Algebraic Multigrid for Graphs and Images

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Overview

• discretized PDE matrices on unstructured grids can be interpreted as weighted graph matrices (with the graph edges corresponding to the grid edges)

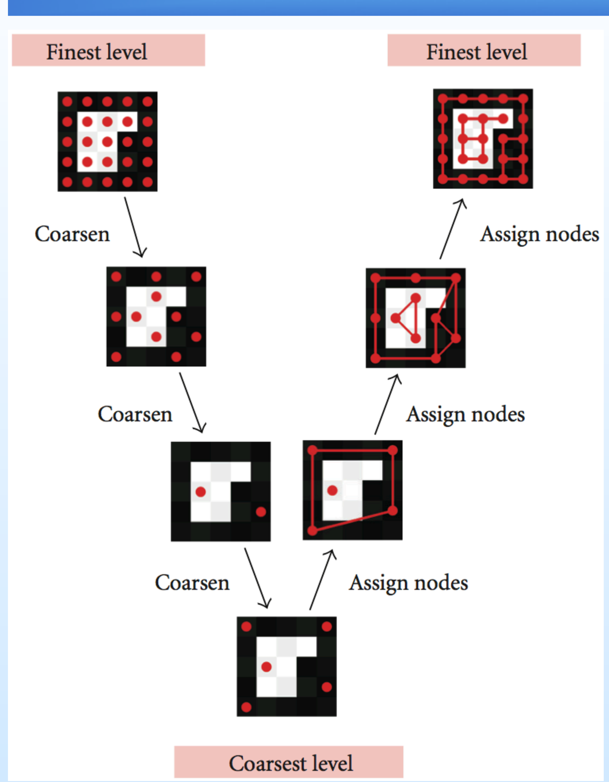
• this analogy opens the door to applying ideas from multilevel numerical methods for PDEs on unstructured grids to graph problems

• in the context of AMG, this was first done by Brandt and co-workers [1,2,3], for problems in image segmentation [1,2] and in clustering and manifold detection [3]

• AMG coarsening, interpolation, and variational coarse operator definition can be applied directly to graph problems, to create hierarchical overlapping groupings of graph nodes (nested image segments or graph clusters) and detect salient groupings

• in [4], we extend the 'Segmentation by weighted aggregation' algorithm of [1,2,3] to handle sequences of images and to employ a scale-invariant saliency measure, and apply it to microscope movies of cell division; we also consider satellite images here

AMG for Image Segmentation



• weighted adjacency matrix based on intensity differences of neighbouring pixels:

$$A_{ij}^{[1]} = \begin{cases} e^{-\alpha|I_i^{[1]}-I_j^{[1]}|} & \text{if } i, j \text{ are neighbours,} \\ 0, & \text{otherwise,} \end{cases}$$

• we use standard AMG Ruge-Stueben coarsening (1 pass) using standard AMG strength of connection: $A_{ij}^{[r]} \geq \theta \sum_{k \neq i} A_{ik}^{[r]}$

• we use a standard interpolation matrix:

$$P_{ij}^{[1,2]} = \begin{cases} 1 & \text{if } i \in C^{[1]},\ i = C_j^{[1]} \\ 0 & \text{if } i \in C^{[1]},\ i \neq C_j^{[1]} \\ \dfrac{A_{iC_j^{[1]}}^{[1]}}{\sum_{k \in C^{[1]}} A_{ik}^{[1]}} & \text{if } i \notin C^{[1]}. \end{cases}$$

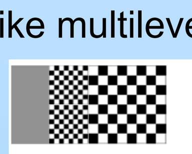
• we use standard Galerkin coarsening to define the coarse weight matrix: $A^{[2]} = P^{[1,2]T} A^{[1]} P^{[1,2]}$

Additional Algorithmic Components

The SWA algorithm from [1,2,3] introduces:

• rescale coarse weights based on average intensity: $A_{ij}^{[r+1]} \leftarrow A_{ij}^{[r+1]} e^{-\tilde{\alpha}|I_i^{[r+1]}-I_j^{[r+1]}|}$

• rescale coarse weights based on coarse features like multilevel variance, shape, and orientation:



$$var(X) = E((X - E(X))^2) = E(X^2) - E(X)^2$$

$$A_{ij}^{[r+1]} \leftarrow A_{ij}^{[r+1]} e^{-\beta||\mathbf{s}_i^{[r+1]}-\mathbf{s}_j^{[r+1]}||_2}$$

• detect salient segments in upward stage using saliency measure, and sharpen segment boundaries

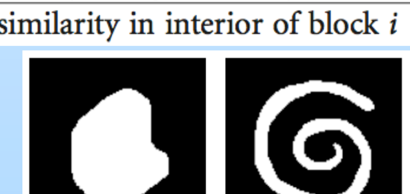
In [4] we add:

• scale-invariant saliency measure:

$$\Gamma_i^{[r]} = \frac{\text{average similarity along boundary of block } i}{\text{average similarity in interior of block } i}$$

$$\Gamma_i^{[r]} = \frac{L_{ii}^{[r]}/G_{ii}^{[r]}}{W_{ii}^{[r]}/V_{ii}^{[r]}}$$

$$L_{ij}^{[r]} = \begin{cases} -A_{ij}^{[r]} & \text{if } i \neq j, \\ \sum_{k \neq i} A_{ik}^{[r]} & \text{if } i = j, \end{cases}$$

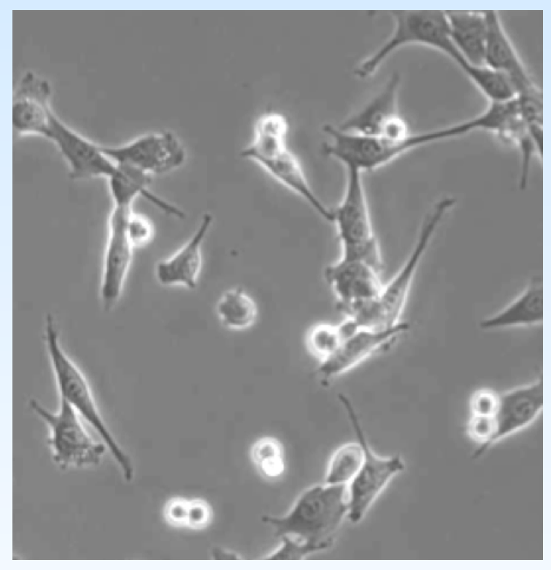


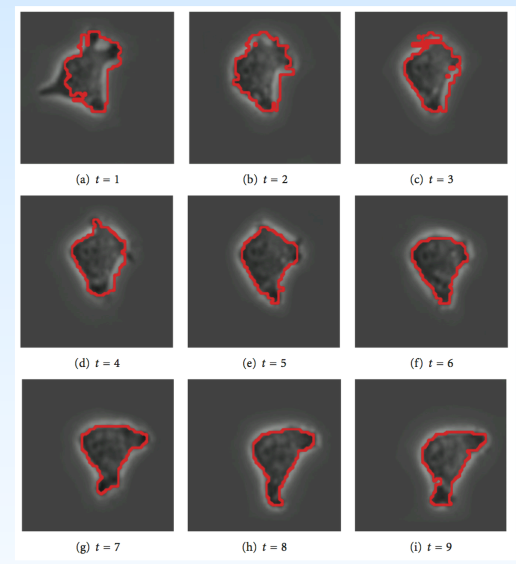
(a)Small L_{ii}, large W_{ii}. (b)Large L_{ii}, small W_{ii}.

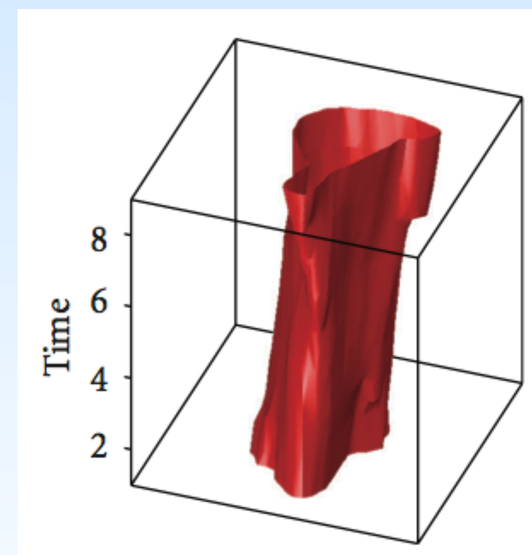
$W^{[r]} = A^{[r]}$

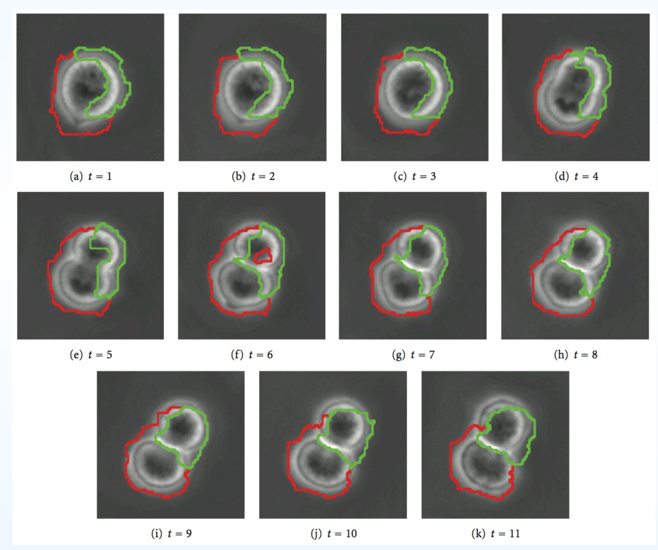
• 3D segmentation for images stacked in time

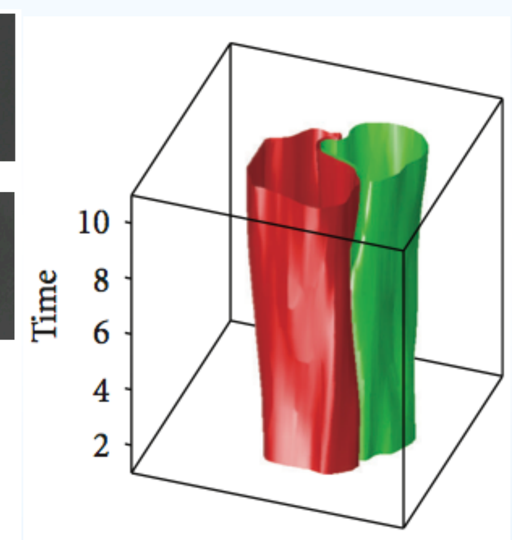
Numerical Results

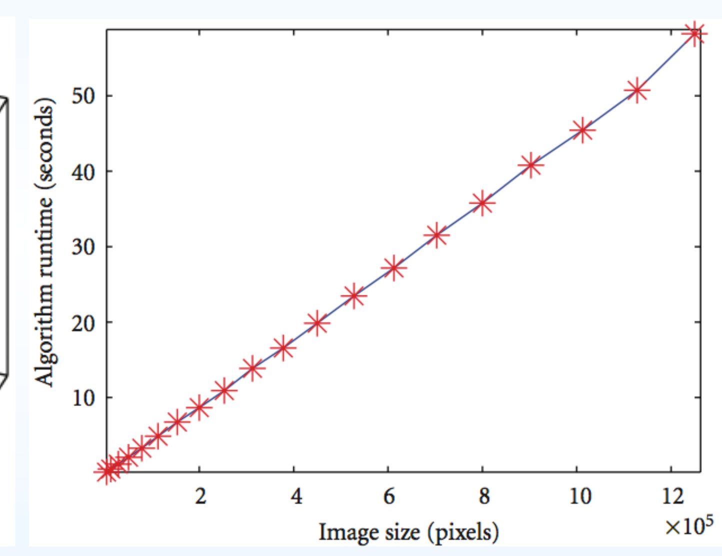


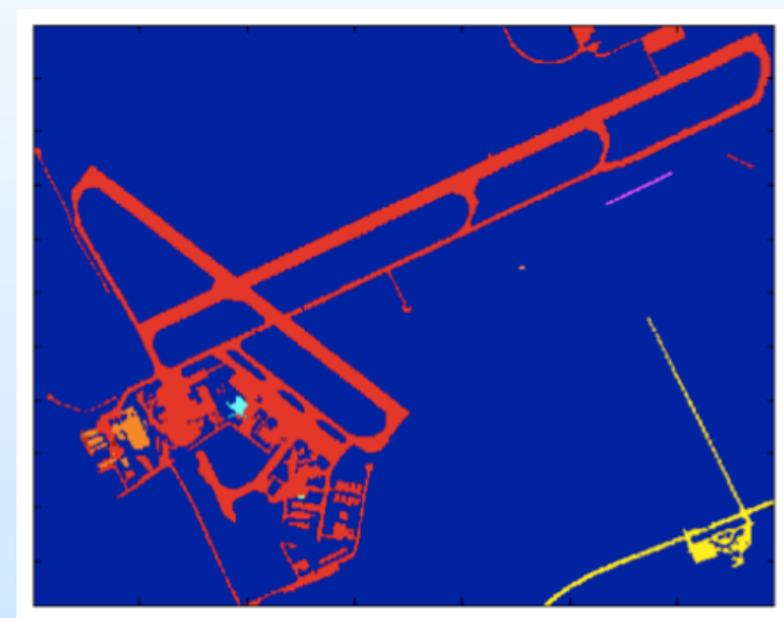












References

[1] Eitan Sharon, Achi Brandt, and Ronen Basri. "Fast multiscale image segmentation." Proc. IEEE Computer Society Conference on Computer Vision and Pattern Recognition, pp. 70-71, 2000.

[2] Eitan Sharon, Meirav Galun, Dahlia Sharon, Ronen Basri, and Achi Brandt. "Hierachy and adaptivity in segmenting visual scenes." Nature 442:810-813, 2006.

[3] Dan Kushnir, Meirav Galun, and Achi Brandt. "Fast multiscale clustering and manifold identification." Pattern Recognition 39:1876-1891, 2006.

[4] Tiffany Inglis, Hans De Sterck, Geoffrey Sanders, Haig Djambazian, Robert Sladek, Saravanan Sundararajan, and Thomas J. Hudson. "Multilevel Space-Time Aggregation for Bright Field Cell Microscopy Segmentation and Tracking." International Journal of Biomedical Imaging, vol. 2010, Article ID 582760, 21 pages, 2010.