Extending Noether's Theorem

Poisson-Lie Symmetries in Field Theory

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Abstract

We extend Noether's theorem to symmetries governed by Poisson-Lie groups (the classical picture of quantum groups). We show they are not symplectomorphisms but instead transform the symplectic form in a characteristic way, and arise as the natural symmetries of a class of curved phase spaces. The abstract formalism is developed in the covariant phase space framework and applied to the case of a particle, a string, and 3D gravity.

1. What Is A "Symmetry", Really?

Physical Requirement: Dynamics are preserved

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Main Use: Scalar conserved quantities

1a. Noether Symmetries In The Covariant Phase Space Formalism

Input

$$\varphi \in \mathcal{F}$$
$$\mathcal{L}(\varphi, \partial\varphi, \dots)$$
$$\mathfrak{g} : \mathfrak{g} \mapsto \mathfrak{X}(\mathcal{F})$$

$\mathbb{d}\mathcal{L} = \mathcal{E}_I \mathbb{d}\varphi^I + \nabla_\mu \theta^\mu$ ⟹

CPS Data

$$(\mathcal{F}_{\mathrm{EL}}, \Omega)$$
$$\Omega := \int_C n_\mu \mathbb{d}\theta^\mu$$
$$\mathbb{i}_{\mathfrak{g}(\alpha)}\Omega =: \langle \mathbb{Q}, \alpha \rangle$$

$\mathbb{L}_{\mathfrak{g}(\alpha)}\mathcal{L} = \nabla_\mu \langle R^\mu, \alpha \rangle$
Noethers Theorem ⟹

Output

$$\mathbb{Q} \approx \mathbb{d}q$$
$$j^\mu := \mathbb{i}_{\mathfrak{g}(\cdot)}\theta^\mu - R^\mu$$
$$q = \int_C n_\mu j^\mu$$

Key Points

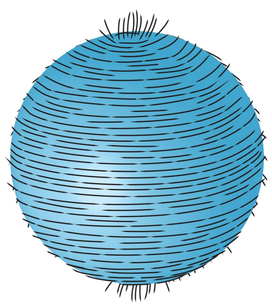
- Noether symmetries are symplectic:
  $\mathbb{i}_{\mathfrak{g}(\alpha)}\Omega \approx \mathbb{d}\langle q, \alpha \rangle \implies \mathbb{L}_{\mathfrak{g}(\alpha)}\Omega \approx 0$
- Noether implies $\mathbb{L}_{\mathfrak{g}(\alpha)}\mathcal{E}_I \approx 0$ but the converse is not necessarily true!

2. Going Beyond Noether: Poisson-Lie Symmetries

2a. Poisson-Lie (PL) Groups

Idea

A deformation of a Lie group in the Moyal sense. (classical quantum group)



Poisson-Lie Group

Lie group $\mathcal{G}$ with Poisson bivector $\pi_{\mathcal{G}}$ compatible (equivariant) with the group multiplication.

Example

$\mathcal{G} = (\mathbb{R}^3, +)$ with $\{x_i, x_j\} = \epsilon_{ijk}x_k$, is Poisson-Lie, so equivariance means:

$$\{x_i + a_i, x_j + a_j\} = \epsilon_{ijk}(x_k + a_k)$$

Key Points

$\mathcal{G}$ has a natural non-Abelian dual PL group, $\mathcal{G}^*$

$\langle \alpha, [X_1, X_2]_* \rangle := \langle \delta(\alpha), X_1 \otimes X_2 \rangle$, $\alpha \in \mathfrak{g}, X_i \in \mathfrak{g}^*$

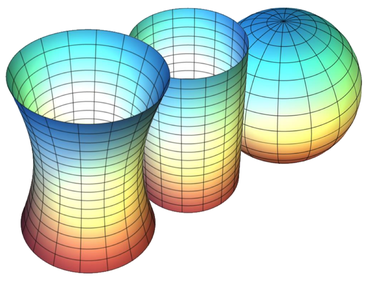
where

$$\delta := d\pi_{\mathcal{G}}|_{g=1} : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$$

2b. The Heisenberg Double & Drinfel'd Double

Idea

Curved phase spaces and their natural symmetries



Heisenberg Double Phase Space

$\mathcal{D}_{\mathrm{H}} := \mathcal{G} \bowtie \mathcal{G}^*$ is symplectic!

$$\Omega_{\mathrm{H}} = \frac{1}{2}\left(\langle \ell^{-1}\mathbb{d}\ell, \mathbb{d}hh^{-1} \rangle + \langle \mathbb{d}\tilde{\ell}\tilde{\ell}^{-1}, \tilde{h}^{-1}\mathbb{d}\tilde{h} \right)$$

where $\mathcal{D}_H \ni G = \ell h = \tilde{h}\tilde{\ell} = (\ell \triangleright h)(\ell \triangleleft h)$

Drinfeld Double

$\mathcal{D}_{\mathrm{D}} := \mathcal{G} \bowtie \mathcal{G}^*$ is Poisson-Lie!

Examples

- If $\mathcal{G}^*$ is Abelian, we recover, $\mathcal{D}_{\mathrm{H}} \cong T^*\mathcal{G} \cong \mathcal{G} \ltimes \mathfrak{g}^*$
- Lorentz group is a Drinfel'd double: $\mathcal{D}_{\mathrm{D}} := SL(2,\mathbb{C}) \cong SU(2) \bowtie AN(2,\mathbb{C})$

Key Points

- $\mathcal{D}_{\mathrm{D}}$ naturally acts on $\mathcal{D}_{\mathrm{H}}$ from the left & right:

$$\mathbb{i}_{\mathfrak{r}(\alpha)}\Omega_{\mathrm{H}} = -\langle \alpha, \tilde{\ell}^{-1}\mathbb{d}\tilde{\ell} \rangle, \quad \mathbb{i}_{\mathfrak{r}(Y)}\Omega_{\mathrm{H}} = \langle Y, h^{-1}\mathbb{d}h \rangle$$
$$\mathbb{i}_{\mathbb{I}(\alpha)}\Omega_{\mathrm{H}} = -\langle \alpha, \mathbb{d}\ell\ell^{-1} \rangle, \quad \mathbb{i}_{\mathbb{I}(Y)}\Omega_{\mathrm{H}} = \langle Y, \mathbb{d}\tilde{h}\tilde{h}^{-1} \rangle$$

where $\alpha \in \mathfrak{g}, Y \in \mathfrak{g}^*$

Poisson-Lie Symmetries

1. $\mathbb{L}_{\mathfrak{g}(\alpha)}\mathcal{E}_I \approx 0$
2. $\mathbb{i}_{\mathfrak{g}(\alpha)}\Omega \approx \langle Q^{-1}\mathbb{d}Q, \alpha \rangle$

$\implies Q : \mathcal{F}_{\mathrm{EL}} \to \mathcal{G}^*$ is conserved!

Remarks

- Recovers Noether $\iff$ $\mathcal{G}^*$ is Abelian ($\iff$ $\pi_{\mathcal{G}} = 0$)
- Genuinely goes beyond Noether! $\mathbb{L}_{\mathfrak{g}(\alpha)}\Omega = -\frac{1}{2}\langle [\mathbb{Q}, \mathbb{Q}]_*, \alpha \rangle$
- Lagrangian invariance *does not* matter.

3. Some Poisson-Lie Symmetric Field Theories

0+1D: Deformed Spinning Top

A particle with a generically curved phase space, $\mathcal{D}_{\mathrm{H}} := \mathcal{G} \bowtie \mathcal{G}^*$

1+1D: Klimčík-Ševera Open String

A string with Poisson-Lie group target space. T-duality in $\mathcal{G} \leftrightarrow \mathcal{G}^*$

2+1D: Gravity with C.C.

Palatini-Cartan formulation of 3D gravity with C.C. Λ

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4. Future Research Questions

- What lessons can we learn about non-commutative spacetime?
- What do these conserved quantities say about integrability?
- How do our results from 3D gravity extend to 4D gravity?
- In what other contexts can we find these symmetries? E.g. asymptotic symmetries in gravity?