\tilde{\theta}$, since this is the amplitude of the nonlinear solution. In our previous attempt we set

$$t = \sqrt{\frac{\ell}{g}}\tau,$$

i.e. we used a time scale $T_c = \sqrt{\ell/g}$, which was independent of a, and proportional to the period of the linearized solution. We need a time scale which is relevant to the nonlinear solution, one which depends on a. Since we do not know how the period depends on a we are forced to introduce an unknown function $\sigma(a)$ via

$$t = \sqrt{\frac{\ell}{g}}\frac{\tau}{\sigma(a)}. \tag{4.17}$$

This is known as the **method of strained coordinates (MSC)** (we have 'strained' time by an unknown function $\sigma(a)$). We will return to this method later.

Since in the limit $a \to 0$ the period does go to $\sqrt{\ell/g}$ we can take $\sigma(0) = 1$. With this time scaling the nondimensionalized problem is

$$\begin{aligned} \sigma^2(a)\frac{d^2\tilde{\theta}}{d\tau^2} + \frac{\sin a\tilde{\theta}}{a} &= 0, \\ \tilde{\theta}(0) &= 1, \\ \frac{d\tilde{\theta}}{d\tau}(0) &= 0. \end{aligned} \tag{4.18}$$

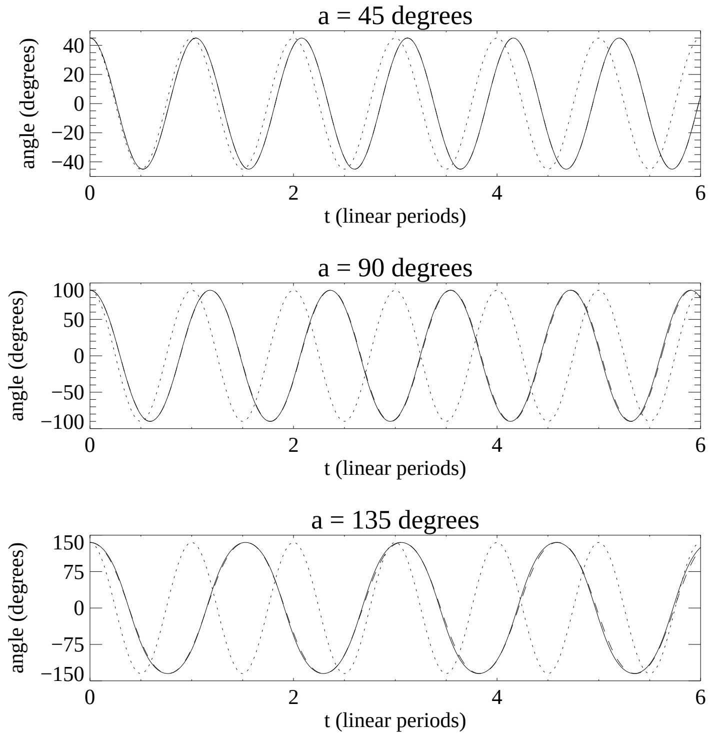

Figure 4.2: Comparison of singular perturbation theory solution with linear and nonlinear solutions for different initial angles. Dotted curves: linear solution. Solid curves: nonlinear solution. Dashed curves: singular perturbation theory solution. The dashed curves are almost identical to the nonlinear solution.

We now expand both $\tilde{\theta}$ and σ in powers of a^2, via

$$\begin{aligned} \tilde{\theta} &= \theta_0(\tau) + a^2\theta_1(\tau) + a^4\theta_2(\tau) + \cdots, \\ \sigma(a) &= 1 + a^2\sigma_1 + a^4\sigma_2 + \cdots. \end{aligned} \tag{4.19}$$

Substituting the series into the differential equation gives

$$\begin{aligned} &\Big(1 + 2\sigma_1 a^2 + (2\sigma_2 + \sigma_1^2)a^4 + \cdots\Big)\Big(\frac{d^2\theta_0}{d\tau^2} + a^2\frac{d^2\theta_1}{d\tau^2} + \cdots\Big) \\ &\quad + \big(\theta_0 + a^2\theta_1 + a^4\theta_2 + \cdots\big) - \frac{a^2}{6}\big(\theta_0 + a^2\theta_1 + a^4\theta_2\big)^3 + \mathcal{O}(a^4) = 0. \end{aligned} \tag{4.20}$$

$\mathcal{O}(1)$ **Problem:** The leading-order problem is unchanged

$$\left.\begin{aligned} &\tfrac{d^2\theta_0}{d\tau^2} + \theta_0 = 0, \\ &\theta_0(0) = 1, \\ &\tfrac{d\theta_0}{d\tau}(0) = 0. \end{aligned}\right\} \quad \Rightarrow \quad \theta_0 = \cos\tau$$

$\mathcal{O}(a^2)$ **Problem:** At $\mathcal{O}(a^2)$ we have

$$\begin{aligned} 2\sigma_1\frac{d^2\theta_0}{d\tau^2} + \frac{d^2\theta_1}{d\tau^2} + \theta_1 - \frac{1}{6}\theta_0^3 &= 0, \\ \theta_1(0) &= 0, \\ \frac{d\theta_1}{d\tau}(0) &= 0. \end{aligned} \tag{4.21}$$

$$\Rightarrow \frac{d^2\theta_1}{d\tau^2} + \theta_1 = \underbrace{\frac{1}{24}\cos 3\tau + \frac{1}{8}\cos\tau.}_{\text{We had this before}} + 2\sigma_1\cos\tau$$

There is a new resonant forcing term, namely $2\sigma_1 \cos\tau$. By *choosing* $\sigma_1 = -1/16$ the resonant forcing terms are eliminated. There is in fact no choice about this. The only way to eliminate the secular growth in the $O(\epsilon)$ solution is be eliminating the resonant forcing term. This reduces the problem to

$$\frac{d^2\theta_1}{d\tau^2} + \theta_1 = \frac{1}{24}\cos 3\tau, \tag{4.22}$$

which, with the initial conditions, gives

$$\theta_1 = -\frac{1}{192}\left(\cos\tau - \cos 3\tau\right). \tag{4.23}$$

The total solution, so far, is

$$\begin{aligned} \tilde{\theta} &= \cos\tau + \frac{a^2}{192}\left(\cos\tau - \cos 3\tau\right) + \mathcal{O}_F(a^4), \\ \sigma(a) &= 1 - \frac{a^2}{16} + \mathcal{O}_F(a^4), \end{aligned} \tag{4.24}$$

where

$$\tau = \sqrt{\frac{g}{\ell}}\sigma(a)t. \tag{4.25}$$

The dimensional solution is

$$\theta(t) = a\tilde{\theta}(\tau) = a\tilde{\theta}\left(\sqrt{\frac{g}{\ell}}\sigma(a)t\right), \tag{4.26}$$

or

$$\begin{aligned} \theta(t) = a\cos&\left(\sqrt{\frac{g}{\ell}}\left(1 - \frac{a^2}{16} + \cdots\right)t\right) \\ &+ \frac{a^3}{192}\left[\cos\left(\sqrt{\frac{g}{\ell}}\left(1 - \frac{a^2}{16} + \cdots\right)t\right) - \cos\left(3\sqrt{\frac{g}{\ell}}\left(1 - \frac{a^2}{16} + \cdots\right)t\right)\right] \\ &+ \mathcal{O}_F(a^5). \end{aligned} \tag{4.27}$$

The nonlinear solution frequency is $\sigma(a)\sqrt{g/\ell} = \left(1 - \frac{a^2}{16} + \cdots\right)\sqrt{g/\ell} < \sqrt{g/\ell}$ which makes sense because we know that the period of the nonlinear solution must be larger than the period of the linear solution because the forcing in the nonlinear problem, $(g/l)\sin\theta$, is smaller that the forcing in the linear problem, $(g/l)\theta$ (i.e., the acceleration of the nonlinear pendulum is smaller than for the linear pendulum). The SPT solution (4.27) is shown in figure 4.2 showing excellent agreement with the full nonlinear solution over six linear periods for very large initial angles.

Chapter 5

Asymptotic Series

5.1 Asymptotics: large and small terms

Notation

For order of magnitude of a number of function we will use the symbol O_M:

$$\begin{aligned} 90 &= \mathcal{O}_M(100) \\ 0.0072 \sin x &= \mathcal{O}_M(10^{-2}) \end{aligned}$$

Definition 5.1.1 (The $\mathcal{O}$ "big-oh" Symbol) *Let f and g be two functions defined on a region $\mathcal{D}$ in $\mathbb{R}^n$ or $\mathbb{C}^n$. Then*

$$f(x) = \mathcal{O}(g(x)) \quad \text{on } \mathcal{D} \tag{5.1}$$

means that

$$|f(x)| \leq k|g(x)| \quad \forall x \in \mathcal{D} \tag{5.2}$$

for some constant k.

We will usually be interested in the relative behaviour of two functions in the neighbourhood of a point x_0. In that case when we write

$$f(x) \;=\; \mathcal{O}(g(x)) \text{ as } x \to x_0$$

we mean there exists a constant k and a neighbourhood of x_0, $\mathcal{U}$, such that

$$|f(x)| \leq k|g(x)| \text{ for } x \in \mathcal{U}$$

Remarks

1. If $g(x) \neq 0$ then $f(x) = \mathcal{O}(g(x))$ in $\mathcal{D}$ or $f(x) = \mathcal{O}(g(x))$ as $x \to x_0$ can be written as $\frac{f(x)}{g(x)} < \infty$ in $\mathcal{D}$, or $\frac{f(x)}{g(x)}$ is bounded as $x \to x_0$.

2. $\mathcal{O}(g(x))$ on its own has no meaning. The equals sign in "$f(x) = \mathcal{O}(g(x))$" is an abuse of notation.

$$f(x) = \mathcal{O}(g(x)) \Rightarrow 2f(x) = \mathcal{O}(g(x)) \tag{5.3}$$

   but this does not mean that $2f(x) = f(x)$.

3. $f(x) = \mathcal{O}(g(x))$ does not imply that $g(x) = \mathcal{O}(f(x))$. For example, $x^2 = \mathcal{O}(x)$ as $x \to 0$ since $|x^2| < 5|x|$ for $|x| < 5$, but $x \neq \mathcal{O}(x^2)$ as $x \to 0$ because it is not true that $|x| < k|x^2|$ for some constant k in a neighbourhood of 0.

4. An expression containing $\mathcal{O}$ is to be considered a class of functions. For example, $\mathcal{O}(1)+\mathcal{O}(x^2)$ in $0 < x < \infty$ denotes the class of all functions of the type $f+g$ where $f = \mathcal{O}(1)$ and $g = \mathcal{O}(x^2)$.

5. If $f(x) = c$ is a constant, $f = \mathcal{O}(1)$ no matter what the value of c is.

$$\begin{aligned} 10^{-9} &= \mathcal{O}(1), \\ 1 &= \mathcal{O}(1), \\ 10^{9} &= \mathcal{O}(1). \end{aligned}$$

Example 5.1.1

- $x^2 = \mathcal{O}(x)$ on [-2,2] since $x^2 < 5|x|$ on $[-2, 2]$.
- $x^2 \neq \mathcal{O}(x)$ on $[1, \infty]$ since $\frac{|x^2|}{|x|} = |x|$ is unbounded on $[1, \infty]$.
- $\sin(x) = \mathcal{O}(1)$ on $\mathbb{R}$.
- $x^2 = \mathcal{O}(x)$ as $x \to 0$ since $\frac{x^2}{x} = x$ is bounded as $x \to 0$.
- $e^x - 1 = \mathcal{O}(x)$ as $x \to 0$ since $\frac{|e^x-1|}{|x|}$ is bounded as $x \to 0$.

Definition 5.1.2 (The o "little-oh" symbol) *Let f and g be functions defined on a region $\mathcal{D}$ and let x_0 be a limit point of $\mathcal{D}$. Then*

$$f(x) = \mathrm{o}(g(x)) \text{ as } x \to x_0,$$

means that

$$\frac{f(x)}{g(x)} \to 0 \text{ as } x \to x_0.$$

Example 5.1.2

- $x^3 = \mathrm{o}(x^2)$ as $x \to 0$.
- $x^3 = \mathrm{o}(x^4)$ as $x \to \infty$.
- $x^n = \mathrm{o}(e^x)$ as $x \to \infty$.

Note that $f(x) \ll g(x)$ as $x \to x_0$ is the same as $f = \mathrm{o}(g(x))$ as $x \to x_0$.

Definition 5.1.3 (Asymptotic Equivalence) *Let f and g be defined in a region $\mathcal{D}$ with limit point x_0. We write*

$$f \sim g \text{ as } x \to x_0 \tag{5.4}$$

to mean that

$$\frac{f(x)}{g(x)} \to 1 \text{ as } x \to x_0 \tag{5.5}$$

Note:

1. x_0 could be $\pm\infty$.
2. $f \sim g$ as $x \to x_0$ implies that $f = \mathcal{O}(g(x))$ and $g = \mathcal{O}(f(x))$. The converse is not true. For example, $f(x) = x$, $g(x) = 5x$.

Example 5.1.3

-
$$x + \frac{1}{x} \sim \frac{1}{x} \quad \text{as } x \to 0,$$
since
$$\frac{x + \frac{1}{x}}{\frac{1}{x}} = x^2 + 1 \to 1 \quad \text{as } x \to 0.$$

-
$$x + \frac{1}{x} \sim x \quad \text{as } x \to \infty.$$

-
$$x^3 + 9x^4 - \frac{3}{2}x^5 \sim \begin{cases} x^3 & \text{as } x \to 0; \\ -\frac{3}{2}x^5 & \text{as } x \to 0. \end{cases}$$

-
$$e^{x-9/x} \sim \begin{cases} e^{-9/x} & \text{as } x \to 0; \\ e^x & \text{as } x \to \infty. \end{cases}$$

Note: $f \sim g$ as $x \to x_0 \Rightarrow g \sim f$ as $x \to x_0$.

Note: $f \sim g$ means that $f - g \ll g$.

Example 5.1.4 *The functions $f = e^x + x$ and $g = e^x$ are asymptotic to one another as $x \to \infty$ as*

$$\frac{f-g}{g} = \frac{x}{e^x} \to 0 \text{ as } x \to \infty$$

Note that the difference $f - g$ *does not* go to 0! The difference goes to infinity as $x \to \infty$. Saying $f \sim g$ as $x \to x_0$ does not mean that f and g get close in an absolute sense, it only means that they get close in a relative sense: $f - g$ can blow up but $f - g$ gets small relative to f or g (i.e., gets small in the sense that $(f - g)/g \to 0$). Saying something is large or small can only be done in comparison with something else. You shouldn't say 0.0000001 is small. It is small compared to 1 (which, if someone says 0.0000001 is small, is what they mean implicitely), but it is large compared with 10^{-20}.

Definition 5.1.4 (Asymptotic Series) *To say that*

$$g(x) \sim x^4 - 3x^2 - 2x + \cdots \text{ as } x \to \infty,$$

means the following:

1. *$g \sim x^4$, i.e. $\frac{g}{x^4} \to 1$ as $x \to \infty$,*

2. $g - x^4 \sim -3x^2$*, i.e.* $\frac{g-x^4}{-3x^2} \to 1$ *as* $x \to \infty$*,*

3. $g - x^4 + 3x^2 \sim -2x$*, i.e.* $\frac{g-x^4+3x^2}{-2x} \to 1$ *as* $x \to \infty$*,*

etc. The series on the right hand side is an example of an asymptotic series. In the series the fastest growing term comes first. Each successive term must grow more slowly than the preceding term.

Asymptotic series are very useful for finding approximate values of integrals and functions, which we consider next.

5.2 Asymptotic Expansions

We begin by finding an asymptotic expression for an integral.

5.2.1 The Exponential Integral

The exponential integral function $\mathrm{Ei}(x)$ is defined by:

$$\mathrm{Ei}(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt \quad \text{for } x > 0. \tag{5.6}$$

This is not very useful as it stands – can we find a useful approximation? Successively integrating by parts gives

$$\mathrm{Ei}(x) = \underbrace{e^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots + \frac{(-1)^{n-1}(n-1)!}{x^n}\right)}_{S_n(x)} + \underbrace{(-1)^n n! \int_x^\infty \frac{e^{-t}}{x^{n+1}}\,dt}_{R_n(x)}. \tag{5.7}$$

As $n \to \infty$, $S_n(x)$ gives a divergent series as is easily seen from the ratio test. The ratio of the (m+1)st and mth terms is

$$\frac{\frac{(-1)^m m!}{x^{m+1}}}{\frac{(-1)^{m-1}(m-1)!}{x^m}} = \frac{m}{x} \to \infty \text{ as } m \to \infty \tag{5.8}$$

for fixed x. Suppose we change the question from "What is the limit of $S_n(x)$ as $n \to \infty$ for fixed x?", to "What is the limit as $x \to \infty$ for fixed n".

Have

$$\begin{aligned} |\mathrm{Ei}(x) - S_n(x)| &= |R_n(x)| \\ &\le n! \int_x^\infty \frac{e^{-t}}{t^{n+1}}\,dt \\ &\le \frac{n!}{x^{n+1}} \int_x^\infty e^{-t}\,dt \end{aligned}$$

so

$$|\mathrm{Ei}(x) - S_n(x)| \le \frac{n!}{x^{n+1}} e^{-x} \to 0 \text{ as } x \to \infty. \tag{5.9}$$

Hence for fixed n, $S_n(x)$ gives a good approximation to $\mathrm{Ei}(x)$ if x is sufficiently large. An alternative derivation of this result is the following. Because the error term R_n alternates in sign $S_{2n-1} <$

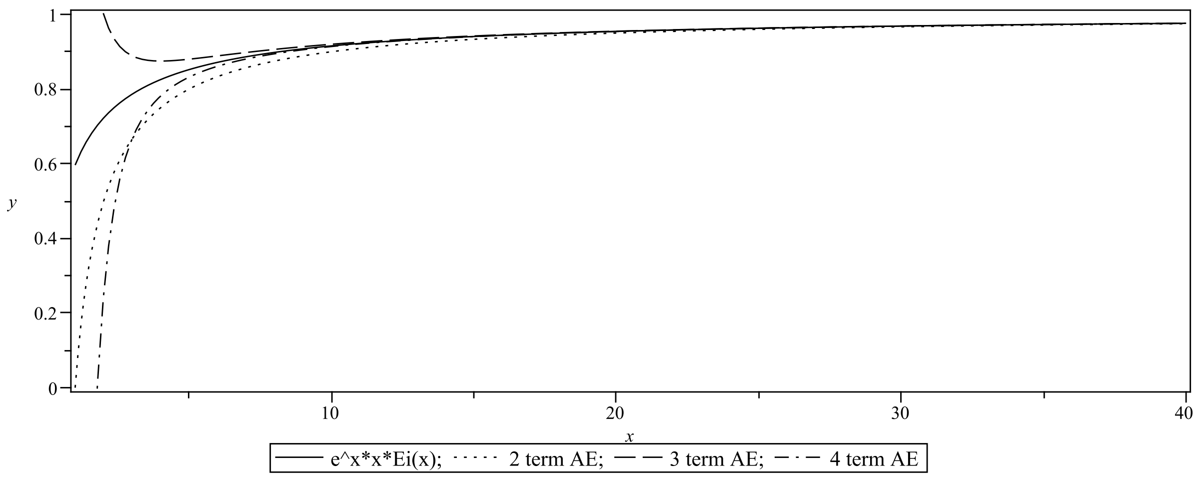

Figure 5.1: Comparison of $xe^x Ei(x)$ and asymptotic approximations using two (dots), three (dashes) and four (dash-dot) terms of the Asymptotic Expansion.

$Ei(x) < S_{2n}$ so the magnitude of the error is less than the magnitude of the first omitted term, namely $e^{-x}n!/x^{n+1}$, as above.

Because of this result we can write

$$\mathrm{Ei}(x) \sim e^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots\right) \text{ as } x \to \infty. \tag{5.10}$$

This is an asymptotic expansion of $\mathrm{Ei}(x)$. Figure 5.1 compares $xe^x\mathrm{Ei}(x)$ with $xe^xS_i(x)$ for $i = 1, 2, 3$. The first two terms of the Asymptotic Expansion, $1 - 1/x$, is within 1% of the exact value for x larger than about 13.3. Using the first four terms the error is less than 1% for x larger than about 6.3.

Now we can ask the question "For a given value of x for what value of n, call it $N(x)$, does $S_n(x)$ give the best approximation to $\mathrm{Ei}(x)$?". The answer to this question is difficult to determine precisely as we only have an upper bound on the magnitude of the error which is easy to use. We can approximate the answer by minimizing our bound on the error. This means choosing n so the first neglected term in the alternating series is minimized. As shown above the ratio of the magnitudes of the (n+1)st and nth terms is

$$\frac{n}{x} < 1 \quad \text{if } n \le x. \tag{5.11}$$

The terms decrease until $n > x$ thus the minimum is at $nN(x) = \lfloor x \rfloor$, the greatest integer less than x. This implies that as a function of n, $|\mathrm{Ei}(x) - S_n(x)|$ initially decreases monotonically until n exceeds x after which it increases monotonically. This is illustrated in Figure 5.2 which compares $S_n(x)$ with $Ei(x)$ as a function of n for $x = 5$ and 10. Alternatively,

$$|R_n(x)| \le \frac{n!e^{-x}}{x^{n+1}} = \frac{e^{-x}}{x} \cdot \frac{1}{x} \cdot \frac{2}{x} \cdot \frac{3}{x} \cdot \cdots \cdot \frac{n}{x}. \tag{5.12}$$

The factors $1/x$, $2/x$, ... are less than 1, hence decrease R_n until n becomes larger than x.

In summary, for the exponential integral, for fixed x our upper bound on the error is minimized when $n = \lfloor x \rfloor$. Hence, $S_{\lfloor x \rfloor}(x)$ is an estimate $\mathrm{Ei}(x)$ with error $R_{\lfloor x \rfloor}(x) < \frac{e^{-x}\lfloor x \rfloor!}{x^{\lfloor x \rfloor+1}}$.

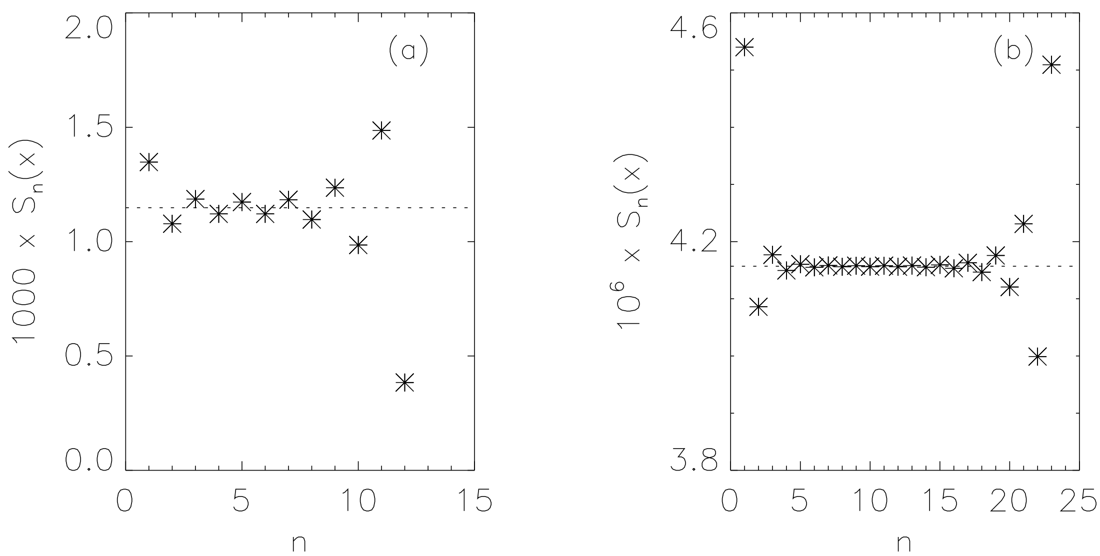

Figure 5.2: Comparison of $\mathrm{Ei}(x)$ (dotted line) with values of partial sums S_n as a function of n. (a) $x = 5.0$. (b) $x = 10.0$.

Rule of Thumb: For an alternating divergent series use $S_N(x)$ where the $(N+1)^{\mathrm{st}}$ term in the asymptotic series is the smallest.

The rule of thumb is a rough guide. In practice we can often take n much less than $n_{\mathrm{opt}} = \lfloor x \rfloor$, depending on the level of accuracy required. This is particularly true for large x as shown in Figure 5.2. Here it can be seen that the $S_n(x)$ are very close to $\mathrm{Ei}(x)$ over a much broader range of values of n when $x = 10$ than when $x = 5$.

Example 5.2.1 *For $x = 10$, $R_4(10) \leq \frac{4!e^{-10}}{10^5} \approx 1.1 \times 10^{-8}$. The error bound gives an approximate error of*

$$\left|\frac{R_4(10)}{S_4(10)}\right| \times 100\% \quad = \quad 0.26\%,$$

whereas using the optimal value of n the approximate error is

$$\left|\frac{R_{10}(10)}{S_{10}(10)}\right| \times 100\% \quad = \quad 0.04\%.$$

The actual error using S_4 is

$$\left|\frac{\mathrm{Ei}(10) - S_4(10)}{\mathrm{Ei}(10)}\right| \times 100\% \quad = \quad 0.18\%$$

and it is 0.0193% using S_{10} and -0.0202% using S_{11}.

Important point: For a given x there is a *minimum* error (which is less than the error bound, in this case $|R_{\lfloor x \rfloor}| \leq \lfloor x \rfloor! e^{-x}/x^{\lfloor x \rfloor + 1}$) that can be made. In contrast, for a convergent power series the error can be made arbitrarily small if we are prepared to sum enough terms. In this example the minimum error decreases as x increases.

5.2.2 Asymptotic Sequences and Asymptotic Expansions (Poincaré 1886)

Definition 5.2.1 *A set of functions* $\{\varphi_n(x)\}, n = 1, 2, 3, \ldots$ *for* $x \in \mathcal{D}(= \mathbb{R}, \mathbb{R}^n, \mathbb{C})$ *is an asymptotic sequence (AS) as* $x \to x_0$ *if for each* n, $\varphi_n(x)$ *is defined on* $\mathcal{D}$ *and* $\varphi_{n+1}(x) = \mathrm{o}(\varphi_n(x))$ *as* $x \to x_0$.

Example 5.2.2

- $\{(x - x_0)^n\}$ is an asymptotic sequence as $x \to x_0$, but is **not** an asymptotic sequence as $x \to \infty$.
- $\{e^{-x}x^{-a_n}\}$ is an asymptotic sequence as $x \to \infty$ where $a_n \in \mathbb{R}$ with $a_n + 1 > a_n$.
- $\{\ln(x)^{-n}\}$ is an asymptotic sequence as $x \to \infty$.

Definition 5.2.2 *Let* x, x_0 *and* $\mathcal{D}$ *be defined as above and let* $f(x)$ *be a function on* $\mathcal{D}$. *Let* $\{\varphi_n(x)\}$ *be an asymptotic series as* $x \to x_0$. *The '***formal***' series*

$$f = \sum_{n=1}^{N} a_n \varphi_n(x) \tag{5.13}$$

is said to be an asymptotic expansion of f *as* $x \to x_0$ *to* N *terms provided*

$$f(x) - \sum_{n=1}^{N} a_n \varphi_n(x) = \left\{ \begin{array}{c} \mathrm{o}(\varphi_N(x)) \\ \text{or} \\ \mathcal{O}(\varphi_{N+1}(x)) \end{array} \right\} \text{ as } x \to x_0. \tag{5.14}$$

Note that (5.14) gives some information about the error, i.e.

$$\text{error} = f(x) - \sum_{n=1}^{N} a_n \varphi_n(x) \to 0$$

faster than $\varphi_N(x) \to 0$ as $x \to x_0$ or it blows up more slowly. This means that the error is small compared to $\varphi_N(x)$. Of course this may only be useful if $\varphi_N(x) \to 0$ as $x \to x_0$ and only for x sufficiently close to x_0.

Important Point: The accuracy of an asymptotic approximation is limited. It has nothing to do with ordinary convergence. In the case of a function $f(x)$ expressed as a convergent power series we can make the error arbitrarily small if we are prepared to sum enough terms. In an asymptotic expansion the potential accuracy is limited.

Example 5.2.3

For $\mathrm{Ei}(x)$ the smallest we can guarantee the error to be less than

$$\frac{n!e^{-x}}{x^{n+1}},$$

with $n = \lfloor x \rfloor$ for any given x. This is an upper bound on the error, so the actual error might be a lot smaller but without further analysis we can't say any more about the error. Thus, there is nothing we can do to reduce the error using this asymptotic expansion (a function has many asymptotic expansions, a different one may give a better error estimate).

Example 5.2.4 *The Bessel function $J_o(x)$ has the power series expansion*

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots . \tag{5.15}$$

which converges to $J_0(x)$ for all x. The power series is completely useless unless x is small. For example,

$$J_0(4) = 1 - 4 + 4 - \frac{16}{9} + \cdots , \tag{5.16}$$

and 8 terms are need to get three digits of accuracy. An asymptotic expansion for $J_0(x)$ is

$$\begin{aligned} J_0(x) \sim \sqrt{\frac{2}{\pi x}} \Bigg\{ & \left(1 - \frac{9}{128x^2} + \cdots\right) \cos\left(x - \frac{\pi}{4}\right) \\ & + \left(\frac{1}{8x} - \frac{75}{1024x^2} + \cdots\right) \sin\left(x - \frac{\pi}{4}\right) \Bigg\} \quad \text{as } x \to \infty. \end{aligned} \tag{5.17}$$

This series is divergent for all x. This non-divergent asymptotic series is, however, extremely useful. The leading order term

$$\sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right) \tag{5.18}$$

gives $J_0(x)$ to three digit accuracy for all $x \geq 4$! Example approximations are shown in Figure 5.3. There it can be seen that the leading-order asymptotic approximation is very good for $x \geq 1$ whereas the 4, 10 and 20-term power series approximations are useful for $x <\approx 3$, 7.5 and 15 respectively. Finding $J_0(99)$ using the power-series would clearly be difficult but easy using the asymptotic approximation! We will discuss finding the asymptotic expansion for the Bessel function in the next chapter.

Claim 5.2.1 *If $f(x)$ and $\{\varphi_n(x)\}$ are known where $\{\varphi_n\}$ is an asymptotic series, then the asymptotic expansion for f in terms of the φ_n is unique.*

Proof: Need to find a_n's such that

$$f \sim a_1\varphi_1 + a_2\varphi_2 + \cdots \text{ as } x \to x_0. \tag{5.19}$$

This means that

$$\begin{aligned} f - a_1\varphi_1 = \mathrm{o}(\varphi_1(x)) & \text{ as } x \to x_0, \\ \Rightarrow \frac{f - a_1\varphi_1}{\varphi_1} &= \frac{f}{\varphi_1} - a_1 \to 0 \text{ as } x \to x_0. \end{aligned}$$

Thus, take

$$a_1 = \lim_{x \to x_0} \frac{f}{\varphi_1}. \tag{5.20}$$

Next

$$\begin{aligned} f - a_1\varphi_1 - a_2\varphi_2 &= \mathrm{o}(\varphi_2), \\ \Rightarrow \frac{f - a_1\varphi_1 - a_2\varphi_2}{\varphi_2} &= \frac{f - a_1\varphi_1}{\varphi_2} - a_2 \to 0 \text{ as } x \to x_0. \end{aligned}$$

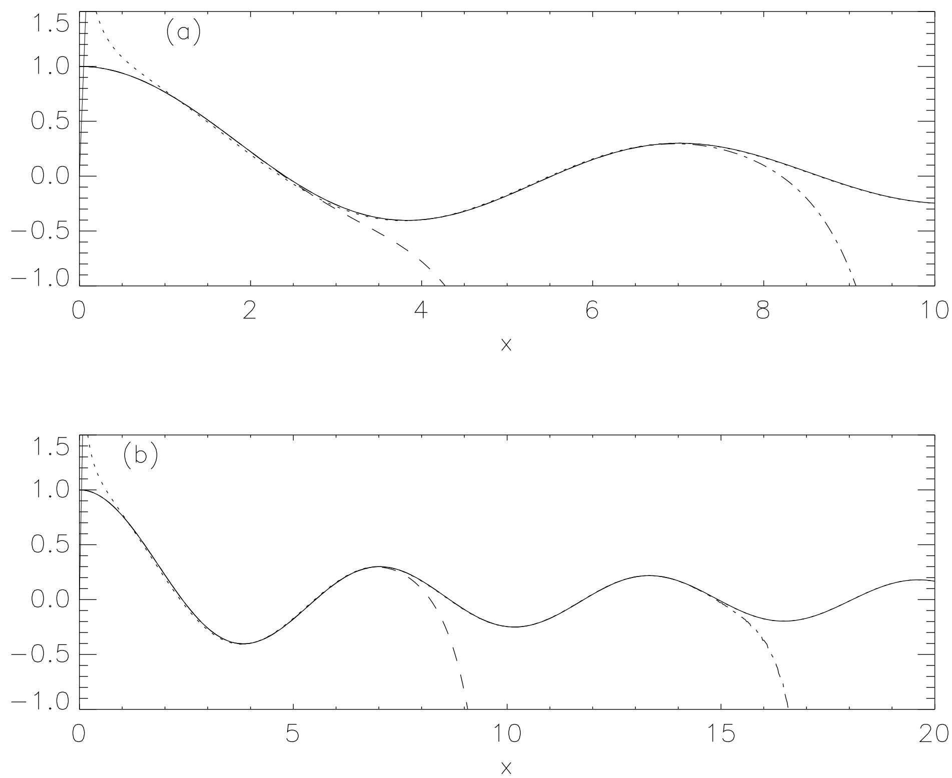

Figure 5.3: Comparison of approximation of $J_0(x)$ with power series or asymptotic series. In both panels the solid curve is $J_0(x)$ and the dotted curve is the leading-order term of the asymptotic expansion. (a) Dashed: 4 term power series approximation. Dash-dot: 10 term power series approximation. (b) Dashed: 10 term power series approximation. Dash-dot: 20 term power series approximation.

Therefore take

$$a_2 = \lim_{x \to x_0} \frac{f - a_1 \varphi_1}{\varphi_2}. \tag{5.21}$$

The pattern is clear.

Note:

1. This might give something useless, such as all a_n's are zero, as would happen, for example, if $f = e^{-x}$ and $\varphi_n(x) = \frac{1}{x^n}$ as $x \to \infty$.

2. If the asymptotic series is not known, there will be many possible asymptotic expansions for f. For example,

$$\begin{aligned}
\sin 2\epsilon &\sim 2\epsilon - \frac{4}{3}\epsilon^3 + \frac{4}{15}\epsilon^5 + \cdots \text{ as } \epsilon \to 0, \\
\sin 2\epsilon &\sim 2\tan\epsilon - 2\tan^3\epsilon + 2\tan^5\epsilon + \cdots \text{ as } \epsilon \to 0, \\
\sin 2\epsilon &\sim 2\left(\frac{3\epsilon}{3+2\epsilon^2}\right) - \frac{7}{12}\left(\frac{3\epsilon}{3+2\epsilon^2}\right)^5 + \cdots \text{ as } \epsilon \to 0.
\end{aligned}$$

5.2.3 The Incomplete Gamma Function

Example 5.2.5 *The incomplete Gamma function is defined as*

$$\gamma(a,x) \;=\; \int_0^x e^{-t}t^{a-1}\,dt \tag{5.22}$$

for $a, x > 0$.

1. *Derive a power series expansions which converges for all* x. *Show it is useless for large* x.
2. *Find an asymptotic expansion for* γ *by writing (5.22) as*

$$\begin{aligned}\gamma(a,x) &= \int_0^\infty e^{-t}t^{a-1}dt - \int_x^\infty e^{-t}t^{a-1}\,dt \\ &= \Gamma(a) - \mathrm{Ei}_{a-1}(x).\end{aligned}$$

Solution:

1. Using the convergent power series expansion of e^{-t} write

$$\begin{aligned}e^{-t}t^{a-1} &= t^{a-1}\sum_{n=0}^{\infty}\frac{(-1)^n t^n}{n!} \\ &= \sum_{n=0\infty}\frac{(-1)^n t^{n+a-1}}{n!}.\end{aligned} \tag{5.23}$$

The partial sums converge uniformly on any interval $[0, x]$ so we can integrate term by term to get

$$\gamma(a,x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{x^{n+a}}{n+a} = \sum_{n=0}^{\infty}a_n, \tag{5.24}$$

where

$$a_n = \frac{(-1)^n}{n!}\frac{x^{n+a}}{n+a}. \tag{5.25}$$

Applying the ratio test,

$$\frac{a_{n+1}}{a_n} = \frac{x}{(a+n+1)(n+1)} \to 0 \text{ as } n \to \infty, \tag{5.26}$$

showing that the series converges for all x. For any fixed N the partial sum

$$S_N(x) = x^a\sum_{n=0}^{N}\frac{(-1)^n x^n}{(a+n)n!} \to \infty \text{ as } x \to \infty. \tag{5.27}$$

Thus, for large x a large number of terms from the power series are needed to obtain a reasonably accurate approximation. This makes the power series useless for large x.

2. Proceeding as for $\mathrm{Ei}(x)$, several integration by parts yields

$$\begin{aligned}\mathrm{Ei}_{a-1}(x) = x^a e^{-x}&\left(\frac{1}{x} + \frac{(a-1)}{x^2} + \cdots + \frac{(a-1)^{[n-1]}}{x^n}\right) \\ &+ (a-1)^{[n]}\int_x^\infty e^{-t}t^{a-(n+1)}dt,\end{aligned} \tag{5.28}$$

where $k^{[n]} = k(k-1)(k-2)\cdots(k-n+1)$.
Set

$$\begin{aligned} S_n(x,a) &= x^a e^{-x}\left(\frac{1}{x} + \frac{(a-1)}{x^2} + \cdots + \frac{(a-1)^{[n-1]}}{x^n}\right), \\ R_n(x,a) &= (a-1)^{[n]}\int_x^\infty e^{-t}t^{a-n+1}\,dt. \end{aligned} \tag{5.29}$$

As before $S_n(x,a)$ is divergent as $n \to \infty$. For fixed x the integral in R_n converges for all $a > 0$ and $\lim_{x\to\infty} R_n(x,a) = 0$. Have

$$\mathrm{Ei}_{a-1}(x) \sim x^a e^{-x}\left(\frac{1}{x} + \frac{a-1}{x^2} + \cdots + \frac{(a-1)^{[n-1]}}{x^n} + \cdots\right), \tag{5.30}$$

as $x \to \infty$.

Chapter 6

Asymptotic Analysis for 2^{nd} order ODEs

6.1 Introduction

We now consider asymptotic methods for finding approximate solutions of ordinary differential equations of the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0. \tag{6.1}$$

A good reference is Bender and Orzag and much of the following is based on material in that text.

Definition 6.1.1 *$x_0 \neq \infty$ is an ordinary point if p and q are analytic in a neighbourhood of x_0.*

Definition 6.1.2 *$x_0 \neq \infty$ is a regular singular point if it is not an ordinary point and if $(x - x_0)^2 q(x)$ and $(x - x_0)p(x)$ are analytic in a neighbourhood of x_0.*

Definition 6.1.3 *The point $x_0 \neq \infty$ is called an irregular singular point if it is not an ordinary point or a regular singular point. $\Rightarrow$ $(x-x_0)^2 q(x)$ or $(x-x_0)p(x)$ are not analytic in a neighbourhood of x_0.*

For $x_0 = \infty$ set $t = \frac{1}{x}$ and classify the point $t = 0$.

Fact 6.1.1 *Any solution of (6.1) is analytic in a neighbourhood of an ordinary point. More over, the solution can be expanded in a convergent power series about x_0. The radius of convergence is at least as large as the distance from x_0 to the nearest singularity of p and q on the complex plane.*

Example 6.1.1

$$y'' + \frac{4x}{1+x^2}y' - \frac{2}{1+x^2}y = 0 \tag{6.2}$$

has an ordinary point at $x = 0$.

One solution is

$$y = \frac{x}{1+x^2} = x(1 - x^2 + x^4 - x^6 + \cdots) \tag{6.3}$$

which has radius of convergence 1, which is equal to the distance from $x = 0$ to $x = \pm i$, the singularity of p and q.

Fact 6.1.2 *A solution of (6.1) may be analytic at a regular singular point x_0. If it is not, its singularity must be either a pole or an algebraic or logarithmic branch point. At a regular singular point*

1. *there is always a solution of the form $y_1 = (x - x_0)^\alpha A(x)$ where $A(x)$ is analytic at x_0.*
2. *the second solution has one of the following forms*

$$y_2 = (x - x_0)^\beta B(x), \tag{6.4}$$

or

$$y_2 = (x - x_0)^\alpha A(x) \ln(x - x_0) + (x - x_0)^\beta C(x), \tag{6.5}$$

where B and C are analytic.

Fact 6.1.3 *At an irregular singular point at least one solution does not have the form of (6.4) or (6.5).*

Example 6.1.2 *Consider*

$$y'' + \frac{3}{2x} y' + \frac{1}{4x^3} y = 0. \tag{6.6}$$

1. $x = 0$ is an irregular singular point since $q(x)x^2 = \frac{1}{4x}$ is not analytic.
2. Letting $t = \frac{1}{x}$ and $y(x) \to \tilde{y}(t)$ the ODE becomes

$$\tilde{y}'' - \frac{1}{t}\tilde{y}' + \frac{1}{4t}\tilde{y} = 0. \tag{6.7}$$

Neither p and q are analytic, so $t = 0$ ($x = \infty$) is not an ordinary point. Both $t^2 q = t/4$ and $tp = -1$ are analytic so $t = 0$ ($x = \infty$) is a regular singular point.

Example 6.1.3 *Consider the ODE*

$$(x - 1)(2x - 1)y'' + 2xy' - 2y = 0. \tag{6.8}$$

First, put it in standard form,

$$\to y'' + \frac{x}{(x-1)(x-\frac{1}{2})} y' - \frac{1}{(x-1)(x-\frac{1}{2})} y = 0. \tag{6.9}$$

From this we see that $x = 1$ and $1/2$ are regular singular points. $x = \infty$ is also a regular singular point. One solution of the ODE is $y_1 = 1/(x - 1)$ which has a Taylor Series expansion about $x = 0$ which converges for $|x| < 1$, where $x = 0$ is an ordinary point. Note that the radius of convergence goes beyond the singularity at $x = \frac{1}{2}$. A second linearly independent solution is $y_2 = x$ which has a Taylor Series expansion about $x = 0$ that converges everywhere.

Example 6.1.4 *Consider the ODE*

$$y'' - \frac{1+x}{x} y' + \frac{1}{x} y = 0. \tag{6.10}$$

- $x = 0$ is a regular singular point.
- $x = \infty$ is an irregular singular point.
- Two linearly independent solutions are $y_1 = e^x$ and $y_2 = 1 + x$.
  - Both are analytic at $x = 0$
  - y_1 has an essential singularity at $x = \infty$. y_2 has a pole at ∞.

6.2 Finding the behaviour near an Irregular Singular Points: Method of Carlini-Liouville-Green

The method of Frobenius can be used to find the solution in a neighbourhood of ordinary or regular singular points. More interesting for our purposes are irregular singular points for which asymptotic methods yield useful solutions.

Example 6.2.1 *Consider*

$$x^3 y'' = y \to y'' - \frac{1}{x^3} y = 0. \tag{6.11}$$

Find the behaviour as $x \to 0^+$, an irregular singular point.

6.2.1 Finding the Leading Behaviour

Attempt 1: Method of Frobénius:

Try find a solution in the form of a power series

$$y = \sum_{n=0}^{\infty} a_n x^{n+\alpha}, \; a_0 \neq 0. \tag{6.12}$$

Have

$$y'' = \sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n x^{n+\alpha-2}, \tag{6.13}$$

so

$$\begin{aligned} x^3 y'' - y &= \sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n x^{n+\alpha+1} - a_n x^{n+\alpha} \\ &= -a_0 x^{\alpha} + (-a_1 + \alpha(\alpha-1)a_0) x^{\alpha+1} \\ &\quad + (-a_2 + (1+\alpha)\alpha a_1) x^{\alpha+2} + \cdots = 0. \end{aligned} \tag{6.14}$$

The coefficient of each distinct power of x must be zero, hence all the a_n's are zero! Thus there is no solution in power series form.

Attempt 2: The behaviour of solutions of ODEs as x approaches an irregular singular point usually involves some sort of exponential behaviour. This suggests looking for a solution of the form $y = e^{S(x)}$, (Carlini-Liouville-Green Method).

From

$$y'' + p(x)y' + q(x)y = 0, \tag{6.15}$$

we get

$$S'' + (S')^2 + p(x)S' + q(x) = 0. \tag{6.16}$$

We now proceed by **assuming** that $S'' \ll (S')^2$ as $x \to x_0$. If this is true we can write

$$(S')^2 \sim -pS' - q \text{ as } x \to x_0. \tag{6.17}$$

We now replace the '$\sim$' by '=' and solve for S' giving

$$S' \sim \frac{-p \pm \sqrt{p^2 - 4q}}{2} \quad \text{as } x \to x_0. \tag{6.18}$$

Integrating this gives an approximation to S. We then need to verify that $S'' \ll (S')^2$ as $x \to x_0$.

For our problem
$$y'' - \frac{1}{x^3}y = 0, \tag{6.19}$$
giving
$$S'' + (S')^2 - \frac{1}{x^3} = 0. \tag{6.20}$$
Ignoring S'^2,
$$(S')^2 \sim \frac{1}{x^3} \text{ or } S' \sim \pm\frac{1}{x^{3/2}} \text{ as } x \to 0^+. \tag{6.21}$$
This means that
$$S'(x) \quad = \quad \pm x^{-3/2} + C'(x),$$
where $C'(x) = \mathrm{o}(x^{-3/2})$ as $x \to 0^+$. Differentiating gives
$$S'' = \mp\frac{3}{2}x^{-5/2} + C''(x). \tag{6.22}$$
Assuming $C''(x) = \mathrm{o}(x^{-5/2})$ as $x \to 0^+$ (no quarantee!), $S'' \ll S'^2$ as we assumed. So far so good — everything is consistent .

Integrating (6.22) gives
$$S(x) \sim \mp 2x^{-1/2} \text{ as } x \to 0^+. \tag{6.23}$$
Here we have assumed that if $f \sim g$ as $x \to x_0$, then
$$\int f(x)\,dx \sim \int g(x)\,dx, \tag{6.24}$$
i.e.,
$$\int f(x)\,dx = \int g(x)\,dx + h(x), \tag{6.25}$$
where $h(x) = o(\int g\,dx)$. This is not always the case, but usually OK. It is possible to have $f(x) = g(x) + h(x)$ where $h = \mathrm{o}(g)$ as $x \to x_0$ (hence $f \sim g$) for which $\int h\,dx$ is not $\mathrm{o}(\int g\,dx)$. A trivial case is $g = x$ and $h = 0$ where $\int h = c \gg \int g = x^2/2$ as $x \to 0$ where c is a constant. As long as $\int f$ blows up we are usually OK (but not always!).

So far our approximate behaviour for y as $x \to 0^+$ is
$$y = e^{S(x)}, \tag{6.26}$$
where
$$S \sim \pm 2x^{-1/2} \text{ as } x \to 0^+, \tag{6.27}$$
or
$$S = \pm 2x^{-1/2} + C(x) \tag{6.28}$$
where $C(x) = \mathrm{o}(x^{-1/2})$ as $x \to 0^+$. We now need to improve the solution. We now consider one case, i.e., in particular, take
$$S = 2x^{-1/2} + C(x). \tag{6.29}$$

Subsituting (6.29) into (6.20) gives

$$\begin{aligned}\frac{3}{2}x^{-5/2} + C'' + \frac{1}{x^3} - 2x^{-3/2}C' + (C')^2 - \frac{1}{x^3} = 0 \\ \Rightarrow \frac{3}{2}x^{-5/2} + C'' - 2x^{-3/2}C' + (C')^2 = 0.\end{aligned} \tag{6.30}$$

Here we have made no approximations. Two independent solutions of this second order, nonlinear ODE will provide two linearly independent solutions of the original ODE for $y(x)$. To proceed we assume that

$$C \ll x^{-1/2} \Rightarrow C' \ll x^{-3/2} \Rightarrow (C')^2 \ll x^{-3/2}C'. \tag{6.31}$$

We also assume that $C'' \ll x^{-5/2}$. With these reasonable assumptions the nonlinear ODE for C gives

$$\begin{aligned}2x^{-3/2}C' &\sim \frac{3}{2}x^{-5/2} \quad \text{as } x \to 0^+, \\ \Rightarrow C' &\sim \frac{3}{4}x^{-1} \quad \text{as } x \to 0^+, \\ \Rightarrow C &\sim \frac{3}{4}\ln x \quad \text{as } x \to 0^+,\end{aligned}$$

or

$$C(x) = \frac{3}{4}\ln x + D(x)$$

where $D(x) \ll \ln x$ as $x \to 0^+$. This latter equation is exact. Substituting into the exact equation for C gives

$$D'' - \frac{3}{16x^2} + \frac{3}{2x}D' + (D')^2 - \frac{2}{x^{3/2}}D' = 0. \tag{6.32}$$

Using $D' \ll \frac{1}{x}$, which is the correct behaviour for the solution we are chasing down, the dominant balance gives

$$-2x^{-3/2}D' \sim \frac{3}{16x^2} \quad \text{as } x \to 0^+. \tag{6.33}$$

Integrating gives

$$D \sim \frac{-3}{16}x^{1/2} + d, \quad \text{as } x \to 0^+. \tag{6.34}$$

Note that here, because $x^{1/2} \to 0$ as $x \to 0^+$ we've have to include a constant of integration. Because $d \gg x^{1/2}$ as $x \to 0^+$ we can write $D(x)$ as

$$D(x) = d + \delta(x), \tag{6.35}$$

where $\delta \to 0$ as $x \to 0^+$ (and in fact $\delta(x) \sim -3x^{1/2}/16$ as $x \to 0$). Thus

$$\begin{aligned}y &= e^{2x^{-1/2} + \frac{3}{4}\ln x + d + \delta(x)}, \\ &= C_1 x^{3/4} e^{2x^{-1/2}} e^{\delta(x)}, \\ &\sim C_1 x^{3/4} e^{2x^{-1/2}} \quad \text{as } x \to 0^+,\end{aligned} \tag{6.36}$$

since $e^{\delta(x)} \to 1$ as $x \to 0^+$.

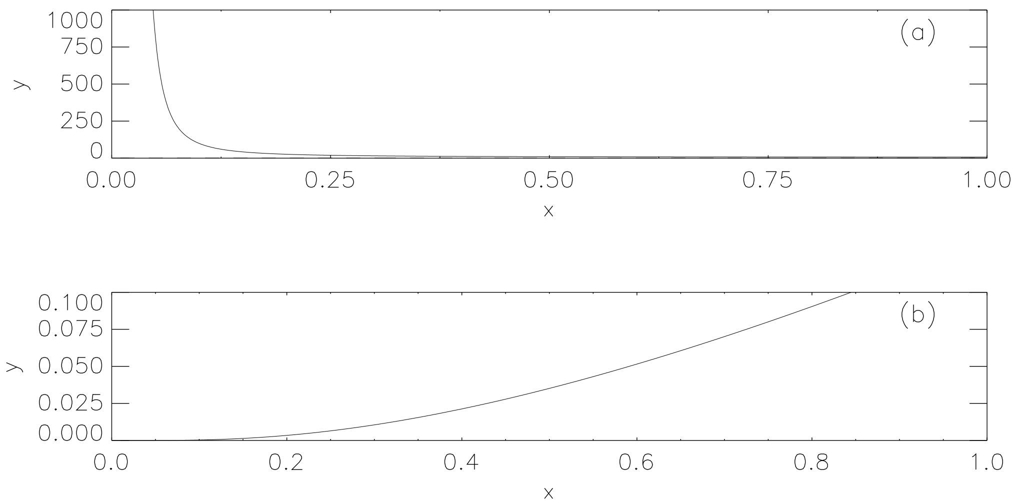

Figure 6.1: Leading behaviours of the solutions of $y'' - y/x^3 = 0$ as $x \to 0^+$. (a) $x^{3/4}e^{2x^{-\frac{1}{2}}}$. (b) $x^{3/4}e^{-2x^{-\frac{1}{2}}}$.

Definition 6.2.1 *The contributions to $S(x)$ that do not vanish as $x \to x_0$, some irregular singular point, gives the* **leading behaviour** *of the solution of the differential equation.*

In this case $C_1 x^{3/4} e^{2x^{-\frac{1}{2}}}$ is the leading behaviour of one of the solutions of $x^3 y'' - y = 0$. Note that because the differential equation is linear and homogeneous, the constant C_1 is arbitrary. The second solution, with $S \sim -2x^{-\frac{1}{2}}$ as $x \to 0^+$, has the leading behaviour $C_2 x^{3/4} e^{-2x^{-\frac{1}{2}}}$. This function goes to zero as $x \to 0+$ so rapidly than all its derivatives go to zero. The two leading behaviours are illustrated in Figure 6.1.

6.2.2 Further Improvements: corrections to the leading behaviour.

Let

$$y = x^{3/4} e^{2x^{-1/2}} [1 + g(x)] \quad \text{where } g \to 0 \quad \text{as } \; x \to 0^+. \tag{6.37}$$

Substituting into the ODE gives

$$g'' + \left(\frac{3}{2x} - \frac{2}{x^{3/2}} \right) g' - \frac{3}{16x^2} - \frac{3}{16x^2} g = 0. \tag{6.38}$$

This is a linear second order differential equation. It has two linearly independent solutions which implies two consistent dominant balances. We want the one satisfying $g \to 0$ as $x \to 0^+$. Hence

$$\frac{3}{16x^2} g \ll \frac{3}{16x^2} \quad \text{as } x \to 0^+. \tag{6.39}$$

In addition,

$$\frac{3}{2x} \ll \frac{2}{x^{3/2}} \quad \text{as } x \to 0^+, \tag{6.40}$$

so we must have

$$g'' - \frac{2}{x^{3/2}} g' \sim \frac{3}{16x^2} \quad \text{as } x \to 0^+. \tag{6.41}$$

To further simplify, consider four possibilities:

1. $-3x^{-3/2}g'$ is negligible $\Rightarrow g \sim \frac{-3}{16}\ln x$, which is inconsistent with $g \to 0$ as $x \to 0^+$.
2. $\frac{3}{16}x^2$ is negligible $\Rightarrow g' \sim e^{2x^{-3/2}}$ which is inconsistent with $g \to 0$ as $x \to 0^+$.
3. All three terms are needed. Hopefully this is not the case.
4. g'' is negligible. This gives

$$\begin{aligned} g' &\sim -\frac{3}{32}x^{-1/2} \text{ as } x \to 0^+, & (6.42)\\ \Rightarrow g &\sim -\frac{3}{16}x^{1/2} \text{ as } x \to 0^+. & (6.43) \end{aligned}$$

This is consistent since this gives $g'' \sim 3x^{-3/2}/64 \ll 3x^{-2}/16 \sim -2x^{-3/2}g'$ as $x \to 0^+$.

Next let

$$g = -\frac{3}{16}x^{1/2} + g_1(x), \tag{6.44}$$

where

$$g_1 \ll x^{1/2} \quad \text{as } x \to 0^+. \tag{6.45}$$

Proceeding as above we find that

$$g_1 \sim -\frac{15}{512}x \text{ as } x \to 0^+. \tag{6.46}$$

We see that the first two terms of the asymptotic expansion for $g(x)$ are proportional to $x^{1/2}$ abd x. Taking this as a pattern we guess that $g(x)$ has the form

$$g(x) \sim \sum_{n=1}^{\infty} a_n x^{n/2} \tag{6.47}$$

where $a_1 = 3/16$ and $a_2 = -15/512$. Substituting into the ODE for g gives

$$\begin{aligned} a_{n+1} &= \frac{(2n-1)(2n+3)}{16(n+1)}a_n \\ \Rightarrow a_n &= \frac{-\Gamma(n-\frac{1}{2})\Gamma(n+\frac{3}{2})}{\pi 4^n n!} \end{aligned} \tag{6.48}$$

hence

$$\therefore y \sim \underbrace{x^{3/4}e^{2x-\frac{1}{2}}}_{\substack{\text{leading}\\ \text{behaviour}}} \underbrace{\sum_{n=0}^{\infty} \frac{\Gamma(n-\frac{1}{2})\Gamma(n+\frac{3}{2})x^{\frac{n}{2}}}{\pi 4^n n!}}_{\text{divergent}} \quad \text{as } x \to 0^+ \tag{6.49}$$

gives the complete asymptotic expansion of one solution of the differential equation.

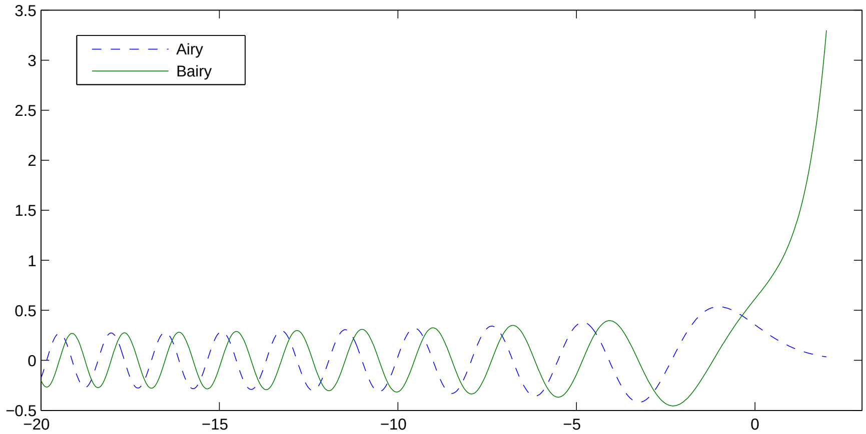

Figure 6.2: Airy and Bairy functions: solutions of $y'' = xy$.

6.3 The Airy Equation

Example 6.3.1 (Airy Equation)

$$y'' = xy \tag{6.50}$$

has an irregular singular point at $x = \infty$.

The solution of the Airy equation is exponential for $x > 0$ and oscillatory for $x < 0$. Why? One well known solution of the Airy equation, the Airy function $\mathrm{Ai}(x)$, decays rapidly for large positive x. A second solution, the Bairy function $\mathrm{Bi}(x)$ grows rapidly. The Airy and Bairy functions are shown in Figure 6.2. The Airy function is named after George Airy who used this function in his study of optics (1838). It also arises in leading-order descriptions of dispersive wave fronts (e.g., surface gravity waves). We will encounter it below when we consider the turning point problem which arises in many areas of physics.

Exercise 6.3.1 *Show that the two possible leading asymptotic behaviours as $x \to +\infty$ are*

$$y_1 \sim C_1 x^{-1/4} e^{-\frac{2}{3}x^{3/2}} \text{ as } x \to +\infty, \tag{6.51}$$

$$y_2 \sim C_2 x^{-1/4} e^{\frac{2}{3}x^{3/2}} \text{ as } x \to +\infty. \tag{6.52}$$

The Airy function $\mathrm{Ai}(x)$ is the unique solution to (6.50) that satisfies (6.51) with $C_1 = \frac{1}{2}\pi^{-1/2}$. This uniquely defines $\mathrm{Ai}(x)$. Why? The leading behaviour of $\mathrm{Bi}(x)$ is given by (6.52) with $C_2 = 1/\sqrt{\pi}$. This does not uniquely define $\mathrm{Bi}(x)$. Why? In Figure 6.3 the leading behaviours for large positive x are shown and compared with $x^{-\frac{1}{4}} e^{\pm 2x^1.5/3}$.

Full Asymptotic Expansion for $\mathrm{Ai}(x)$

Let

$$\mathrm{Ai}(x) = \frac{1}{2\sqrt{\pi}} x^{-1/4} e^{-\frac{2}{3}x^{3/2}} w(x) \text{ as } x \to +\infty. \tag{6.53}$$

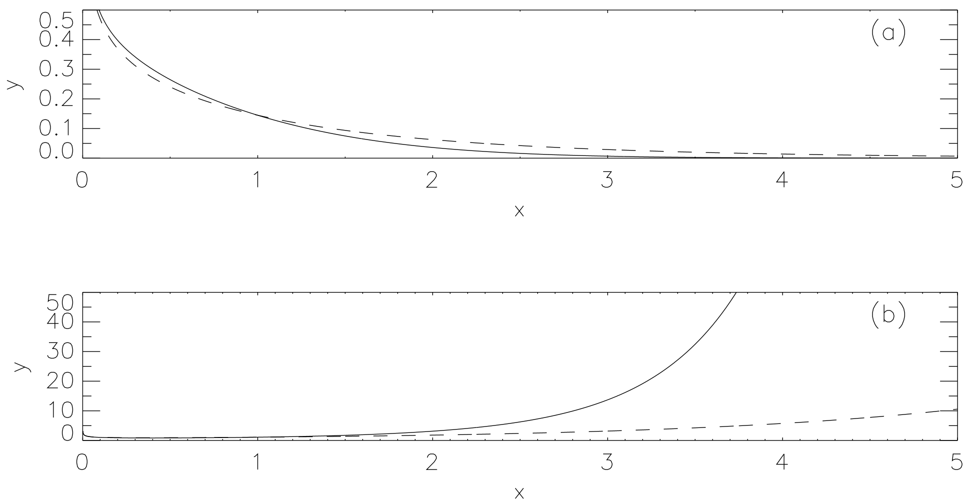

Figure 6.3: Leading behaviours of the Airy and Bairy functions for large positive x. (a) Airy function leading behaviour $\frac{1}{2\sqrt{\pi}}x^{-\frac{1}{4}}e^{-2x^{3/2}/3}$ (solid) compared with the more slowly decaying function $\frac{1}{2\sqrt{\pi}}x^{-\frac{1}{4}}e^{-2x/3}$ (dashed). (a) Leading behaviour of the Bairy function, $\frac{1}{\sqrt{\pi}}x^{-\frac{1}{4}}e^{2x^{3/2}/3}$ (solid), compared with the more slowly growing function $\frac{1}{\sqrt{\pi}}x^{-\frac{1}{4}}e^{2x/3}$ (dashed).

Assume we can find $w(x) \sim \sum_{n=0}^{\infty} a_n x^{\alpha n}$ with $a_0 = 1$ and $\alpha < 0$. Subsituting into (6.50), gives an ODE for w,

$$x^2 w'' - \left(2x^{5/2} + \frac{1}{2}x\right) w' + \frac{5}{16} w = 0. \tag{6.54}$$

Exercise 6.3.2 *Show $\alpha = -\frac{3}{2}$ and*

$$\mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}} x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{3/2}} \sum_n \frac{x^{-\frac{3n}{2}}}{2\pi} \left(\frac{-3}{4}\right)^n \frac{\Gamma\left(n+\frac{5}{6}\right)\Gamma\left(n+\frac{1}{6}\right)}{n!}, \tag{6.55}$$

where

$$\begin{aligned} \Gamma(n) &= (n-1)! \quad \text{for } n \in \mathbb{Z}, & (6.56)\\ \Gamma(x+1) &= x\Gamma(x), & (6.57)\\ \Gamma(x) &= \int_0^\infty e^{-t} t^{x-1}\, dt. & (6.58) \end{aligned}$$

6.4 Asymptotic Relations for Oscillatory Functions

Solutions of the Airy equation:

$$y'' - xy = 0, \tag{6.59}$$

have the leading asymptotic behaviour

$$y \sim C x^{-1/4} e^{\pm \frac{2}{3} x^{3/2}} \text{ as } x \to +\infty. \tag{6.60}$$

For $x \to -\infty$, proceeding as before we would obtain

$$y \sim C(-x)^{-1/4} e^{\pm \frac{2}{3} i (-x)^{3/2}} \text{ as } x \to -\infty. \tag{6.61}$$

Now for the real valued solutions Ai(x) and Bi(x) we need a linear combination of the real and imaginary parts, so it is tempting to write

$$y \sim C_1(-x)^{-\frac{1}{4}} \sin\left(\frac{2}{3}(-x)^{\frac{3}{2}}\right) + C_2(-x)^{-\frac{1}{4}} \cos\left(\frac{2}{3}(-x)^{\frac{3}{2}}\right), \tag{6.62}$$

as $x \to -\infty$.

Fact 6.4.1 *For large negative x*

$$\text{Ai} \approx \frac{1}{\sqrt{\pi}}(-x)^{-\frac{1}{4}} \sin\left(\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}\right), \tag{6.63}$$

$$\text{Bi} \approx \frac{1}{\sqrt{\pi}}(-x)^{-\frac{1}{4}} \cos\left(\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}\right). \tag{6.64}$$

It is, however, incorrect to write[1]

$$\text{Ai} \sim \frac{1}{\sqrt{\pi}}(-x)^{-\frac{1}{4}} \sin\left(\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}\right), \tag{6.65}$$

because both sides have zeros which, while very close together for large x, do not exactly coincide. Hence

$$\lim_{x \to -\infty} \frac{\text{Ai}(x)}{\text{R.H.S.}}, \tag{6.66}$$

does not exist. To say the functions in the numerator and denominator are asymptotic to one another requires that this limit exists and that it is equal to one.

How can we fix this?

Example 6.4.1 *Consider $f(x) = \sin x$ and $g(x) = \sin\left(x + \frac{1}{x}\right)$. For large x the graphs of f and g are almost identical (see Figure 6.4), however, the zeros of f and g do not coincide so $\lim_{x\to\infty}(f/g)$ is undefined. Thus we cannot say that $f \sim g$ as $x \to \infty$ even though the difference between the two functions clearly goes to zero since $1/x$, the phase shift of g relative to f goes to zero.*

How can we fix this?

Idea 1: $f(x) = \sin(r(x))$ and $g(x) = \sin(q(x))$ where $r \sim q$ as $x \to \infty$.

Idea 2: We can write

$$\begin{aligned} \sin\left(x + \frac{1}{x}\right) &= \cos\frac{1}{x}\sin x + \sin\frac{1}{x}\cos x, \\ &= w_1(x)\sin x + w_2(x)\cos x, \end{aligned} \tag{6.67}$$

where

$$w_1(x) = \cos\frac{1}{x} \sim 1 \text{ as } x \to \infty, \tag{6.68}$$

and

$$w_2(x) = \sin\frac{1}{x} \sim \frac{1}{x} \text{ as } x \to \infty. \tag{6.69}$$

Be careful not to say $\sin(1/x) \sim 0$ as $x \to \infty$!

[1]Although you will often see this written in a sloppy use of the notation '$\sim$'.

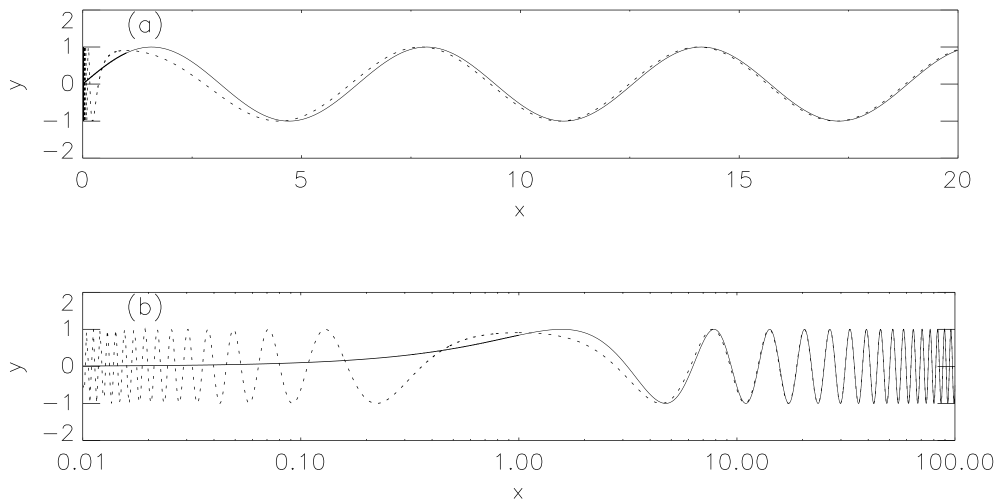

Figure 6.4: Comparison of $\sin(x)$ (solid) and $\sin(x + 1/x)$ (dots).

Returning to the Airy Function, for $x \to -\infty$, write $\mathrm{Ai}(x)$ in the form

$$y = w_1(x)(-x)^{-\frac{1}{4}} \sin\theta + w_2(x)(-x)^{-\frac{1}{4}} \cos\theta, \tag{6.70}$$

where $\theta \equiv \frac{2}{3}(-x)^{3/2} + \frac{\pi}{4}$, and then seek the asymptotic behaviour of $w_1(x)$ and $w_2(x)$. The reason for the introduction of the phase shift $\pi/4$ will be pointed out later, although we will not show how one can tell this is convenient.

Substitution into $y'' - xy = 0$ gives

$$\begin{aligned} &\left[w_1'' + \frac{1}{2}x^{-1}w_1' + 2(-x)^{\frac{1}{2}}w_2' + \frac{5}{16}x^{-2}w_1\right]\sin\theta \\ &\quad + \left[w_2'' - \frac{1}{2}x^{-1}w_2' - 2(-x)^{\frac{1}{2}w_1' + \frac{5}{16}x^{-2}w_2}\right]\cos\theta = 0. \end{aligned} \tag{6.71}$$

This gives one equation for two unknowns. We need two equations. There is a lot of freedom but the simplest choice is to assume that the coefficients of $\sin\theta$ and $\cos\theta$ are both zero. Next let

$$\begin{aligned} w_1(x) &\sim \textstyle\sum_{n=0}^{\infty} a_n(-x)^{-\frac{3}{2}n} \\ w_2(x) &\sim \textstyle\sum_{n=0}^{\infty} b_n(-x)^{-\frac{3}{2}n} \end{aligned} \quad \text{as } x \to -\infty. \tag{6.72}$$

Substituting these expansions into the two couple ODEs for w_1 and w_2 then gives

$$\begin{aligned} a_{2n} &= a_0(-1)^n c_{2n}, \\ a_{2n+1} &= b_0(-1)^n c_{2n+1}, \\ b_{2n} &= b_0(-1)^n c_{2n}, \\ b_{2n+1} &= a_0(-1)^{n+1} c_{2n+1} \end{aligned} \tag{6.73}$$

for $n = 0, 1, \ldots$ with $c_0 = 1$ and

$$\begin{aligned} c_n &= \frac{(2n+1)(2n+3)\cdots(6n-1)}{144^n n!} \\ &= \frac{1}{2\pi}\left(\frac{3}{4}\right)^n \frac{\Gamma\left(n+\frac{5}{6}\right)\Gamma\left(n+\frac{1}{6}\right)}{n!}, \end{aligned} \tag{6.74}$$

for $n = 1, 2, 3, \ldots$. For the Airy and Bairy functions (because of the introduction of the phase shift $\pi/4$)

$$\mathrm{Ai}(x): \qquad a_0 = \frac{1}{\sqrt{\pi}},\ b_0 = 0,$$
$$\mathrm{Bi}(x): \qquad a_0 = 0,\ b_0 = \frac{1}{\sqrt{\pi}}.$$

Note: $w_1(x)$ and $w_2(x)$ are **not** oscillating functions.

6.5 The Turning Point Problem

The turning point problem is a classical problem in mathematical physics. It arises in many physical contexts in which wave-like behaviour occurs in one part of a domain and not in another. In Quantum mechanics it arises in the context of the Schrödinger Equation

$$\left[\frac{d^2}{dz^2} + E - V(z)\right]\phi(z) = 0, \tag{6.75}$$

where ϕ is the wave function, E is the energy and $V(z)$ is a potential well. A similar equation arises in the context of internal gravity waves in a density stratified fluid (a useful example because I will show you animations of internal waves impinging on a turning point and of internal wave tunneling). Here the governing equation for the velocity stream function for two-dimensional waves (vertical-horizontal plane), excluding the effects of the Earth's rotation, is

$$\nabla^2\Psi_{tt} + N^2(z)\Psi_{xx} = 0. \tag{6.76}$$

Here $N(z)$, given by

$$N^2(z) = -\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz}(z), \tag{6.77}$$

is called the buoyancy frequency, $\bar{\rho}(z)$ is the fluid density, g is the acceleration of gravity and ρ_0 is a reference density. Since the partial differential equation for Ψ is linear and the coefficients are independent of x and t one can look for solutions of the form

$$\Psi = e^{i(kx-\omega t)}\phi(z), \tag{6.78}$$

which leads to

$$\left[\frac{d^2}{dz^2} + \frac{N^2(z) - \omega^2}{\omega^2}k^2\right]\phi(z) = 0. \tag{6.79}$$

Both of equations (6.75) and (6.79) have the form

$$\phi'' + Q(z)\phi = 0. \tag{6.80}$$

If $Q(z) > 0$ the solution ϕ is oscillatory, i.e., has wave-like behaviour. If $Q(z) < 0$ the solution is exponential, or non-wave-like. If $Q(z)$ changes sign then the behaviour of the solution is different in different regions. A boundary between the wave-like and non wave-like regions, i.e., points where $Q(z) = 0$, is called a turning point in 1-D. Tunneling, a classical problem in Quantum Mechanics, occurs when two turning points are present, with a narrow non-wave-like region separating two wave-like regions. Waves can 'tunnel' from one wave-like region to another through the barrier separating them.

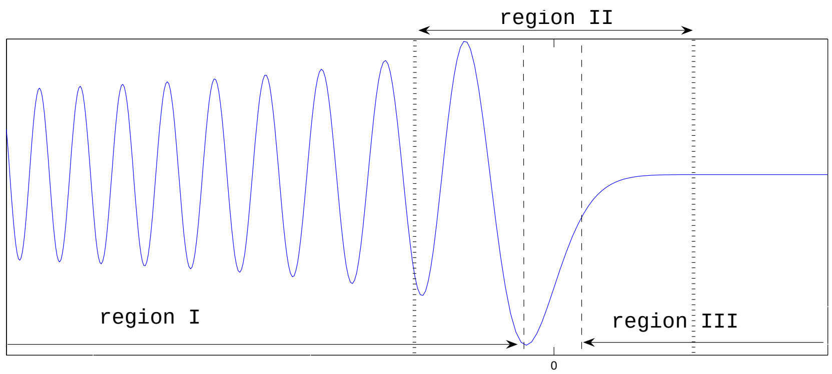

Figure 6.5: Schematic of the solution of the turning point problem.

Goal: Our goal is to find a uniformly valid asymptotic solution to

$$\psi_{zz} + Q(z)\psi = 0, \tag{6.81}$$

where

$$Q(z) \quad \text{is} \begin{cases} > 0, & \text{for} \quad z < 0; \\ < 0, & \text{for} \quad z > 0; \end{cases} \tag{6.82}$$

and has a simple root at $z = 0$ (i.e., Q is linear near $z = 0$).

When $z < 0$ ψ has an oscillatory behaviour (i.e., wave-like), while for $z > 0$ ψ behaves exponentially (non-wave-like). We assume that 'far' from $z = 0$, Q varies slowly compared to the scale on which ψ varies. Thus we assume

$$Q = Q(\zeta), \tag{6.83}$$

where

$$\zeta = \epsilon z, \tag{6.84}$$

is the slow space variable. Rewriting (6.81) in terms of ζ gives

$$\epsilon^2 \frac{d^2\psi}{d\zeta^2} + Q(\zeta)\psi = 0. \tag{6.85}$$

The asymptotic solution of (6.85) will be found in three different regions (see Figure 6.5). Solutions in adjacent regions must be matched in overlapping regions of validity. In regions I and III the solution is given by WKB Theory. These solutions are invalid in region II. The solution in region I can be interpreted as the combination of incident and reflected waves and we will determine the relationship between these two waves.

6.5.1 WKB Theory: Outer Solution

In WKB theory we seek a solution of the form

$$\psi = e^{i\left(\frac{S_0(\zeta)}{\epsilon} + S_1(\zeta) + \epsilon S_2(\zeta) + \cdots\right)}, \tag{6.86}$$

such that

$$\frac{S_0}{\epsilon} \gg S_1 \gg \epsilon S_2 \gg \cdots, \tag{6.87}$$

and

$$\epsilon S_2 \ll 1, \tag{6.88}$$

as $\epsilon \to 0$. If (6.87) and (6.88) are satisfied, $e^{i(\epsilon S_2+\cdots)} \sim 1$ and $\psi \sim e^{i\left(\frac{S_0}{\epsilon}+S_1\right)}$ as $\epsilon \to 0$.

Substituting (6.86) into (6.85) we obtain

$$\left[Q(\zeta) - S_0'^2 + \epsilon(iS_0'' - 2S_0'S_1') + \epsilon^2(iS_1'' - S_1'^2 - 2S_0'S_2') + \cdots\right]\psi = 0. \tag{6.89}$$

We now proceed by setting the coefficient of ϵ^n $(n = 0, 1, 2, \ldots)$ to zero.

$\mathcal{O}(1)$ **Problem:**

At $\mathcal{O}(1)$ we have

$$S_0'^2 = Q(\zeta). \tag{6.90}$$

There are two cases to consider:

case (a): $\zeta < 0 \to S_0'(\zeta) = \pm\sqrt{Q(\zeta)} \Longrightarrow S_0 = \pm\int_0^\zeta \sqrt{Q(t)}dt.$
case (b): $\zeta > 0 \to S_0'(\zeta) = \pm i\sqrt{-Q(\zeta)} \Longrightarrow S_0 = \pm i\int_0^\zeta \sqrt{-Q(t)}dt.$

Constants of integration result in a multiplicative factor of ψ. At the moment we are interested in finding ψ up to a constant factor so these constants of integration are not of interest. Thus, the lower limit of integration is arbitrary and we take it to be zero.

$\mathcal{O}(\epsilon)$ **Problem:**

At $O(\epsilon)$ we have

$$S_1' = \frac{i}{2}\frac{S_0''}{S_0'} = \frac{i}{2}\frac{d}{d\zeta}\ln(|S_0'|) \to S_1 = \frac{i}{2}\ln(|S_0'|) = i\ln(|Q(\zeta)|^{1/4}). \tag{6.91}$$

WKB Solution:

Combining the first two terms of the solution gives

$$\psi \sim \begin{cases} Q(\zeta)^{-1/4}e^{\pm\frac{i}{\epsilon}\int_0^\zeta \sqrt{Q(t)}dt}, & \text{for } \zeta < 0; \\ (-Q(\zeta))^{-1/4}e^{\pm\frac{i}{\epsilon}\int_0^\zeta \sqrt{-Q(t)}dt}, & \text{for } \zeta > 0; \end{cases} \tag{6.92}$$

as $\epsilon \to 0$. This is what is meant by the WKB solution. Note that it predicts an amplitude $|Q(\zeta)|^{-1/4}$ which becomes infinite as $\zeta \to 0$, i.e., as the turning point is approached. This will render the WKB solution invalid when ζ becomes sufficiently small. A 'vertical' wavenumber

$$m = \frac{\partial}{\partial\zeta}\int_0^\zeta \sqrt{Q(t)}dt = \sqrt{Q(\zeta)}$$

can be defined in the region $\zeta < 0$. The wavelength of the oscillations in the region $\zeta < 0$ varies with ζ and is approximately $2\pi/m(\zeta)$. The amplitude is then proportional to $m^{-1/2}$. Note that this wavenumber goes to zero as the turning point is approached which implies that the length scale of the oscillations goes to infinity. Our solution violates our assumption that Q varies slowly compared with the scale that ψ varies on. In the region $\zeta > 0$ $\sqrt{-Q(\zeta)}$ defines the decay scale for the exponential behaviour. Similar comments apply to the breakdown of the solution.

6.5.2 Region of Validity for WKB Solution

The asymptotic approximation for ψ given by (6.92) is valid only in regions where (6.87) and (6.88) are satisfied. We need to find these regions. For $|\zeta| \ll 1$

$$Q \approx a\zeta + b\zeta^2, \tag{6.93}$$

where, by assumption $a < 0$, so for $\zeta < 0$ we have

$$\begin{aligned}\frac{1}{\epsilon}\int_0^\zeta \sqrt{Q(t)}\,dt &\approx \frac{1}{\epsilon}\int_0^\zeta (at)^{1/2}\left(1+\frac{b}{a}t\right)^{1/2} dt \approx \frac{1}{\epsilon}\int_0^\zeta (at)^{1/2}\left(1+\frac{1}{2}\frac{bt}{a}+\cdots\right) dt \\ &= -\frac{1}{\epsilon}\left(\frac{2}{3}\sqrt{-a}(-\zeta)^{3/2} - \frac{1}{5}\frac{b}{\sqrt{-a}}(-\zeta)^{5/2}+\cdots\right).\end{aligned} \tag{6.94}$$

For $\zeta > 0$

$$\frac{1}{\epsilon}\int_0^\zeta \sqrt{-Q(t)}dt \approx \frac{1}{\epsilon}\left(\frac{2}{3}\sqrt{-a}\zeta^{3/2} - \frac{1}{5}\frac{b}{\sqrt{-a}}\zeta^{5/2}+\cdots\right). \tag{6.95}$$

Thus

$$\frac{S_0}{\epsilon} \sim \begin{cases} \pm\frac{2}{3}\sqrt{-a}\frac{(-\zeta)^{3/2}}{\epsilon} & \text{as} \quad \zeta \to 0^-, \\ \pm\frac{2}{3}i\sqrt{-a}\frac{\zeta^{3/2}}{\epsilon} & \text{as} \quad \zeta \to 0^+. \end{cases} \tag{6.96}$$

For future reference note that

$$e^{i\frac{S_0}{\epsilon}} \approx e^{i\left(\pm\frac{2}{3}\sqrt{|a|}\frac{|\zeta|^{3/2}}{\epsilon}\right)} \iff |\zeta|^{5/2}/\epsilon \ll 1 \iff |\zeta| \ll \epsilon^{2/5}. \tag{6.97}$$

Similarly we find

$$S_1 \sim -\frac{1}{4}|a|^{-1/4}\ln|\zeta| \tag{6.98}$$

and

$$\epsilon S_2 \sim \pm\frac{5}{48}\frac{\epsilon}{\sqrt{|a|}}|\zeta|^{-3/2}, \tag{6.99}$$

as $\zeta \to 0$. The WKB solution is valid if $\epsilon S_2 \ll 1$, $S_1 \ll \frac{S_0}{\epsilon}$ and $\epsilon S_2 \ll S_1$. The first of these requires

$$\epsilon S_2 \ll 1 \Rightarrow \epsilon|\zeta|^{-3/2} \ll 1 \iff |\zeta| \gg \epsilon^{2/3}, \tag{6.100}$$

and the second requires

$$\frac{S_0}{\epsilon} \gg S_1 \iff \frac{|\zeta|^{3/2}}{\epsilon}/\ln|\zeta| \gg 1. \tag{6.101}$$

Setting $|\zeta| = \epsilon^{2\alpha/3}$ with $\alpha > 0$, we have $\ln|\zeta| = \frac{2}{3}\alpha\ln\epsilon$ and $|\zeta|^{3/2} = \epsilon^\alpha$. Thus we need $\frac{\epsilon^{\alpha-1}}{\ln\epsilon} \gg 1$ as $\epsilon \to 0$ which requires $\alpha < 1$. Thus $\frac{S_0}{\epsilon} \gg S_1$ if $|\zeta| \gg \epsilon^{2/3}$. For the third condition, $S_1 \gg \epsilon S_2$, we also find that $|\zeta| \gg \epsilon^{2/3}$ is again necessary.

Therefore, the WKB Solution is valid if

$$|\zeta| \gg \epsilon^{2/3}. \tag{6.102}$$

The approximation to the WKB Solution, given by (6.96), (6.98) and (6.99) is valid if $|\zeta| \ll \epsilon^{2/5}$ so the approximate WKB solution is valid if $\epsilon^{2/3} \ll |\zeta| \ll \epsilon^{2/5}$.

6.5.3 Inner Solution

When $|\zeta| = 0(\epsilon^{2/3})$ the WKB solution is invalid. We need a new solution that is valid for smaller values of $|\zeta|$. Since $\epsilon \ll \epsilon^{2/3} \ll 1$ we will find a solution valid for $|\zeta| \ll 1$. Then the WKB solution and the new solution will both be valid in $\epsilon^{2/3} \ll |\zeta| \ll 1$. It will, however, be more convenient to match the solutions using the approximate WKB solution. To do this we simply refine the matching region to $\epsilon^{2/3} \ll |\zeta| \ll \epsilon^{2/5}$.

For $|\zeta| \ll 1$, $Q(\zeta) \approx a\zeta$ in which case ψ can be approximated by ψ_{II} which is a solution of

$$\frac{d^2\psi_{\text{II}}}{d\zeta^2} = -\frac{a\zeta}{\epsilon^2}\psi_{\text{II}}. \tag{6.103}$$

This is a scaled version of Airy's equation.

The Airy Equation:

$y''(x) = xy(x)$ is the Airy equation. Two independent solutions $A_i(x)$ and $B_i(x)$ have been defined. The general solution of (6.103) is

$$\psi_{\text{II}} = CA_i\Big(\big(\frac{-a}{\epsilon^2}\big)^{1/3}\zeta\Big) + DB_i\Big(\big(\frac{-a}{\epsilon^2}\big)^{1/3}\zeta\Big). \tag{6.104}$$

Since we want to match ψ_{II} to an exponentially decaying solution in region III we will have to take $D = 0$ (on physical grounds if there is no energy source to the right of the turning point this must be the case if the solution is to remain bounded. If there is an energy source or a second turning point $Bi(x)$ may need to be included). $A_i(x)$ is the unique solution of the Airy equation with the asymptotic behaviour

$$A_i(x) \sim \frac{1}{2\sqrt{\pi}}x^{-1/4}e^{-2/3x^{3/2}} \quad \text{as} \quad x \to +\infty. \tag{6.105}$$

For $x < 0$ we have

$$A_i(x) = w_1(x)\sin\left(\tfrac{2}{3}(-x)^{3/2} + \tfrac{\pi}{4}\right) - w_2(x)\cos\left(\tfrac{2}{3}(-x)^{3/2} + \tfrac{\pi}{4}\right), \tag{6.106}$$

where

$$w_1(x) \sim \frac{1}{\sqrt{\pi}}(-x)^{-1/4} \quad \text{and} \quad w_2 \sim \frac{(-x)^{7/4}}{\sqrt{\pi}} \quad \text{as} \quad x \to -\infty. \tag{6.107}$$

6.5.4 Matching

We now know the asymptotic behaviour for ψ in each of the three regions.

$$\textbf{Region III:} \quad \zeta \gg \epsilon^{2/3} \qquad \psi \sim \psi_{\text{III}} = B(-Q(\zeta))^{-1/4}e^{-\frac{1}{\epsilon}\int_0^\zeta \sqrt{-Q(t)}dt} \tag{6.108}$$

$$\textbf{Region II:} \quad |\zeta| \ll 1 \qquad \psi \approx \psi_{\text{II}} = CA_i\left(\left(\frac{-a}{\epsilon^2}\right)^{1/3}\zeta\right) \tag{6.109}$$

$$\textbf{Region I:} \quad -\zeta \gg \epsilon^{2/3} \qquad \psi \approx \psi_{\text{I}} = Q(\zeta)^{-1/4}\left[\tilde{E}e^{\frac{i}{\epsilon}\int_0^\zeta \sqrt{Q(t)}dt} + \tilde{F}e^{-\frac{i}{\epsilon}\int_0^\zeta \sqrt{Q(t)}dt}\right]. \tag{6.110}$$

Here we have already used the fact that we require ψ to decay exponentially as $\zeta \to +\infty$. We now need to relate the constants $B, C, \tilde{E}$ and $\tilde{F}$.

Matching ψ_{III} and ψ_{II}:

When $\zeta \gg \epsilon^{2/3}$, $(-a)^{1/3}\frac{\zeta}{\epsilon^{2/3}} \gg 1$ and

$$\psi \sim \psi_{II} \sim \frac{C}{2\sqrt{\pi}}\left(\left(\frac{-a}{\epsilon^2}\right)^{1/3}\zeta\right)^{-1/4} e^{-\frac{2}{3}\left(\left(\frac{-a}{\epsilon^2}\right)^{1/3}\zeta\right)^{3/2}} = \frac{C}{2\sqrt{\pi}}\frac{(-a)^{-1/12}}{\epsilon^{-1/6}}\zeta^{-1/4}e^{-\frac{2}{3}\frac{(-a)^{1/2}}{\epsilon}\zeta^{3/2}}. \tag{6.111}$$

On the other hand

$$\psi \sim \psi_{III} \sim B(-a)^{-1/4}\zeta^{-1/4}e^{-\frac{2}{3}\frac{\sqrt{-a}}{\epsilon}\zeta^{3/2}} \quad \text{if} \quad \zeta \ll \epsilon^{2/5} \quad \text{(see 6.95)}. \tag{6.112}$$

The asymptotic expressions (6.111) and (6.112) are both valid when $\epsilon^{2/3} \ll \zeta \ll \epsilon^{2/5}$. Thus we must have

$$\frac{C}{2\sqrt{\pi}}\frac{(-a)^{-1/12}}{\epsilon^{-1/6}} = B(-a)^{-1/4} \Rightarrow C = \frac{2\sqrt{\pi}}{\epsilon^{1/6}(-a)^{1/6}}B. \tag{6.113}$$

Matching ψ_{II} and ψ_{I}:

As above the matching region is $\epsilon^{2/3} \ll |\zeta| \ll \epsilon^{2/5}$. For $\epsilon^{2/5} \gg -\zeta \gg \epsilon^{2/3}$,

$$\begin{aligned}\psi_{II} &= \frac{2\sqrt{\pi}}{(-\epsilon a)^{1/6}}BA_i\Big(\big(\frac{-a}{\epsilon^2}\big)^{1/3}\zeta\Big)\\ &\approx \frac{2B}{(-\epsilon a)^{1/6}}\left(\big(\frac{-a}{\epsilon^2}\big)^{1/3}(-\zeta)\right)^{-1/4}\sin\Big(\tfrac{2}{3}\Big(\big(\frac{-a}{\epsilon^2}\big)^{1/3}(-\zeta)\Big)^{3/2}+\tfrac{\pi}{4}\Big)\\ &= 2B(a\zeta)^{-1/4}\sin\Big(\tfrac{2}{3}\frac{(-a)^{1/2}}{\epsilon}(-\zeta)^{3/2}+\tfrac{\pi}{4}\Big).\end{aligned} \tag{6.114}$$

Next, on $\epsilon^{2/3} \ll -\zeta \ll \epsilon^{2/5}$, we also have

$$\begin{aligned}\psi_I &\approx (a\zeta)^{-1/4}\big(\tilde{E}e^{-\frac{i}{\epsilon}\frac{2}{3}(-a)^{1/2}(-\zeta)^{3/2}} + \tilde{F}e^{\frac{i}{\epsilon}\frac{2}{3}(-a)^{1/2}(-\zeta)^{3/2}}\big),\\ &= (a\zeta)^{-1/4}\Big(E\sin\Big(\tfrac{2}{3}\frac{\sqrt{-a}}{\epsilon}(-\zeta)^{3/2}+\tfrac{\pi}{4}\Big)\\ &\qquad + F\cos\Big(\tfrac{2}{3}\frac{\sqrt{-a}}{\epsilon}(-\zeta)^{3/2}+\tfrac{\pi}{4}\Big)\Big).\end{aligned} \tag{6.115}$$

Evidently, comparing (6.114) and (6.115) $F = 0$ and $E = 2B$.

6.5.5 Summary of asymptotic solution

$$\psi \approx \begin{cases} B[-Q(\zeta)]^{-1/4}e^{-\frac{1}{\epsilon}\int_0^\zeta \sqrt{-Q(t)}dt}; & \zeta \gg \epsilon^{2/3},\\ \frac{2\sqrt{\pi}}{(-a\epsilon)^{1/6}}BA_i\left(\epsilon^{-2/3}(-a)^{1/3};\zeta\right) & |\zeta| \ll 1,\\ 2B[Q(\zeta)]^{-1/4}\sin\left(\frac{1}{\epsilon}\int_\zeta^0 \sqrt{Q(t)}dt + \frac{\pi}{4}\right); & -\zeta \gg \epsilon^{2/3}.\end{cases} \tag{6.116}$$

[Note in third expression integral now goes from ζ to 0 because $\frac{1}{\epsilon}\int_0^\zeta \sqrt{Q(t)}dt \approx -\frac{1}{\epsilon}\frac{2}{3}\sqrt{-a}(-\zeta)^{3/2}$ so $\frac{1}{3}\int_\zeta^0 \sqrt{Q(t)}dt \approx \frac{1}{\epsilon}\frac{2}{3}\sqrt{-a}(-\zeta)^{3/2}$ as in (6.115).]

6.5.6 Physical Interpretation

Consider the region $-\zeta \gg \epsilon^{2/3}$. Using $w = \cos(kx - \sigma t)\psi$ where $w = -\Psi_x$ vertical velocity of fluid in the presence of IGWs then

$$
\begin{aligned}
w = \cos(kx - \sigma t)\psi &\sim 2Bm^{-1/2}\cos(kx - \sigma t)\sin\left(\tfrac{1}{\epsilon}\int_\zeta^0 m(t)dt + \tfrac{\pi}{4}\right) \\
&= Bm^{-1/2}\Big\{\sin\left(kx - \tfrac{1}{\epsilon}\int_0^\zeta m(t)dt - \sigma t + \tfrac{\pi}{4}\right) \\
&\qquad + \sin\left(-kx - \tfrac{1}{\epsilon}\int_0^\zeta m(t)dt + \sigma t + \tfrac{\pi}{4}\right)\Big\}.
\end{aligned}
$$

Using $z = \zeta/\epsilon$ and redefining $m(z) = \sqrt{Q(\epsilon z)}$ we can rewrite this as

$$
w = \frac{B}{m^{1/2}}\left\{\sin\left(kx - \int_0^z m(z')\,dz' - \sigma t + \tfrac{\pi}{4}\right) - \sin\left(kx + \int_0^z m(z')\,dz' - \sigma t - \tfrac{\pi}{4}\right)\right\} \tag{6.117}
$$

If $m = m_0$ is constant for $|z| > z_0$ then

$$
\int_0^z m(t)dt = \int_0^{z_0} m(t)dt + \int_{z_0}^z m(t)dt = \hat{\phi} + m_0(z - z_0) = \phi + m_0 z \tag{6.118}
$$

where $\hat{\phi}$ and $\phi = \hat{\phi} - mz_0$ are constants. Thus we have

$$
w = Bm^{-1/2}\Big\{\sin\left(kx - m_0 z - \sigma t - \phi + \tfrac{\pi}{4}\right) - \sin\left(kx + m_0 z - \sigma t + \phi - \tfrac{\pi}{4}\right)\Big\}. \tag{6.119}
$$

The first term represents a wave with phase speed $-\sigma/m_0$ in the z direction while the second term represents a wave with phase speed σ/m_0. These are reflected and incident waves respectively and they both have the same amplitude. The two waves have different phases, hence at the turning point there is perfect reflection with a phase shift. Note that physically we should expect perfect reflection. As $\zeta \to \infty$ the solution ψ goes to zero exponentially fast, hence there can be no energy far to the right. Since the system has no dissipation all the incident energy must be reflected.

6.6 Tunneling

Chapter 7

Singular Perturbation Theory: Examples and Techniques

7.1 More examples of problems from Singular Perturbation Theory

We have seen a few problems where Regular Perturbation Theory fails:

1. Solving equations of the form
$$\epsilon x^3 + 2x - 1 = 0. \tag{7.1}$$
RPT fails in this case because when $\epsilon = 0$ one of the roots is lost. One root goes to $-\infty$ as $\epsilon \to 0$, i.e., $\epsilon = 0$ is a singular point of one of the solutions.

2. ODEs such as
$$\begin{aligned} \epsilon y'' + y' &= 0, \\ y(0) &= a, \\ y(1) &= b. \end{aligned} \tag{7.2}$$
As we've seen solutions have a thin boundary layer in the vicinity of one of the boundaries where the solution changes very rapidly. In this boundary layer $\epsilon y''$ is important: the dominant balance in the equation must include this term. Looking for exponential solutions of the form $y = e^{\lambda x}$ results in the polynomial $\epsilon\lambda^2 + \lambda = 0$, showing an intimate connection of this ODE problem with the polynomial problem above.

3. The nonlinear pendulum. Here nonlinearity introduces secular terms.

Definition 7.1.1 (Uniformly Ordered Asymptotic Expansion) *The Asymptotic Expansion*
$$f(x, y; \epsilon) \sim \sum_n a_n(x, y)\varphi_n(\epsilon) \text{ as } \epsilon \to 0^+, \tag{7.3}$$
for an Asymptotic Sequence $\{\varphi_n\}$ *and for* $(x, y) \in \mathcal{D}$ *is uniformly ordered if*
$$a_n(x, y)\varphi_n(\epsilon) = \mathcal{O}(\varphi_n(\epsilon)), \tag{7.4}$$
uniformly on $\mathcal{D}$*, for all* n*.*

Theorem 7.1.1 *The Asymptotic Expansion is uniformly ordered if the $a_n(x,y)$ are all bounded.*

Example 7.1.1

$$f(\tau,\epsilon) \sim 1 - \frac{\epsilon}{1-\tau} - \frac{\tau}{(1-\tau)^3}\epsilon^2 \text{ as } \epsilon \to 0 \tag{7.5}$$

for $1 < \tau < 2$ is not uniformly ordered.

Uniform ordering requires that

$$\left|\frac{\epsilon}{1-\tau}\right| \leq A\epsilon \text{ for all } \tau \in (1,2) \tag{7.6}$$

which is not true since the left hand side blows up as $\tau \to 1$.

Example 7.1.2 (Simple Pendulum) *Regular Perturbation Theory gave*

$$\theta = \cos\tau + \left(\frac{1}{192}(\cos\tau - \cos 3\tau) + \frac{\tau}{16}\sin\tau\right)\epsilon + \mathcal{O}_F(\epsilon^2).$$

This is not uniformly ordered on $\tau \in [0,\infty)$ since there is no constant A such that

$$\left|\left(\frac{1}{192}(\cos\tau - \cos 3\tau) + \frac{\tau}{16}\sin\tau\right)\epsilon\right| < A\epsilon, \tag{7.7}$$

for all τ. It is uniformly ordered on any finite interval.

Definition 7.1.2 *An Asymptotic Expansion that is not uniformly ordered is said to be disordered (or to become disordered), or non-uniform. Singular Perturbation Theory is used to deal with problems with non-uniformities.*

Common sources of non-uniformity

1. Infinite domains.
2. Singularities in the DE.
3. Small parameters multiplying the highest derivative in a DE.
4. Change in type of PDE.
   → many others

Example 7.1.3 *Special case of an example of Lighthill's:*

$$\begin{aligned}(x+\epsilon y)y' + y &= 1,\\ y(1) &= 2,\end{aligned} \tag{7.8}$$

for $0 \leq x \leq 1$ and $0 < \epsilon \ll 1$.

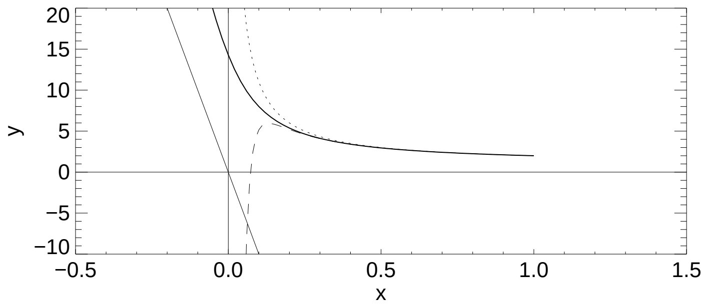

Figure 7.1: Comparison of exact and RPT solutions of Lighthill's example. Solid curve: exact solution. $O(1)$ RPT solution: dotted curve. $O(\epsilon)$ RPT solution: dashed curve. The sloping straight line is the singular line $y = -x/\epsilon$.

This has the exact solution

$$y = -\frac{x}{\epsilon} + \sqrt{\left(\frac{x}{\epsilon}\right)^2 + \frac{2(1+x)}{\epsilon} + 4}. \tag{7.9}$$

Trying Regular Perturbation Theory methods by setting

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots , \tag{7.10}$$

leads to

$$\to y = \frac{1+x}{x} - \frac{(1-x)(1+3x)}{2x^3}\epsilon + \mathcal{O}_F(\epsilon^2). \tag{7.11}$$

The Asymptotic Expansion is disordered. For x close to zero the $\mathcal{O}(\epsilon)$ term becomes larger than the leading term. What is the source of the non-uniformity? Note that each term in the asymptotic expansion is singular at $x = 0$. In contrast the exact solution is well behaved at $x = 0$. The exact solution, however, is singular at $\epsilon = 0$. Regular Perturbation Theory assumes the solution is well behaved at $\epsilon = 0$, and hence has a power series expansion about $\epsilon = 0$.

The DE has a line of singularities along $x + \epsilon y = 0$. In the Regular Perturbation Theory solution the singularity is at $x = 0$. The shift of the singularity from $x + \epsilon y = 0$ to $x = 0$ is the source of the difficulty. A Perturbation Theory scheme should keep singularities as close as possible to their original location.

Example 7.1.4 *Consider the Boundary Value Problem*

$$u_{xx} + \epsilon u_{yy} - u_y = 0, \tag{7.12}$$

where $\epsilon > 0$ *and* $0 \le x, y \le 1$ *with boundary conditions*

$$\begin{aligned} u(0, y) &= a(y) \quad u(x, 0) = b(x), \\ u(1, y) &= c(y) \quad u(x, 1) = d(x). \end{aligned} \tag{7.13}$$

This is a Dirichlet problem if $\epsilon > 0$. It is well posed in the sense of Hadamard (the solution exists, is unique, and depends continuously on the boundary conditions).

Trying Regular Perturbation Theory, at leading order we have

$$u_{0xx} - u_{0y} = 0, \tag{7.14}$$

$u_0(x, y)$ satisfies the same boundary conditions as $u(x, y)$ does, however the PDE is no longer elliptic, it is parabolic and cannot in general satisfy all the boundary conditions. Problems where the PDEs change type in limiting cases of interest are common. One example is fluid flow. Compressible fluid flow is of hyperbolic character. There is a maximum speed at which information can travel, namely the sound speed. In the incompressible limit, a limiting case commonly studied, the sound speed goes to infinity. The governing equations are now of elliptic character.

Example 7.1.5

$$\begin{aligned} \nabla^4 u &= 0 \quad \text{on} \quad \mathcal{D} \\ u &= f \quad \text{on} \quad \partial\mathcal{D} \\ \epsilon\frac{\partial u}{\partial n} + u &= g \quad \text{on} \quad \partial\mathcal{D}. \end{aligned} \tag{7.15}$$

Setting $\epsilon = 0$ to gives the reduced problem in RPP, which, if $f \neq g$ has no solution. If $f = g$ then we have insufficient boundary conditions and the solution is non-unique.

7.2 The linear damped oscillator

The equations for the linear damped oscillator, which arise in many contexts (e.g., mass-spring system, RLC-circuit), are

$$\begin{aligned} m\ddot{x} + 2\beta\dot{x} + kx = 0, \\ x(0) = x_0, \\ \dot{x} = v_0 = 0, \end{aligned} \tag{7.16}$$

where m, β and k are all positive. For a damped mass-spring system they represent the mass, damping due to friction, and the linear spring constant respectively, while x is the amount the spring has been stretched. For an RLC circuit the coefficients represent the inductance, resistance and inverse capacitance, while x represents the charge on the capacitor. This linear second-order ODE can be easily solved analytically and its complete solution and behaviour should be familiar to you. We will consider it in some detail from a perturbation theory perspective for which we need to introduce a small parameter. We will consider several approaches, some of which will fail. Knowledge of the exact solution will allow us to understand why those that fail do, giving some insight into possible pitfalls that can occur in more difficult problems for which exact solutions are unknown.

Nondimensionalization

The units of the various terms in the problem are

$$[t] = T, \quad [x] = L, \quad [m] = M, \quad [\beta] = \frac{M}{T}, \quad [k] = \frac{M}{T^2}. \tag{7.17}$$

1. $L_C = x_0$ is the only possible choice for the length scale since x is the only variable that involves dimensions of length. For all solutions the maximum value of $|x|$ is x_0. Hence, define the non-dimensional displacement
$$y = \frac{x}{x_0}. \tag{7.18}$$

2. There are several choices for the time scale:

   (a) $T_c = T_c^{(1)} = \sqrt{\frac{m}{k}}$

(b) $T_c = T_c^{(2)} = \frac{\beta}{k}$

(c) $T_c = T_c^{(3)} = \frac{m}{\beta}$.

We'll consider each in turn. All three non-dimensionalizations will include a single nondimensional positive parameter

$$\epsilon = \frac{\beta}{\sqrt{mk}}. \tag{7.19}$$

For weak damping ϵ is the small parameter in our non-dimensional problem. For strong damping ϵ^{-1} is the small parameter. Note that the time scales are related by

$$T_c^{(1)} = \epsilon^{-1} T_c^{(2)} = \epsilon T_c^{(3)}. \tag{7.20}$$

Case (a): Set $t = T_c^{(1)}\tau = \sqrt{\frac{m}{k}}\tau$. This gives

$$\begin{aligned} y_{\tau\tau} + 2\epsilon y_\tau + y &= 0, \\ y(0) &= 1, \\ y_\tau(0) &= 0. \end{aligned} \tag{7.21}$$

1. If $\epsilon \ll 1$ we can try to apply the methods of Regular Perturbation Theory by setting $y = y_0(\tau) + \epsilon_1(\tau) + \epsilon^2 y_2(\tau) + \cdots$. At leading order we obtain $y_0 = \cos\tau$. At the next order we get

$$\begin{aligned} y_{1\tau\tau} + y_1 &= 2\sin\tau, \\ y_1(0) = y_{1\tau} &= 0. \end{aligned} \tag{7.22}$$

   This has a resonant forcing. Therefore we need methods of Singular Perturbation Theory if we are interested in the solution for times of $\mathcal{O}(\epsilon^{-1})$ or longer. For much shorter times this method gives a useful approximation.

2. If $\epsilon \gg 1$ write the ODE as

$$\frac{1}{\epsilon} y_{\tau\tau} + 2y_\tau + \frac{1}{\epsilon} y = 0, \tag{7.23}$$

   and set $y = y_0 + \frac{1}{\epsilon} y_1 + \frac{1}{\epsilon^2} y_2 + \cdots$.

   $\mathcal{O}(1)$:

$$\begin{aligned} y_{0\tau} &= 0, \\ y_0(0) = y_{0\tau}(0) &= 0 \end{aligned} \tag{7.24}$$

   The solution is $y_0(\tau) = 1$. We are lucky that the two initial conditions for a first-order DE are satisfied. For general initial conditions there would be no solution.

   $\mathcal{O}(\epsilon^{-1})$: At the next order the problem for y_1 is

$$\begin{aligned} y_{1\tau} &= -\frac{1}{2}\left(y_{0\tau\tau} + y_0\right) = -\frac{1}{2}, \\ y_1(0) = y_{1\tau}(0) &= 0. \end{aligned} \tag{7.25}$$

   The DE and the second initial condition are inconsistent. Hence, the $\mathcal{O}(\epsilon^{-1})$ problem has no solution.

Case (b): Next consider the time scale β/k. Setting $t = T_c^{(2)} s = \frac{\beta}{k} s$ we have

$$\begin{aligned} y_{ss} + 2\epsilon^2 y_s + \epsilon^2 y &= 0, \\ y(0) &= 1, \\ y_s(0) &= 0. \end{aligned} \tag{7.26}$$

1. If $\epsilon \ll 1$ set $y = y_0 + \epsilon^2 y_1 + \epsilon^4 y_2 + \cdots$.

   $\mathcal{O}(1)$**:**

$$\left.\begin{array}{rcl} y_{ss} &=& 0 \\ y_0(0) &=& 1 \\ y_{0s}(0) &=& 0 \end{array}\right\} \Rightarrow y_0(s) = 1. \tag{7.27}$$

   $\mathcal{O}(\epsilon)$**:**

$$\left.\begin{array}{rcl} y_{1ss} &=& -2y_{0s} - y_0 = -1 \\ y_0(0) &=& -y_{1s}(0) = 0 \end{array}\right\} \Rightarrow y_1(s) = -\frac{s^2}{2} \tag{7.28}$$

   $\therefore\ y = 1 - s^2\epsilon^2/2 + \cdots$ which becomes disordered when $s = \mathcal{O}(\epsilon^{-1})$. Note also that small $\epsilon \ll 1$ corresponds to weak damping for which the solution is oscillatory. This does not look promising however it it a valid approximation for sufficiently short times.

2. If $\epsilon \gg 1$ write ODE as

$$\frac{1}{\epsilon^2} y_{ss} + 2y_s + y = 0, \tag{7.29}$$

   and set $y = y_0 + \frac{1}{\epsilon^2} y_1 + \frac{1}{\epsilon^4} y_2 + \cdots$.

   $\mathcal{O}(1)$**:**

$$\left\{\begin{array}{l} 2y_{0s} + y_0 = 0 \\ y_0(0) = 1, y_{0s}(0) = 0 \end{array}\right. \Rightarrow \text{no solution} \tag{7.30}$$

Case (c): Set $t = T_c^{(3)} \xi = \frac{m}{\beta} \xi$. We now have

$$\begin{aligned} \epsilon^2 y_{\xi\xi} + 2\epsilon^2 y_\xi + y &= 0 \\ y(0) &= 1, \\ y_\xi(0) &= 0. \end{aligned} \tag{7.31}$$

1. If $\epsilon^2 \ll 1$, $\mathcal{O}(1)$ problem is $y_0(\xi) = 0$ which can't satisfy the initial conditions $y_0(0) = 1$. Hence, there is no solution.

2. If $\epsilon \gg 1$ write the DE as

$$y_{\xi\xi} + 2y_\xi + \frac{1}{\epsilon^2} y = 0, \tag{7.32}$$

   and set $y = y_0 + \frac{1}{\epsilon^2} y_1 + \frac{1}{\epsilon^4} y_2 + \cdots$.

   $\mathcal{O}(1)$**:**

$$\begin{aligned} y_{0\xi\xi} + 2y_{0\xi} &= 0, \\ y_0(0) &= 1, \\ y_{0\xi}(0) &= 0. \end{aligned} \tag{7.33}$$

   Integrating the DE $y_{0\xi} = \tilde{A} e^{-2\xi} \Rightarrow y_0(\xi) = A e^{-2\xi} + C$. The initial conditions give $A = 0$ and $C = 1$. Hence $y_0(\xi) = 1$.

$\mathcal{O}(\epsilon^{-2})$**:**

$$
\begin{aligned}
y_{1\xi\xi} + 2y_{1\xi} &= -y_0 = -1, \\
y_1(0) &= y_{1\xi}(0) = 0.
\end{aligned}
\tag{7.34}
$$

Integrating the DE gives $y_{1\xi} + 2y_1 = -\xi + C$. From the two initial conditions $C = 0$. Integrating again gives $y_1 = Ae^{-2\xi} + \frac{1}{4} - \frac{\xi}{2}$. The initial condition $y_1(0) = 0$ gives $A = -\frac{1}{4}$.

Thus, the RPT solution to $\mathcal{O}(\epsilon^{-2})$ is

$$
y = 1 + \left(\frac{1}{4}(1 - e^{-2\xi}) - \frac{\xi}{2}\right)\frac{1}{\epsilon^2} + \cdots , \tag{7.35}
$$

which becomes disordered after a time of $\mathcal{O}(\epsilon^2)$.

Summary of RPT solutions:

Case (a): $T_c = T_c^{(1)} = \sqrt{\frac{m}{k}}$:

1. weak damping $\to$ Regular Perturbation Theory fails at times of $\mathcal{O}(\epsilon^{-1})$ due to secular terms.
2. strong damping $\to$ no solution

Case (b): $T_c = T_c^{(2)} = \frac{\beta}{k}$:

1. weak damping $\to$ disordered by $s = \mathcal{O}(\epsilon^{-1})$
2. strong damping $\to$ no solution

Case (c): $T_c = T_c^{(1)} = \frac{m}{\beta}$:

1. weak damping $\to$ no solution
2. strong damping $\to$ disordered when $\xi = \mathcal{O}(\epsilon^{-2})$.

Exact Solution & Discussion for $\epsilon \gg 1$

For the exact solution the choice of nondimensionalization is irrelevant. Using the time scale $T_c^{(1)}$ we have

$$
\begin{aligned}
y_{\tau\tau} + 2\epsilon y_\tau + y &= 0, \\
y(0) &= 1, \\
y_\tau(0) &= 0,
\end{aligned}
$$

where $\epsilon = \beta/\sqrt{mk}$.

The solution has three different forms depending on the size of ϵ.

Case I: $\epsilon < 1$ (underdamped).

In dimensional time the exact solution is

$$
y = e^{-t/T_c^{(3)}}\left(\cos\left(\sqrt{1-\epsilon^2}\frac{t}{T_c^{(1)}}\right) + \frac{\epsilon}{\sqrt{1-\epsilon^2}}\sin\left(\sqrt{1-\epsilon^2}\frac{t}{T_c^{(1)}}\right)\right) \tag{7.36}
$$

The solution consists of oscillations on the time scale $T_c^{(1)}$ with an amplitude that decays on the time scale $T_c^{(3)}$. Since $T_c^{(1)} = \epsilon T_c^{(3)}$, for $\epsilon \ll 1$ the time scale of the oscillations is much shorter than the time scale of the amplitude decay.

Regular Perturbation Theory runs into difficulty for two reasons.

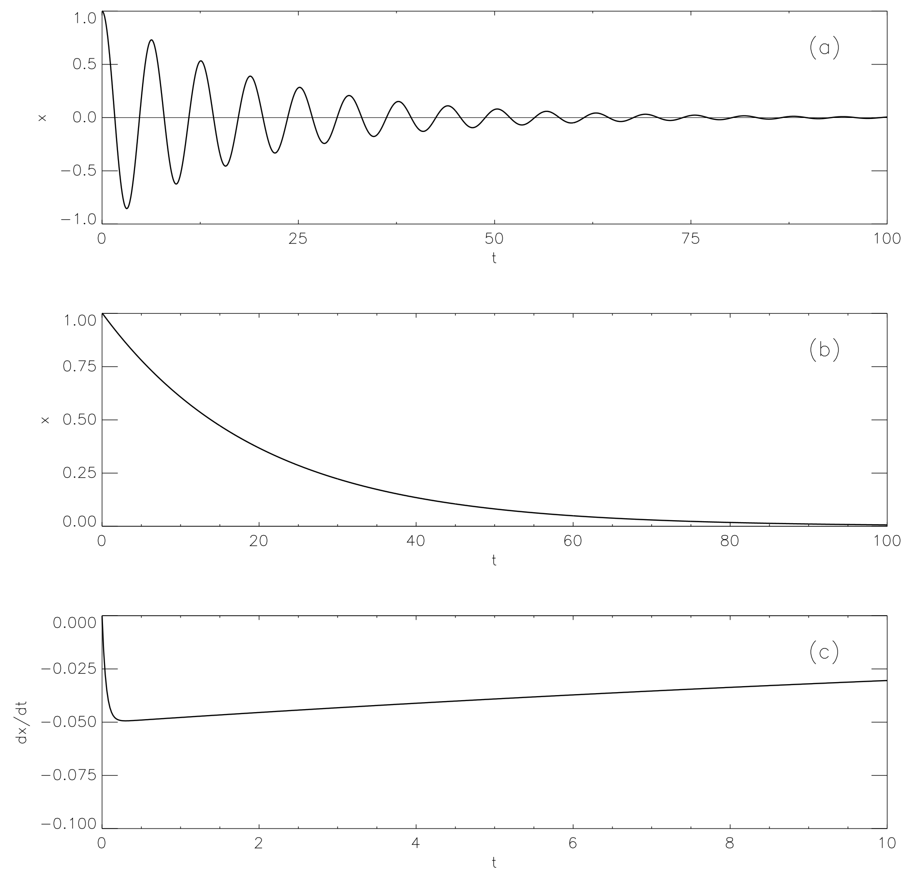

Figure 7.2: Exact solution of the linear damped oscillator for $m = k = 1$. (a) Underdamped case with $\epsilon = 0.05$. (b) Overdamped case with $\epsilon = 10$. (c) $\dot{x}$ for overdamped case.

1. The frequency of the oscillations depends on ϵ. This results in secular forcing terms and the solution breaks down after a time of $\mathcal{O}(\epsilon^{-1})$. We have seen this behaviour before when we consider the simple nonlinear pendulum.

2. The solution includes behaviour on two very different time scales: the amplitude decays on a slow time scale which significantly modifies the solution after times of $\mathcal{O}(T_c^{(3)})$. Introducing a time scale implicitely assumes the solution evolves on a single time scale. In reality we need two time scales for the weakly damped case. We will return to this when we study the method of multiple scales.

Case II: $\epsilon = 1$ (critically damped).

$$y = e^{-t/T_c^{(3)}} \left(\frac{t}{T_c^{(3)}} + 1 \right). \tag{7.37}$$

Since ϵ is neither large or small there is no small parameter to exploit. All time scales are identical and all three terms in the ODE are important. Can't use perturbation methods.

Case III: $\epsilon > 1$ (overdamped).

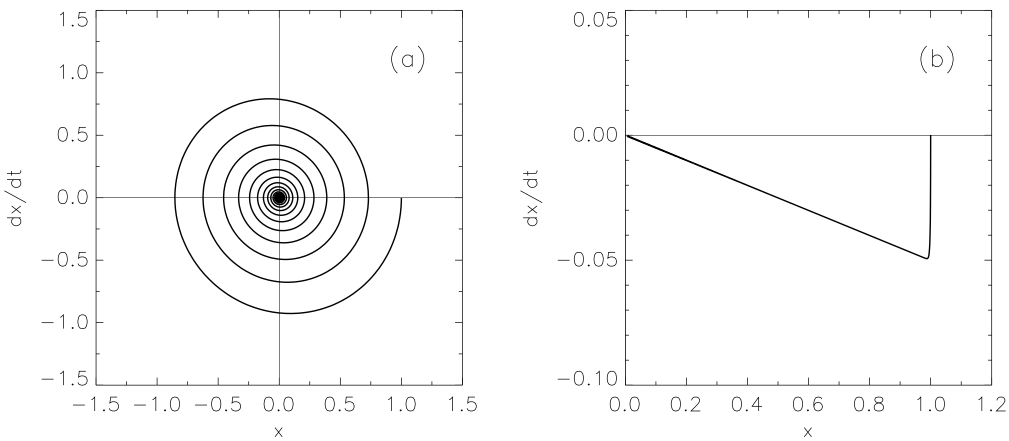

Figure 7.3: Exact solution of the linear damped oscillator for $m = k = 1$ on the phase plane. (a) Underdamped case with $\epsilon = 0.05$. (b) Overdamped case with $\epsilon = 10$.

The exact solution now has the form

$$\begin{aligned} y = &\frac{1}{2}\left(1+\frac{\epsilon}{\sqrt{\epsilon^2-1}}\right)e^{-(\epsilon-\sqrt{\epsilon^2-1})t/T_c^{(1)}} \\ &+\frac{1}{2}\left(1-\frac{\epsilon}{\sqrt{\epsilon^2-1}}\right)e^{-(\epsilon+\sqrt{\epsilon^2-1})t/T_c^{(1)}}. \end{aligned} \tag{7.38}$$

For $\epsilon \gg 1$, $\epsilon - \sqrt{\epsilon^2-1} = \epsilon - \epsilon\sqrt{1-\frac{1}{\epsilon^2}} \approx \frac{1}{2\epsilon}$ and $\epsilon + \sqrt{\epsilon^2-1} \approx 2\epsilon$. Hence, for large ϵ

$$\begin{aligned} y &\approx e^{-t/(2\epsilon T_c^{(1)})} - \frac{1}{4\epsilon^2}e^{-2\epsilon t/T_c^{(1)}}, \\ &= e^{-t/(2T_c^{(2)})} - \frac{1}{4\epsilon^2}e^{-2t/T_c^{(3)}}. \end{aligned} \tag{7.39}$$

The first term decays on the slow time scale $T_c^{(2)}$ while the second term decays on the fast time scale $T_c^{(3)}$ (since $T_c^{(2)} = \epsilon^2 T_c^{(3)} \gg T_c^{(3)}$).

When we nondimensionalized using time scale $T_c^{(1)}$ we could not obtain a solution. The solution does not involve this time scale!. In terms of τ the exact solution is

$$y \approx e^{-\tau/(2\epsilon)} - \frac{1}{4\epsilon^2}e^{-2\epsilon\tau}. \tag{7.40}$$

As $\epsilon \to \infty$ the second term goes to zero faster than any power of ϵ^{-1}

When we nondimensionalized using time scale $T_c^{(2)}$, (Case (b)), the reduced problem was

$$\begin{aligned} 2y_{0s} + y_0 &= 0, \\ y_0(0) &= 1, \\ y_{0s}(0) &= 0. \end{aligned} \tag{7.41}$$

which has no solution.

In terms of s, (7.39) is

$$y \approx e^{-s/2} - \frac{1}{4\epsilon^2}e^{-2\epsilon^2 s}. \tag{7.42}$$

The second term is exponentially small for large ϵ, ($s > 0$), and is lost in the Regular Perturbation Theory problem. Yet, it is needed at $s = 0$ to satisfy the two initial conditions.

Nondimensionalizing using $T_c^{(3)}$, the exact solution is

$$y = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1-\frac{1}{\epsilon^2}}}\right) e^{-\left(1-\sqrt{1-\frac{1}{\epsilon^2}}\right)\xi} + \frac{1}{2}\left(1 - \frac{1}{\sqrt{1-\frac{1}{\epsilon^2}}}\right) e^{-\left(1+\sqrt{1-\frac{1}{\epsilon^2}}\right)\xi}. \tag{7.43}$$

Expanding $\sqrt{1-\frac{1}{\epsilon^2}}$ and $1/\sqrt{1-\frac{1}{\epsilon^2}}$ in powers of $\frac{1}{\epsilon^2}$,

$$\begin{aligned} y &= \left(1 + \frac{1}{4\epsilon^2} + \cdots\right) e^{-\xi/(2\epsilon^2)+\cdots} \\ &\quad + \left(-\frac{1}{4\epsilon^2} + \cdots\right) e^{-2\xi} e^{\left(\frac{1}{2\epsilon^2}+\cdots\right)\xi}, \\ &= 1 + \left(\frac{1}{4}\left(1 - e^{-2\xi}\right) - \frac{\xi}{2}\right)\frac{1}{\epsilon^2} + \mathcal{O}_F(\frac{1}{\epsilon^4}). \end{aligned} \tag{7.44}$$

which recovers the Regular Perturbation Theory solution. The solution becomes disordered because the Taylor Series expansion

$$e^{-\frac{\xi}{2\epsilon^2}} = 1 - \frac{\xi}{2\epsilon^2} + \frac{1}{8}\frac{\xi^2}{\epsilon^4} + \cdots, \tag{7.45}$$

which converges as the number of terms goes to ∞ for fixed ξ, is a disordered Asymptotic Expansion when ξ is $\mathcal{O}(\epsilon^2)$.

7.3 Method of Multiple Scales

The linear damped oscillator is an example of a physical system that varies on more than one time scale. There many other such examples - far too numerous to list. The simple nonlinear pendulum is one such example. To see this consider the approximate solution we obtained using the Method of Strained Coordinates:

$$\theta(t) \approx a\cos\left(\sqrt{\frac{g}{\ell}}\left(1 - \frac{a^2}{16}\right)t\right) + \frac{a^3}{192}\left[\cos\left(\sqrt{\frac{g}{\ell}}\left(1 - \frac{a^2}{16}\right)t\right) - \cos\left(3\sqrt{\frac{g}{\ell}}\left(1 - \frac{a^2}{16}\right)t\right)\right]. \tag{7.46}$$

Using $\cos(A - B) = \cos A\cos B + \sin A\sin B$ we can write this as

$$\begin{aligned} \theta(t) \approx\ & a\cos\left(\sqrt{\frac{g}{\ell}}\frac{a^2}{16}t\right)\cos\left(\sqrt{\frac{g}{\ell}}t\right) + \sin\left(\sqrt{\frac{g}{\ell}}\frac{a^2}{16}t\right)\sin\left(\sqrt{\frac{g}{\ell}}t\right) \\ & + \frac{a^3}{192}\left[\cos\left(\sqrt{\frac{g}{\ell}}\frac{a^2}{16}t\right)\cos\left(\sqrt{\frac{g}{\ell}}t\right) + \sin\left(\sqrt{\frac{g}{\ell}}\frac{a^2}{16}t\right)\sin\left(\sqrt{\frac{g}{\ell}}t\right)\right. \\ & \left. - \cos\left(3\sqrt{\frac{g}{\ell}}\frac{a^2}{16}t\right)\cos\left(3\sqrt{\frac{g}{\ell}}t\right) - \sin\left(3\sqrt{\frac{g}{\ell}}\frac{a^2}{16}t\right)\sin\left(3\sqrt{\frac{g}{\ell}}t\right)\right]. \end{aligned} \tag{7.47}$$

Introducing a slow time scale $\tau = a^2 t$ (slow because τ changes by $\mathcal{O}(1)$ when t changes by $\mathcal{O}(1/a^2$ which is very large for small a) we can write this as

$$
\begin{aligned}
\theta(t) \approx a \cos\left(\sqrt{\frac{1}{16}\frac{g}{\ell}}\tau\right)\cos\left(\sqrt{\frac{g}{\ell}}t\right) + \sin\left(\frac{1}{16}\sqrt{\frac{g}{\ell}}\tau\right)\sin\left(\sqrt{\frac{g}{\ell}}t\right) \\
+ \frac{a^3}{192}\Bigg[\cos\left(\frac{1}{16}\sqrt{\frac{g}{\ell}}\tau\right)\cos\left(\sqrt{\frac{g}{\ell}}t\right) + \sin\left(\frac{1}{16}\sqrt{\frac{g}{\ell}}\tau\right)\sin\left(\sqrt{\frac{g}{\ell}}t\right) \\
- \cos\left(\frac{3}{16}\sqrt{\frac{g}{\ell}}\tau\right)\cos\left(3\sqrt{\frac{g}{\ell}}t\right) - \sin\left(\frac{3}{16}\sqrt{\frac{g}{\ell}}\tau\right)\sin\left(3\sqrt{\frac{g}{\ell}}t\right)\Bigg].
\end{aligned}
\tag{7.48}
$$

We can interpret this as fast oscillations on the time scale t multiplied by slowly varying amplitudes which are functions of the slow time scale τ.

We will now solve the simple nonlinear pendulum using the method of multiple scales by assuming at the outset that the behaviour of the system depends on more than one time scale. This is the method of multiple scales. It is an extremely powerful method with widespread applications.

7.3.1 The Simple Nonlinear Pendulum

Recall the scaled problem

$$
\begin{aligned}
\theta'' + \frac{\sin(a\theta)}{a} &= 0, \\
\theta(0) &= 1, \\
\theta'(0) &= 0,
\end{aligned}
\tag{7.49}
$$

The nonlinearity in the system causes a slow drift (period not exactly 2π) so we can think of θ varying on two time scales: one corresponding to the fast oscillations and one corresponding to the time scale of the drift. We assume the solution depends on two time scales t and $\tau = a^2 t$. With this in mind we try looking for a solution of the form

$$
\theta = f(t, \epsilon t; a^2) = f(t, \tau; a^2) \tag{7.50}
$$

The idea is to treat t and τ as two independent variables. The chain rule gives

$$
\begin{aligned}
\frac{d\theta}{dt} &= f_t + a^2 f_\tau, \\
\frac{d^2\theta}{dt^2} &= f_{tt} + 2a^2 f_{t\tau} + a^4 f_{\tau\tau},
\end{aligned}
\tag{7.51}
$$

so the the problem in terms of f becomes

$$
\begin{aligned}
f_{tt} + 2a^2 f_{t\tau} + a^4 f_{\tau\tau} + f - a^2\frac{f^3}{6} + \cdots &= 0, \\
f(0, 0; a^2) &= 1, \\
f_t(0, 0; a^2) + a^2 f_\tau(0, 0; a^2) &= 0.
\end{aligned}
\tag{7.52}
$$

As usual look for solutions of the form

$$
f = f_0(t, \tau) + a^2 f_1(t, \tau) + a^4 f_2(t, \tau) + \cdots . \tag{7.53}
$$

$\mathcal{O}(1)$ Problem:

$$\begin{aligned} f_{0_{tt}} + f_0 &= 0, \\ f_0(0,0) &= 1, \\ f_{0_t}(0,0) &= 0, \end{aligned} \tag{7.54}$$

which has the solution

$$f_0 = A(\tau)\cos(t) + B(\tau)\sin(t) \tag{7.55}$$

where

$$A(0) = 1 \qquad\qquad B(0) = 0. \tag{7.56}$$

$\mathcal{O}(a^2)$ Problem:

$$\begin{aligned} f_{1_{tt}} + f_1 &= -2f_{0_{t\tau}} + \frac{1}{6}f_0^3, \\ f_1(0,0) &= 0, \\ f_{1_t}(0,0) + f_{0_\tau}(0,0) &= 0. \end{aligned} \tag{7.57}$$

After some algebra to evaluate the forcing terms we get

$$\begin{aligned} f_{1_{tt}} + f_1 = &\left[2A'(\tau) + \frac{1}{8}(A^2B + B^3)\right]\sin t \\ &+ \left[-2B'(\tau) + \frac{1}{8}(AB^2 + A^3)\right]\cos t \\ &+ \frac{1}{24}(A^3 - 3AB^2)\cos 3t - \frac{1}{24}(B^3 - 3A^2B)\sin 3t. \end{aligned} \tag{7.58}$$

We must now eliminate the secular terms. This gives

$$\begin{aligned} A'(\tau) &= -\frac{1}{16}(A^2 + B^2)B, \\ B'(\tau) &= \frac{1}{16}(A^2 + B^2)A, \end{aligned} \tag{7.59}$$

We now have two coupled nonlinear ODEs to solve!
Multiplying the first by A, the second by B and adding gives

$$\frac{d}{d\tau}\left(\frac{1}{2}(A^2 + B^2)\right) = 0, \tag{7.60}$$

so $A^2 + B^2$ is constant. From the initial conditions $A^2 + B^2 = 1$ so

$$\begin{aligned} A'(\tau) &= -\frac{1}{16}B, \\ B'(\tau) &= \frac{1}{16}A, \end{aligned} \tag{7.61}$$

We now have a couple set of linear ODEs which are easily solved. Eliminating B gives

$$A''(\tau) + \frac{1}{16^2}A = 0. \tag{7.62}$$

The solution is

$$A(\tau) = \cos(\frac{\tau}{16}),$$
$$B(\tau) = \sin(\frac{\tau}{16}), \tag{7.63}$$

The $\mathcal{O}(1)$ solution is

$$\begin{aligned} f_0 &= \cos(\frac{\tau}{16})\cos t + \sin(\frac{\tau}{16})\sin t, \\ &= \cos\Big(t - \frac{\tau}{16}\Big), \\ &= \cos\Big((1 - \frac{a^2}{16})t\Big). \end{aligned} \tag{7.64}$$

Using the method of strained coordinates we obtained

$$\begin{aligned} \theta(t) = \cos&\left((1 - \frac{a^2}{16} + \cdots)t\right) \\ &+ \frac{a^2}{192}\left[\cos\left((1 - \frac{a^2}{16} + \cdots)t\right) - \cos\left(3\left(1 - \frac{a^2}{16} + \cdots\right)t\right)\right] \\ &+ \mathcal{O}_F(a^4). \end{aligned} \tag{7.65}$$

The method of multiple scales has recovered the same first term with identical frequencies to $\mathcal{O}(a^2)$.

With the resonant forcing terms eliminated in the $\mathcal{O}(a^2)$ problem the problem for f_1 simplifies to

$$\begin{aligned} f_{1_{tt}} + f_1 &= \frac{1}{24}(A^3 - 3AB^2)\cos 3t - \frac{1}{24}(B^3 - 3A^2B)\sin 3t, \\ f_1(0,0) &= 0, \\ f_{1_t}(0,0) + f_{0_\tau}(0,0) &= 0. \end{aligned} \tag{7.66}$$

Using the known forms for A and B this can be rewritten as

$$\begin{aligned} f_{1_{tt}} + f_1 &= \frac{1}{24}\cos(3\tau)\cos 3t + \frac{1}{24}\sin(3\tau)\sin 3t, \\ f_1(0,0) &= 0, \\ f_{1_t}(0,0) &= 0, \end{aligned} \tag{7.67}$$

which has the general solution

$$\begin{aligned} f_1(t,\tau) = -&\frac{1}{192}\cos(3\tau)\cos 3t - \frac{1}{192}\sin 3\tau \sin 3t \\ &+ A(\tau)\cos t + B(\tau)\sin t, \end{aligned} \tag{7.68}$$

where A and B are new unknown functions whose initial conditions are determined by the initial conditions at $(t,\tau) = (0,0)$.

Note that the first two terms combine to give

$$-\frac{1}{192}\cos(3\tau)\cos 3t - \frac{1}{192}\sin 3\tau \sin 3t = -\frac{1}{192}\cos\Big((1 - \frac{a^2}{16})3t\Big) \tag{7.69}$$

which is again in agreement with our previous solution.

The new unknown function A and B can be determined by eliminating the resonant forcing in the $\mathcal{O}(a^4)$ problem. To go to higher order, however, will require the introduction of an even longer time scale $\tau_1 = a^4 t$, consistent with our previously obtain amplitude dependent frequency $\sigma = 1 - a^2/16 + \mathcal{O}(a^4)$.

7.4 Methods for Singular Perturbation Problems

7.4.1 Method of Strained Coordinates (MSC)

This method, which we have already seen, is used to deal with secular terms. It is the technique we used to solve the nonlinear pendulum problem. Here we consider a similar example, the only real twist being the use of non-zero initial conditions for the first derivative $\dot{x}$ which results in the initial conditions making non-zero contributions to the higher-order problems.

Example 7.4.1 (The free Duffing oscillator)

$$\begin{aligned} \ddot{x} + x + \epsilon x^3 &= 0, \\ x(0) &= 0, \\ \dot{x}(0) &= v, \end{aligned} \tag{7.70}$$

where $0 < \epsilon \ll 1$.

This obeys the conservation law

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{\epsilon x^4}{4}\right) = 0, \tag{7.71}$$

hence

$$E(t) = E(0) = \frac{1}{2}v^2. \tag{7.72}$$

It follows that

$$\frac{1}{2}x^2 + \frac{\epsilon}{4}x^4 \leq \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{\epsilon}{4}x^4 = \frac{1}{2}v^2, \tag{7.73}$$

hence x is bounded for all time.

Regular Perturbation Theory Solution:

$\mathcal{O}(1)$: The leading-order problem is

$$\left.\begin{array}{rcl} \ddot{x}_0 + x_0 &=& 0 \\ x_0(0) &=& 0 \\ \ddot{x}_0(0) &=& v \end{array}\right\} \Rightarrow x_0 = v \sin t. \tag{7.74}$$

$\mathcal{O}(\epsilon)$: At the next order we get

$$\begin{aligned} \ddot{x}_1 + x_1 &= -x_0^3 \\ &= -v^3 \sin^3 t, \\ &= \frac{v^3}{4}(\sin 3t - 3\sin t), \\ x_1(0) = \dot{x}_1(0) &= 0. \end{aligned} \tag{7.75}$$

The $\sin t$ term is a secular term which leads to unbounded growth in the amplitude of the oscillations. The solution fails (becomes disordered) when $t = \mathcal{O}(\epsilon^{-1})$.

Method of Strained Coordinates: We now allow the time scale to depend on ϵ, as we did for the nonlinear pendulum problem. Thus, we seek a solution of the form

$$x = x(\sigma;\epsilon) = x_0(\sigma) + \epsilon x_1(\sigma) + \epsilon^2 x_2(\sigma) + \cdots, \tag{7.76}$$

where

$$\sigma(\epsilon) = \omega(\epsilon)t = (1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)t, \tag{7.77}$$

i.e., let the frequency ω be a function of ϵ. Because the energy in the system is $v^2/2$, we should expect the frequency of the oscillations to also depend on v. The ODE becomes

$$\begin{aligned} (1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)^2 x_{\sigma\sigma} + x + \epsilon x^3 &= 0, \\ x(0) &= 0, \\ (1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)x_\sigma(0) &= v. \end{aligned} \tag{7.78}$$

Because the initial velocity is non-zero, there will be non-zero contributions to the higher-order initial conditions.

$\mathcal{O}(1)$: The leading-order problem is

$$\left.\begin{array}{rcl} x_{0\sigma\sigma} + x_0 &=& 0 \\ x_0(0) &=& 0, \\ x_{0\sigma}(0) &=& v \end{array}\right\} \Rightarrow x_0 = v\sin\sigma. \tag{7.79}$$

$\mathcal{O}(\epsilon)$: At the next order we have

$$\begin{aligned} x_{1\sigma\sigma} + x_1 &= -x_0^3 - 2\omega_1 x_{0\sigma\sigma}, \\ &= \frac{v^3}{4}\left(\sin(3\sigma) - 3\sin\sigma\right) + 2\omega_1 v\sin\sigma, \\ &= \frac{v^3}{4}\sin(3\sigma) + \left(2\omega_1 v - \frac{3}{4}v^3\right)\sin\sigma. \end{aligned} \tag{7.80}$$

The secular terms can be eliminated by setting

$$\omega_1 = \frac{3}{8}v^2, \tag{7.81}$$

leaving us with

$$x_{1\sigma\sigma} + x_1 = \frac{v^3}{4}\sin 3\sigma, \tag{7.82}$$

which has the general solution

$$x_1 = -\frac{v^3}{32}\sin 3\sigma + A\sin\sigma + B\cos\sigma. \tag{7.83}$$

The initial conditions are

$$\begin{aligned} x_1(0) &= 0, \\ x_{1\sigma}(0) + \omega_1 x_{0\sigma}(0) &= 0, \end{aligned} \tag{7.84}$$

the latter of which is

$$x_{1\sigma}(0) = \omega_1 v = \frac{3}{8}v^3. \tag{7.85}$$

From these $A = -9v^3/32$ and $B = 0$, hence

$$x_1 = -\frac{v^3}{32}\sin 3\sigma - \frac{9}{32}v^3 \sin\sigma. \tag{7.86}$$

$\mathcal{O}(\epsilon^2)$: At the next order the DE and initial conditions give

$$\begin{aligned} x_{2\sigma\sigma} + x_2 &= -(2\omega_2 + \omega_1^2)x_{0\sigma\sigma} - 2\omega_1 x_{1\sigma\sigma} - 3x_0^2 x_1, \\ x_2(0) &= 0, \\ x_{2\sigma}(0) + \omega_1 x_{1\sigma}(0) + \omega_2 x_{0\sigma}(0) &= 0. \end{aligned} \tag{7.87}$$

Using the previous solutions the DE is

$$x_{2\sigma\sigma} + x_2 = \left(2\omega_2 v + \frac{69}{128}v^5\right)\sin\sigma - \frac{75}{128}v^5\sin 3\sigma - \frac{3}{128}v^5 \sin 5\sigma. \tag{7.88}$$

The secular terms are eliminated by setting

$$\omega_2 = -\frac{69}{256}v^4. \tag{7.89}$$

The solution of the resulting DE satisfying the boundary conditions is

$$x_2(\sigma) = \frac{75}{1024}v^5\sin(3\sigma) + \frac{3}{3072}v^5\sin(5\sigma) + \frac{47}{512}\sin\sigma. \tag{7.90}$$

The full solution to $\mathcal{O}(\epsilon^2)$ is

$$\begin{aligned} x = \sin\omega t &- \frac{v^3}{32}\Big(\sin(3\omega t) + 9\sin\omega t\Big)\epsilon \\ &+ \left(\frac{75}{1024}v^5\sin(3\omega t) + \frac{3}{3072}v^5\sin(5\omega t) + \frac{47}{512}\sin\omega t\right)\epsilon^2 \\ &+ \mathcal{O}_F(\epsilon^3), \end{aligned} \tag{7.91}$$

where

$$\omega = \omega(\epsilon) = 1 + \frac{3}{8}v^2\epsilon - \frac{69}{256}v^4\epsilon^2 + \mathcal{O}_F(\epsilon^3). \tag{7.92}$$

7.4.2 The Linstedt-Poincaré Technique

The Linstedt-Poincaré Technique

- is a generalization of the previous method;
- was first used in theory of nonlinear waterwaves by G. G. Stokes in 1847;
- was proved by Poincaré to have a uniformly valid asymptotic expansion in 1892. He credited idea to an obscure paper by Linstedt;
- **only works for periodic motion**;

- is useful for finding oscillatory (periodic) solutions of DEs of the form

$$\ddot{x} + \omega_0^2 x = \epsilon f(t, x, \dot{x}; \epsilon). \tag{7.93}$$

A common problem amenable to this technique is free oscillations in a conservative systems. Consider a regulating equation of the form

$$\ddot{x} + F(x) = 0, \tag{7.94}$$

which has constant energy

$$E(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + V(x), \tag{7.95}$$

where

$$V = \int_0^x F(\xi) d\xi. \tag{7.96}$$

The solution of

$$\begin{aligned} \ddot{x} + F(x) &= 0, \\ x(0) &= \alpha, \\ \dot{x}(0) &= \beta, \end{aligned} \tag{7.97}$$

is given by writing the conservation law as

$$\begin{aligned} \dot{x}^2 &= 2(E - V), \\ \Rightarrow \frac{dt}{dx} &= \pm \frac{1}{\sqrt{2(E-V)}}, \\ \Rightarrow t &= \pm \int^x \frac{d\xi}{\sqrt{2(E-V)}}. \end{aligned} \tag{7.98}$$

Using the initial conditions this give $t(x)$ which in principle can be inverted to get $x(t)$. This is usually impossible to do analytically. Perturbation Theory is useful for finding an approximate solution provided $F(x)$ involves a small parameter and the reduced problem can be solved..

Example 7.4.2 *Consider F of the form*

$$F(x) = \omega_0^2 x - \epsilon f(x). \tag{7.99}$$

Our problem is

$$\begin{aligned} \ddot{x} + \omega_0^2 x &= \epsilon f(x), \\ x(0) &= \alpha, \\ \dot{x}(0) &= \beta, \end{aligned} \tag{7.100}$$

for which we wish to find periodic solutions. For simplicity we will assume $\beta = 0$ for now. We will consider non-zero β later. Following standard MSC methods we introduce a scaled time scale

$$\tau = \omega(\epsilon) t. \tag{7.101}$$

giving

$$\begin{aligned} \omega^2(\epsilon) x_{\tau\tau} + \omega_0^2 x &= \epsilon f(x), \\ x(0) &= \alpha, \\ x_\tau(0) &= \beta = 0, \end{aligned} \tag{7.102}$$

and as usual expand x and ω via

$$\begin{aligned} x &= x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots, \\ \omega &= \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots. \end{aligned} \tag{7.103}$$

Since $x(\tau)$ is periodic, so are the $x_n(\tau)$. Substituting the expansions in the DE gives

$$\begin{aligned} &(\omega_0^2 + 2\omega_0\omega_1\epsilon + (2\omega_0\omega_2 + \omega_1^2)\epsilon^2 + \cdots)(x_{0\tau\tau} + \epsilon x_{1\tau\tau} + \epsilon^2 x_{2\tau\tau} + \cdots) \\ &\quad + \omega_0^2(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots) = \epsilon f(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots). \end{aligned} \tag{7.104}$$

or

$$\begin{aligned} &\omega_0^2(x_{0\tau\tau} + x_0) + \Big(\omega_0^2(x_{1\tau\tau} + x_1) + 2\omega_0\omega_1 x_{0\tau\tau}\Big)\epsilon + \cdots \\ &\quad = \epsilon\Big(f(x_0) + f'(x_0)(\epsilon x_1 + \epsilon^2 x_2 + \cdots) + \cdots\Big). \end{aligned} \tag{7.105}$$

$\mathcal{O}(1)$: The leading-order problem is

$$\begin{aligned} x_{0\tau\tau} + x_0 &= 0, \\ x_0(0) &= \alpha, \\ x_{0\tau}(0) &= 0, \end{aligned} \tag{7.106}$$

which has the solution

$$x_0 = \alpha\cos\tau. \tag{7.107}$$

$\mathcal{O}(\epsilon)$: At the next order we have

$$\begin{aligned} x_{1\tau\tau} + x_1 &= -\frac{2\omega_1}{\omega_0}x_{0\tau\tau} + \frac{1}{\omega_0^2}f(x_0), \\ &= \frac{2\omega_1}{\omega_0}\alpha\cos\tau + \frac{1}{\omega_0^2}f(\alpha\cos\tau), \\ x_1(0) = x_{1\tau}(0) &= 0 \end{aligned} \tag{7.108}$$

Since $\cos\tau$ is an even periodic function, so is $g(\tau) = f(\alpha\cos\tau)$. Hence we can expand $f(\alpha\cos\tau)$ in a cosine series:

$$f(\alpha\cos\tau) = a_0 + a_1\cos\tau + a_2\cos 2\tau + \cdots, \tag{7.109}$$

where

$$a_0 = \frac{2}{\pi}\int_0^{2\pi} f(\alpha\cos\tau)\,d\tau, \tag{7.110}$$

and

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \cos(n\pi) f(\alpha\cos\tau)\,d\tau. \tag{7.111}$$

Therefore

$$x_{1\tau\tau} + x_1 = \frac{a_0}{\omega_0^2} + \underbrace{\left(\frac{2\omega_1\alpha}{\omega_0} + \frac{a_1}{\omega_0^2}\right)\cos\tau}_{\text{secular term}} + \sum_{n=2}^{\infty}\frac{a_n}{\omega_0^2}\cos(n\tau). \tag{7.112}$$

To eliminate the secular terms take

$$\omega_1 = -\frac{a_1}{2\omega_0\alpha}, \tag{7.113}$$

after which we can solve for x_1 to get

$$x_1(\tau) = \frac{a_0}{\omega_0^2} - \sum_{n=2}^{\infty} \frac{a_n \cos n\tau}{\omega_0^2(n^2-1)} + A\cos\tau + B\sin\tau. \tag{7.114}$$

The initial conditions then give

$$\begin{aligned} &B = 0, \\ &A = -\frac{a_0}{\omega_0^2} + \sum_{n=2}^{\infty} \frac{a_n}{\omega_0(n^2-1)}. \end{aligned} \tag{7.115}$$

Note: If $x'(0) = \beta \neq 0$ then at leading order we would have $x_0 = \alpha\cos\tau + \beta\sin\tau$ and then we would have $f(\alpha\cos\tau + \beta\sin\tau)$ at $\mathcal{O}(\epsilon)$ which has to be expanded in a full Fourier Series. This would result in

$$f(x_0) = a_0 + a_1\cos\tau + b_1\sin\tau + \cdots, \tag{7.116}$$

and

$$x_{1\tau\tau} + x_1 = \frac{a_0}{\omega_0^2} + \left(\frac{a_1}{\omega_0^2} + \frac{2\omega_1\alpha}{\omega_0}\right)\cos\tau + \left(\frac{b_1}{\omega_0^2} + \frac{2\omega_1\beta}{\omega_0}\right)\sin\tau + \cdots. \tag{7.117}$$

To remove the secular terms we need both

$$\omega_1 = -\frac{a_1}{2\omega_0\alpha}, \tag{7.118}$$

and

$$\omega_1 = -\frac{b_1}{2\omega_0\beta}, \tag{7.119}$$

which is only possible if $\beta a_1 = \alpha b_1$. This is always true! This is easy to prove. The difference is

$$\begin{aligned} \beta a_1 - \alpha b_1 &= \beta\frac{1}{\pi}\int_0^{2\pi} f(\alpha\cos\tau + \beta\sin\tau)\cos\tau\, d\tau \\ &\quad - \alpha\frac{1}{\pi}\int_0^{2\pi} f(\alpha\cos\tau + \beta\sin\tau)\sin\tau\, d\tau, \\ &= \frac{1}{\pi}\int_0^{2\pi} f(\alpha\cos\tau + \beta\sin\tau)(-\alpha\sin\tau + \beta\cos\tau)\, d\tau, \\ &= \frac{1}{\pi}\int_0^{2\pi} f(\alpha\cos\tau + \beta\sin\tau)\left(\frac{d}{d\tau}(\alpha\cos\tau + \beta\sin\tau)\right) d\tau, \\ &= \frac{1}{\pi}\int_\alpha^\alpha f(u)\, du = 0. \end{aligned} \tag{7.120}$$

Theorem 7.4.1 (Periodicity of Solutions) *For sufficiently small ϵ solutions to*

$$\ddot{x} + \omega_0^2 x = \epsilon f(x) \tag{7.121}$$

are periodic and admit a Linsted-Poincaré expansion, i.e. a uniformly ordered Asymptotic Expansion (see text by Murdock).

7.4.3 Free Self Sustained Oscillations in Damped Systems

Free self sustained oscillations in damped system arise through a combination of damping and forcing. A general 1-D problem is

$$\ddot{x} + \omega_0^2 x = \epsilon f(x, \dot{x}), \tag{7.122}$$

along with initial conditions.

Example 7.4.3 *A special case of Rayleigh's equation is*

$$\ddot{x} + x = \epsilon\left(\dot{x} - \frac{1}{3}\dot{x}^3\right). \tag{7.123}$$

Multiplying by $\dot{x}$ gives the energy equation

$$\frac{d}{dt}\Big(\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2\Big) = \epsilon\dot{x}^2\Big(1 - \frac{1}{3}\dot{x}^2\Big). \tag{7.124}$$

The energy increases in time if $\dot{x}^2 < 3$ and decreases in time if $\dot{x}^2 > 3$. It is plausible that periodic solutions exist. Let's assume a periodic solution exists and try to find it. We do not know appropriate initial conditions for a periodic solution. In fact, only special initial conditions give rise to periodic solutions so we will have to determine them as part of the solution.

Without loss of generality we can assume $\dot{x}(0) = 0$. For a periodic solution $\dot{x}$ must be zero at some time. Let this time be $t = 0$. Thus, let

$$\begin{aligned} x(0) &= \alpha(\epsilon) = \alpha_0 + \epsilon\alpha_1 + \epsilon^2\alpha_2 + \cdots, \\ \dot{x}(0) &= 0. \end{aligned} \tag{7.125}$$

Here we have used the fact that the periodic orbit will depend on ϵ, hence we must allow α too as well. As usual let

$$\tau = \omega(\epsilon)t = (1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)t, \tag{7.126}$$

giving the DE

$$\omega^2(\epsilon)x_{\tau\tau} + x = \epsilon(\omega x_\tau - \frac{1}{3}\omega^3 x_\tau^3). \tag{7.127}$$

$\mathcal{O}(1)$: The leading-order problem is

$$\left.\begin{aligned} &x_{0\tau\tau} + x_0 = 0, \\ &x_0(0) = \alpha_0, \\ &x_{0\tau}(0) = 0, \end{aligned}\right\} \Rightarrow x_0 = \alpha_0\cos\tau, \tag{7.128}$$

where α_0 is undetermined. This is a reflection of the fact that when $\epsilon = 0$ we get circular orbits (on the x-$\dot{x}$ plane) with arbitrary radius.

$\mathcal{O}(\epsilon)$: At the next order the DE is

$$x_{1\tau\tau} + x_1 = 2\omega_1\alpha_0\cos\tau + \left(\frac{1}{4}\alpha_0^3 - \alpha_0\right)\sin\tau - \frac{1}{12}\alpha_0^3\sin 3\tau. \tag{7.129}$$

There are two secular terms we must make equal to zero. Since $\alpha_0 = 0$ is uninteresting (it gives the zero solution) setting the coefficient of $\cos\tau$ to zero gives

$$\omega_1 = 0, \tag{7.130}$$

while setting the coefficient of $\sin\tau$ to zero gives

$$\alpha_0 = \pm 2. \tag{7.131}$$

These two points are in fact on the same orbit: there are two locations where $\dot{x} = 0$, one at $x = 2$ and at the other $x = -2$ to $\mathcal{O}(1)$. We can choose to start at either one. Taking

$$\alpha_0 = 2, \tag{7.132}$$

gives

$$\begin{aligned} x_{1\tau\tau} + x_1 &= -\frac{2}{3}\sin 3\tau, \\ x_1(0) &= \alpha_1, \\ x_{1\tau}(0) &= 0, \end{aligned} \tag{7.133}$$

the solution of which is

$$x_1(\tau) = \frac{1}{12}\sin 3\tau - \frac{1}{4}\sin\tau + \alpha_1\cos\tau. \tag{7.134}$$

$\mathcal{O}(\epsilon^2)$: At the next order, using $\omega_1 = 0$, we have

$$x_{2\tau\tau} + x_2 = \left(4\omega_2 + \frac{1}{4}\right)\cos\tau - \frac{1}{2}\cos 3\tau + \frac{1}{4}\cos 5\tau + 2\alpha_1\sin\tau - \alpha_1\sin 3\tau. \tag{7.135}$$

To cancel the secular terms set

$$\begin{aligned} \omega_2 &= -\frac{1}{16}, \\ \alpha_1 &= 0, \end{aligned} \tag{7.136}$$

giving

$$\begin{aligned} x_{2\tau\tau} + x_2 &= -\frac{1}{2}\cos 3\tau + \frac{1}{4}\cos 5\tau, \\ x_2(0) &= \alpha_2, \\ x_{2\tau}(0) &= 0. \end{aligned} \tag{7.137}$$

which we will leave unsolved. The solution will involve an undetermined constant α_2 the value of which is obtained by the solvability condition for the $\mathcal{O}(\epsilon^3)$ problem.

So far we have

$$\begin{aligned} x(0) &= 2 + \mathcal{O}_F(\epsilon^2), \\ \omega &= 1 - \frac{1}{16}\epsilon^2 + \mathcal{O}_F(\epsilon^3) \\ x &= 2\cos\tau + \epsilon\Big(\frac{1}{12}\sin 3\tau - \frac{1}{4}\sin\tau\Big) + \mathcal{O}_F(\epsilon^2), \end{aligned} \tag{7.138}$$

with $\tau = \omega t$.

We have found one periodic orbit and in fact it can be shown that the exact solution only has one. Figure 7.4 compares numerical solutions of (7.123) using $\epsilon = 0.5$ for $x(0) = 1$, 2, and 3. In Figure 7.5 the solutions are shown on a phase plane and the solution starting at $(x, \dot{x}) = (2, 0)$ is compared with a circle of radius 2 and with the periodic orbit (7.138) obtained using perturbation theory. The orbit for $x(0) = 2$ appears closed in the figure after going around the origin over six times (see Figure 7.4). This shows that perturbation theory gives an excellent solution for ϵ as large as 0.5.

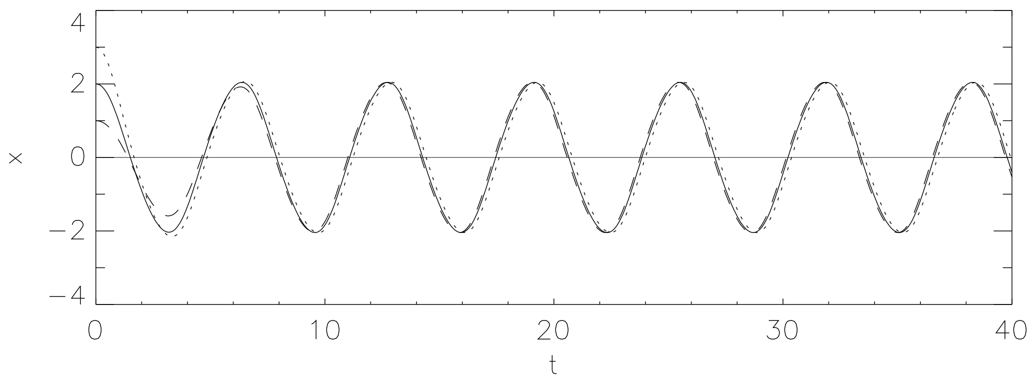

Figure 7.4: Numerical solutions of Rayleigh's equation (7.123) for $\epsilon = 0.5$.

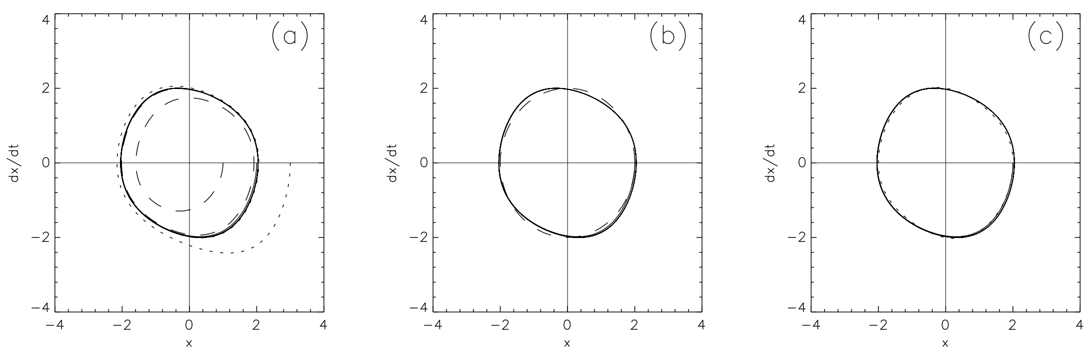

Figure 7.5: (a) Numerical solutions of Rayleigh's equation (7.123) for $\epsilon = 0.5$. (b) Comparison of numerical solution starting at $(x, \dot{x}) = (2, 0)$ with circle of radius 2. (c) Comparison of numerical solution starting at $(x, \dot{x}) = (2, 0)$ with perturbation theory solution.

7.4.4 MSC: The Lighthill Technique (Lighthill, 1949, *Phil. Mag.* 40)

Example 7.4.4 *Find an Asymptotic Expansion of the solution to*

$$
\begin{aligned}
(x + \epsilon y)\frac{dy}{dx} + y &= 1 \\
y(1) &= 2,
\end{aligned}
\tag{7.139}
$$

for $0 \le x \le 1$, $0 < \epsilon \ll 1$.

Recall that Regular Perturbation Theory gives

$$
\begin{aligned}
y_0 &= 1 + \frac{1}{x}, \\
y_1 &= \frac{(3x+1)(x-1)}{2x^3}, \\
y_2 &= \frac{(3x+1)(1-x^2)}{2x^5},
\end{aligned}
\tag{7.140}
$$

As $x \to 0$ y_2 is more singular than y_1 which is more singular than y_0, so the solution becomes disordered.

Lighthill suggested looking for a solution of the form

$$y(x) = \widetilde{y}(z) = Y_0(z) + \epsilon Y_1(z) + \epsilon^2 Y_2(z) + \cdots , \tag{7.141}$$

where the new strained coordinate z is introduced via

$$x = z + \epsilon x_1(z) + \epsilon^2 x_2(z) + \cdots . \tag{7.142}$$

The functions $x_n(z)$ must be determined as part of the solution. They are chosen by invoking what is now known as Lighthill's Rule.

Definition 7.4.1 (Lighthill's Rule) *Choose $x_n(z)$ to get rid of singularities in the higher order problems, i.e., $Y_n(z)$ should not be more singular than $Y_{n-1}(z)$.*

In terms of our new variables we have

$$(x + \epsilon y)\frac{dy}{dx} + y = (x + \epsilon \tilde{y})\frac{d\tilde{y}}{dz}\Big/\frac{dx}{dz} + \tilde{y}, \tag{7.143}$$

so the DE can be written as

$$(x + \epsilon \tilde{y})\frac{d\tilde{y}}{dz} + \frac{dx}{dz}\tilde{y} = \frac{dx}{dz}. \tag{7.144}$$

Expanding gives

$$\begin{aligned}\Big(z + \epsilon(x_1(z) + Y_0) + \epsilon^2(x_2(z) + Y_1) + \cdots\Big)\Big(Y_0' + \epsilon Y_1' + \cdots\Big)& \\ + \Big(1 + \epsilon x_1' + \epsilon^2 x_2' + \cdots\Big)\Big(Y_0 + \epsilon Y_1 + \cdots\Big)& \\ = \Big(1 + \epsilon x_1' + \epsilon^2 x_2' + \cdots\Big).&\end{aligned} \tag{7.145}$$

With the change of variables the boundary condition at $x = 1$ is applied at the value of z that satisfies $z + \epsilon x_1(z) + \cdots = 1$, which is unknown until the $x_j(z)$ are found. Since the boundary condition will be applied at $z = 1$ in the limit $\epsilon \to 0$, let $x = 1$ correspond to

$$z = 1 + a_1\epsilon + a_2\epsilon^2 + \cdots . \tag{7.146}$$

Assuming all the $x_n(z)$ have Taylor Series expansions about $z = 1$ (which is reasonable since the only known singularity is at $x = 0$, i.e., near $z = 0$) we have

$$[a_1 + x_1(1)]\epsilon + [a_2 + x_1'(1)a_1 + x_2(1)]\epsilon^2 + \cdots = 0. \tag{7.147}$$

The boundary condition is

$$\tilde{y}(1 + a_1\epsilon + a_2\epsilon^2 + \cdots) = 2, \tag{7.148}$$

or

$$Y_o(1) + \Big(Y_1(1) + Y_0'(1)a_1\Big)\epsilon + \cdots = 2, \tag{7.149}$$

assuming the $Y_n(z)$ also have Taylor Series expansions about $z = 1$.

$\mathcal{O}(1)$: At leading order we have

$$
\begin{aligned}
z\frac{dY_0}{dz} + Y_0 &= 1 \\
Y_0(1) &= 2
\end{aligned}
\tag{7.150}
$$

The general solution of the DE is $1 + A/z$. Applying the boundary condition gives

$$
Y_0 = 1 + \frac{1}{z}. \tag{7.151}
$$

$\mathcal{O}(\epsilon)$: The $\mathcal{O}(\epsilon)$ problem is

$$
\begin{aligned}
z\frac{dY_1}{dz} + Y_1 &= -(Y_0(z) + x_1(z))\frac{dY_0}{dz} + \frac{dx}{dz}(1 - Y_0) \\
&= -\left(1 + \frac{1}{z} + x_1\right)\frac{1}{z^2} - \frac{1}{z}\frac{dx_1}{dz}, \\
Y_1(1) &= -a_1 Y_0'(1).
\end{aligned}
\tag{7.152}
$$

We now invoke Lighthill's Criterion: chose $x_1(z)$ so that $Y_1(z)$ is not more singular that $Y_0(z)$. This is easily done by choosing x_1 to make the right hand side of the DE for Y_1 equal to zero, for then $Y_1(z) = B/z$. Thus, $x_1(z)$ is chosen as a solution of the DE

$$
x_1' - \frac{1}{z}x_1 = \frac{1}{z} + \frac{1}{z^2}, \tag{7.153}
$$

which has the general solution

$$
x_1(z) = -1 - \frac{1}{2z} + cz, \tag{7.154}
$$

where c is an arbitrary constant. We are free to choose a value for c, however it is convenient to choose it's value so that the location where boundary condition is applied is not moved from $z = 1$, i.e., choose c so that $a_1 = 0$. Since $a_1 = -x_1(1)$, we choose c so that $x_1(1) = 0$ which gives $c = 3/2$. Thus

$$
x_1(z) = -1 - \frac{1}{2z} + \frac{3}{2}z. \tag{7.155}
$$

The boundary condition for Y_1 is now determined, namely $Y_1(1) = 0$. This gives

$$
Y_1(z) = 0. \tag{7.156}
$$

So far we have

$$
\begin{aligned}
\tilde{y}(X) &= 1 + \frac{1}{z} + \mathcal{O}(\epsilon^2) + \mathcal{O}_F(\epsilon^2), \\
x = z + \left(\frac{3}{2}z - 1 - \frac{1}{2z}\right)\epsilon &+ \mathcal{O}_F(\epsilon^2).
\end{aligned}
\tag{7.157}
$$

as $\epsilon \to 0$. If we ignore the $\mathcal{O}(\epsilon^2)$ terms we can solve for $z(x)$, giving

$$
z(x) = \frac{x + \epsilon \pm \sqrt{x^2 + 2\epsilon x + 4\epsilon^2 + 2\epsilon}}{2 + 3\epsilon}. \tag{7.158}
$$

We must take the positive sign so that $x = z$ when $\epsilon = 0$ (since $x \geq 0$). Hence

$$\rightarrow y(x) = -\frac{x}{\epsilon} + \sqrt{\left(\frac{x}{\epsilon}\right)^2 + 2\left(\frac{x+1}{\epsilon}\right) + 4}. \tag{7.159}$$

This is in fact the exact solution, however this is just luck. In general we will not find the exact solution.

Comment: Different choices of c are possible. For example, taking $c = 0$ gives

$$x_1(z) = -1 - \frac{1}{2z} \tag{7.160}$$

in which case

$$Y_1(z) = \frac{3}{2z}. \tag{7.161}$$

Now

$$\begin{aligned} \tilde{y} &= 1 + \frac{1}{z} + \frac{3}{2z}\epsilon + \mathcal{O}_F(\epsilon^2) \text{ as } \epsilon \to 0 \\ x &= z - \epsilon\left(1 + \frac{1}{2z}\right) + \mathcal{O}_F(\epsilon^2) \text{ as } \epsilon \to 0 \end{aligned} \tag{7.162}$$

which is no longer the exact solution.

7.4.5 The Pritulo Technique

Lighthill's technique involves solving a differential equation for $x(z)$. Here is an alternative procedure, introduced by Pritulo (1962, *J. App. Math. Mech.* **26**) that avoids the introduction of a second set of differential equations to solve.

Basic Methodology:

1. First find a Regular Perturbation Theory expansion:

$$y = y_0(x) + \epsilon y_1(x) + \cdots . \tag{7.163}$$

   If the expansion is uniformly ordered you're done.

2. Next, strain the coordinates:

$$x = z + \epsilon x_1(z) + \cdots \tag{7.164}$$

   This is identical to Lighthill's method.

3. Next, substitute (7.164) into (7.163) and do a Taylor Series expansion of the Y_i's about $x = z$. This gives

$$\begin{aligned} y &= Y_0(z) + \epsilon\Big(Y_0'(z)x_1(z) + Y_1(z)\Big) \\ &\quad + \epsilon^2\left(\frac{1}{2}Y_0''(z)x_1^2(z) + Y_0'(z)x_2(z) + Y_1'(z)x_1(z) + Y_2(z)\right) + \cdots \end{aligned} \tag{7.165}$$

4. Choose the $x_n(z)$ to eliminate more singular terms according to Lighthill's Rule.

Example 7.4.5 (Lighthill's example revisited) *We will demonstrate this method be reconsidering Lighthill's example*

$$\begin{aligned} (x + \epsilon y)y' + y &= 1, \\ y(1) &= 2, \end{aligned} \tag{7.166}$$

for $0 \le x \le 1$ *and* $0 < \epsilon \ll 1$.

Regular Perturbation Theory gives

$$y \sim 1 + \frac{1}{x} + \frac{(x-1)(3x+1)}{2x^3}\epsilon + \frac{(1-x)(3x+1)}{2x^5}\epsilon^2 + \cdots , \tag{7.167}$$

as $\epsilon \to 0$. Set $x = z + \epsilon x_1(z) + \epsilon^2 x_2(z) + \cdots$. In terms of z we get

$$\begin{aligned} \frac{1}{x} &= \frac{1}{z} - \frac{x_1(z)}{z^2}\epsilon + \mathcal{O}(\epsilon^2) \\ \frac{1}{x^3} &= \frac{1}{z^3} - \frac{3x_1(z)}{z^4}\epsilon + \mathcal{O}(\epsilon^2) \end{aligned} \tag{7.168}$$

so

$$y = 1 + \frac{1}{z} - \underbrace{\left(\frac{x_1(z)}{z^2} + \frac{(z-1)(3z+1)}{2z^3}\right)}_{Y_1(z)}\epsilon + \mathcal{O}_F(\epsilon^2) \tag{7.169}$$

Invoking Lighthill's rule, we now choose $x_1(X)$ to eliminate the part of $Y_1(z)$ which is more singular than $Y_0(z)$. That is, we want to eliminate the $1/z^3$ singularity. This can be done by choosing $x_1(z)$ so that $Y_0(z) = 0$. This is of course only one of an infinity of choices. Doing this gives

$$x_1(X) = \frac{(z-1)(3z+1)}{2z}. \tag{7.170}$$

This gives a solution which is identical to $x_1(X)$ using Lighthill's method, so as before we have obtained the exact solution.

7.4.6 Comparison of Lighthill and Pritulo Techniques

Lighthill's method: Set $y = Y_0(z) + \cdots$ where $x = z + \epsilon x_1(z) + \cdots$. Solve a DE for $x_1(z)$ and then solve a simplified DE for $Y_1(z)$.

Pritulo's method: No DE to solve for the $x_n(z)$, however in the RPT expansion $y = y_0(x) + \epsilon y_1(x) + \cdots$ we need to first find the $y_n(x)$ and then expand these functions in a Taylor Series about $x = z$.

Pritulo's Technique takes an AE

$$y \sim y_0(x) + y_1(x)\epsilon + y_2(x)\epsilon^2 + \cdots, \tag{7.171}$$

which is not uniformly ordered and gives an asymptotic expansion

$$\tilde{y} \sim \tilde{y}_0(z) + \tilde{y}_1(z)\epsilon + \tilde{y}_2(z)\epsilon^2 + \cdots, \tag{7.172}$$

which is uniformly ordered.

How is this possible? It is only possible because the two solutions are only equivalent as an infinite series. The truncated series are not equivalent. For example, assuming $y_0(x)$ and $y_1(x)$ can be expanded in a Taylor Series about z,

$$\begin{aligned} y_0(x) + \epsilon y_1(x) &= y_0(z) + [y_1(z) + y_0'(z)x_1(z)]\epsilon \\ &\quad + [y_0''(z)\frac{x_1^2(z)}{2} + y_0'(z)x_2(z) + y_1'(z)x_1(z)]\epsilon^2 + \cdots, \\ &= \tilde{y}_0(z) + \tilde{y}_1(z)\epsilon + \tilde{y}_2(z)\epsilon^2 + \cdots. \end{aligned} \tag{7.173}$$

i.e.,

$$\begin{aligned} y_0(x) + \epsilon y_1(x) &= \tilde{y}_0(z) + \tilde{y}_1(z)\epsilon + \tilde{y}_2(z)\epsilon^2 + \cdots \\ &\sim \tilde{y}_0(z) + \tilde{y}_1(z)\epsilon \end{aligned} \tag{7.174}$$

as $\epsilon \to 0$. The difference between $y_0(x) + \epsilon y_1(x)$ and $\tilde{y}_0(x) + \epsilon \tilde{y}_1(z)$ is $\mathcal{O}(\epsilon^2)$ and the difference may be singular in x or z. In particular, the difference can be larger than the retained terms as x goes to the singular point. For fixed x, as $\epsilon \to 0$ the difference is negligible.

The Pritulo technique amounts to a rearrangement of the terms in an RPT series. We will return to this when we compare a couple of nonlinear wave equations, derived with asymptotic methods.

Chapter 8

Matched Asymptotic Expansions

In this chapter we study the method of matched asymptotic expansions. This is a technique used for problems where the solution behaves on very different scales in two overlapping regions. This type of behaviour is typical of ODEs for which the highest derivative is multiplied by a small parameter, in which case the solution will in general have a thin boundary layer adjacent to a boundary in which the solution varies very rapidly. Outside of the boundary layer the solution typically varies on a much longer length scale. If the dependent variable is time the boundary layer is sometimes called an initial layer. In other problems the thin layer of rapid variation can occur in the interior of the domain (interior or transition layers).

Example 8.0.1 *Consider the problem*

$$\begin{aligned} \epsilon y''(x) + y'(x) &= a, \\ y(0) &= 0, \\ y(1) &= 1, \end{aligned} \tag{8.1}$$

where $0 < a < 1$ *and* $0 < \epsilon \ll 1$.

Trying Regular Perturbation Theory we let

$$y = y_0 + \epsilon y_1(x) + \epsilon^2 y_2(x) + \cdots , \tag{8.2}$$

resulting in the $\mathcal{O}(1)$ problem

$$y_0' = a \tag{8.3}$$

which has general solution $y_0(x) = ax + b$. Unfortunately we have two boundary conditions and unless $a = 1$ they can't both be satisfied.

The exact solution of the full problem is

$$y = ax + (1-a)\frac{1 - e^{-x/\epsilon}}{1 - e^{-1/\epsilon}}. \tag{8.4}$$

The solution for $a = 0.5$ and $\epsilon = 0.02$ is shown in Figure 8.1 along with the straight line $y_0(x) = ax + 1 - a$, which is the RPT solution obtained by applying the boundary condition at $x = 1$. We can see that in a thin region near $x = 0$ the solution varies very rapidly while outside this thin layer the solution is approximated very accurately by the RPT solution $y_0(x) = ax + 1 - a$.

The region in which y_0 provides a good approximation of the solution is called the outer region. It is the region in which the presence of the small parameter ϵ in the differential equation correctly

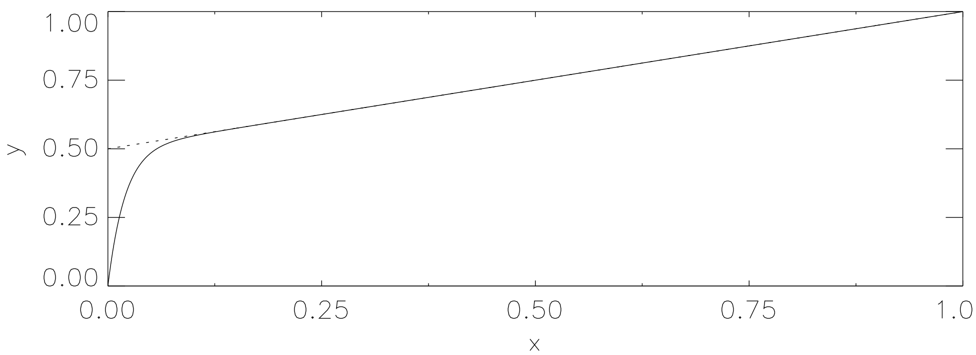

Figure 8.1: Solution of 8.1 (solid curve) for $a = 0.5$ and $\epsilon = 0.02$, compared with the outer solution $y = ax + 1 - a$ (dots).

indicates which term can be neglected. In the thin layer near the origin the solution varies very rapidly. Here y'' is very large and $\epsilon y''$ is not negligible compared to the other two terms in the differential equation. The region in which $\epsilon y''$ is important is called a boundary layer or the inner region.

Obvious questions are: How can we find a useful approximate solution in the inner region? How do we know which boundary condition to apply to the outer solution obtained using RPT? We then have to somehow match these solutions as we did in the turning point problem.

For the solution in the inner region we need to rescale x. Let

$$\chi = \frac{x}{\delta(\epsilon)}, \tag{8.5}$$

where $\delta(\epsilon) \to$ as $x \to 0$ is a measure of the currently unknown boundary layer thickness. Let $y(x) = \tilde{y}(\chi)$. In terms of $\tilde{y}(\chi)$ the ODE becomes

$$\tilde{y}'' + \frac{\delta}{\epsilon}\tilde{y}' = \frac{\delta^2}{\epsilon}a. \tag{8.6}$$

In the outer region the dominant balance in the differential equation was between y' and a. In the inner region the dominant balance must include $\tilde{y}''$. If it is between $\tilde{y}''$ and $\frac{\delta^2}{\epsilon}a$ we should take $\delta = \epsilon^{1/2}$. Then $\tilde{y} = A\chi + B$ where A and B are constants. Unless $A = 0$ $\tilde{y}'$ is $\mathcal{O}(1)$ and $\delta/\epsilon\tilde{y}' = \mathcal{O}(\epsilon^{-1/2}$ is larger than the retained terms. If $A = 0$ the boundary condition gives $B = 0$ and hence $\tilde{y} = 0$. This can't be correct. Thus the dominant balance must be between $\tilde{y}''$ and $\frac{\delta}{\epsilon}\tilde{y}'$. Hence we should take

$$\delta(\epsilon) = \epsilon, \tag{8.7}$$

giving

$$\tilde{y}'' + \tilde{y}' = \epsilon a. \tag{8.8}$$

We now use RPT. Setting $\tilde{y} = \tilde{y}_0 + \epsilon\tilde{y}_1 + \epsilon^2\tilde{y}_2 + \cdots$, at leading order we have

$$\tilde{y}_0'' + y_0' = 0, \tag{8.9}$$

which has the general solution

$$\tilde{y}_0 = Ae^{-\chi} + B. \tag{8.10}$$

We know from the exact solution that the boundary layer is in a thin neighbourhood near $\chi = 0$. Thus, we should apply the boundary condition at $x = 0$, since the other boundary point $x = 1$ is not in the boundary layer. For many problems it may not be obvious which boundary condition to apply, although this can often be determined from a consideration of the physical problem being considered. For now we will use our knowledge of the location of the boundary layer. If we assumed that the boundary layer was near $x = 1$ we would not be able to match the inner and outer solutions. We will discuss this later.

Applying the boundary condition $\tilde{y}_0(0) = 0$ gives

$$\tilde{y}_0(\chi) = A(1 - e^{-\chi}). \tag{8.11}$$

We have the following approximations of the solution:

(i) In the inner region: $y \approx y_{in} = y_0 = a(x - 1) + 1$.

(ii) In the outer region: $y \approx y_{out} = \tilde{y}_0(\chi) = A(1 - e^{-\chi})$.

The solutions involve an unknown constant A. We now need to match the two solutions in order to determine its value. To do this we need to find a matching region where both the inner and outer solutions are valid. That is we need to find $\alpha > \beta > 0$ such that

$$\begin{aligned} y &\sim y_0(x) \text{ as } \epsilon \to 0 \quad \text{for some region } x > \epsilon^\alpha, \\ y &\sim \tilde{y}_0(\chi) \text{ as } \epsilon \to 0 \quad \text{for some region } x < \epsilon^\beta. \end{aligned} \tag{8.12}$$

Note that $\epsilon^\beta \ll \epsilon^\alpha \ll 1$ and that both approximations are valid in the matching region $\epsilon^\beta \ll x \ll \epsilon^\beta$.

Proving the existence of a matching region is often the most difficult part of matched asymptotic expansions. In general we want $x \to 0$ and $\chi \to \infty$ in the matching region as $\epsilon \to 0$ and we will often just assume this.

For our particular problem we can use $\epsilon \ll x \ll \epsilon^\gamma$ as the matching region for any $0 < \gamma < 1$. Taking $\gamma = 1/4$, if $\epsilon \ll x \ll \epsilon^{1/4}$ we have $y_0(x) = ax + 1 - a \sim 1 - a$ as $\epsilon \to 0$ and $\chi = x/\epsilon \gg 1$ as $\epsilon \to 0$ hence $\tilde{y}_0(\chi) = A(1 - e^{-\chi}) \sim A$ as $\epsilon \to 0$ and $\chi \to \infty$. The solutions agree iff $A = 1 - a$.

Thus we have

$$y \sim \begin{cases} a(x-1) + 1, & \text{if } x \ll \epsilon^{1/4}; \\ (1-a)(1 - e^{-x/\epsilon}), & \text{if } x \gg \epsilon. \end{cases} \tag{8.13}$$

Note: The crucial point in matched asymptotic expansions is that the two solutions agree asymptotically in an interval, not at a point.

Now suppose we had assumed that $x = 0$ was in the outer region and that $x = 1$ was in the boundary layer. Then our outer solution would have been

$$y_0(x) = ax, \tag{8.14}$$

and we'd have use the inner variable

$$\chi = \frac{1 - x}{\delta(\epsilon)}. \tag{8.15}$$

Note that $0 \le x \le 1$ implies that $\chi \ge 0$ and the boundary $x = 1$ corresponds to $\chi = 0$. The ODE for $\tilde{y}(\chi) = y(x)$ is

$$\tilde{y}'' - \frac{\delta}{\epsilon}\tilde{y} = \frac{\delta^2}{\epsilon}a. \tag{8.16}$$

As before we should take $\delta = \epsilon$ leading to the $\mathcal{O}(1)$ solution $\tilde{y}_0 = Ae^{\chi} + B$. Applying the boundary condition $\tilde{y}_0(0) = 1$ (recall $x = 1$ corresponds to $\chi = 0$), gives

$$\tilde{y}_0 = A(e^{\chi} - 1) + 1. \tag{8.17}$$

Now we can't match the solutions because as $\chi \to \infty$, which corresponds to x moving away from $x = 1$ into the interior of the domain $0 \leq x \leq 1$, the inner solution blows up while the outer solution $y_0 \to a$ as $x \to 1$.

Note also that $\epsilon < 0$ changes the location of the boundary layer. If $\epsilon < 0$ then assuming the boundary layer is near $x = 0$ leads again to $\chi = x/\epsilon$. This time as $\epsilon \to 0$ with x in the matching region, $\chi \to -\infty$ now $e^{-\chi}$ blows up. Assuming the boundary layer is near $x = 1$ leads to solutions we can properly match.

Example 8.0.2 *Example 2: (From Bender & Orzag). Find a leading-order approximation to $y(0)$ where $y(x)$ is the solution of*

$$\begin{aligned}(x - \epsilon y)y' + xy &= e^{-x},\\ y(1) &= 1/e.\end{aligned} \tag{8.18}$$

In the outer region, where can set $\epsilon = 0$, $y \approx y_{out}$ where

$$xy'_{out} + xy_{out} = e^{-x}, \tag{8.19}$$

which has the general solution

$$y_{out} = Ae^{-x} + \ln xe^{-x}. \tag{8.20}$$

Now we need to determine whether or not $x = 1$ is in the outer region. If so we can use the initial condition $y(1) = 1/e$ to determine A. From the original ODE

$$y' = \frac{e^{-x} - xy}{x - \epsilon y} \tag{8.21}$$

so, at $x = 1$

$$y'(1) = \frac{e^{-1} - 1/e}{x - \epsilon/e} = 0. \tag{8.22}$$

In particular, $y'(1)$ is $\mathcal{O}(1)$, i.e., $y'(1)$ does not blow up as $\epsilon \to$. Hence we can conclude that $x = 1$ is in the outer region. Setting $y_{out}(1) = 1/e$ gives

$$y_{out}(x) = (1 + \ln x)e^{-x}. \tag{8.23}$$

Where is the inner region? In the outer region $\epsilon yy'$ is negligible compared with the other terms. You leave the outer region when it is no longer negligible, e.g., when $\epsilon yy'$ is comparable to e^{-x}. We can use y_{out} as an approximation and see when $\epsilon y_{out}y'_{out}$ is comparable to e^{-x}. Thus, we need to determine when

$$\epsilon y_{out}y'_{out} = \epsilon(1 + \ln x)e^{-x}\left[\frac{1}{x}e^{-x} - (1 + \ln x)e^{-x}\right] \approx e^{-x}. \tag{8.24}$$

If $x = \mathcal{O}(1)$ the l.h.s is $\mathcal{O}(\epsilon)$ while the r.h.s. is $O(1)$ so we clearly need $x \ll 1$. If $x \ll 1$ we have $1/x \gg |\ln x|$ and $e^{-x} \approx 1$, so we need to find x such that

$$\epsilon\left|\frac{\ln x}{x}\right| \approx 1. \tag{8.25}$$

This gives $x = \mathcal{O}(\epsilon \ln \epsilon)$ (exercise).

In the inner region x is small so $e^{-x} \approx 1$ and $xy \ll xy'$ so we expect that the inner solution is governed by

$$(x - \epsilon y_{in}) y_{in}' = 1. \tag{8.26}$$

More systematically, let

$$\chi = \frac{x}{\delta(\epsilon)} \tag{8.27}$$

where $\delta \to 0$ as $\epsilon \to 0$. The ODE gives

$$\Big(\delta\chi - \epsilon\tilde{y}\Big)\frac{1}{\delta}\tilde{y}' + \delta\chi\tilde{y} = e^{-\delta\chi}. \tag{8.28}$$

The right hand side is approximately 1. Since we need to retain the $\tilde{y}\tilde{y}'$ term take $\delta = \epsilon$. This results in

$$\Big(\chi - \tilde{y}\Big)\tilde{y}' \approx 1, \tag{8.29}$$

as above. To solve this write it as

$$\frac{d\chi}{d\tilde{y}} = \chi - \tilde{y}. \tag{8.30}$$

We now have a linear equation for $\chi(\tilde{y})$ which can be solved. This results in

$$\chi = \tilde{y} + 1 + ae^{\tilde{y}}, \tag{8.31}$$

or

$$x = \epsilon(1 + y_{in}) + ce^{y_{in}}. \tag{8.32}$$

This implicitely gives $y_{in}(x)$ if we can determine the value of c. We do this by matching the inner and outer solutions.

Claim:

1. $\frac{\epsilon^{1/2}}{\epsilon|\ln \epsilon|} \to \infty$ as $\epsilon \to 0$ so $x = \mathcal{O}(\epsilon^{1/2})$ is in the outer region.

2. The inner region is given by $x \ll 1$ so $x = \mathcal{O}(\epsilon^{1/2})$ is in the inner region as well.

When $x \ll 1$ the outer solution is

$$y_{out} \approx 1 + \ln x, \tag{8.33}$$

while for $x = \mathcal{O}(\epsilon^{1/2})$ we have $x \approx ce^{y_{in}}$ so $y_{in} \approx \ln x - \ln c$. Matching gives $c = e^{-1}$ so in the inner region $x \approx \epsilon(1 + y_{in}) + e^{y_{in}-1}$, which can be solved implicitly for $y_{in}(0)$.

Chapter 9

Asymptotics used to derive model equations: derivation of the Korteweg-de Vries equation for internal waves

9.1 Introduction

So far we have concentrated on using perturbation and asymptotic methods to find approximate solutions of difficult problems. An equally important use of asymptotic methods is to derive approximate mathematical models, the solution of which provides a good approximation to some of the solutions of the original set of equations. The word *some* in the preceding sentence is important as the asymptotic procedure used to derive the simplified set of equations is based on the introduction of small parameters that come from scaling the original variables. The scaling is determined by the type of phenomena one wishes to investigate and it identifies terms which are not important in a first approximation. The scaling choice leads to approximate asymptotic models that are simpler and which isolate certain types of behaviour in the original system.

The Korteweg-de Vries, or KdV, equation was originally derived in 1895 by Diederik Johannes Korteweg and his PhD student Gustav de Vries in the context of surface water waves. The KdV equation is restricted to uni-directional wave propagation (this is one way it simplies your problem: it eliminates waves propagating in other directions). It has been derived in many different physical contexts, such as waves in beams or rods, nonlinear electric lines, blood pressure waves, large scale planetary Rossby waves in the atmosphere and oceans, nonlinear spring-mass systems and many others. We will consider the case of horizontally propagating internal gravity waves as illustrated in Figure 9.1.

Consider an ideal inviscid, incompressible, density stratified fluid in a non-rotating reference frame. The equations of motion are

$$\begin{aligned} \rho\frac{D\vec{u}}{Dt} &= -\vec{\nabla}p - \rho g\hat{k}, \\ \frac{D\rho}{Dt} &= 0, \\ \vec{\nabla}\cdot\vec{u} &= 0. \end{aligned} \tag{9.1}$$

Here $\rho(\vec{x},t)$ is the fluid density, $\vec{u} = (u(\vec{x},t), v(\vec{x},t), w(\vec{x},t))$ is the fluid velocity, p is the pressure

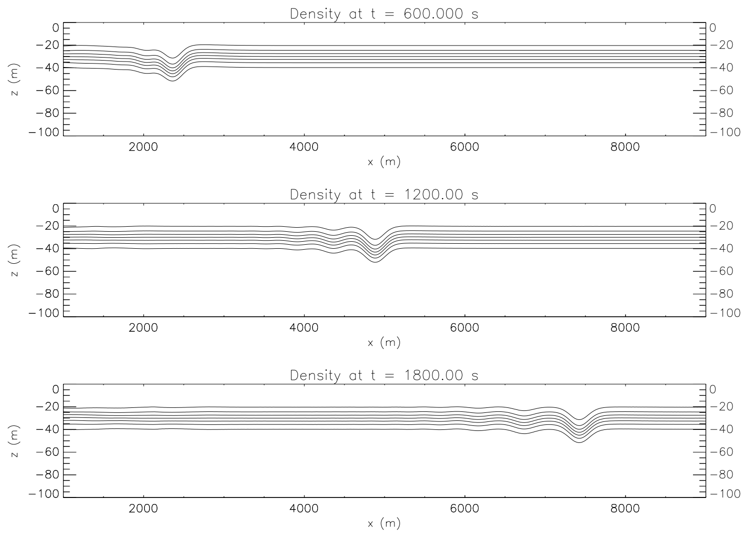

Figure 9.1: Internal gravity waves propagating horizontally in a density stratified fluid. Shown are density contour plots at different times.

field, g is the gravitational acceleration and $\hat{k}$ is the unit vector in the positive z direction with z pointing upward. The first of these equations is called the momentum equation. It is a statement of Newton's second law applied to a 'fluid particle': mass per unit volume (ρ) times the fluid particle's acceleration ($D\vec{u}/Dt$) is equal to the net force per unit volume acting on the fluid particle. The forces include those associated with pressure gradients ($\vec{\nabla}p$) and gravitational forces ($\rho g\hat{k}$). The third equation is the incompressibility condition. It says that the volume of a parcel of fluid is constant in time. The intermediate equation states that the density of a fluid particle is constant in time, which is a consequence of conservation of mass combined with incompressibility and the fact that we are neglecting diffusion processes (e.g., diffusion of heat and, in the ocean, salts).

The material derivative $D/Dt = \partial/\partial t + \vec{u}\cdot\vec{\nabla}$ gives the rate of change moving with a particle. To understand how it arises, consider a particle at position $\vec{x}(t)$ at time t which has velocity $\vec{u}_L(t) = \vec{u}(\vec{x}(t), t)$. Its acceleration is

$$\begin{aligned}\vec{a}_L(t) &= \frac{d\vec{u}_L}{dt}(t) = \Big(\frac{d\vec{x}}{dt}\cdot\vec{\nabla}\Big)\vec{u}(\vec{x}(t),t) + \frac{\partial\vec{u}}{\partial t},\\ &= \frac{D\vec{u}}{Dt}(\vec{x}(t),t),\end{aligned} \tag{9.2}$$

To simplify matters we will consider two-dimensional motion in a vertical plane. Taking $\vec{u} = (u, w)$ and $\vec{x} = (x, z)$ the equations of motion in component form are

$$\begin{aligned}\rho\frac{Du}{Dt} &= -p_x,\\ \rho\frac{Dw}{Dt} &= -p_z - \rho g,\\ \frac{D\rho}{Dt} &= 0,\\ u_x + w_z &= 0.\end{aligned} \tag{9.3}$$

There is one more simplification we will make before deriving the KdV equation. In the oceans and lakes the density ρ varies only slightly from its mean value ρ_o. If we set $\rho = \rho_o + \rho_v$, ρ_v/ρ_o

has a typical value of about 0.02 in the oceans and is much smaller in lakes. Hence $\rho D\vec{u}/Dt = (\rho_o + \rho_v)D\vec{u}/Dt \approx \rho_o D\vec{u}/Dt$. No matter what the solution of the equations is, $\rho_v D\vec{u}/Dt$ will be at most a few percent of $\rho_o D\vec{u}/Dt$. This leads to the final set of equations

$$\rho_o \frac{Du}{Dt} = -p_x, \tag{9.4}$$

$$\rho_o \frac{Dw}{Dt} = -p_z - \rho g, \tag{9.5}$$

$$\frac{D\rho}{Dt} = 0, \tag{9.6}$$

$$u_x + w_z = 0. \tag{9.7}$$

9.1.1 Streamfunction Formulation

From (9.7) there exists a streamfunction ψ such that $(u, w) = (\psi_z, -\psi_x)$. The curl of the momentum equation then gives the vorticity equation

$$\frac{\partial}{\partial t}\nabla^2\psi - \frac{g\rho_x}{\rho_o} = J(\psi, \nabla^2\psi), \tag{9.8}$$

where $\nabla^2\psi = u_z - w_x$ is the vorticity and the Jacobian operator J is defined by $J(A, B) = A_x B_z - A_z B_x$. Note that $J(\psi, A) = -(u, w,) \cdot \vec{\nabla} A$, so $\partial(\cdot)/\partial t - J(\psi, \cdot) = D(\cdot)/Dt$.

The density equation can be written as

$$\frac{\partial}{\partial t}\rho = J(\psi, \rho), \tag{9.9}$$

We now we split the velocity and density fields into a background, undisturbed, state plus a perturbation. We assume the undisturbed state to be a state of rest ($\vec{u} = 0$) with a stable stratification $\rho = \bar{\rho}(z)$. Thus, letting $\rho = \bar{\rho}(z) + \rho'(x, z, t)$, the density becomes

$$\frac{\partial \rho'}{\partial t} - \frac{d\bar{\rho}}{dz}\psi_x = J(\psi, \rho'). \tag{9.10}$$

Defining

$$b = \frac{g\rho'}{\rho_o}, \tag{9.11}$$

we can write the vorticity and density equations as

$$\frac{\partial}{\partial t}\nabla^2\psi - b_x = J(\psi, \nabla^2\psi), \tag{9.12}$$

$$\frac{\partial b}{\partial t} + N^2(z)\psi_x = J(\psi, b), \tag{9.13}$$

where

$$N^2(z) = -\frac{g}{\rho_o}\frac{d\bar{\rho}}{dz} \geq 0. \tag{9.14}$$

N is called the *buoyancy frequency*. For a stable stratification, with $\bar{\rho}$ a non-increasing function, N is real. It gives the frequency of oscillation of a fluid parcel that is displaced vertically an infinitesimally small distance (i.e., in linear theory).

9.1.2 Boundary conditions

We will need some boundary conditions. We assume the fluid is bounded by rigid horizontal boundaries at $z = -H$ and 0. The latter condition eliminated the complication of surface waves. The vertical velocity $w = -\psi_x$ at the boundaries must be zero, hence ψ is constant along $z = -H, 0$. Now $\int_{-H}^{0} u \, dz = \int_{-H}^{0} \psi_z \, dz = \psi(x, 0, t) - \psi(x, -H, t)$ which is independent of x as ψ is independent of x along $z = -H, 0$. Assuming our disturbance is of finite extent, as $x \to \infty$ $u \to 0$ from which we can conclude that $\psi(x, 0, t) = \psi(x, -H, t)$. Since physically only derivatives of ψ mean something, we can take

$$\psi = 0 \text{ on } z = -H, 0. \tag{9.15}$$

For the density field, since fluid at the boundaries stays there ($w = 0$), the density perturbation at the boundaries will be zero. Hence

$$b = 0 \text{ on } z = -H, 0. \tag{9.16}$$

9.2 Nondimensionalization and introduction of two small parameters

We will now work with equations (9.12)–(9.14). To scale them we need to make some decisions regarding the type of solutions we are interested in. There are many observations of internal waves propagating in the horizontal direction (see Figure 9.1). They often have the property that their horizontal length scale is long compared with the water depth and their wave amplitude is not too large. We also consider uni-directional propagating wave, i.e., all waves are propagating in the positive x direction. This will be the phenomena we focus on.

Nondimensionalization:

- **Vertical length scale**: water depth H.
- **Horizontal length scale**: typical wavelength L
- **Buoyancy frequency**: typical value N_o.
- **Fact**: wave propagation speeds are determined by $N(z)$ and H. Thus, $c_o = N_o H$ is the appropriate scaling for the wave propagation speed.
- **Time scale**: $T = L/c_o = L/(N_o H)$ is the time to travel distance L at phase speed c_o.

Thus, we set

$$\begin{aligned} (x, z, t) &= (L\tilde{x}, H\tilde{z}, \frac{L}{N_o H}\tilde{t}), \\ N(z) &= N_o \tilde{N}(\tilde{z}), \end{aligned} \tag{9.17}$$

- **Horizontal velocity scale**: We assume a small amplitude perturbation to the undisturbed state. By small, we mean the wave induced horizontal velocity u is small compared with the horizontal propagation speed. Hence we set

$$u = \epsilon c_o \tilde{u} \tag{9.18}$$

where ϵ is a small nondimensional parameter measuring the wave amplitude.

- **Streamfunction**: Setting $\psi = \epsilon\Psi\tilde{\psi}$ and using $u = \psi_z$, we find

$$u = \epsilon c_o \tilde{u} = \psi_z = \frac{\epsilon\Psi}{H}\tilde{\psi}_{\tilde{z}}, \tag{9.19}$$

hence we should choose

$$\Psi = c_o H, \tag{9.20}$$

so that $\tilde{u} = \tilde{\psi}_{\tilde{z}}$.

- **Vertical velocity scale**: From $u_x + w_z = 0$ the vertical velocity should be scaled by

$$w = \epsilon \frac{H}{L} c_o \tilde{w}. \tag{9.21}$$

It is easily verified that $\tilde{w} = -\tilde{\psi}_{\tilde{x}}$.

- **Scaling for b**: Since b is proportional to the density perturbation which is small, let $b = \epsilon B\tilde{b}$. In (9.13) the two linear terms on the left hand side should dominate (this is a choice – quadratic terms should be negligible at leading order since the perturbation is small), thus, we need

$$\frac{B}{T} = N_o^2 \frac{\Psi}{L}, \tag{9.22}$$

which gives

$$b = \epsilon N_o^2 \frac{L}{c_o}\frac{c_o H}{H}\tilde{b} = \epsilon N_o^2 H\tilde{b}. \tag{9.23}$$

With these scalings, the governing equations (9.12)–(9.13) become, after dropping the tildes,

$$\frac{\partial}{\partial t}\psi_{zz} - b_x = \epsilon J(\psi, \psi_{zz}) - \mu\frac{\partial}{\partial t}\psi_{xx} + \epsilon\mu J(\psi, \psi_{xx}), \tag{9.24}$$

$$\frac{\partial}{\partial t}b + N^2(z)\psi_x = \epsilon J(\psi, b), \tag{9.25}$$

where $\mu = (H/L)^2$. In Figure 9.1 $\mu \approx 0.04$. In the following we will assume that the horizontal length scale is large compared with the water depth, that is, we *assume* that $H \ll L$, i.e., $\mu \ll 1$. Thus, we have two small parameters, ϵ and μ.

9.3 Asymptotic expansion

The KdV equation is an evolution for small amplitude, long waves, i.e., waves which are long compared to the water depth. Since there are two small parameters, ϵ and μ in the nondimensionalized equations, we expand ψ and b in powers of both ϵ and μ:

$$\begin{aligned} \psi &\sim \psi^{(0)} + \epsilon\psi^{(1,0)} + \mu\psi^{(0,1)} + \cdots + \epsilon^i\mu^j\psi^{(i,j)} + \cdots, \\ b &\sim b^{(0)} + \epsilon b^{(1,0)} + \mu b^{(0,1)} + \cdots + \epsilon^i\mu^j b^{(i,j)} + \cdots, \end{aligned} \tag{9.26}$$

as $\epsilon, \mu \to 0$.

9.3.1 The $O(1)$ problem

At leading order we have

$$\frac{\partial}{\partial t}\psi^{(0)}_{zz} - b^{(0)}_x = 0, \tag{9.27}$$

$$\frac{\partial}{\partial t}b^{(0)} + N^2(z)\psi^{(0)}_x = 0, \tag{9.28}$$

from which we get

$$\frac{\partial^2}{\partial t^2}\psi^{(0)}_{zz} + N^2(z)\psi^{(0)}_{xx} = 0. \tag{9.29}$$

We now look for separable solutions of the form

$$\psi^{(0)} = B(x,t)\phi(z). \tag{9.30}$$

This is motivated by our interest in waves which are propagating in the x direction along the wave guide bounded by the surface and the bottom. The function $\phi(z)$ will give the vertical structure of the waves while $B(x,t)$ will satisfy a wave equation describing the propagation and evolution of the wave in the x direction.

Substituting (9.30) into (9.29) results in

$$B_{tt}\phi'' + N^2 B_{xx}\phi = 0,$$

or

$$\frac{B_{tt}}{B_{xx}} = -N^2\frac{\phi}{\phi_{zz}}. \tag{9.31}$$

This says that a function of x and t is equal to a function of z. Hence both sides must be equal to a constant, say c^2, giving

$$B_{tt} - c^2 B_{xx} = 0, \tag{9.32}$$

$$\phi'' + \frac{N^2}{c^2}\phi = 0. \tag{9.33}$$

The boundary conditions (see (9.15)) are

$$\phi(-1) = \phi(0) = 0, \tag{9.34}$$

since the nondimensional water depth is 1. Equations (9.33)–(9.34) represent an eigenvalue problem for ϕ with eigenvalue c. Since $N^2 \geq 0$ there are an infinite number of discrete solutions (ϕ_i, c_i) with $c_i^2 > 0$ which can be ordered such that $c_1 > c_2 > c_3 > \cdots > 0$ with 0 as a cluster point. Solutions for different values of c correspond to different wave modes. From (9.32) we see that c is the propagation speed of the waves. At this order all waves propagate with speed c and waves can propagate in either direction. We will now restrict attention to rightward propagating waves only, so that $B_t + cB_x = 0$. From (9.28)

$$b^{(0)}_t = -N^2 B_x\phi = \frac{N^2}{c}B_t\phi, \tag{9.35}$$

hence

$$b^{(0)} = \frac{B}{c}N^2(z)\phi(z) = -\frac{B}{c}\phi''. \tag{9.36}$$

9.3.2 The $O(\epsilon)$ problem

At $O(\epsilon)$ equations (9.24)–(9.25) give

$$\frac{\partial}{\partial t}\psi^{(1,0)}_{zz} - b^{(1,0)}_x = J(\psi^{(0)}, \psi^{(0)}_{zz}), \tag{9.37}$$

$$\frac{\partial}{\partial t}b^{(1,0)} + N^2(z)\psi^{(1,0)}_x = J(\psi^{(0)}, b^{(0)}). \tag{9.38}$$

From the solution to the $O(1)$ problem

$$J(\psi^{(0)}, \psi^{(0)}_{zz}) = BB_x\Big(\phi\phi''' - \phi'\phi''\Big), \tag{9.39}$$

$$J(\psi^{(0)}, b^{(0)}) = -cBB_x\Big(\phi\phi''' - \phi'\phi''\Big). \tag{9.40}$$

We again look for separable solutions, i.e., we assume $\psi^{(1,0)}$ and $b^{(1,0)}$ can both be expressed as a function of x and t multiplying a function of z. The form of the nonlinear forcing terms suggests the ansatz

$$\begin{aligned} \psi^{(1,0)} &= B^2\phi^{(1,0)}, \\ b^{(1,0)} &= B^2D^{(1,0)}. \end{aligned} \tag{9.41}$$

Substituting into (9.37)–(9.38) gives

$$\begin{aligned} 2BB_t\phi^{(1,0)}_{zz} - 2BB_xD^{(1,0)} &= BB_x\Big(\phi\phi''' - \phi'\phi''\Big), \\ 2BB_tD^{(1,0)} + 2BB_xN^2\phi^{(1,0)} &= -cBB_x\Big(\phi\phi''' - \phi'\phi''\Big), \end{aligned} \tag{9.42}$$

Using $B_t = -cB_x$:

$$\begin{aligned} -c\phi^{(1,0)}_{zz} - D^{(1,0)} &= \frac{1}{2}\Big(\phi\phi''' - \phi'\phi''\Big), \\ -cD^{(1,0)} + N^2\phi^{(1,0)} &= -\frac{c}{2}\Big(\phi\phi''' - \phi'\phi''\Big), \end{aligned} \tag{9.43}$$

from which, after eliminating $D^{(1,0)}$ we have

$$\phi^{(1,0)}_{zz} + \frac{N^2}{c^2}\phi^{(1,0)} = -\frac{1}{c}\Big(\phi\phi''' - \phi'\phi''\Big). \tag{9.44}$$

The boundary conditions are again $\phi^{(1,0)} = 0$ at $z = -1, 0$. This is an inhomogeneous version of (9.33), the ODE we had to solve in the $O(1)$ problem.

9.3.3 The problem

Now suppose we multiply the left handside of (9.44) by ϕ and integrate from -1 to 0. We have

$$\begin{aligned} \int_{-1}^{0} \phi\Big(\phi^{(1,0)}_{zz} + \frac{N^2}{c^2}\phi^{(1,0)}\Big)\,dz &= \int_{-1}^{0}\Big(\phi\phi^{(1,0)}_{zz} + \frac{N^2}{c^2}\phi\phi^{(1,0)}\Big)\,dz, \\ &= \int_{-1}^{0}\Big(\phi\phi^{(1,0)}_{zz} - \phi_{zz}\phi^{(1,0)}\Big)\,dz, \\ &= -\int_{-1}^{0}\Big(\phi'\phi^{(1,0)}_{z} - \phi'\phi^{(1,0)}_{z}\Big)\,dz, \\ &= 0 \end{aligned} \tag{9.45}$$

where we have used (9.33) in the secon step and have integrated by parts in the last step using the fact that ϕ and $\phi^{(1,0)}$ are zero at the boundaries.

If we multiply the right-hand side of (9.44) by ϕ and integrate from -1 to 0 we obtain

$$\int_{-1}^{0} \frac{1}{c}(\phi\phi''' - \phi'\phi'')\phi \, dz = \frac{3}{2c}\int_{-1}^{0} \phi'^3 \, dz, \tag{9.46}$$

after a couple of integrations by parts. In general, this is not going to be zero (it is zero if N is symmetric about the mid-depth). Thus, in general, there is no solution to (9.44). This is an example of resonant forcing and to proceed we need to find a way to eliminate the resonant part of the forcing.

9.3.4 The fix

To fix this problem we have to reconsider the $O(1)$ problem. At leading order we found that $B(x,t)$ satisfies the linear long wave equation $B_t + cB_x = 0$. This says that a wave with any shape propagates without changing shape. In reality the wave will change shape due to nonlinear and dispersive effects. This can be seen in Figure 9.1. Thus, we need to modify the wave equation to introduce small corrections via

$$B_t \sim -cB_x - \epsilon R(x,t) - \mu Q(x,t) + h.o.t., \quad \text{as } \epsilon, \mu \to 0, \tag{9.47}$$

Consider the $O(1)$ problem again. The governing equations are

$$\frac{\partial}{\partial t}\psi_{zz}^{(0)} - b_x^{(0)} = 0, \tag{9.48}$$

$$\frac{\partial b^{(0)}}{\partial t} + N^2(z)\psi_x^{(0)} = 0. \tag{9.49}$$

Using $\psi^{(0)} = B(x,t)\phi$ and $b^{(0)} = \frac{N^2}{c}B\phi$ and using (9.47) these equations give,

$$-(c\phi'' + \frac{N^2}{c}\phi)B_x - \epsilon R\phi'' - \mu Q\phi'' + \cdots = 0, \tag{9.50}$$

$$(-N^2\phi + N^2\phi)B_x - \epsilon\frac{N^2}{c}R\phi - \mu\frac{N^2}{c}Q\phi + \cdots = 0. \tag{9.51}$$

We have not changed ϕ so it still satisfies (9.33). Thus, the leading-order terms are zero. The higher-order terms become part of the higher-order problems. In particular, the $O(\epsilon)$ term belongs in the $O(\epsilon)$ problem and the $O(\mu)$ term belongs in the $O(\mu)$ problem.

9.3.5 The $O(\epsilon)$ problem revisited

Incorporating the left-over $O(\epsilon)$ terms from the $O(1)$ problem into the $O(\epsilon)$ problem gives

$$-R\phi'' + \frac{\partial}{\partial t}\psi_{zz}^{(1,0)} - b_x^{(1,0)} = BB_x\Big(\phi\phi''' - \phi'\phi''\Big), \tag{9.52}$$

$$-\frac{N^2}{c}R\phi + \frac{\partial b^{(1,0)}}{\partial t} + N^2(z)\psi_x^{(1,0)} = -cBB_x\Big(\phi\phi''' - \phi'\phi''\Big). \tag{9.53}$$

As before, we use the ansatz (9.41) and clearly we should take $R \propto BB_x$. Setting $R = \alpha BB_x$ we have

$$-c\phi_{zz}^{(1,0)} - D^{(1,0)} = \frac{\alpha}{2}\phi'' + \frac{1}{2}\Big(\phi\phi''' - \phi'\phi''\Big), \tag{9.54}$$

$$-cD^{(1,0)} + N^2(z)\phi^{(1,0)} = \frac{\alpha}{2c}N^2\phi - \frac{c}{2}\Big(\phi\phi''' - \phi'\phi''\Big), \tag{9.55}$$

from which, after using $N^2\phi = -c^2\phi''$, we get the revised ODE for $\phi^{(1,0)}$

$$\phi_{zz}^{(1,0)} + \frac{N^2}{c^2}\phi^{(1,0)} = -\frac{\alpha}{c}\phi'' - \frac{1}{c}\Big(\phi\phi''' - \phi'\phi''\Big). \tag{9.56}$$

We still have the solvability condition that the integral of the right-hand side multiplied by ϕ should be zero. This is used to determine the value of α. The result is

$$\alpha = \frac{3}{2}\frac{\int_{-1}^{0}\phi'^3\,dz}{\int_{-1}^{0}\phi'^2\,dz}. \tag{9.57}$$

A similar procedure for the $O(\mu)$ problem yields $Q = \beta B_{xxx}$ where

$$\beta = \frac{c}{2}\frac{\int_{-1}^{0}\phi^2\,dz}{\int_{-1}^{0}\phi'^2\,dz}, \tag{9.58}$$

which is always positive. The final equation for B, to $O(\epsilon,\mu)$, is

$$B_t + cB_x + \epsilon\alpha BB_x + \mu\beta B_{xxx} = 0, \tag{9.59}$$

which is the KdV equation. Attempts to solve higher-order problems lead to the introduction of higher-order corrections to the evolution equation for B.

Appendix A: USEFULL FORMULAE

Trigonometric Identities:

$$\begin{aligned}
\sin^3(t) &= \frac{3}{4}\sin(t) - \frac{1}{4}\sin(3t),\\
\cos^3(t) &= \frac{3}{4}\cos(t) + \frac{1}{4}\cos(3t),\\
\sin^5(t) &= \frac{5}{8}\sin(t) - \frac{5}{16}\sin(3t) + \frac{1}{16}\sin(5t),\\
\cos^5(t) &= \frac{5}{8}\cos(t) + \frac{5}{16}\cos(3t) + \frac{1}{16}\cos(5t),\\
(A\cos t + B\sin t)^3 &= \frac{3}{4}A(A^2+B^2)\cos t + \frac{3}{4}B(A^2+B^2)\sin t\\
&\quad + \frac{1}{4}A(A^2-3B^2)\cos 3t - \frac{1}{4}B(B^2-3A^2)\sin 3t
\end{aligned}$$

$$\begin{aligned}
\sin(nt)\cos(mt) &= \frac{\sin((n+m)t) + \sin((n-m)t)}{2},\\
\sin(nt)\sin(mt) &= \frac{\cos((n-m)t) - \cos((n+m)t)}{2},\\
\cos(nt)\cos(mt) &= \frac{\cos((n+m)t) + \cos((n-m)t)}{2},
\end{aligned}$$

Solutions of homogeneous ODEs for $y(x)$:

$$\begin{aligned}
y'' + \frac{a}{x}y' + \frac{b}{x^2}y = 0 \qquad &\rightarrow \qquad \text{try } y \propto x^n,\\
y' = \frac{1}{4}y(4-y) \qquad &\rightarrow \qquad \frac{dx}{dy} = (\frac{1}{y} + \frac{1}{4-y})
\end{aligned}$$

Particular solutions of common forced ODEs:

$$\begin{aligned}
y'' + \lambda^2 y = \sin\lambda t \qquad & y_p = -\frac{1}{2\lambda} t\cos\lambda t, \\
y'' + \lambda^2 y = \cos\lambda t \qquad & y_p = \frac{1}{2\lambda} t\sin\lambda t, \\
y'' + \lambda^2 y = \sin\alpha t \qquad & y_p = \frac{1}{\lambda^2-\alpha^2}\sin\alpha t \quad \text{for } \lambda \neq \alpha, \\
y'' + \lambda^2 y = \cos\alpha t \qquad & y_p = \frac{1}{\lambda^2-\alpha^2}\cos\alpha t \quad \text{for } \lambda \neq \alpha, \\
y' - \lambda y = e^{\lambda t} \qquad & y_p = te^{\lambda t}, \\
y'' - \lambda y' = 1 \qquad & y_p = -\frac{t}{\lambda}, \\
y'' - \lambda y' = e^{\lambda t} \qquad & y_p = \frac{t}{\lambda}e^{\lambda t} - \frac{1}{\lambda^2}e^{\lambda t}
\end{aligned}$$

Taylor Series:

$$\tanh(x) = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots, \tag{9.60}$$

Expansions:

$$\begin{aligned}
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3, \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4, \\
(a+b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5, \\
(a_o + a_1\mu + a_2\mu^2 + \cdots)^2 &= a_o^2 + 2a_0a_1\mu + (2a_oa_2 + a_1^2)\mu^2 + \cdots, \\
(a_o + a_1\mu + a_2\mu^2 + \cdots)^3 &= a_o^3 + 3a_0^2a_1\mu + (3a_o^2a_2 + 3a_oa_1^2)\mu^2 + \cdots, \\
(a_o + a_1\mu + a_2\mu^2 + \cdots)^4 &= a_o^4 + 4a_0^3a_1\mu + (4a_o^3a_2 + 6a_o^2a_1^2)\mu^2 + \cdots, \\
(a_o + a_1\mu + a_2\mu^2 + \cdots)^5 &= a_o^5 + 5a_0^4a_1\mu + (5a_o^4a_2 + 10a_o^3a_1^2)\mu^2 + \cdots.
\end{aligned}$$

Methods:

- Lighthill: $y = Y(X)$ is replaced with

$$x = X + \epsilon x_1(X) + \cdots.$$

- Pritulo: $y = y_o(x) + \epsilon y_1(x) + \cdots$ is replaced by

$$Y_o(X) + \epsilon Y_1(X) + \cdots \quad \text{with} \quad x = X + \epsilon x_1(X) + \cdots.$$

- MSC and Poincaré-Linstedt: $\tau = \omega(\epsilon)t$

Solutions to Selected Problems

Problems from chapter 2

1(a). Have a second order polynomial, hence two roots to find. Setting $\epsilon = 0$ gives two distinct roots -6 and 1 hence expand in powers of ϵ. Get

$$x^{(1)} = -6 - \frac{3}{7}\epsilon - \frac{12}{7^3}\epsilon^2 + O(\epsilon^3),$$
$$x^{(2)} = -6 - \frac{4}{7}\epsilon + \frac{12}{7^3}\epsilon^2 + O(\epsilon^3).$$

1(c). Polynomial of degree three, hence need to find three roots. Setting $\epsilon = 0$ gives a double root at $x_0 = 1$ and a single root $x_0 = -2$. Near the single root expand in powers of ϵ to find $x^{(1)} = -2 + \epsilon/9 + (2/243)\epsilon^2 + O(\epsilon^3)$. Near the double root expand in powers of $\epsilon^{1/2}$ to get $x^{(2,3}) = 1 \pm i\epsilon^{1/2}/\sqrt{3} + \epsilon/18 + O(\epsilon^{3/2})$.

1(e). Need to find three roots. Setting $\epsilon = 0$ gives $x_0 = -1$ as a double root. To find the two roots near $x_0 = -1$ expand in powers of $\epsilon^{1/2}$. Find $x^{1/2} = -1 \pm \epsilon^{1/2} - 3\epsilon/2 + O(\epsilon^{3/2})$. For the third root dominant balance is between ϵx^3 and x^2 so $\epsilon x^3 \approx -x^2$ or $x \approx -1/\epsilon$. Thus set $x = -1/\epsilon + x_1 + x_2\epsilon + x_3\epsilon^2 + \cdots$. Fine $x^{(3)} = -1/\epsilon + 2 + 3\epsilon + O(\epsilon^2)$.

1(g). Need to find four roots. Setting $\epsilon = 0$ give a quadratic equation with two distinct roots. For these expand in powers of ϵ giving $x^(1) = 1+2\epsilon+18\epsilon^2+O(\epsilon^3)$ and $x^(2) = 2-24\epsilon+488\epsilon^2+O(\epsilon^3)$. For the other two roots the dominant balance is between ϵx^4 and x^2 which gives $x \approx \pm i\epsilon^{-1/2}$. Let $\mu = \epsilon^{1/2}$ and $y = \mu x = y_0 + y_1\mu + y_2\mu^2 + \cdots$. Get $y^4 + \mu y^3 + y^2 - 3\mu y + 2\mu^2 = 0$. The leading order problem gives $y_0 = \pm i$ and $y_0 = 0$ as a double root. Only first two of interest. Since $\pm i$ are distinct single roots expand in powers of μ. Find $y = \pm i - 2\mu \pm 3i\mu^2 + O(\mu^3)$ or $x^{3,4} = \pm i/\epsilon^{1/2} - 2 \pm 3i\epsilon^{1/2} + O(\epsilon)$.