Transfer Functions of Distributed Parameter Systems: A Tutorial

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Abstract

In this tutorial article the rich variety of transfer functions for systems described by partial differential equations is illustrated by means of several examples under various boundary conditions. An important feature is the strong influence of the choice of boundary conditions on the dynamics and on system theoretic properties such as pole and zero locations, properness, relative degree and minimum phase. It is sometimes possible to design a controller using the irrational transfer function and several such techniques are outlined. More often, the irrational transfer function is approximated by a rational one for the purpose of controller design. Various approximation techniques and their underlying theory are briefly discussed.

Key words: infinite-dimensional systems, partial differential equations, control theory, distributed parameter systems, relative degree, minimum phase

Most of the analysis of control systems and the development of algorithms for controller design has been for systems modeled by ordinary differential equations, which provide adequate models for RLC circuits, rigid robots and many other systems. However, in many systems the physical quantity of interest depends on several independent variables. For instance, the temperature of an object depends on both position and time, as does the deflection produced by structural vibration. When there is more than one independent variable, the equation modeling the dynamics involves partial derivatives and is thus a partial-differential equation (PDE). The aim of this article is to acquaint the reader with a number of examples of systems with dynamics modeled by a partial-differential equation and in particular, to derive and analyze their transfer functions. Although the theory is generally applicable, for simplicity of exposition, all the examples are single-input-single-output and in one space dimension.

Since the solution of the PDE reflects the distribution of a physical quantity such as the temperature of a rod or the deflection of a beam, these systems are often called distributed-parameter systems (DPS).

As is well known, the transfer functions of systems modeled by ordinary differential equations, often called lumped-parameter systems, are rational functions. In contrast, the transfer functions of distributed-parameter systems are irrational functions. Another difference is that the state space is infinite dimensional, usually a Hilbert space. Consequently, DPS are also called infinite-dimensional systems. In this article we do not discuss state-space realizations of these systems, but instead concentrate on the frequency-domain description in terms of the transfer function.

The analysis of rational and irrational transfer functions differ in a number of important aspects. The most obvious differences between rational and irrational transfer functions are the poles and zeros. Irrational transfer functions often have infinitely many poles and zeros, as is the case for $\frac{\cosh\sqrt{s}}{\sinh\sqrt{2s}}$, $\frac{e^{-2s}}{s(1+e^{-s})}$ and $\frac{\sinh s}{\cosh 2s}$, but e^{-s} and many other functions have neither poles nor zeros. "Dead-beat" transfer functions, such as $\frac{se^{-s}}{1+s}$, have finitely many poles and zeros. In this article, we concen-

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trate on transfer functions derived from PDEs. In these systems the location of the poles and zeros depends crucially on the choice of boundary conditions. Another striking difference between rational and irrational functions is that of limits of the transfer functions at infinity; there can be several distinct limits at infinity depending on the path taken in the complex plane. For example, if you approach infinity along the real axis, e^{-s} has the limit zero. However, if you approach infinity along the imaginary axis, the values oscillate and so this limit does not exist. While concepts like relative degree and minimum phase are important for irrational functions, the definitions used for rational functions are no longer appropriate. For example, e^{-s} has infinite relative degree and is not minimum phase, despite the fact that it has no zeros. Consequently, it is necessary to generalize the usual definitions of the above concepts, as well as other concepts, to make them meaningful for irrational functions.

Since the best way to appreciate the rich diversity of infinite-dimensional behavior is through studying examples derived from typical PDEs, we derive the transfer functions of some representative examples of controlled systems under various boundary conditions: a heated rod, a vibrating string, noise in a duct, and a flexible beam. For all these examples we also analyze the main properties relevant to control design, including the pole and zero locations, the stability properties, the relative degree and the existence of limits of the transfer functions at infinity. In addition, we check whether or not the system is minimum phase or positive real. Precise definitions are given in the appendices.

If a transfer function model is available, it is possible to design a controller using frequency domain techniques. We briefly indicate some of the relevant theory that has been developed for irrational transfer functions. Since practical controller design is usually based on rational approximations, we briefly discuss the main approximation techniques, and the underlying theory. In particular, balanced order reduction is a common approach and conditions under which this is justified are given.

This paper is intended as a tutorial paper, not a survey, and no attempt is made to list the extensive literature on control of distributed parameter systems.

1 Heat Flow in a Rod

One of the simplest examples of a system modelled by a partial-differential equation is the problem of controlling the temperature profile in a rod of length L with constant thermal conductivity K_0, mass density ρ and specific heat C_p. Applying of the principle of conservation of energy to arbitrarily small volumes in the bar leads to the following partial-differential equation for the temperature $z(x,t)$ at time t at position x from the left-hand end [18, sect. 1.3]

$$C_p\rho\frac{\partial z(x,t)}{\partial t} = K_0\frac{\partial^2 z(x,t)}{\partial x^2}, \qquad x\in(0,L), \qquad t\geq 0. \tag{1}$$

In addition to modeling heat flow, this equation also models other types of diffusion, such as chemical diffusion and neutron flux. To fully determine the temperature, one needs to specify the initial temperature profile $z(x,0)$ as well as the boundary conditions at each end. We show how different boundary conditions for the heat equation (1) influence the transfer function.

1.1 Neumann boundary control

Consider the situation where the rod is insulated at the end $x=0$. At the other end, $x=L$, the rate of heat flow into the rod is controlled. Applying Fourier's law of heat conduction we obtain the boundary conditions

$$\frac{\partial z}{\partial x}(0,t)=0, \qquad K_0\frac{\partial z}{\partial x}(L,t)=u(t), \tag{2}$$

where $u(t)$ is the rate of heat flow into the rod.

The temperature is measured at a fixed point x_0 on the rod, where $0<x_0\leq L$ and so the observation is given by

$$y(t)=z(x_0,t). \tag{3}$$

An expression for the transfer function can be derived in two ways. Perhaps the best known is to take Laplace transforms with respect to the time variable t, assuming an initial condition of zero. The Laplace-transformed PDE yields an ordinary differential equation with the solution $\hat{y}(s)=G(s)\hat{u}(s)$, where $\hat{u},\hat{y}$ denote the Laplace transforms and $G(s)$ is the transfer function. (For the mathematical justification of taking Laplace transforms in this manner see Example 4.3.12 in Curtain & Zwart [10, sect.4.3] and Cheng & Morris [9] .) By the nature of the Laplace transform, the relationship between $\hat{u}$ and $\hat{y}$ is defined only on the region of analyticity of G which is $\{s\in\mathbb{C} : \operatorname{Re}s>\alpha\}$ for some real α. In other words, $\hat{y}(s)=G(s)\hat{u}(s)$ for s in this region, but not necessarily elsewhere. A counterexample is given in Curtain & Zwart [10, Ex. 4.3.8].

An alternative approach is to seek a solution of the form $z(x,t)=e^{st}z_0(x)$ with the input $u(t)=e^{st}$, which has a signal interpretation Zwart [44]. This approach leads to the same ordinary differential equation and the same solution, but this approach needs no special justification and, moreover, it yields a transfer function that is well-defined for nearly all s.

Suppose we take Laplace transforms of (1), (2) with zero initial condition $z(\cdot,0)=0$. Denoting the Laplace transforms by $\hat{z}, \hat{u}$, the resulting boundary value problem is

$$K_0 \frac{d^2\hat{z}(x,s)}{dx^2} = C_p \rho s \hat{z}(x,s), \tag{4}$$

$$\frac{d\hat{z}}{dx}(0,s) = 0, \qquad K_0 \frac{d\hat{z}}{dx}(L,s) = \hat{u}(s). \tag{5}$$

The general solution of the differential equation (4) is

$$\hat{z}(x,s) = A \sinh\left(\frac{\sqrt{s}x}{\alpha}\right) + B \cosh\left(\frac{\sqrt{s}x}{\alpha}\right),$$

where α is the thermal diffusivity and $\alpha 2 = \frac{K_0}{C_p \rho}$. Applying the boundary conditions (5), we obtain the solution

$$\hat{z}(x,s) = \frac{\alpha \cosh\left(\frac{\sqrt{s}x}{\alpha}\right)}{K_0 \sqrt{s} \sinh\left(\frac{\sqrt{s}L}{\alpha}\right)} \hat{u}(s),$$

and since

$$\hat{y}(s) = \hat{z}(x_0, s),$$

the transfer function $G_{heat1}(x_0,s) = \frac{\hat{y}(s)}{\hat{u}(s)}$ of the system is given by

$$G_{heat1}(x_0,s) = \frac{\alpha \cosh\left(\frac{\sqrt{s}x_0}{\alpha}\right)}{K_0 \sqrt{s} \sinh\left(\frac{\sqrt{s}L}{\alpha}\right)}. \tag{6}$$

The function G_{heat1} should be considered a function of s, with x_0 a parameter indicating the dependence of the transfer function on the choice of observation point.

The poles of G_{heat1} are the zeros of the denominator and are the real nonnegative numbers $-(k\pi\alpha/L)^2$, while the zeros are the zeros of the numerator and are the real negative numbers $-((k\pi + \pi/2)\alpha/x_0)^2$, where $k = 0, 1, \ldots$. Note that, due to the pole at 0, G_{heat1} is not stable, but it is proper and well-posed. (See Theorem A.2 and Definitions B.1 and B.2.)

We can rewrite G_{heat1} as

$$G_{heat1}(x_0,s) = \frac{\alpha e^{\frac{\sqrt{s}(x_0 - L)}{\alpha}}}{K_0 \sqrt{s}} \frac{(1 + e^{-\frac{2\sqrt{s}x_0}{\alpha}})}{(1 - e^{-\frac{2\sqrt{s}L}{\alpha}})}.$$

In the case that $x_0 = L$, we have, for real λ,

$$\lim_{\lambda \to \infty} \sqrt{\lambda} G_{heat1}(\lambda, s) = \frac{\alpha}{K_0},$$

which means that $G_{heat1}(s)$ has finite relative degree. (See definition C.1.) However, in the case that $x_0 \neq L$, there does not exist an integer k such that for real λ $\lim_{\lambda\to\infty} \lambda^k G_{heat1}(\lambda)$ exists and is non-zero, which means that G_{heat1} does not have finite relative degree.

As explained in Appendix D (or Curtain & Zwart [10, Example 4.3.12]) we can express $G_{heat1}(x_0,s)$ as the infinite partial-fraction expansion

$$G_{heat1}(x_0,s) = \frac{\alpha^2}{K_0 L s} + \frac{2L}{K_0} \sum_{k=1}^{\infty} \frac{(-1)^k \cos(k\pi x_0/L)}{L^2 s + (k\pi\alpha)^2}. \tag{7}$$

When $x_0 = L$, it is easily seen that the transfer function

$$G_{heat1}(L,s) = \frac{\alpha^2}{K_0 L s} + \frac{2L}{K_0} \sum_{k=1}^{\infty} \frac{1}{L^2 s + (k\pi\alpha)^2},$$

is positive real (Definition E.1). Consequently, $G_{heat1}(L,s)$ can be stabilized by the feedback $u(t) = -\kappa y(t)$ for any positive gain κ (Theorem E.2).

1.2 Dirichlet boundary control

Suppose now that we can vary the temperature $u(t)$ at the end point $x = L$ and we can measure the temperature at an interior point x_0 of the rod. The boundary conditions become

$$z(0,t) = 0, \qquad z(L,t) = u(t). \tag{8}$$

We still measure the temperature at a point x_0 and so the observation is

$$y(t) = z(x_0, t).$$

We derive a closed-form expression for the transfer function, as before, by solving the ordinary differential equation (4) with the boundary conditions

$$\hat{z}(0,s) = 0, \qquad \hat{z}(L,s) = \hat{u}(s),$$

which are the Laplace transforms of (8), to obtain the transfer function

$$G_{heat2}(x_0,s) = \frac{\sinh\left(\frac{\sqrt{s}x_0}{\alpha}\right)}{\sinh\left(\frac{\sqrt{s}L}{\alpha}\right)}.$$

Note that the zero at $s = 0$ of $\sinh\left(\frac{\sqrt{s}L}{\alpha}\right)$ in the denominator cancels with the zero of $\sinh\left(\frac{\sqrt{s}x_0}{\alpha}\right)$ in the numerator. Thus the poles of G_{heat2} are the same as G_{heat1}, except that there is no pole at $s = 0$. The transfer function is analytic and uniformly bounded in the open right half-plane and, since, the poles are all in the open left half-plane, G_{heat2} is stable. (See Theorem A.2.) Hence G_{heat2} is well-posed and proper. In fact, unless $x_0 = L$, G_{heat2} is strictly proper. As is the case for G_{heat1}, for $x_0 \neq L$, G_{heat2} has infinite relative degree (Definition C.2). The zeros are at $s = -(k\pi\alpha/x_0)^2, k = 1, 2...$, and, using the same technique as in [8], it can be shown that the system is minimum phase (Definition F.1). As for G_{heat1}, we can calculate an infinite partial-fraction expansion. The infinite partial-fraction expansion for G_{heat2} is

$$G_{heat2}(x_0, s) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} 2k\pi \sin(k\pi x_0/L)}{L^2 s + (k\pi\alpha)^2}.$$

1.3 Mixed boundary conditions

Suppose that again we consider heat flow in a rod described by (1) with the control action the heat flow at $x = L$ as in (2), and with the measurement of the temperature at $x = x_0$. However, instead of an insulated end at $x = 0$ as in (2), we suppose that the temperature is kept constant at $x = 0$, which leads to the observation (3) and the boundary conditions

$$z(0, t) = 0, \quad K_0 \frac{\partial z}{\partial x}(L, t) = u(t).$$

Taking Laplace transforms as above, we obtain the transfer function

$$G_{heat3} = \frac{\alpha \sinh\left(\frac{\sqrt{s}x_0}{\alpha}\right)}{K_0 \sqrt{s} \cosh\left(\frac{\sqrt{s}L}{\alpha}\right)}.$$

This transfer function has zeros at $-(\frac{k\pi\alpha}{x_0})^2$, $k = 1, 2...$ and poles at $-(\frac{(k\pi + \pi/2)\alpha}{L})^2$, $k = 0, 1, 2...$, which are different from those of G_{heat1} and G_{heat2}. The transfer function G_{heat3} is analytic and uniformly bounded in the open right half-plane and since the poles are all in the open left half-plane, it is stable. Hence G_{heat3} is well-posed and also strictly proper. For $x_0 \neq L$, G_{heat3} has infinite relative degree and it is minimum phase. The infinite partial-fraction expansion for G_{heat3} is

$$G_{heat3}(x_0, s) = \frac{1}{K_0} \sum_{k=1}^{\infty} \frac{(-1)^k 2L \sin\left(\left(k + \frac{1}{2}\right) \pi x_0/L\right)}{L^2 s + ((k\pi + \pi/2)\alpha)^2}.$$

2 Vibrating string

2.1 Undamped vibrations

The application of Newton's law to a vibrating string of length L yields the partial-differential equation, known as the wave equation,

$$\frac{\partial^2 z(x,t)}{\partial t^2} = c^2 \frac{\partial^2 z(x,t)}{\partial x^2}, \qquad 0 < x < L, \qquad t \geq 0, \tag{9}$$

for the deflection $z(x, t)$ at time t at the position x along the string. The constant $c = \frac{\tau}{\rho}$ where τ is the string tension and ρ the density, Towne [39, chap.1]. The wave equation models many other situations such as acoustic plane waves, lateral vibrations in beams, and electrical transmission lines.

To complete the model, appropriate boundary conditions as well as initial conditions $z(\cdot, 0)$ and $\frac{\partial z}{\partial t}(\cdot, 0)$ need to be defined. Suppose that the ends are fixed with control and observation both distributed along the string. For simplicity we normalize the units so that $L = 1$ and $c = 1$ to obtain the equations

$$\frac{\partial^2 z(x,t)}{\partial t^2} = \frac{\partial^2 z(x,t)}{\partial x^2} + b(x)u(t), \tag{10}$$

$$z(0,t) = 0, \qquad z(1,t) = 0, \tag{11}$$

$$y(t) = \int_0^1 \frac{\partial z(x,t)}{\partial t} b(x) \mathrm{d}x, \tag{12}$$

where $b \in \mathbf{L}_2(0, 1)$ describes both the control and observation action, which is a type of distributed colocation. As in the previous examples, the transfer function can be calculated by taking Laplace transforms of (10) – (12) with respect to t with initial conditions $z(x, 0) = 0$ and $\dot{z}(x, 0) = 0$. Alternatively, the transfer function can be found by seeking a solution of the form $z(x, t) = e^{st} z_0(x)$ with $u(t) = e^{st}$. With either approach, the resulting boundary value problem is

$$\frac{d^2 \hat{z}(x,s)}{dx^2} = s^2 \hat{z}(x,s) - b(x)\hat{u}(s); \qquad \hat{z}(0,s) = 0 = \hat{z}(1,s). \tag{13}$$

This equation is a forced, linear ordinary differential equation in the variable x, which can be solved by standard methods, such as variation of parameters. Another approach is to put the differential equation into first-order form and then calculate the exponential matrix of the resulting system. The first-order form is

$$\frac{d}{dx} \begin{bmatrix} \hat{z} \\ \frac{d\hat{z}}{dx} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ s^2 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \hat{z} \\ \frac{d\hat{z}}{dx} \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} b(x)\hat{u}(s).$$

The matrix exponential of A with respect to x is given by

$$E(x, s) = E^{Ax} = \begin{bmatrix} \cosh(sx) & \frac{1}{s}\sinh(sx) \\ s\sinh(sx) & \cosh(sx) \end{bmatrix},$$

and the general solution of the system of equations is, for initial conditions $\hat{z}(0,s)=\alpha$ and $\frac{d\hat{z}}{dx}(0,s)=\beta$,

$$\begin{bmatrix} \hat{z} \\ \frac{d\hat{z}}{dx} \end{bmatrix} = E(x,s)\begin{bmatrix} \alpha \\ \beta \end{bmatrix} - \int_0^x E(\xi,s)\begin{bmatrix} 0 \\ 1 \end{bmatrix} b(x-\xi)\hat{u}(s)\mathrm{d}\,\xi.$$

Applying the boundary conditions from (13) we obtain

$$\begin{aligned}\hat{z}(x,s) = \tfrac{\hat{u}(s)}{s} \Big(& \tfrac{\sinh(sx)}{\sinh(s)} \textstyle\int_0^1 \sinh\big((1-\xi)\,s\big)b(\xi)\,\mathrm{d}\xi ... \\ & - \textstyle\int_0^x \sinh\big((1-\xi)\,s\big)b(\xi)\,\mathrm{d}\xi\Big).\end{aligned} \tag{14}$$

After taking the Laplace transform of the observation (12) and dividing by $\hat{u}(s)$ we obtain the system transfer function

$$G_{wave1}(s) = \int_0^1 b(x)a(x,s)\,\mathrm{d}x$$

where

$$\begin{aligned} a(x,s) = & \frac{\sinh(sx)}{\sinh(s)} \int_0^1 \sinh\big((1-\xi)\,s\big)b(\xi)\mathrm{d}\,\xi ... \\ & - \int_0^x \sinh\big((1-\xi)\,s\big)b(\xi)\mathrm{d}\,\xi. \end{aligned}$$

With the choice

$$b(x) = \begin{cases} 1, & 0 \le x \le 1/2, \\ 0, & 1/2 < x \le 1, \end{cases}$$

the transfer function is

$$G_{wave1}(s) = \frac{1}{2s} + \frac{2\cosh\left(\frac{s}{2}\right) - \cosh^2\left(\frac{s}{2}\right) - \cosh(s)}{s^2\sinh(s)}.$$

The poles are at $\imath\pi k$, for integers k, and so G_{wave1} is neither stable nor proper. Moreover, G_{wave1} has finite relative degree. By the same process as for (7), we can calculate an infinite partial-fraction expansion

$$\begin{aligned} G_{wave1}(s) &= \frac{1}{2s} + \sum_{k=1}^{\infty}\left(\frac{\mathrm{Res}(\imath k\pi)}{s-\imath k\pi} + \frac{\mathrm{Res}(-\imath k\pi)}{s+\imath k\pi}\right) \\ &= \frac{1}{2s} + 2s\sum_{k=1}^{\infty}\frac{\mathrm{Res}(\imath k\pi)}{s^2+k^2\pi^2}, \end{aligned}$$

where the residue at $\imath k\pi$ is $\mathrm{Res}(\imath k\pi)$=$\mathrm{Res}(-\imath k\pi) = \nu_k$, $\nu_{4r} = 0, \nu_{4r+1} = \frac{1}{(4r+1)^2\pi^2}, \nu_{4r+2} = \frac{4}{(4r+2)^2\pi^2}, \nu_{4r+3} = \frac{1}{(4r+3)^2\pi^2}$. It is readily verified that $G_{wave1}(s)$ is positive real and so it can be stabilized by the feedback $u(t) = -\kappa y(t)$ for any positive κ.

If instead of the velocity measurement (12) we measure position so that

$$y(t) = \int_0^1 z(x,t)b(x)\mathrm{d}x, \tag{15}$$

the analysis is very similar. The calculation of the transfer function is identical up to (14). Then using (15) instead of (12) we obtain that the transfer function is now $\frac{1}{s}G_{wave1}(s)$, which has finite relative degree, but is not positive real.

2.2 Damped vibrations

The model (10) with boundary conditions (11) does not include any dissipation, an unrealistic assumption. To capture the effects of damping we introduce some light damping into the model, as in Russell [37]. Letting ε indicate a small positive constant, the wave equation becomes a type of telegrapher's equation Guenther & Lee [18, chap. 1].

$$\frac{\partial^2 z(x,t)}{\partial t^2} + \varepsilon\langle\frac{\partial z(\cdot,t)}{\partial t}, b\rangle b(x) = \frac{\partial^2 z(x,t)}{\partial x^2} + b(x)u(t),$$

where $b \in \mathbf{L}_2(0,1)$ with the same boundary conditions (11) and observation (12) as before.

To obtain the transfer function of this system we note that this system is the result of applying the feedback $u(t) = -\varepsilon y(t) + v(t)$ to the undamped positive-real system (10)). The function $v(t)$ indicates the uncontrolled signal. The transfer function of the damped system is thus

$$G_{wave2}(s) = G_{wave1}(I + \varepsilon G_{wave1})^{-1}$$

and it is stable. In this way we obtain that G_{wave2} is given by

$$\frac{\frac{s}{2}\sinh(s) + 2\cosh\left(\frac{s}{2}\right) - 3\cosh^2\left(\frac{s}{2}\right) + 1}{s(s+\frac{\varepsilon}{2})\sinh(s) + \varepsilon(2\cosh\left(\frac{s}{2}\right) - 3\cosh^2\left(\frac{s}{2}\right) + 1)}.$$

Since G_{wave2} is stable, we know that all the poles lie in the open left half-plane. To obtain more precise information we recall from Russell [37] that if $b = \sum_{k=1}^{\infty}\gamma_k\sqrt{2}\sin(k\pi x)$, $0 < |\gamma_k| \le M/k$, then the poles asymptote to the imaginary axis according to

$$\mu_k = \imath\pi k - \frac{\varepsilon}{2}\gamma_{|k|}^2 + O(\frac{\varepsilon^2}{k^2\pi^2}),$$

where $k = \pm 1, 2, 3, ...$ Our choice of b above satisfies these conditions with

$$\gamma_k = \frac{\sqrt{2}}{k\pi}\left(1 - \cos(\frac{k\pi}{2})\right).$$

Although all poles lie in the open left half-plane, the real parts of the poles asymptote rapidly to zero, as displayed in Figure 2 .

Since G_{wave1} is positive real and $\varepsilon > 0$, Theorem E.2 implies that G_{wave2} is also positive real. Furthermore, it is stable. It is readily verified that G_{wave2} has finite relative degree and so by Theorem F.3 we conclude that it is minimum phase. Consequently G_{wave2} has no zeros in the open right half-plane.

3 Acoustic Waves in a Duct

Consider the modeling of the acoustic waves in a duct such as that shown in Figure 3. The pressure in a duct $p(x,t)$ and the particle velocity $v(x,t)$ are functions of space and time. Denoting the air density by ρ_0 and the speed of sound by c, the following well-known partial-differential equations describe the propagation of sound in a one-dimensional duct Towne [39, chap. 2]

$$\frac{1}{c^2}\frac{\partial p(x,t)}{\partial t} = -\rho_0 \frac{\partial v(x,t)}{\partial x}, \tag{16}$$

$$\rho_0 \frac{\partial v(x,t)}{\partial t} = -\frac{\partial p(x,t)}{\partial x}. \tag{17}$$

We can combine the above system of two first-order equations to obtain the well known wave equation

$$\frac{\partial^2 z(x,t)}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2},$$

where z can indicate the pressure, the velocity, or the displacement. (The latter equation is obtained by using $v = \frac{\partial z}{\partial t}, p = -\rho_0 c^2 \frac{\partial z}{\partial x}$.) The same equation also describes the motion of an undamped string. (See (9).)

Let L indicate the length of the duct. There are various ways to model the boundary conditions at the ends $x = 0$ and $x = L$. Let us consider the simplest first.

3.1 Semi-infinite duct

A loudspeaker acting as a source of noise is mounted at $x = 0$, and it can be regarded as a velocity source $u(t)$ so that

$$v(0,t) = u(t). \tag{18}$$

The simplest way to model the effect of the other end of the duct is to assume that the duct is long enough so that the duct is effectively infinite in length and to neglect the effect of the end.

Taking Laplace transforms of the equations (16) – (18) leads to the ordinary differential equations

$$\begin{aligned} s\hat{p}(x,s) &= -c^2 \rho_0 \frac{d\hat{v}}{dx}(x,s), \\ \rho_0 s \hat{v}(x,s) &= -\frac{d\hat{p}}{dx}(x,s), \\ \hat{v}(0,s) &= \hat{u}(s). \end{aligned}$$

Imposing the requirement that velocity and pressure remain bounded for all x leads to the the solution

$$\hat{v}(x,s) = e^{-x\frac{s}{c}}\hat{u}(s), \qquad \hat{p}(x,s) = \rho_0 c e^{-x\frac{s}{c}}\hat{u}(s).$$

The transfer function relating the pressure measured at point x_0 to the acoustic velocity applied at $x = 0$ is therefore

$$G_{duct0}(x_0,s) = \rho_0 c e^{-x_0\frac{s}{c}}.$$

Although this system is stable, and has no zeros, it is not minimum phase and has infinite relative degree.

This pure time-delay transfer function predicts that a wave travels in one direction, with no attenuation or reflection. In reality, both occur at the end of a duct. The speed of transmission of a wave is c, which means that sound waves in air at $20°C$ will travel from one end to the other end of a 100 meter long duct in about 1/3 seconds. Therefore it is necessary to consider the behaviour of waves at the end of the duct.

3.2 Zero impedance

Since the end $x = L$ is open it is reasonable to assume that the pressure at the end is equal to the exterior pressure,

$$p(L,t) = 0. \tag{19}$$

The impedance Z_L at $x = L$ is defined as

$$Z_L = \frac{\hat{p}(L,s)}{\hat{v}(L,s)},$$

so this boundary condition is equivalent to assuming that the impedance is zero.

Taking Laplace transforms of the partial-differential equations (16), (17) subject to the boundary conditions (18), (19), we obtain the transfer function of the velocity source $u(t)$ to the pressure measured at a point x_0 to be

$$\begin{aligned} G_{duct1}(x_0,s) &= \rho_0 c \frac{\sinh\left((L-x_0)\,\frac{s}{c}\right)}{\cosh\left(\frac{Ls}{c}\right)} \\ &= \rho_0 c\, e^{-x_0\frac{s}{c}}\, \frac{1 - e^{2(x_0-L)\frac{s}{c}}}{1 + e^{-2L\frac{s}{c}}}. \end{aligned}$$

The poles are at $\frac{ic}{L}(k+1/2)\pi$ and the zeros are at $\frac{ikc\pi}{L-x_0}$, for integers k. All poles and zeros lie on the imaginary axis, and so this system is not stable nor is it minimum phase. If the observation is at a point $x_0 \neq 0$, for real λ we have

$$\lim_{\lambda \to \infty} G_{duct1}(x_0, \lambda) = 0.$$

(If the observation is at $x_0 = 0$, then this limit is $\rho_0 c$.) The above limit occurs exponentially fast and the system has infinite relative degree. The pure delay term $e^{-x_0 \frac{s}{c}}$ indicates that for observation points $x_0 \neq 0$, a step input will not be observed immediately. This is an effect of the time of transmission of waves. However, the term $e^{-x_0 \frac{s}{c}}$ only decays as $\mathrm{Re}\, s \to \infty$ and so the transfer function has a different limit depending on how the limit is taken. If we are interested in the energy transmitted at high frequencies we should consider the behaviour of $|G_{duct1}(x_0, \imath\omega)|$ for large ω. This transfer function exhibits oscillatory behaviour and so there is no attenuation in the energy transmitted at higher frequencies.

3.3 Constant impedance

The boundary condition $p(L,t) = 0$ is only an approximation to the behavior of the acoustic waves at an open end. In reality, not all of the energy in the acoustic waves is reflected. A model that includes dissipation at the open end is obtained by assuming that the impedance at the boundaries lies between $p(L,t) = 0$ (zero impedance) and $v(L,t) = 0$ (infinite impedance). The boundary condition at $x = L$ is then

$$p(L,t) = Z_L v(L,t),$$

where the constant impedance $Z_L > \rho_0 c$. Taking Laplace transforms as before with this new boundary condition, we obtain that the transfer function relating the pressure measured at x_0 to the acoustic velocity $u(t)$ applied at $x = 0$ to be

$$G_{duct2}(x_0, s) = e^{-x_0 \frac{s}{c}} \rho_0 c \, \frac{1 + \alpha_L e^{2(x_0 - L)\frac{s}{c}}}{1 - \alpha_L e^{-2L\frac{s}{c}}}, \qquad (20)$$

where the *reflectance*

$$\alpha_L = \frac{Z_L - \rho_0 c}{Z_L + \rho_0 c}.$$

(Note that as the impedance $Z_L \to 0$, the reflectance $\alpha_L \to -1$ and we recover the original transfer function G_{duct1}.) The poles of this system are, for integers k,

$$p_k = \frac{c}{2L} \ln(\alpha_L) + \imath \frac{kc\pi}{L}.$$

The poles of this system still lie on a vertical line, but now, since $\alpha_L < 1$, they are in the left half-plane and the system is stable. As for the previous transfer functions for the duct it is not minimum phase and it has infinite relative degree.

3.4 A more realistic model

A more accurate, frequency-dependent, model of the behaviour of the acoustic waves at an open end was obtained in Beranek [5]. Letting a indicate the duct radius, define $R_2 = \rho_0 c/\pi a^2$, $R_1 = 0.504 R_2$, $C = 5.44 a^3/\rho_0 c^2$ and $M = 0.1952 \rho_0/a$. The end impedance is closely approximated by Beranek [5, pg. 127]

$$Z_L(s) = \pi a^2 \frac{(R_1 + R_2) M s + R_1 R_2 M C s^2}{(R_1 + R_2) + (M + R_1 R_2 C) s + R_1 M C s^2}, \qquad (21)$$

with reflectance

$$\alpha_L(s) = \frac{Z_L(s) - \rho_0 c}{Z_L(s) + \rho_0 c},$$

and the new boundary condition,

$$\hat{p}(L,s) = Z_L(s) \hat{v}(L,s).$$

As before, we assume that the control is produced by a velocity source at $x = 0$,

$$v(0,t) = u(t).$$

The transfer function relating the pressure measured at x_0 to the velocity $u(t)$ applied at $x = 0$ is now

$$G_{duct3}(x_0, s) = \rho_0 c \, e^{-x_0 \frac{s}{c}} G_o(x_0, s),$$

where

$$G_o(x_0, s) = \frac{1 + \alpha_L(s) e^{2(x_0 - L)\frac{s}{c}}}{1 - \alpha_L(s) e^{-2L\frac{s}{c}}}.$$

(Note that as the impedance $Z_L(s) \to 0$, we recover the original transfer function G_{duct1}.) This model leads to a more accurate prediction of the system frequency response than the simpler model G_{duct1} (20) with zero impedance Z_L, see Zimmer, Lipshitz, Morris, Vanderkooy & Obasi [43]; and it is also more accurate than models with constant impedance, see Zimmer [42].

The impedence transfer function $Z_L(s)$ is positive real and so $|\alpha_L(s)| < 1$ for all Re $(s) \geq 0$. Thus, all the poles of the transfer function $G_{duct3}(x_0, s)$ have negative real parts and the system is stable . Since $\alpha_L(s) \approx -\frac{\beta}{s}$ for large $|s|$, where $\beta > 0$, for large $|s|$ the poles asymptote to the exponential curve: $|y| = \beta e^{-2L\frac{s}{c}}$. (See Figure 4.)

The zeros also have negative real parts and a similar qualitative behavior for large $|s|$. The function G_o is analytic on an open set containing the closed right half-plane, it has no zeros in the closed right half-plane and

$$\lim_{|s| \to \infty} G_o(x_0, s) = \rho_0 c.$$

Hence, by Theorem F.2 it is a minimum-phase function and G_{duct3} is the product of a minimum-phase function and $e^{-x_0 \frac{s}{c}}$. The transfer function G_{duct3} has infinite relative degree. Although G_{duct3} has no zeros in the closed right half-plane, unless the observation is at $x_0 = 0$, that is, at the same location as the control, there is a time delay and the system is not minimum-phase.

The effect of step inputs on the system is found by calculating for $x_0 \neq 0$ and real λ

$$\lim_{\lambda \to \infty} G_{duct3}(x_0, \lambda) = 0.$$

Moreover, $G_{duct3}(x_0, \lambda)$ converges to zero exponentially fast. This property is consistent with the fact that due to the time of transmission, a step input is not immediately observed. However, the limit of $|G_{duct3}(x_0, \imath\omega)|$ along the imaginary axis does not exist, which shows that energy is transmitted at high frequencies.

3.5 Including actuator dynamics

More sophisticated models that include the loudspeaker dynamics are analyzed in Zimmer, Lipshitz, Morris, Vanderkooy & Obasi [43]. The boundary condition

$$v(0, t) = u(t)$$

implies that, when undriven, the loudspeaker acts as a perfectly rigid end with zero velocity. Thus, the impedance

$$Z_0(s) = \frac{\hat{p}(0, s)}{\hat{v}(0, s)}$$

is infinite. In fact, a loudspeaker has stiffness, mass and damping, even when undriven. So it will not act as a perfectly rigid end and has finite impedance. At low frequencies the loudspeaker can be modeled as a simple mass-spring-damper system. Indicating the driving voltage of the loudspeaker by $V_D(t)$, the governing equations of the loudspeaker are

$$m_D \ddot{x}_D(t) + d_D \dot{x}_D(t) + k_D x_D(t) = BV_D(t) - A_D p(0, t), \tag{22}$$

where m_D, d_D, k_D, B and A_D are loudspeaker parameters (Beranek [5]). The loudspeaker is coupled to the duct by the pressure at the end $p(0, t)$ and also by

$$A_D \dot{x}_D(t) = \pi a^2 v(0, t), \tag{23}$$

where $v(x, t)$ is the particle velocity in the duct. Taking Laplace transforms of the loudspeaker model in (22), we obtain

$$A_D \hat{p}(0, s) = \frac{Bl}{R_{coil}} \hat{V}_D(s) - Z_0(s) \hat{v}\,(0, s)\,,$$

where

$$Z_0(s) = \frac{\pi a^2}{A_D s}(m_D s^2 + d_D s + k_D) \tag{24}$$

is the mechanical impedance of the loudspeaker, and $\hat{V}_D(s)$ is the Laplace transform of the driving voltage $V_D(t)$. Even when the loudspeaker is undriven ($\widehat{V}_D(s) = 0$), since the impedance Z_0 is not infinite for all frequencies, the velocity at $x = 0$ is only zero if the pressure is zero. Defining the reflectance at $x = 0$ by

$$\begin{aligned} \alpha_0(s) &= \frac{Z_0(s) - \rho_0 c A_D}{Z_0(s) + \rho_0 c A_D} \\ &= \frac{\pi a^2(m_D s^2 + d_D s + k_D) - \rho_0 c A_D^2 s}{\pi a^2(m_D s^2 + d_D s + k_D) + \rho_0 c A_D^2 s}, \end{aligned} \tag{25}$$

we obtain the transfer function

$$G_{duct4}(x_0, s) = \rho_0 c e^{-x_0 \frac{s}{c}} S(s) G_{o2}(s)$$

where

$$G_{o2}(s) = \frac{(1 + \alpha_L(s) e^{2(x_0 - L)\frac{s}{c}})}{(1 - \alpha_0(s)\alpha_L(s) e^{-2L\frac{s}{c}})},$$

$$S(s) = \frac{Bl\big(1 + \alpha_0(s)\big)}{2R_{coil} Z_0(s)}.$$

Using (24,25) , $S(s)$ can be rewritten as

$$S(s) = \frac{Bl A_D s}{R_{coil}\big(\pi a^2(m_D s^2 + \alpha_d s + k_D) + \rho_0 c A_D^2 s\big)}.$$

Thus the zeros are the same as those of G_{duct3} plus one at $s = 0$. The poles are similar to those of G_{duct3} plus additional poles due to the loudspeaker dynamics: the two roots of $\pi a^2(m_D s^2 + \alpha_d s + k_D) + \rho_0 c A_D^2 s$. Both these additional roots are in the left half-plane. Since Z_0 and Z_L are positive real transfer functions, $|\alpha_0(s)| < 1$ and $|\alpha_L(s)| < 1$ for all Re $(s) \geq 0$. All the poles of G_{duct4} are in the left half-plane and the system is stable. Since $\alpha_0(s) \to 1$ as $|s| \to \infty$ the asymptotic behaviour of the poles is the same as for G_{duct3}. The system is stable and strictly proper, but it is not minimum phase due to the exponential decay term and it has infinite relative degree. However, $G_{duct4}(x_0, s) \to 0$ as $|s| \to \infty$. Including a model of the actuator dynamics introduces a low-pass filter into the transfer function and now

$$\lim_{\omega \to \infty} G_{duct4}(x_0, \imath\omega) = 0.$$

Unlike the simpler models of the duct, no energy is transmitted at high frequencies.

4 Vibrating Beams

The simplest example of transverse vibrations in a structure is a beam, where the vibrations can be considered to occur only in one dimension. A photograph of an experimental set-up for a vibrating beam is shown in Figure 4. The analysis of beam vibrations is useful in calculating the dynamics of flexible links, and also in obtaining an understanding of the dynamics of vibrations in more complex geometries, such as large structures.

Consider a homogeneous beam of length L experiencing small transverse vibrations. For small deflections the plane cross-sections of the beam remain planar during bending. Under this assumption, we obtain the classic Euler-Bernoulli beam model for the deflection $w(x,t)$,

$$\frac{\partial^2 w(x,t)}{\partial t^2} + EI\frac{\partial^4 w(x,t)}{\partial x^4} = 0,$$

where E, I are material constants Guenther & Lee [18, chap. 6]. This simple model does not include any damping, and predicts that a beam, once disturbed, would vibrate forever. To model more realistic behaviour, damping should be included. The most common model of damping is Kelvin-Voigt damping, sometimes referred to as Rayleigh damping which leads to the PDE

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w(x,t)}{\partial x^2} + c_d I\frac{\partial^3 w(x,t)}{\partial x^2 \partial t}\right) = 0, \quad (26)$$

where c_d is the damping constant.

4.1 Clamped-free beam with shear force control: a naive approach

Assume that the beam is clamped at $x = 0$ and free at the tip $x = L$, with control of the shear force at the tip. The boundary conditions are

$$\begin{aligned} w(0,t) &= 0, & \tfrac{\partial w}{\partial x}(0,t) = 0, \\ \tfrac{\partial^2 w}{\partial x^2}(L,t) &= 0, & -EI\tfrac{\partial^3 w}{\partial x^3}(L,t) = u(t), \end{aligned}$$

where $u(t)$ represents an applied force at the tip, with positive direction the same as positive direction for the deflection w. If we measure the tip velocity, we have the observation

$$y(t) = \frac{\partial w}{\partial t}(L,t). \quad (27)$$

As in the previous examples, the transfer function can be calculated by taking Laplace transforms of (26) – (4.1) with respect to t and then solving the resulting boundary value problem. Defining

$$m(s) = \left(\frac{-s^2}{EI + sc_d I}\right)^{\frac{1}{4}}, \quad (28)$$

and taking the Laplace transform of (27) we obtain the transfer function

$$G_{beam1}(s) = \frac{s\,N(s)}{EIm^3(s)D(s)}$$

where

$$N(s) = \cosh\big(Lm(s)\big)\sin\big(Lm(s)\big) - \sinh\big(Lm(s)\big)\cos\big(Lm(s)\big), \quad (29)$$

$$D(s) = 1 + \cosh\big(Lm(s)\big)\cos\big(Lm(s)\big). \quad (30)$$

The poles are the solutions $\mu_k, \mu_{-k}, k \geq 1$ to the quadratic equation in s

$$s^2 + c_d I\alpha_k^4 s + EI\alpha_k^4 = 0,$$

where α_k are the roots of the equation in α

$$1 + \cosh(L\alpha)\cos(L\alpha) = 0. \quad (31)$$

The solutions to (31) are either real or pure imaginary and since only α_k^4 occurs in (31) it suffices to consider only the real, positive solutions. Since for large real α, $\cosh(\alpha L) \approx \frac{1}{2}e^{\alpha L}$, these solutions converge to $\frac{(2k+1)\pi}{2L}$ as k approaches infinity. The poles of G_{beam1} are

$$\begin{aligned} \mu_k &= \frac{-c_d I\alpha_k^4 + \sqrt{(c_d I)^2\alpha_k^8 - 4EI\alpha_k^4}}{2}, \\ \mu_{-k} &= \frac{-c_d I\alpha_k^4 - \sqrt{(c_d I)^2\alpha_k^8 - 4EI\alpha_k^4}}{2}. \end{aligned}$$

All poles lie in the left half-plane and $\lim_{k\to\infty}\mu_k = -\frac{E}{c_d}$ and $\lim_{k\to\infty}\mu_{-k} = -c_d I\alpha_k^4 \approx -c_d I(\frac{(2k+1)\pi}{2L})^4$. The transfer function is analytic on an open set containing the closed right half-plane. All poles are in the left half-plane, but although the transfer function is analytic on the open right half-plane, it is not bounded there. Using the exponential formulas for sinh, cos and so on, we obtain that there are constants c_1, c_2 such that for Re $s > 0$,

$$\begin{aligned} &\lim_{|s|\to\infty} e^{-\sqrt{\frac{2}{i}}Lm(s)}N(s) = c_1, \\ &\lim_{|s|\to\infty} e^{-\sqrt{\frac{2}{i}}Lm(s)}D(s) = c_2. \end{aligned}$$

Thus, for positive Re$s > 0$ we have

$$\lim_{|s|\to\infty}\frac{N(s)}{D(s)} = constant.$$

But $|\frac{s}{m^3(s)}|$ is unbounded as $s \to \infty$ and so although all poles are in the left half-plane, the system is not well-posed (Definition B.2). Hence it is unstable. In particular, the magnitude of the frequency response increases with frequency, which does not reflect the physics. Consequently this model is incorrect. The mistake lies in the omission of the effect of damping on the moment in the boundary conditions (4.1).

4.2 Clamped-free beam with shear force control: an improved model

If M denotes the moment, then for a free end, we have

$$M(L,t) = 0, \qquad \frac{\partial M}{\partial x}(L,t) = 0.$$

The moment M of an Euler-Bernoulli beam is equal to $EI\frac{\partial w^2}{\partial x^2}$ only when there is no damping in the system. Kelvin-Voigt damping affects the moment, which becomes

$$M(x,t) = EI\frac{\partial^2 w}{\partial x^2} + c_d I\frac{\partial^3 w}{\partial x^2 \partial t}.$$

The correct boundary conditions for a free end are therefore

$$EI\frac{\partial^2 w}{\partial x^2}(L,t) + c_d I\frac{\partial^3 w}{\partial x^2 \partial t}(L,t) = 0, \qquad t \geq 0, \tag{32}$$

$$-EI\frac{\partial^3 w}{\partial x^3}(L,t) - c_d I\frac{\partial^4 w}{\partial x^3 \partial t}(L,t) = u(t), \qquad t \geq 0. \tag{33}$$

With these boundary conditions at the tip and the original ones at the fixed end (4.1), we obtain the transfer function

$$G_{beam2}(s) = \frac{sN(s)}{m^3(s)\big(EI + sc_d I\big)D(s)},$$

where m is as in (28), N in (29) and D in (30).

This transfer function has the same poles as G_{beam1} except for an extra pole at $-E/c_d$ and so it is analytic in the closed right half-plane. But now, since

$$\lim_{|s|\to\infty} \frac{s}{m^3(s)(EI + sc_d I)} = 0,$$

$G_{beam2}(s)$ is stable and strictly proper . In fact, for some nonzero constant c, and real λ,

$$\lim_{\lambda\to\infty} \lambda^{3/4} G_{beam2}(\lambda) = c$$

and the system has finite relative degree .

The zeros are the solutions to the quadratic equation in s

$$s^2 + c_d I\gamma_k^4 s + EI\gamma_k^4 = 0,$$

where γ_k are the roots of the equation in γ

$$\cosh(L\gamma)\sin(L\gamma) - \sinh(L\gamma)\cos(L\gamma) = 0.$$

The roots γ_k are either real or pure imaginary and so it is sufficient to consider only the positive solutions. These solutions converge to $\frac{(4k+1)\pi}{4L}$ as $k \to \infty$. It follows that, except for a zero at 0, all the zeros are in the open left half-plane and they converge to $-\frac{E}{c_d}$ and $-c_d I\left(\frac{(4k+1)\pi}{4L}\right)^4$. Since the transfer function also has finite relative degree, by Theorem F.2 we can conclude that it is minimum phase.

As explained in Appendix D, the transfer function has the infinite partial-fraction expansion

$$\begin{aligned}
G_{beam2}(s) &= \sum_{k=1}^{\infty} \frac{\text{Res}(\mu_k)}{s-\mu_k} + \frac{\text{Res}(\mu_{-k})}{s-\mu_{-k}} \\
&= \sum_{k=1}^{\infty} \frac{4}{\mu_{-k}-\mu_k}\left(\frac{\mu_k}{s-\mu_k} - \frac{\mu_{-k}}{s-\mu_{-k}}\right) \\
&= \sum_{k=1}^{\infty} \frac{4s}{s^2 + c_d I\alpha_k^4 s + EI\alpha_k^4}
\end{aligned}$$

which shows that G_{beam2} is positive real .

If, instead of the velocity measurement, we take the position measurement $y(t) = w(L,t)$, then in an similar fashion as above we obtain that its transfer function is

$$G_{beam3}(s) = \frac{1}{s} G_{beam2}(s).$$

The transfer function $G_{beam3}(s)$ is not positive real, which demonstrates that physical collocation of actuators and sensors is not enough to obtain a positive-real system. However, G_{beam3} is stable and minimum phase and it has finite relative degree.

4.3 Clamped-free beam with torque control and measurement of angle at tip

If the beam is still clamped at 0, but we exert a torque $u(t)$ at $x = L$ the boundary conditions are (4.1) at $x = 0$ and the tip boundary condition becomes

$$\begin{aligned}
&EI\frac{\partial^2 w}{\partial x^2}(L,t) + c_d I\frac{\partial^3 w}{\partial x^2 \partial t}(L,t) = u(t), \\
&EI\frac{\partial^3 w}{\partial x^3}(L,t) + c_d I\frac{\partial^4 w}{\partial x^3 \partial t}(L,t) = 0, \qquad t \geq 0.
\end{aligned}$$

Suppose we now measure the angular velocity at the tip and so the observation is

$$y(t) = \frac{\partial^2 w}{\partial t \partial x}(L,t).$$

By following our previous steps we obtain the transfer function

$$\begin{aligned}
G_{beam4}(s) &= \frac{sN_2(s)}{m(s)\big(EI + sc_d I\big)D(s)} \\
&= -\frac{m^3(s)N_2(s)}{sD(s)},
\end{aligned}$$

where $m(s)$ is as in (28) , D in (30) and

$$N_2(s) = \cosh\big(Lm(s)\big)\sin\big(Lm(s)\big) + \sinh\big(Lm(s)\big)\cos\big(Lm(s)\big).$$

The poles are the same as those of a clamped-free beam with force control, G_{beam2}. Furthermore, as before, the transfer function is also proper and therefore the system is stable. However, the zeros are very different from before. They are the roots of the quadratic equation in s

$$s^2 + c_d I\beta_k^4 s + EI\beta_k^4 = 0,$$

where β_k are the roots of the equation in β

$$\cosh(L\beta)\sin(L\beta) + \sinh(L\beta)\cos(L\beta) = 0.$$

The solutions β are either real or pure imaginary and we only need to consider the real, positive solutions, which converge to $\frac{(4k+3)\pi}{4L}$ as $k \to \infty$. Except for a zero at 0, the zeros are all the the open left half-plane and they converge to $-\frac{E}{c_d}$

and $-c_d I\left(\frac{(4k+3)\pi}{4L}\right)^4$. The transfer function also has finite relative degree and so by Theorem F.3 it is minimum phase. The infinite partial-fraction expansion is

$$G_{beam4}(s) = \sum_{k=1}^{\infty} \frac{4s\alpha_k^2}{s^2 + c_d I\alpha_k^4 s + EI\alpha_k^4} \times ... \\ \frac{\sinh^2(\alpha_k L)\sin^2(\alpha_k L)}{L\left[\sinh(\alpha_k L) + \sin(\alpha_k L)\right]^2},$$

where α_k are the roots of (31). Since the coefficients are all positive, the transfer function is positive real.

If, instead of measuring $\frac{\partial^2 w}{\partial t \partial x}(L,t)$, we measure the angle of rotation $y(t) = \frac{\partial w}{\partial x}(L,t)$, then in an similar fashion we obtain that its transfer function is $G_{beam5}(s) = \frac{1}{s}G_{beam4}(s)$. The function G_{beam5} is not positive real, but it is stable and minimum phase and it has finite relative degree.

4.4 Pinned-free beam with shear force control

Suppose that instead of the clamped boundary conditions (4.1) at $x = 0$, the beam is pinned at $x = 0$. The boundary conditions at $x = 0$ become

$$w(0,t) = 0, \qquad \frac{\partial^2 w}{\partial x^2}(0,t) = 0, \qquad t \geq 0. \tag{34}$$

The end $x = L$ is free, with control of the shear force at the tip as in Section 4.2 and the boundary conditions at $x = L$ are (32), (33). Assume that we are again measuring the tip position and the observation is

$$y(t) = w(L,t).$$

Taking Laplace transforms of the same beam equation (26) and solving the resulting ordinary differential equation with the boundary conditions (34), (32), (33), we obtain the transfer function

$$G_{beam6}(s) = \frac{-2\sinh\big(Lm(s)\big)\sin\big(Lm(s)\big)}{m^3(s)\,(EI + sc_d I)\,N(s)}$$

where N is defined in (29). The poles are at the zeros of $G_{beam2}(s)$ plus $-\frac{E}{C_d}$ and the zeros of $m(s)$. Hence they are all in the open left half-plane. Using our previous estimates, we see that G_{beam6} behaves like $\frac{constant}{m^3(s)(EI+sc_d I)}$ as $|s| \to \infty$. Hence it is stable and strictly proper and it has finite relative degree. The zeros of G_{beam6} are given by

$$s = -\frac{k^2\pi^2}{2L^2}(C_d I)\left(1 \pm \sqrt{1 - \frac{4EIL^2}{(k\pi C_d)^2}}\right) \qquad k \geq 1,$$

plus another zero at the origin. By Theorem F.2, it is minimum phase.

Although the only change in the model is that the clamped condition at $x = 0$ was changed to a pinned condition, the poles and zeros of the transfer function are quite different from those of the clamped beam transfer function G_{beam2}. The first 10 poles of the pinned-free and clamped-free beams are compared in Figure 6. In many applications, the actual boundary condition at $x = 0$ is intermediate between a pinned and a clamped condition, see Bellezza, Lanari &Ulivi [4] and Morris & Taylor [31].

5 Other DPS Examples

These examples illustrate that the transfer functions of PDE's and their properties are sensitive to boundary conditions. In particular, all the examples illustrate that boundary conditions affect both the zeros and the poles dramatically. For other examples of transfer functions of DPS see Curtain & Zwart [10].

The transfer functions of the wave and duct examples are of a delay type: rational functions of s and $e^{a_k s}$ for finite k and $a_k < 0$. Although obtained from partial-differential equation models, these transfer functions are similar to those obtained for delay-differential equations. Such transfer functions are studied in, for instance, Glover, Lam & Partington [15].

On the other hand, the transfer functions for the heat and beam examples have fractional powers of s. The latter belong to the class of *fractional transfer functions* which have recently received increasing attention in the literature [12]. Typical examples are $\frac{1}{s^\alpha}$, $\frac{1}{(1+s^2)^\alpha}$, e^{-s^α}, where $0 < \alpha < 1$. For physical applications we refer to Duarte Ortigueira & Tenreiro Machado [12].

6 Stabilization and Control

If a closed form expression of the transfer function of a system can be obtained, it may be possible to design a controller directly in the frequency domain: the *direct controller design* approach. Many controller design approaches used on finite-dimensional systems can be extended to certain classes of DPS. In particular, the fractional representation approach to controller design (Vidyasagar [40]) extends readily from rational transfer functions to various algebras of irrational transfer functions, see Vidyasagar [40], Foias, Ozbay & Tannenbaum [13] and Logemann [27] .

Examples of these algebras are $\mathcal{T} = \mathbf{H}_\infty/\mathbf{H}_\infty$ and the *Callier-Desoer* class $\mathcal{B}$ from Callier & Desoer [6,7]. Functions in $\mathcal{B}$ have a decomposition

$$G(s) = g(s) + h(s),$$

where g is an unstable proper rational function and h is a stable irrational function in the class $\mathcal{A}_- \subset \mathbf{H}_\infty$. A function $G_{stab} \in \mathcal{A}_-$ if there exist a positive ϵ and a function f_a such that $e^{\epsilon \cdot} f_a(\cdot) \in \mathbf{L}_1(0,\infty)$ and $f_n \in \mathbb{C}$ such that

$$G_{stab}(s) = \hat{f}_a(s) + f_0 + \sum_{n=1}^{\infty} f_n e^{-st_n},$$

where $t_n > 0$ and $\sum_{n=1}^{\infty} |f_n| e^{\epsilon t_n} < \infty$. (The notation $\hat{\cdot}$ denotes the Laplace transform.) Thus, any transfer function

$G_{stab} \in \mathcal{A}_-$ is the Laplace transform of a distribution of the form

$$f(t) = \begin{cases} f_a(t) + f_0\delta(t) + \sum_{n=1}^{\infty} f_n\delta(t-t_n), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Consequently, transfer functions in $\mathcal{B}$ are proper and, they do not have infinitely many poles asymptoting to the imaginary axis (see Appendix B). Our heat flow and flexible beam examples are in the Callier-Desoer class, but the string examples are not. For a detailed treatment of this class with more examples and the solution of various robust stabilization problems, see Curtain & Zwart [10, Chap. 9].

Stability of a closed loop system is defined for systems with irrational transfer functions exactly as for those with rational transfer functions: for $G, K \in \mathcal{T}$ (or $\mathcal{B}$), K stabilizes G if $(I+KG)^{-1}, G(I+KG)^{-1}, (I+KG)^{-1}K, (I+GK)^{-1} \in \mathbf{H}_\infty$ (or $\mathcal{A}_-$). Coprime factorizations are also defined analogously to the rational case, and the Youla-Jabr-Bongiorno parameterization of all stabilizing controllers has a natural generalization. However, while every $G \in \mathcal{B}$ has a coprime factorization, this is not the case for $\mathcal{T}$. In fact, $G \in \mathcal{T}$ can be stabilized if and only if it has a coprime factorization, see Inouye [20] and Smith [38].

Generalizations of many well-known finite-dimensional results such as the small gain theorem and the Nyquist stability criterion exist, see Logemann [26], [27, sec.5]. The latter reference also surveys PI-control solutions to servo-problems for irrational transfer functions and the internal model principle. More recent results on tracking for infinite-dimensional systems can be found in Rebarber & Weiss [35] and Logemann [28]. Just as for finite-dimensional systems, any positive real system can be stabilized by the static output feedback $u = -\kappa y$ for any $\kappa > 0$ (Theorem E.2). Passivity generalizes in a straightforward way to irrational transfer functions (Desoer & Vidyasagar [11, Chapters V, VI]). Note that for some of these results it is not required to know the transfer function in order to ensure stability of the controlled system. It is only required to know whether the transfer function lies in the appropriate class. A theory of robust H_∞-control designs for infinite-dimensional systems with transfer functions in $\mathcal{T}$ is described in Foias, Ozbay & Tannenbaum [13]. More recent results on this approach can be found in Kashima & Yamamoto [22] and references therein. However, the most powerful results for direct controller design are obtained for the Callier-Desoer class.

A drawback of direct controller design is that, in general, the resulting controller is infinite-dimensional and must be approximated by a finite-dimensional system. For this reason, direct controller design is often known as *late lumping* since the final controller is approximated by a finite-dimensional, or lumped parameter, system.

For many practical examples, controller design based on the transfer function is not feasible, since a closed-form expression for the transfer function may not be available. Instead, a finite-dimensional approximation of the system is first obtained and the controller design is based on this finite-dimensional approximation. This approach is known as *indirect controller design*, or *early lumping*. In fact, the most common method of controller design systems modeled by partial differential equations is to first approximate the original system and to then design a controller for the reduced-order model. The hope is that it has the desired effect on the original system. That this method is not always sucessful was first documented in Balas [2], where the term *spillover effect* was coined. Spillover refers to the phenomenon that a controller that stabilizes a reduced-order model need not necessarily stabilize the original model. Systems with infinitely many poles either on or asymptoting to the imaginary axis are notorious candidates for spillover effects. Some issues associated with approximation of systems for the purpose of controller design are discussed in the next section.

7 Approximation Theory and Practice

Approximation or model reduction of a distributed parameter system involves finding a rational approximation to the original irrational transfer function in some appropriate norm. The appropriate choice of norm depends on the purpose for which one intends to use the reduced-order model: simulation, prediction, or as a means to design a controller for the original system. Most infinite-dimensional systems are either stable, or are the sum of a stable system and a finite-dimensional unstable part. Usually, approximation proceeds by a finding an approximation to the stable infinite-dimensional part.

The two most common norms used as a measure of the approximation error are the $\mathbf{H}_2$- and the $\mathbf{H}_\infty$-norms (Appendix A). Approximation in the $\mathbf{H}_2$-norm is linked to the response of the system to initial conditions, or to a particular input. Every input in $\mathbf{L}_2(0,\infty)$ has a Laplace transform u in $\mathbf{H}_2$. Suppose that G is stable and let G_N be the transfer function of a rational approximation. We obtain the estimate

$$\|Gu - G_N u\|_2 \leq \|G - G_N\|_\infty \|u\|_2.$$

Consequently, the error in the $\mathbf{H}_\infty$-norm between the original transfer function G and an approximation G_N yields a uniform bound on the approximation error over all inputs. The *gap* or graph metric (Zames & El-Sakkary [41]) is the generalization of this measure of distance to unstable systems; details can be found in Vidyasagar [40].

The transfer functions in $\mathbf{H}_\infty$ that can be approximated in the $\mathbf{H}_\infty$-norm by rational transfer functions are those that are continuous on the imaginary axis and have a well-defined limit at infinity; that is

$$\lim_{\omega\to+\infty} G(\imath\omega) = \lim_{\omega\to-\infty} G(\imath\omega).$$

The functions G_{wave1}, G_{duct1}, G_{duct2} and G_{duct3} are not approximable by rational functions in the $\mathbf{H}_\infty$-norm. Moreover, transfer functions of the form $e^{-s}G_r(s)$ are only approximable in the $\mathbf{H}_\infty$-norm by a rational transfer function if G_r is a strictly proper stable transfer function. Approximation in the $\mathbf{H}_\infty$-norm is particularly suitable for use in robust $\mathbf{H}_\infty$- or $\mathbf{H}_2/\mathbf{H}_\infty$-controller design, since the original infinite-dimensional plant can be regarded as belonging to an uncertainty set around the finite-dimensional system. See, for example, Curtain & Zwart [10, chap.9].

Frequency-domain approximation methods for delay-type systems, such as Padé, Fourier-Laguerre and many others are surveyed in Partington [34]. Delay-type systems have a transfer function that is a rational function of s and $e^{-\alpha_k s}$, and k belongs to a finite set. For example, the transfer functions G_{wave2}, G_{duct2}, G_{duct3} and G_{duct4} are of this type.

However, approximation is generally based on the partial-differential equation model. Common methods are modal approximations and numerical approximations, such as finite-element schemes or proper orthogonal decomposition (POD).

7.1 Modal approximation

If the eigenvalues and eigenfunctions (or modes) of the partial differential equation problem are known, or if one has an infinite partial fraction expansion of the transfer function available, the most popular way to obtain the high-order approximation is by *modal approximation.*

Consider the heat flow transfer function G_{heat1}. When we truncate the infinite partial-fraction expansion (7), we obtain the N-th order modal approximation

$$G_{heat1,N} = \frac{\alpha^2}{K_0 L s} + \frac{2L}{K_0} \sum_{k=1}^{N-1} \frac{(-1)^k \cos(k\pi x_0/L)}{L^2 s + (k\pi\alpha)^2}.$$

The $\mathbf{H}_\infty$- error can be made arbitrarily small by increasing N, since

$$\|G_{heat1} - G_{heat1,N}\|_\infty \leq \frac{2L}{K_0(\pi\alpha)^2} \sum_{k=N}^{\infty} \frac{1}{k^2}.$$

Modal approximations will also provide good low-order approximations in the H_∞- norm for all the heated rod examples and for all the flexible beam examples. However, it is not a panacea. For example, if a system, for instance G_{wave1}, has infinitely many unstable poles, a modal approximation will not produce a finite $\mathbf{H}_\infty$-error. Moreover, to obtain a reasonable $\mathbf{H}_\infty$-error it may be necessary to use a very high-order approximation, as in the case of G_{wave2}, where the poles asymptote to the imaginary axis.

Although modal approximations reproduce the lower-order poles exactly, and often have good error bounds, unfortunately, they do not necessarily approximate the zeros well, see Cheng & Morris [8] and Linder, Reichard &Tarkenton [25]. The actual zeros and the zeros of a modal approximation for the heat equation with Dirichlet control are shown in Figure 7. The system is minimum phase and all the zeros lie on the negative real axis. However, not only does the approximation have complex zeros, but it has zeros in the open right half-plane. These spurious zeros are likely due to the fact that the original system has infinite relative degree. Since any finite-dimensional approximation will have finite relative degree, the behaviour at high frequencies can only be reproduced by introducing additional zeros into the approximation. Basing controller design on observation of the calculated zeros could lead to a number of erroneous predictions. For instance, most results for adaptive controllers are restricted to minimum-phase systems. Also, achievable sensitivity reduction is limited by the number of right half-plane zeros.

7.2 Other approximation methods

For more complicated PDE models it may not be possible to obtain a series or closed form expression for the transfer function. Moreover, obtaining an approximation of the transfer function using numerical calculations of the eigenvalues and eigenvectors is not efficient.

Current practice is to obtain a numerical approximation of the original PDE, often using a finite-element method, and use this as the approximating system. The use of approximations in design of linear-quadratic controllers has received much attention in the literature, see for example, the survey paper Banks & Fabiano [3] and the two-volume book Lasiecka & Triggiani [24].Conditions for which approximations can be used in state-space design of H_∞-controllers can be found in Morris [30].

Although finite-element approximations perform well in simulations, they may not converge in $\mathbf{H}_\infty$-norm. For distributed observation and control sufficient conditions for the convergence of the transfer functions in the $\mathbf{H}_\infty$-norm for stable systems and in the gap metric for unstable systems are given in Morris [32].

As is the case for modal approximations, the zeros of finite-element approximations may differ widely from the true zeros and this can have an effect on the performance of a controller (Cheng & Morris [8] and Grad & Morris [17]).

7.3 Approximation of Large Scale Systems

For complex systems, numerical PDE methods lead to approximate models of extremely high order. Often an explicit PDE description may not be available, but only an approximate finite-dimensional model of extremely high order (often in the thousands). So a further approximation step is required to obtain a lower-order model that is tractable for controller design. The most common method is to reduce a balanced realisation of the very high-order finite-dimensional system. Recently, the POD approach has gained in popularity. It is interesting to note that in Lall, Marsden &Glavaški [23] it is interpreted as a type of balancing. In Appendix G it is explained that the $\mathbf{H}_\infty$-error incurred in balancing depends on the sum of the singular values of the system to be approximated. Consequently, in this method there is an implicit assumption that the large-scale finite-dimensional model is a good approximation in the $\mathbf{H}_\infty$-norm to the original infinite-dimensional system. It is this initial unknown error that is the Achilles heel in this approach (see the subsection on "Hankel singular values" below). Consequently, no theoretical error bounds are available for the resulting reduced-order models for complex systems. Other finite-dimensional reduction methods are discussed in the recent book Antoulas [1].

Clearly, numerical issues are paramount in the reduction of the high-order model. Much experience in the numerical approximation of large-scale dynamical systems has been developed during the past few years and this is surveyed in Antoulas [1].

7.4 Hankel singular valuse

The Hankel singular values of a rational transfer function determine the error in the $\mathbf{H}_\infty$-norm of the truncated balanced realisation (Appendix G). The theory of truncated balanced approximations for irrational transfer functions is analogous to the rational case, only now there are typically infinitely many Hankel singular values σ_k (Glover, Curtain & Partington [16]). A necessary condition in this theory is that the system is *nuclear*, i.e., $\sum_{k=1}^{\infty} \sigma_k < \infty$. If the system is nuclear, then approximation by truncating a balanced approximation produces a $\mathbf{H}_\infty$-error bound of

$$\|G - G^r\|_\infty \leq 2 \sum_{k=r+1}^{\infty} \sigma_k,$$

where G_r is the reduced model. Hence the implicit assumption underlying the balancing approach is that the original system is nuclear. Unfortunately, the calculation of the Hankel singular values for irrational transfer functions is only feasible for very simple delay systems. However, the following two tests may prove useful.

A necessary and sufficient condition for a stable system to be nuclear is that Partington [33], writing $G''(s) = \frac{d^2 G(s)}{ds^2}$,

$$\int_0^\infty \int_{-\infty}^{\infty} |G''(x + \imath y)| \mathrm{d}y \mathrm{d}x < \infty.$$

Another necessary and sufficient condition for nuclearity is that G has an infinite partial fraction expansion of the form

$$G(s) = \sum_{k=0}^{\infty} \frac{a_k}{s - \mu_k}, \tag{35}$$

valid for Re $(s) > 0$, where Re $(\mu_k) < 0$ and $\sum_{k=0}^{\infty} \frac{|a_k|}{\mathrm{Re}\,|\mu_k|} < \infty$ (see Partington [33]).

Using (35) we can conclude that $G_{heat2}, G_{beami}, i = 1, ..., 6$ are nuclear and so easy to approximate in the $\mathbf{H}_\infty$-norm. (The transfer function G_{heat1} is the sum of $1/s$ and a nuclear transfer function.)

For certain delay systems good estimates can be obtained Glover, Lam & Partington [15]. Using these results one can show that the Hankel singular values of G_{wave2} and G_{duct4} are $\mathcal{O}(1/k)$. Thus these functions are not nuclear.

None of $G_{duct1}, G_{duct2}, G_{duct3}$ can be approximated in the H_∞-norm by rational functions. However, including a model of the actuator dynamics of the duct example introduces a low-pass filter into the transfer function, which leads to smaller Hankel singular values in G_{duct4}. This transfer function can be approximated by rational functions, but not using balanced truncation, because it is not nuclear. The filtered model $\frac{1}{s+1} G_{duct4}$ is nuclear, and so easy to approximate. This illustrates how precompensation by a low-pass filter can be a means of arriving at a better approximation in the $\mathbf{H}_\infty$-norm.

Unfortunately, it is not feasible to calculate balanced realisations of irrational transfer functions. The practical approach is to first obtain a good, possibly very high order, approximation with a small $\mathbf{H}_\infty$-error bound and then approximate this finite-dimensional model to obtain a reduced-order model .

Why is it important to know that a transfer function is nuclear, when we know that this property is not necessary to obtain a rational approximation in the $\mathbf{H}_\infty$-norm? Suppose that we have a sequence G_N of rational approximations of order N of the irrational transfer function G such that $\|G - G_N\|_\infty = \varepsilon_N$ where $\varepsilon_N \to 0$ as $N \to \infty$. We choose N sufficiently large to obtain a small error ε_N, and calculate a truncated balanced approximation of G_N, G_N^r, of order r. Since G_N is finite-dimensional, we obtain the error bound $\|G_N - G_N^r\|_\infty \leq 2 \sum_{k=r+1}^{N} \sigma_N^k$, where σ_N^k, $k = 1..N$, are the singular values of G_N (Appendix G). It follows that G_N^r approximates G with an error

$$\|G - G_N^r\|_\infty \leq \varepsilon_N + 2 \sum_{k=r+1}^{N} \sigma_N^k.$$

The Hankel singular values of G_N converge to those of G as $N \to \infty$ (Glover, Curtain & Partington [16, Appendix 2]). Suppose that G is not nuclear so that $\sum_{k=1}^{\infty} \sigma_k = \infty$, where σ_k are the singular values of G. Although ε_N can be made arbitrarily small by increasing N, the tail term $\sum_{k=r+1}^{N} \sigma_N^k$ will increase without bound as N increases. In other words, balanced truncation of a non-nuclear transfer function may not lead to a low-order approximation that has a small error in the $\mathbf{H}_\infty$-norm.

8 Conclusions

The aim of this paper was to illustrate the main differences between the rational transfer functions obtained from ODE models and the irrational transfer functions that are obtained from PDE models. Boundary conditions are one aspect of PDE models that does not appear in ODE models. Boundary conditions have a strong effect on the dynamics and many properties of the transfer function, such as the location of poles and zeros. We also illustrate how errors in modeling the boundary conditions can lead to an improper transfer function. The main systems theoretic concepts, such as stability, minimum-phase property and relative degree, have extensions to irrational transfer functions. However, extra complexities are introduced into the analysis. Appropriate formal definitions of all concepts are given in appendices. Stability theory and common controller design methods can be extended to PDE models. However, in practice, finite-dimensional approximations generally need to be used in controller synthesis. Often, these finite-dimensional approximations are of very high order and need to be further reduced in order to obtain a practical controller. Some of the common approximation techniques were outlined, along with the relevant approximation theory underlying this step.

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A Stability and norms

Consider the Hilbert space

$$\mathbf{L}_2(0,\infty) = \{u : [0,\infty) \to \mathbb{C} | \int_0^\infty |u(t)|^2 dt < \infty\}.$$

The $\mathbf{L}_2$-norm, $\|u\|_2 = \left[\int_0^\infty |u(t)|^2 dt\right]^{1/2}$, often has a physical interpretation as the energy of the signal and so it is reasonable to consider only inputs u in $\mathbf{L}_2(0,\infty)$ and to ask that outputs y also be in $\mathbf{L}_2(0,\infty)$.

All $u \in \mathbf{L}_2(0,\infty)$ have a well-defined Fourier transform

$$\hat{u}(\imath\omega) = \int_0^\infty u(t)e^{-\imath\omega t}dt\,.$$

The Fourier transform can also be interpreted as the Laplace transform on the imaginary axis, or the system frequency response. If $u \in \mathbf{L}_2(0,\infty)$ then $\hat{u}$ is in the frequency-domain space

$$\mathbf{L}_2(\imath\mathbb{R}) = \{f : \imath\mathbb{R} \to \mathbb{C} | \|f\|_2 = \left[\frac{1}{2\pi}\int_{-\infty}^\infty |f(\imath\omega)|^2 d\omega\right]^{1/2} < \infty\}$$

and $\|\hat{u}\|_2 = \|u\|_2$. The Laplace transform of a function in $\mathbf{L}_2(0,\infty)$ is analytic in the open right half-plane and so the Laplace transform is in

$$\mathbf{H}_2 = \{f : \mathbb{C}_0^+ \to \mathbb{C} | f \text{ analytic } \& \sup_{x>0} \int_{-\infty}^\infty |f(x+\imath y)|^2 \mathrm{d}y < \infty\}.$$

The restriction of a function in $\mathbf{H}_2$ to the imaginary axis is in $\mathbf{L}_2(\imath\mathbb{R})$, and defining the norm on H_2 to be

$$\|f\|_2 = \left[\sup_{x>0} \frac{1}{2\pi}\int_{-\infty}^\infty |f(x+\imath y)|^2 \mathrm{d}y\right]^{\frac{1}{2}},$$

the norms in these two spaces are equal.

Definition A.1 *If a system maps every input u in $\mathbf{L}_2(0,\infty)$ to an output y in $\mathbf{L}_2(0,\infty)$ and*

$$\sup_{u\neq 0} \frac{\|y\|_2}{\|u\|_2} < \infty,$$

the system is **stable**.

The smallest number γ such that for every input u in $\mathbf{L}_2(0,\infty)$, the corresponding output y satisfies

$$\|y\|_2 \leq \gamma \|u\|_2$$

is the $\mathbf{L_2}$**-gain** *of the system.*

The easiest way to check stability comes from the following well-known result.

Theorem A.2 *A linear system is stable if and only if its transfer function G belongs to*

$$\mathbf{H}_\infty = \{G : \mathbb{C}_0^+ \to \mathbb{C} | G \text{ analytic } \& \sup_{\mathrm{Re}s>0} |G(s)| < \infty\}$$

with norm

$$\|G\|_\infty = \sup_{\mathrm{Re}s>0} |G(s)|.$$

In this case, we say that G is a stable transfer function.

*Furthermore, $\|G\|_\infty$ is the L_2-*gain *of the system.*

B Proper transfer functions

For every transfer function G of a lumped system, the limit $\lim_{s\to\infty} G(s)$ exists and furthermore, has the same value regardless of the direction in which s approaches infinity. We say that these functions have a *limit at infinity.* However, unlike transfer functions of lumped systems, transfer functions of DPS may not be bounded as $|s|$ becomes large. Furthermore, even if this is the case, the limit need not be unique. For example, the limit along the imaginary axis $\lim_{w\to\infty} G(i\omega)$ is in general different from the limit along the real axis $\lim_{\lambda\to\infty} G(\lambda)$. For clarity we reserve the notation $|s| \to \infty$ for the limit that is taken in all directions in the closed right half-plane, as $|s|$ increases. More precisely, we say that G has *the limit G_∞ at infinity* if

$$\lim_{|s|\to\infty} |G(s) - G_\infty| := \lim_{\rho\to\infty} \left[\sup_{\operatorname{Re} s\geq 0 \,\cap\, |s|>\rho} |G(s) - G_\infty| \right] = 0.$$

It is natural to define a transfer function to be proper if it has a finite limit at infinity. However, since irrational functions need not have limits at infinity we need the following more general definition.

Definition B.1 *The function G is* **proper** *if for sufficiently large ρ,*

$$\sup_{\operatorname{Re} s\geq 0 \,\cap\, |s|>\rho} |G(s)| < \infty.$$

If the limit of $G(s)$ at infinity exists and is 0 , we say that G is **strictly proper**.

Proper functions are bounded at infinity on the imaginary axis. A function that does not have this property may still satisfy the following definition.

Definition B.2 *The function G is* **well-posed** *if $|G(s)|$ is uniformly bounded on $\{s \in \mathbb{C} : \operatorname{Re} s > \alpha\}$, for some real α.*

The state-space interpretation of a well-posed system is that for every finite $T > 0$, inputs in $L_2(0, T)$ map to outputs in $L_2(0, T)$. This is a property of physical systems.

A function for which we can choose $\alpha = 0$ in Definition B.2 is not only well-posed but also proper and stable (Appendix A). Stable transfer functions are always well-posed and proper, but they need not possess a limit at infinity.

C Relative Degree

Definition C.1 *A scalar transfer function G has* **finite relative degree** *if there exists a positive number k and a nonzero constant K such that such that, for real λ*

$$\limsup_{\lambda\to\infty} |\lambda^k G(\lambda)| = K. \tag{C.1}$$

For rational transfer functions, there is always an integer n and a nonzero constant M such that

$$\lim_{s\to\infty} s^n G(s) = M.$$

This situation may occur with irrational transfer functions. However, it is possible that, even for real λ, a non-zero limit of $\lambda^n G(\lambda)$ does not exist for any integer n. For instance, with observation at $x_0 = L$ the transfer function of the heat equation with Neumann control (6) satisfies, for real λ,

$$\lim_{\lambda\to\infty} \lambda^{\frac{1}{2}} G_{heat1}(L, \lambda) = \frac{\alpha}{K_0}.$$

Thus, although the system has finite relative degree, no integer order of λ^n yields a non-zero finite limit.

It is also possible that $\lim_{\lambda\to\infty} \lambda^k G(\lambda) = 0$ for all positive numbers k. For example, a simple delay e^{-s} has this property; as does the transfer function $G_{heat1}(x_0, s)$ for any $x_0 \neq L$.

Definition C.2 *If, for all positive numbers k,*

$$\lim_{\lambda\to\infty} \lambda^k G(\lambda) = 0,$$

we say that the function has **infinite relative degree**.

The transfer function $G_{heat1}(x_0, s)$ has infinite-relative degree if $x_0 \neq L$. Note that relative degree is only concerned with the behaviour of a transfer function on the real axis; while properness considers the growth of the function in some right half plane. Unlike rational functions, an irrational function can have positive relative degree, and not be strictly proper. An example of this is e^{-s} which has infinite relative degree but is not strictly proper.

D Partial-fraction expansions

Many, but not all, irrational transfer functions admit a partial-fraction expansion of the form

$$G(s) = \sum_{k=0}^{\infty} \frac{\operatorname{Res}(\lambda_k)}{s - \lambda_k},$$

where λ_k; $k \geq 0$ are the poles of G and the residue at λ_k is

$$\operatorname{Res}(\lambda_k) = \lim_{\lambda\to\lambda_k} (s - \lambda_k) G(s).$$

The partial-fraction expansion can be determined either using state-space realizations (see Curtain & Zwart [10, Lemma 4.3.10]) or by a direct approach based on the Cauchy residue theorem as in [10, Example 7.14]. In particular, if G can be written

$$G(s) = \frac{N(s)}{D(s)}$$

where N and D are entire functions and $D(p) = 0$, but $D'(p) \neq 0$, then the residue at p is readily calculated using l'Hopitâl's rule as

$$\begin{aligned} Res(p) &= \lim_{s\to p} (s - p) \frac{N(s)}{D(s)} \\ &= \lim_{s\to p} \frac{[(s - p)N(s)]'}{D'(s)} \\ &= \frac{N(p)}{D'(p)}. \end{aligned}$$

Hence the residue at each pole p is $\frac{N(p)}{D'(p)}$. For example, the transfer function for the heat equation with Neumann boundary conditions is (6)

$$G_{heat1}(x_0, s) = \frac{N(s)}{D(s)}$$

where

$$N(s) = \alpha \cosh(\frac{\sqrt{s}x_0}{\alpha}),\ D(s) = K_0\sqrt{s}\sinh(\frac{\sqrt{s}L}{\alpha}).$$

The residue of the pole at 0 is

$$\frac{N(0)}{D'(0)} = \frac{\alpha^2}{K_0 L}.$$

This is the coefficient of $\frac{1}{s}$ in the partial fraction expansion. The residues at the other poles are calculated similarly.

E Positive real

Definition E.1 *The scalar transfer function G is* **positive real** *if it is analytic on the open right half-plane and it satisfies the following conditions there:*

real: $\overline{G(s)} = G(\bar{s})$

positive: $G(s) + G(s)^* \geq 0.$

The following result is a direct generalization of the corresponding result for rational transfer functions and can be shown by straightforward manipulation of the transfer functions.

Theorem E.2 *Consider the control system defined by*

$$y(s) = G(s)u(s)$$
$$u(s) = -\kappa y(s) + v(s).$$

If $G(s)$ is positive real, then for any positive constant feedback κ, the controlled system with input v and output y is stable and positive real.

F Minimum-Phase

Stable rational transfer functions are *minimum-phase* if they have no zeros in the closed right half-plane. For irrational transfer functions the situation is more complicated.

Definition F.1 *(Rosenblum &Rovnyak [36, pg. 94]) For a scalar transfer function $G(s) \in \mathbf{H}_\infty$ define the multiplication operator $\Lambda_G : \mathbf{H}_2 \to \mathbf{H}_2$ by $\Lambda_G f = Gf$ for all $f \in \mathbf{H}_2$. Then G is* **minimum-phase** *or* **outer** *if the range of Λ_G is dense in $\mathbf{H}_2$.*

Thus, minimum-phase systems are approximately invertible in the following sense: for every $y \in \mathbf{H}_2$ and $\epsilon > 0$ there is $u \in H_2$ with $\|Gu - y\|_2 < \epsilon$. For rational transfer functions this definition corresponds to having no zeros in the open right half-plane: $s/(s+1)$ is outer.

This is not true for irrational transfer functions. Consider a pure time-delay of T seconds which has transfer function e^{-Ts}. Consider its action on functions in $L_2(0,\infty)$. All functions in the range equal 0 on $[0,T]$ and so the range of this operator is not dense in $L_2(0,\infty)$. Since, as explained in Appendix A, H_2 corresponds to the Laplace transforms of functions in $L_2(0,\infty)$, a pure time-delay transfer function is not minimum-phase, even though it does not have any zeros. Note also that the inverse of a time-delay is an advance in time and so this operator cannot be inverted, even approximately.

Any function $G \in \mathbf{H}_\infty$ that does not have any zeros in the open right half-plane can be written (see Hoffman [19, pg. 133])

$$G(s) = e^{\imath\alpha} e^{-\rho s} \exp\left[-\int \frac{ts+\imath}{t+\imath s} \mathrm{d}\mu(t)\right] h(s), \tag{F.1}$$

where ρ, α are real, $\rho > 0$, μ is a finite singular positive measure on the imaginary axis and h is *outer* or minimum-phase. Even if a function $G \in \mathbf{H}_\infty$ has no zeros in the closed right half-plane, it may not be outer due to the presence of a delay $e^{-\rho s}$ or a singular factor $\exp\left[-\int \frac{ts+\imath}{t+\imath s}\mathrm{d}\mu(t)\right]$. However, transfer functions with real coefficients ($\overline{G(s)} = G(\bar{s})$) must have $\alpha = 0$. If, in addition, G is analytic on a set that includes the closed right half-plane, then there is no singular factor. In this case the factorization (F.1) simplifies to $G(s) = e^{-\rho s}h(s)$. One way to check whether or not $\rho = 0$ is to check if G has finite relative degree. We summarize this in the following useful theorem.

Theorem F.2 *(Jacob, Morris & Trunk [21]) The transfer function $G \in \mathbf{H}_\infty$ with real coefficients is minimum phase if it*

(1) has finite relative degree,

(2) is analytic on an open set containing the closed right half-plane and

(3) has no zeros in the open right half-plane.

The following result is also useful.

Theorem F.3 *(Jacob, Morris &Trunk [21]) The transfer function $G \in \mathbf{H}_\infty$ is minimum phase if it*

(1) has finite relative degree

(2) is analytic on an open set containing the closed right half-plane and

(3) is positive real.

G Balanced order reduction

For a stable finite-dimensional system with transfer function and realization (A, B, C) the controllability gramian L_c is the positive semi-definite solution to

$$AL_c + L_cA^* = -BB^*$$

while the observability gramian L_o is the positive semi-definite solution to

$$A^*L_o + L_oA = -C^*C.$$

Definition G.1 *The* **Hankel singular values** *of the system equal the positive square roots of the nonzero eigenvalues of the product L_BL_C.*

The singular values of a system do not depend on the realization; see for instance Antoulas [1].

Definition G.2 *A* **balanced realisation** *for a stable rational transfer function is one for which the controllability gramian and the observability gramian are equal: $L_B = L_C$.*

Truncation of a balanced realisation leads to a reduced order model with a $\mathbf{H}_\infty$-error that depends on the neglected singular values .

Theorem G.3 *(Moore [29]) If the original system (A, B, C) has order n with singular values $\sigma_k, k = 1, ...n$ and G^r is the truncated balanced realisation of order $r < n$, then*

$$\|G - G^r\|_\infty \leq 2 \sum_{k=r+1}^{n} \sigma_k.$$

This method is often used to obtain an approximation with a low order r and a known bound on the H_∞-error. The infinite-dimensional version of this result can be found in Glover, Curtain & Partington [16]. Using these results one can show that the Hankel singular values of G_{wave2} and G_{duct4} are $O(1/k)$. So they are not nuclear, although they can be approximated by rational functions in the $\mathbf{H}_\infty$-norm. We recall that this was not the case for G_{duct1}, G_{duct2} or G_{duct3}. Including a model of the actuator dynamics of the duct example introduces a low-pass filter into the transfer function, thus making it easier to approximate.

Fig. 1. Heat Flow in a Rod. The regulation of the temperature profile of a rod is the simplest example of a control system modelled by a partial differential equation. The control action can be effected by varying the heat flow at one end (Neumann boundary conditions) or by keeping the temperature at one end fixed (Dirichlet boundary conditions). The temperature is measured at some point along the rod. The choice of these boundary conditions and the measurement position affect the location of the poles and zeros of the system significantly.

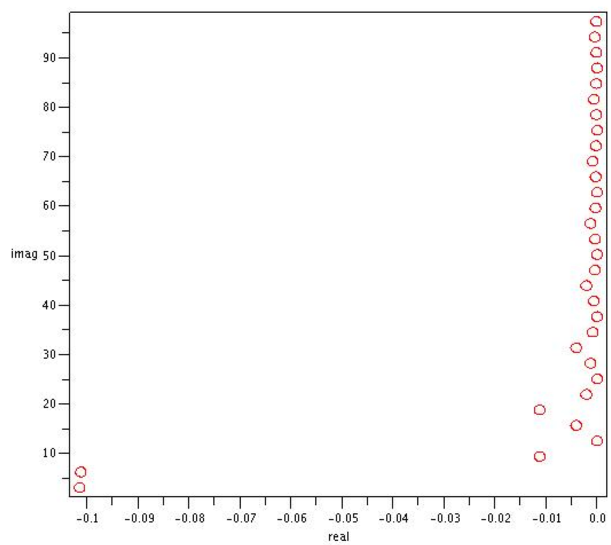

Fig. 2. Poles of a damped string with damping parameter $\varepsilon = 1$. In the idealized model of an undamped string, all poles lie on the imaginary axis. The introduction of damping to the model results in all poles lying in the open left half–plane. However, as $|s|$ increases, the magnitude of the real part decreases, which causes the poles to asymptote to the imaginary axis. The poles were calculated by finding zeros of the denominator of the transfer function.

Fig. 3. Acoustic noise in a duct. A noise signal is produced by a loudspeaker placed at one end of the duct. In this photo a loudspeaker is mounted midway down the duct where it is used to control the noise signal. The pressure at the open end is measured by means of a microphone as shown in the photo. How the behavior of the acoustic waves at the open end is modeled affects the location of the poles significantly. (Photo by courtesy of Prof. S. Lipshitz, University of Waterloo)

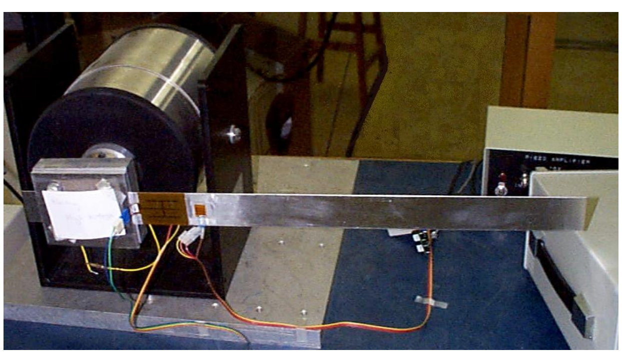

Fig. 4. A flexible beam is the simplest example of transverse vibrations in a structure. It has relevance to control of flexible robots. This photograph shows a beam controlled by means of a motor at one end. (Photo by courtesy of Prof. M.F. Golnaraghi, Simon Fraser University)

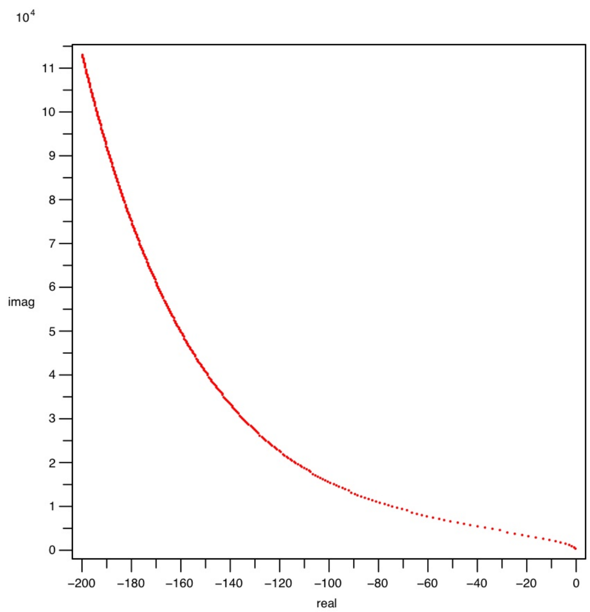

Fig. 5. Poles of Acoustic Noise in a Duct. In an idealized model the open end is assumed to have zero impedence ($p(L,t)=0$), which results in all poles lying on the imaginary axis. In a model assuming a constant nonzero impedence at the open end the poles are located on a vertical line in the left half-plane. However, if a more detailed model is used to obtain the impedence, (21) , the locations of the poles, shown in this figure, are quite different. All poles are all in the left half-plane and as $|s|$ increases they asymptote to the exponential curve $|y| = \beta e^{-2L\frac{s}{c}}$. ($a = .101$ m, $c = 341$ m/s, $L = 3.54$ m, $\rho = 1.20\text{kg/m}^3$) The poles were calculated by finding zeros of the denominator of the transfer function.

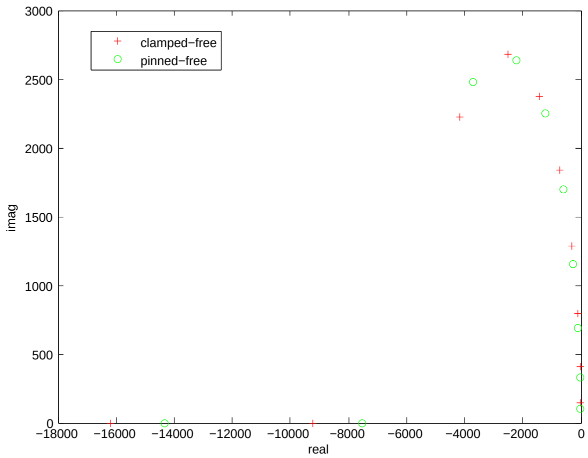

Fig. 6. Poles of Clamped-free and Pinned-free Beams. This figure shows the poles of two beams that are identical except that in one beam one end is clamped, while in the other beam the same end is pinned. (In both beams the other end is free.) The poles were calculated by finding zeros of the denominator of each transfer function. Although the asymptotic behavior of the poles of both beams is similar, the positions of the poles vary considerably, depending on the type of boundary conditions. ($L = 1$m, $E = 2.68 \times 1010$N/m2, $I = 1.64 \times 10^{-9}$m4, $C_v = 2$Ns/m, $Cd = 107$Ns/m2)

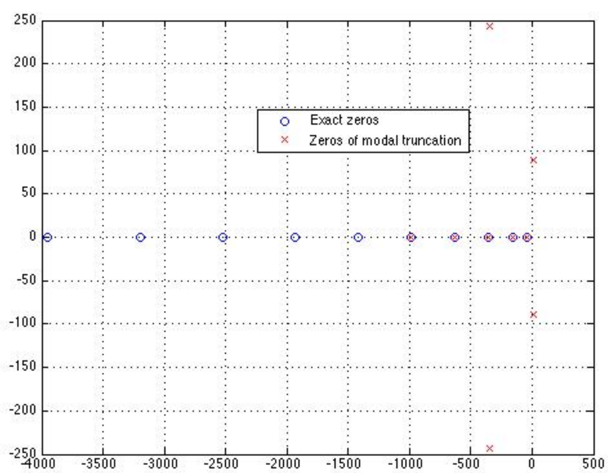

Fig. 7. The zeros of a rational approximation can be quite different from the exact zeros. This figure shows the exact zeros of a heat equation on $[0, 1]$ with Dirichlet control at $x = 1$ and temperature measurement at $x = 1/2$, compared to the zeros of a modal approximation comprising the first 10 terms in the modal expansion. Note that the exact and approximated zeros are remarkably different. The original system is minimum phase and all the zeros are real and negative. However, the rational approximation contains complex zeros, including some in the open right half-plane. Consequently, the rational approximation is not minimum phase.