System and Network Synchronization

by

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Author's Declaration

I hereby declare that I am the sole author of this report. This is a true copy of the report, including any required final revisions, as accepted by my examiners.

I understand that my report may be made electronically available to the public.

Abstract

This report addresses the summary of chaos synchronization on systems as well as networks via various synchronization techniques which are commonly applied. For the system synchronization, multiple methods of controller design are presented to achieve chaos synchronization of systems; in addition, generalized synchronization is considered to handle the synchronization among systems with different orders. For the network synchronization, basic model constructions are introduced, which lead to the techniques of complete or cluster synchronization. The numerical simulations are also provided to support the theoretical analysis for each application.

Acknowledgements

First and foremost, I would like to thank my supervisor Dr Xinzhi Liu. His insight and guidance have made my study exciting, and led this report to intriguing places. I feel very lucky to have been the student of such a great theorist at this interesting time for the subject. I also thank Dr Jun Liu for his time and patience in reviewing my work, and all the colleagues I have met for their generous help to make this possible.

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Chapter 1

Introduction

There are many real-world applications depending on chaotic behaviors. Either chaotic systems or chaotic networks exist in all fields of sciences and humanities, for instance, laser systems, electrical power grids, cellular and metabolic networks, etc. Moreover, many real systems in nature, such as biological and social systems can be described by various chaotic models of complex networks. All of these can also be extended by introducing dynamical elements into the network nodes, and many of these large-scale complex dynamical networks display a collective synchronization motion. In general, there are three categories for synchronization: complete synchronization, partial synchronization and generalized synchronization. As a result, chaos synchronization has been the considerable interest within science and technology communities. For instance, chaos synchronization of horizontal platform system[3], synchronization of fractional order chaotic systems [4], synchronization of the unified chaotic system [5-6],etc.

Generally speaking, the synchronization process can be achieved between so-called drive and response systems (networks), while by designing controllers (u) attached to response system (network) and by determining topological connection (i.e., coupling matrix). Mostly, the synchronization model could be simply constructed as follows:

$$\begin{aligned} \dot{x} &= f(x) \\ \dot{y} &= g(y) + u(x, y) \end{aligned} \tag{1.0.1}$$

$$\begin{aligned} \dot{x}_i &= f(x_i) + \sum_{j=1}^{N} c_{ij} \quad x_j, i = 1, 2, ..., N \\ \dot{y}_i &= g(y_i) + \sum_{j=1}^{N} c_{ij} \quad y_j + u_i, i = 1, 2, ..., N \end{aligned} \tag{1.0.2}$$

for systems and networks respectively; where the error systems (networks) are defined directly as: $e = x \quad y$. Meanwhile, synchronization is one of the stability problems, hence techniques of solving Lyapunov function will be heavily use in the theoretical analysis.

Constructively, this report provides a comprehensive overview of various synchronization patterns demonstrated by several specific chaotic systems and networks. Our contribution on this topic basically lies in the combinatorial aspect of the synchronization types; therefore the chapters are going to be divided into four major aspects: system complete synchronization, system generalized synchronization, network complete synchronization and network cluster synchronization. For each section, we will propose by model construction, theoretical analysis and numerical simulations.

Chapter 2

System Complete Synchronization Techniques

2.1 Global Synchronization with Linear Feedback Control

It is well-known that Chua's system is the first analog circuit to realize chaos in experiments.It is a simple electronic circuit that exhibits classic chaos behavior and an autonomous circuit made from standard components(resistor,capacitors,inductors), which must satisfy three criteria to display chaotic behavior: one or more nonlinear elements;one or more locally active resistors; and three or more energy-storage elements.The general Chua's system is defined as:

$$\begin{aligned} \dot{x} &= p(y - x - g(x)) \\ \dot{y} &= x - y + z \\ \dot{z} &= -qy \end{aligned} \tag{2.1.1}$$

where $x, y, z \in R$ are state variables representing voltage cross capacitors and electric current in the inductor; $g(x) = G_b x + \frac{1}{2}(G_a - Gb)(|x + E| - |x - E|)$ describes the electrical response of nonlinear resistor, G_a, G_b, E are constants; $p > 0, q > 0$ are determined by particular value of circuit components. In particular, we consider the new Chua's system where $g(x) = x\sqrt{sin(x) + |x|}$, and the chaotic behavior will exhibit when choosing p=11,q=14.87:

$$\begin{aligned} \dot{x} &= p(y - x - x\sqrt{sin(x) + |x|}) \\ \dot{y} &= x - y + z \\ \dot{z} &= -qy \end{aligned} \tag{2.1.2}$$

phase potrait.jpg

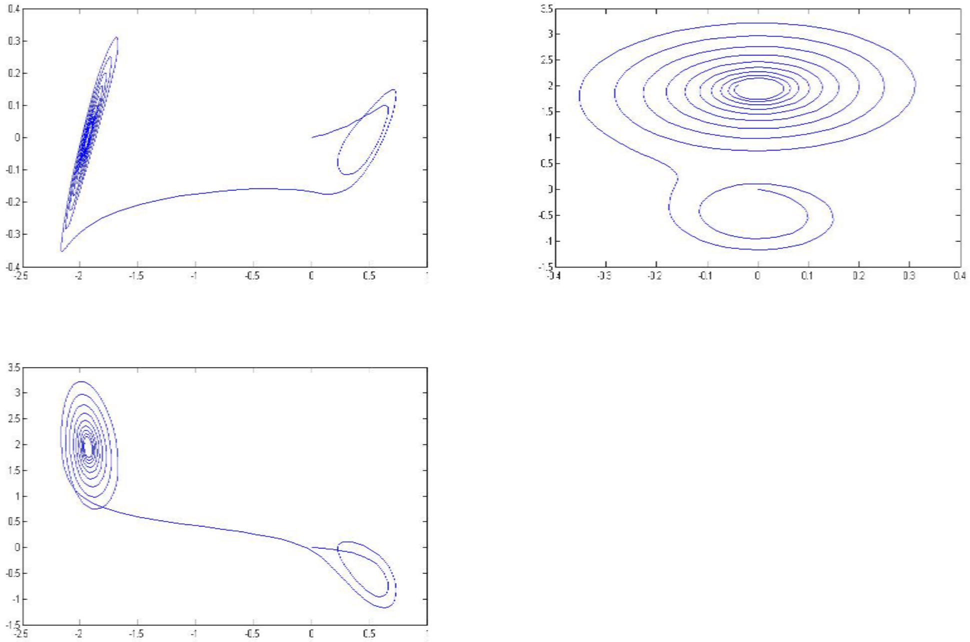

Figure 2.1: Phase Portrait of Chua's system

Based on the new Chua's system (2.1.1), its response system and error system can be described as:

$$\begin{aligned} \dot{x_2} &= p(y \quad 2 \quad x_2 \quad x_2\sqrt{sin(x_2) + |x_2|}) + u_1 \\ \dot{y_2} &= x_2 \quad y_2 + z_2 + u_2 \\ \dot{z_2} &= \quad qy_2 + u_3 \end{aligned} \tag{2.1.3}$$

where $u_i(i = 1, 2, 3)$ is feedback control input which satisfies $u_i(0, 0, 0) = 0$.

$$\begin{aligned} \dot{e_x} &= p(e_x + e_y \quad f(e_x)) + u_1 \\ \dot{e_y} &= e_x \quad e_y + e_z + u_2 \\ \dot{e_z} &= \quad qe_y + u_3 \end{aligned} \tag{2.1.4}$$

Theorem 2.1.1. *If the following linear controller is added to the error system* (2.1.4), $u_1 = \quad p\sigma_x e_x, u_2 = u_3 = 0$ *where* σ_x *is any parameter given beforehand such that* $\sigma_x > 2$. *Then, the zero solution of system* (2.1.4) *is globally exponentially stable.*

Theorem 2.1.2. *If the following linear controller is added to the error system* (2.1.4), $u_1 = \quad p\delta_x e_x, u_2 = \quad \delta_y, u_3 = 0$ *where* $\delta_x and \delta_y$ *is any parameter given beforehand such that* $\delta_x > 1 and \delta_y > \frac{1}{\delta_x \quad 1} \quad 1$*. Then, the zero solution of system* (2.1.4) *is globally exponentially stable.*

Both theorems above provide us the construction of linear controller; the proofs are pretty similar, thus the proof of Theorem 1.1.1 will be presented:

Proof of Theorem 1.1.1:

Consider the following positive definite and radially unbounded Lyapunov function:

$$V_1 = \frac{1}{p}e_x^2 + e_y^2 + \frac{1}{q}e_z^2 \quad \varepsilon e_y e_z = e^T G_1 e \tag{2.1.5}$$

where

$$G_1 = \begin{pmatrix} \frac{1}{p} & 0 & 0 \\ 0 & 1 & \varepsilon/2 \\ 0 & \varepsilon/2 & \frac{1}{q} \end{pmatrix}$$

Let $\lambda_m(G_1) and \lambda_M(G_1)$ be the minimum and maximum eigenvalues of G_1, respectively. Then:

$$\lambda_m(G_1)(e_x^2 + e_y^2 + e_z^2) \leq V_1 \leq \lambda_M(G_1)(e_x^2 + e_y^2 + e_z^2) \tag{2.1.6}$$

Also, $g(x) = x\sqrt{sin(x) + |x|}$ can be showed to be a strictly monotonically increasing function to obtain $0 \leq e_x f(e_x) < \infty$.

Differentiating V_1 with respect to time:

$$\dot{V_1} = e^T G_2 \quad 2e_x f(e_x) \leq e^T G_2 \leq \lambda_M(G_2)\frac{\lambda_M(G_1)}{\lambda_m(G_1)}(e_x^2 + e_y^2 + e_z^2) \leq \lambda_M(G_2)\frac{\lambda_M(G_1)}{V}{}_1 \tag{2.1.7}$$

where

$$G_2 = \begin{pmatrix} 2(1 \quad \sigma_x) & 2 & \varepsilon/2 \\ 2 & \varepsilon q \quad 2 & \varepsilon/2 \\ \varepsilon/2 & \varepsilon/2 & \varepsilon \end{pmatrix}$$

In order to ensure G_1, G_2 are positive definite and negative definite respectively, the following inequalities have to be satisfied:

$$\begin{aligned} &\varepsilon > 0 \\ &\varepsilon < \frac{8}{4q+1} \\ &\sigma_x > \frac{64 \quad 16\varepsilon + \varepsilon^2}{64 \quad 32\varepsilon q \quad 8\varepsilon} + 1 + \frac{\varepsilon}{8} \end{aligned} \tag{2.1.8}$$

which leads to $0 < \varepsilon \leq 1, \sigma_x > 2$.
Then it holds that $V_1(t) \leq V_1(t_0) e^{\lambda_M(G_2) \frac{M \ G_1)}{} t \ \ t_0)}$ and
$e_x^2 + e_y^2 + e_z^2 \leq \frac{V_1(t)}{\lambda_m(G_1)} \leq \frac{V_1(t_0)}{\lambda_m(G_1)} e^{\frac{M \ G_2)}{M \ G_1)}(t \ \ t_0)} \leq \frac{\lambda_M(G_1)}{\lambda_m(G_1)} (e_x^2(t_0) + e_y^2(t_0) + e_z^2(t_0)) e^{\frac{M \ G_2)}{M \ G_1)}(t \ \ t_0)}$.
Consequently, e_x^2, e_y^2, e_z^2 converge to zero exponentially, thus systems (2.1.2) and (2.1.3) are globally exponentially synchronized.

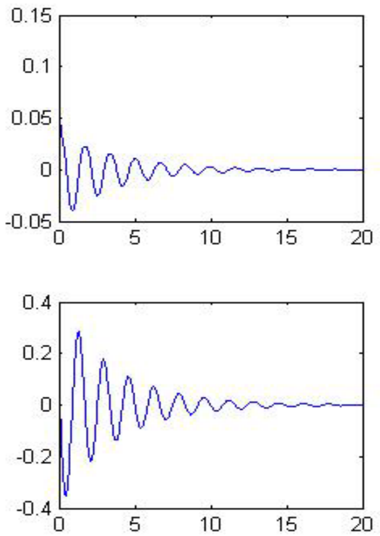

Figure 2.2: Error system with linear controller in Theorem 1.1.1

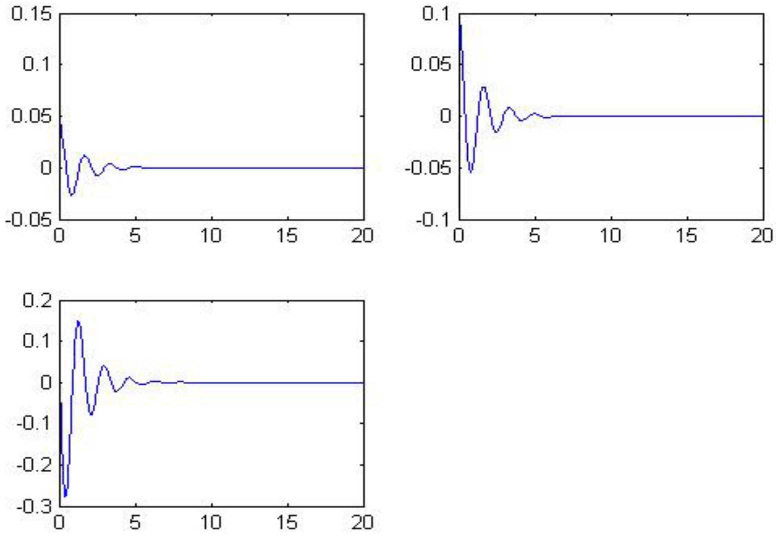

Figure 2.3: Error system with linear controller in Theorem 1.1.2

2.2 Global Synchronization with Decoupling Feedback Control

For the error system (2.1.4), a decoupling feedback controller will be introduced to realize global exponential synchronization. Since g(x) is strictly monotonically increasing and $g(x_2) \quad g(x_1) = k(x_1, x_2)(x_2 \quad x_1)$, then $k(x_1, x_2) \geq 0$ and the error system becomes:

$$
\begin{aligned}
\dot{e_x} &= p(e_x + e_y \quad k(x_1, x_2)(e_x)) + u_1 \\
\dot{e_y} &= e_x \quad e_y + e_z + u_2 \\
\dot{e_z} &= \quad q e_y + u_3
\end{aligned} \tag{2.2.1}
$$

Theorem 2.2.1. *Choose $\gamma_x > p > 0$, apply feedback $u_1 = \quad \gamma_x e_x$ to the first equation of error system above and cross-feedback $u_2 = \quad e_x$ to the second equation, let $u_3 = 0$. Then, the zero solution of system* (2.2.1) *is globally exponentially stable, so that systems* (2.1.2) *and* (2.1.3) *are globally exponentially synchronized.*

Proof. After applying the controller in Theorem 2, system (2.2.1) becomes:

$$
\begin{aligned}
\dot{e_x} &= p(e_x + e_y \quad k(x_1, x_2)(e_x)) \quad \gamma_x e_x \\
\dot{e_y} &= \quad e_y + e_z \\
\dot{e_z} &= \quad q e_y
\end{aligned} \tag{2.2.2}
$$

then the $e_y and e_z$ realize decoupling with e_x and their coefficient matrix is Hurwitz, thus their zero solution is globally exponentially stable (i.e there exist two positive constants $\quad, \beta > 0$ such that $|e_y| \leq \beta e^{\quad t}, |e_z| \leq \beta e^{\quad t}$.

Nest, construct a raidally unbounded and positive definite Lyapunov function $|e_x|$ for the first equation of (2.2.2), and computing the up-right-hand Dini derivative of $|e_x|$ along the trajectory of the first equation of system (2.2.2), one has:

$$
D^+|e_x| \leq (p \quad pk(x_1, x_2) \quad \gamma_x)|e_x| + p|e_y| \leq (p \quad \gamma_x)|e_x| + p|e_y| \tag{2.2.3}
$$

and compare it with the following:

$$
\dot{\xi} = (p \quad \gamma_x)\xi + p|e_y| \tag{2.2.4}
$$

then it is easy to obtain the estimation solution of (2.2.4):

$$
|\xi(t)| \leq |\xi(0)|e^{(p \quad \gamma_x)t} + \int_0^t e^{(p \quad \gamma_x)(t \quad \tau)} p|e_y(\tau)| d\tau \leq |\xi(0)|e^{(p \quad \gamma_x)t} + p\beta e^{(p \quad \gamma_x)t} \int_0^t e^{(p \quad \gamma_x \quad \quad)\tau} d\tau \tag{2.2.5}
$$

If $p + \gamma_x \qquad = 0$, then $|\xi(t)| \leq |\xi(0)|e^{(p \quad \gamma_x)t}$; if $p + \gamma_x \qquad > 0$, it holds that:

$$|\xi(t)| \leq |\xi(0)|e^{(p \quad \gamma_x)t} + p\beta e^{(p \quad \gamma_x)t} \frac{e^{(\quad p+\gamma_x \quad)t \quad 0}}{p + \gamma_x \quad alpha} \leq (|\xi(0)| + \frac{p\beta}{p + \gamma_x})e^{\quad min(\quad , \quad p+\gamma_x)t} \tag{2.2.6}$$

Finally, by comparing (2.2.6) with the theorem in [Liao,2010], we can conclude the zero solution of (2.2.2) is globally exponentially stable; thus systems (2.1.2) and (2.1.3) are globally exponentially synchronized.

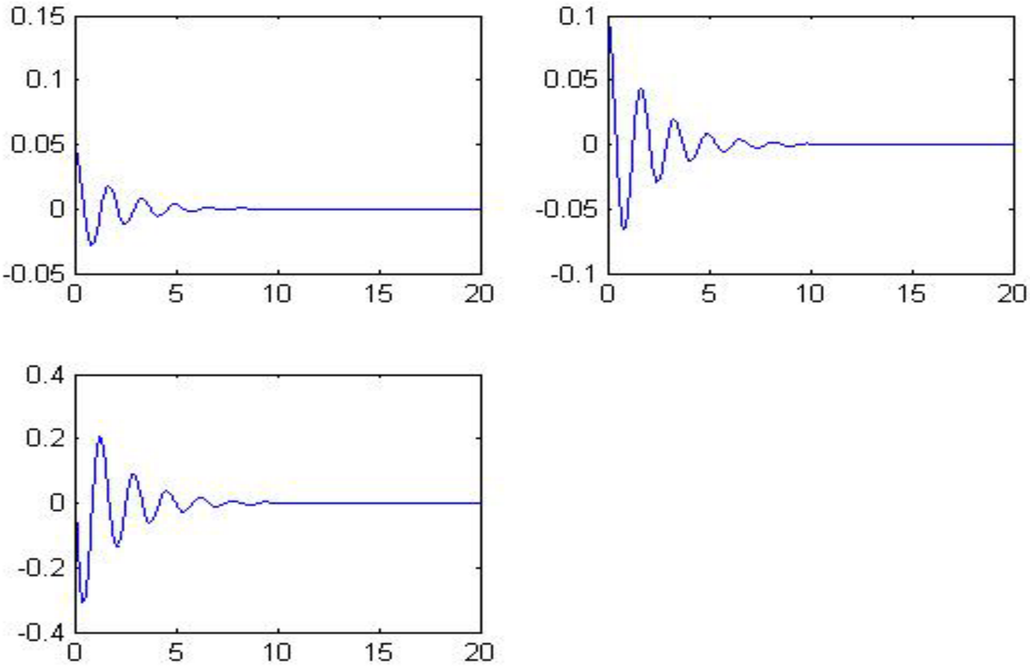

Figure 2.4: Error system by decoupling feedback

2.3 Global Synchronization with Adaptive Control

In general, the parameters p and q in system (2.1.2) are uncertain, thus the following response system is considered:

$$
\begin{aligned}
\dot{x_2} &= \hat{p}(y \quad 2 \quad x_2 \quad x_2\sqrt{sin(x_2) + |x_2|}) + u_1 \\
\dot{y_2} &= x_2 \quad y_2 + z_2 + u_2 \\
\dot{z_2} &= \quad \hat{q}y_2 + u_3
\end{aligned}
\tag{2.3.1}
$$

where $\hat{p}$ and $\hat{q}$ are the estimates of the uncertain p and q. Let $\bar{p} = \hat{p} \quad p$ and $\bar{q} = \hat{q} \quad q$, then the error system of (2.1.2) and (2.3.1) as follows:

$$
\begin{aligned}
\dot{e_x} &= \hat{p}(e_x + e_y \quad x_2\sqrt{sin(x_2) + |x_2|} + x_1\sqrt{sin(x_1) + |x_1|}) + \bar{p}(x_1 + y_1 \quad x_1\sqrt{sin(x_1) + |x_1|}) + u_1 \\
\dot{e_y} &= e_x \quad e_y + e_z + u_2 \\
\dot{e_z} &= \quad \hat{q}e_y \quad \bar{q}y_1 + u_3
\end{aligned}
\tag{2.3.2}
$$

Theorem 2.3.1. *If the following adaptive controller is added to the error system (16):*

$$
\begin{aligned}
u_1 &= \quad \hat{p}(e_x + e_y \quad x_2\sqrt{sin(x_2) + |x_2|} + x_1\sqrt{sin(x_1) + |x_1|}) \quad e_x \\
u_2 &= \quad e_x \quad e_z \\
u_3 &= \hat{q}e_y \quad e_z \\
\dot{\hat{p}} &= \dot{\bar{p}} = \quad e_x(x_1 + y_1 \quad x_1\sqrt{sin(x_1) + |x_1|}) \\
\dot{\hat{q}} &= \dot{\bar{q}} = e_z y_1
\end{aligned}
\tag{2.3.3}
$$

then the equilibrium points of system (2.3.2) *with adaptive control law* (2.3.3) *are globally asymptotically stable. And thus, the systems* (2.1.2) *and* (2.3.1) *are globally synchronized. Also, the parameter estimates* $\hat{p}$ *and* $\hat{q}$ *converge to p and q as t tends to infinity respectively.*

Proof. Applying the same technique, the Lyapunov function candidate is constructed as follows:

$$V = \frac{1}{2}(e_x^2 + e_y^2 + e_z^2 + \bar{p}^2 + \bar{q}^2) \quad (2.3.4)$$

then take the derivative of it:

$$\begin{aligned}
\dot{V} =& e_x(\hat{p}(e_x + e_y \quad x_2\sqrt{sin(x_2) + |x_2|} + x_1\sqrt{sin(x_1) + |x_1|} + u_1)) \\
&+ e_y(e_x \quad e_y + e_z + u_2) \\
&+ e_z(\quad \hat{q}e_y + u_3) \quad \hat{q}(e_z y_1 \quad \dot{\hat{q}}) \\
&+ \bar{p}(e_x(x_1 + y_1 \quad x_1\sqrt{sin(x_1) + |x_1|}) + \dot{\bar{p}}) \\
&= \quad e_x^2 \quad e_y^2 \quad e_z^2 \leq 0
\end{aligned} \quad (2.3.5)$$

By using LaSalle-Yoshizawa Theorem, all the equiliburim points of the closed loop systems are globally stable. Moreover, the error system (2.3.2) converges to zero, and systems (2.1.2) and (2.3.1) are globally synchronized.

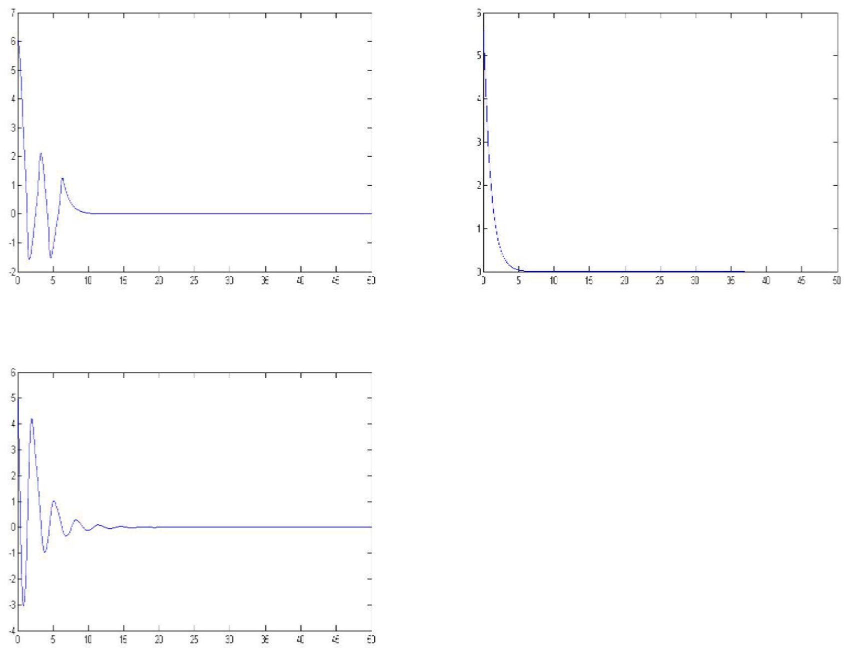

Figure 2.5: Error system with adaptive controller

converge.jpg

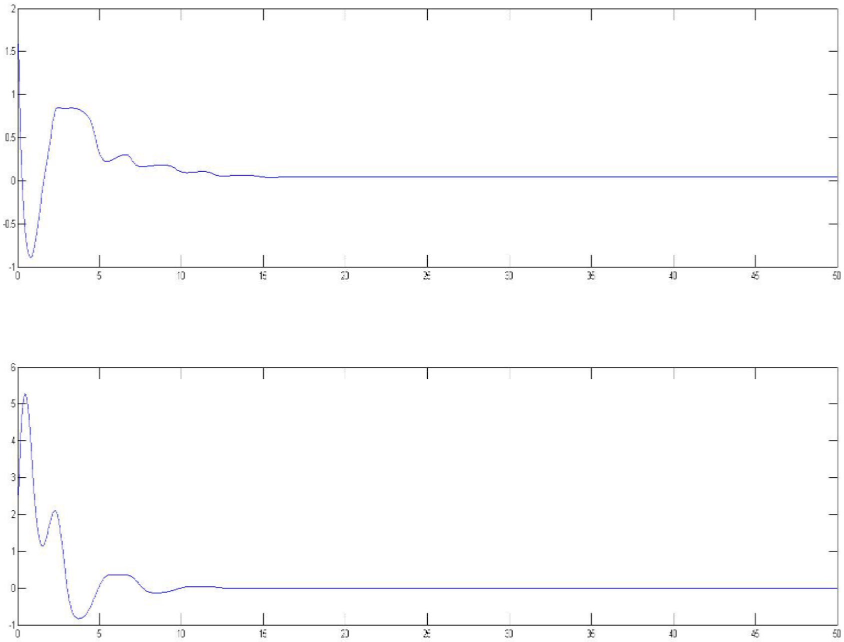

Figure 2.6: Convergence of $\bar{p}$ and $\bar{q}$

(*)Notice that based on the simulation, the values of $\bar{p}$ and $\bar{q}$ are converging to certain value (i.e zero) as time increases; however, $\bar{p}$ and $\bar{q}$ seems not always converge to zero with different choice of estimator $\hat{p}$ and $\hat{q}$.

2.4 Global Synchronization with Time Delay

Based on the general Chua's system (with g(x)) and its response system (with $\bar{g}(x)$), the delay synchronization controller can be constructed as follows:

$$\begin{aligned} u_1 &= p(\bar{g}(x_2)\quad g(x_1)\quad e_x(t)) + e_x(t\quad \tau) \\ u_2 &= e_x(t) \end{aligned} \tag{2.4.1}$$

then the error system can be described as:

$$\begin{aligned} \dot{e_x} &= pe_y(t)\quad 2pe_x(t)\quad e_x(t\quad \tau) \\ \dot{e_y} &= \quad e_y(t) + e_z(t) \\ \dot{e_z} &= \quad qe_y(t) \end{aligned} \tag{2.4.2}$$

Lemma 2.4.1. *The linear system of ODE's* $\dot{E}(t) = AE(t) + BE(t\quad \tau)$ *with* $A, B \in R^{nxn}$ *is asymptotical stable if and only if the matrix A+B is Hurwitz and the matrix* $C(s) = sI\quad (I\quad \tau\Delta(s)B)\ ^{1}(A+B)$ *is nonsingular for all s, where* $\Delta(s) = \frac{e^{\ s\tau}\ 1}{s\tau}$.

Theorem 2.4.2. *Under the delay synchronization controller, the error function* $E = (e_x, e_y, e_z)^T$ *between the drive and response systems satisfies* $\|U\ ^{1}E\|_2 \to 0$ *(i.e asymptotical synchronization) if and only if* $F(\sigma,\tau) \neq 0)$ *for all* $\sigma > 0$ *and* $e^{\ 2\sigma\tau} \geq \sigma^2$, *where the function* $F(\sigma,\tau)$ *is define by:*

$$\begin{aligned} F(\sigma,\tau) =& |\sigma e^{\sigma\tau} + cos(\tau\sqrt{e^{\ 2\sigma\tau}\quad \sigma^2})| \\ &+ |e^{\ 2\sigma\tau}\quad \sigma^2\quad e^{\ \sigma\tau} sin(\tau\sqrt{e^{\ 2\sigma\tau}\quad \sigma^2})| \end{aligned} \tag{2.4.3}$$

Proof. Notice system (2.4.2) can be written in a concise form as:

$$A = \begin{pmatrix} \ 2 & p & 0 \\ 0 & \ 1 & 1 \\ 0 & \ q & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} \ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then we can obtain A+B and all its eigenvalues with negative real part, thus Hurwitz with any q¿0. Choosing

$$\beta(s) = \begin{pmatrix} \frac{\ 3}{1+\tau\ \ (s)} & \frac{p}{1+\tau\ \ (s)} & 0 \\ 0 & \ 1 & 1 \\ 0 & \ q & 0 \end{pmatrix}$$

and its eigenvalues are $\lambda_1 = \frac{3}{1+\tau\ \ (s)}$,$\lambda_2 = \ \ \frac{1}{2}(1 + \sqrt{1\ \ \ 4q})$ and $\lambda_3 = \ \ \frac{1}{2}(1$ $\sqrt{1\ \ \ 4q})$. Clearly, for any s with positive real part, it holds $s \neq \lambda_{2,3}$. Then let $lambda_1 = 3s$ and $s = \sigma + i\omega$ we have:

$$
\begin{aligned}
& s + e^{\ \ s\tau} \quad 1 = \quad 1 \Leftrightarrow s + e^{\ \ s\tau} = 0 \\
\Leftrightarrow & \sigma e^{\sigma\tau} = \quad cos\omega\tau \\
& \omega = e^{\ \ \sigma\tau} sin(\omega\tau)
\end{aligned}
\tag{2.4.4}
$$

Finally we can eliminate ω by using $sin^2(\omega\tau) + cos^2(\omega\tau) = 1$, which is $\omega^2 + \sigma^2 = e^{\ \ 2\sigma\tau} \Rightarrow \omega = \pm\sqrt{e^{\ \ 2\sigma\tau} \quad \sigma^2})$, and substitute into (2.4.4):

$$
\begin{aligned}
& \sigma e^{\sigma\tau} = \quad cos(\tau\sqrt{e^{\ \ 2\sigma\tau} \quad \sigma^2}) \\
& e^{\ \ 2\sigma\tau} \quad \sigma^2 = e^{\ \ \sigma\tau} sin(\tau\sqrt{e^{\ \ 2\sigma\tau} \quad \sigma^2})
\end{aligned}
\tag{2.4.5}
$$

and this is equivalent to $F(\sigma, \tau) = 0$.

For a given $\tau > 0$, $g(\sigma) = e^{\ \ 2\sigma\tau} \quad \sigma^2$ is a decreasing function of σ and g(0)=1, thus it has a unique positive root $\sigma^*(\tau)$. Hence, the condition $F(\sigma, \tau) \neq 0$ only needs to be checked for $\sigma \in [0, \sigma^*(\tau)]$. However, there do not exist σ such that $F(\sigma, \tau) = 0$ when $\tau = 0.5, 2and4$. Consequently, the delay controller synchronizes drive and response systems asymptotically.

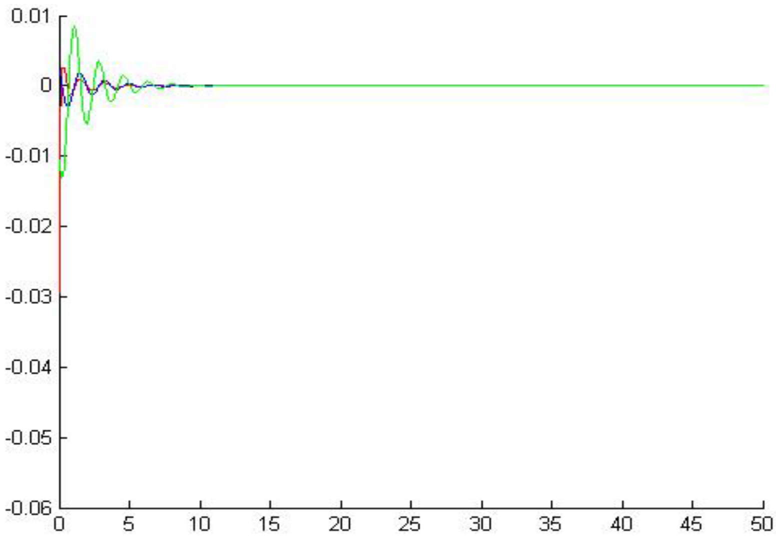

Figure 2.7: Error system with $\tau = 0.5$

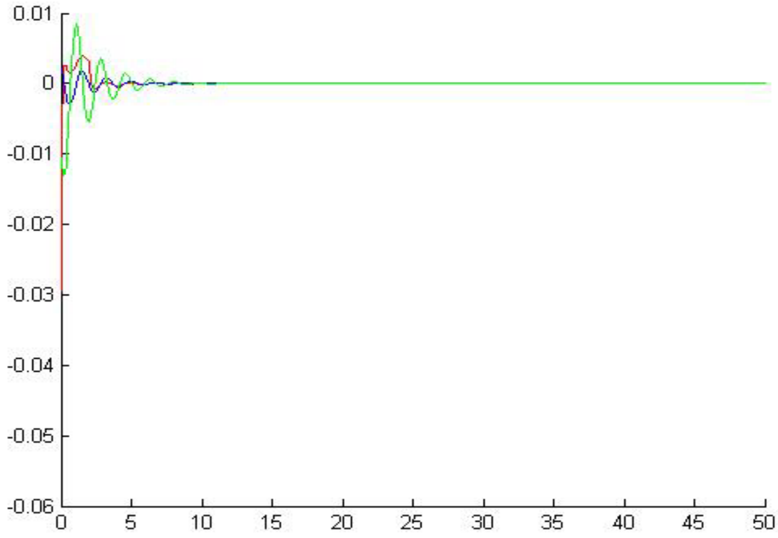

Figure 2.8: Error system with $\tau = 2$

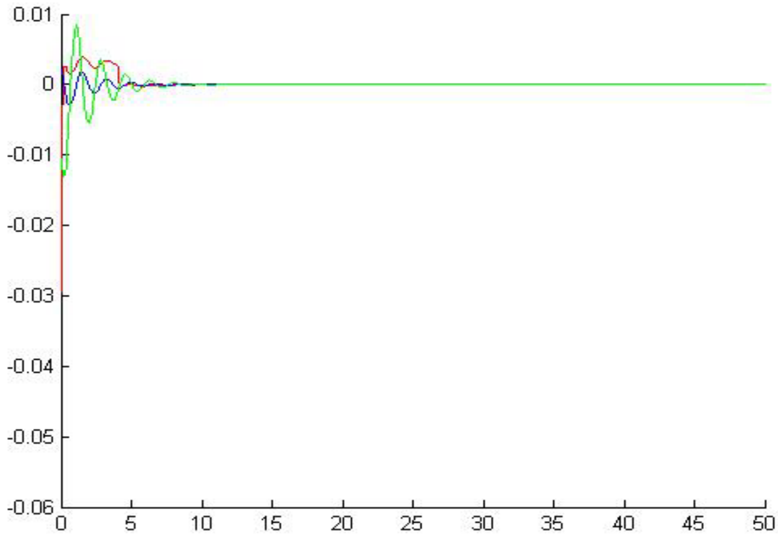

Figure 2.9: Error system with $\tau = 4$

2.5 Global Robust Adaptive Synchronization

Instead of the synchronization among identical systems, a robust adaptive synchronization method is considered to deal with the external perturbation on the response system. Here, we will implement this method on the hyperchaotic finance system, then transfer to the New Chua's system.

Consider the hyperchaotic finance system and its response system described by:

$$\dot{x}_m = f_m(x_m) + G_m(x_m)\theta \tag{2.5.1}$$

$$\dot{x}_s = f_s(x_s) + G_s(x_s)\theta + d_s(x_s, t) + u \tag{2.5.2}$$

where f_s and G_s are know maps, d_s is an unknown disturbance and θ is an unknown parameter vector. Then we can obtain the error system:

$$\dot{e} = f_s \quad f_m + (G_s \quad G_m)\theta + d_s + u \tag{2.5.3}$$

Remark. From the formulation above, the New Chua's system can be implemented without consideration of G_s and θ; also for simplification, f_s and f_m are considered to have similar structure. However, these nonlinear mappings can be unrelated.

Theorem 2.5.1. *Consider the systems* (2.5.1) *and* (2.5.2)*, which satisfies* $\|d_s(x,t)\| \leq d_{s0}$, $(d_{s0} \leq \bar{d}_s$*, both positive constants; and* $\|\theta\|$ *is upper bounded by a known positive constant* $\bar{\theta}$*. The control law*

$$u = \quad Ae \quad (f_s \quad f_m) \quad (G_s \quad G_m)\hat{\theta} + u_r \tag{2.5.4}$$

with

$$u_r = \quad \frac{\gamma_3 e}{\lambda_{min}(K)[\|e\| + \gamma_1 e^{\quad \gamma_0 t}]} \tag{2.5.5}$$

$$\dot{\hat{\theta}} = \quad \gamma_\theta[\gamma_2\|e\|\hat{\theta} + (G_s \quad G_m)^T K e] \tag{2.5.6}$$

where $A^T P + PA = \quad Q$*, -*A *is Hurwitz,* $P = P^T > 0$*,*Q*¿0,*$K = P + P^T$*,*$\gamma_0 \geq 0$*,*$\gamma_1, \gamma_2, \gamma_\theta > 0$*,* $\gamma_3 = 2\|K\|_F \bar{d}_0 + \gamma_2 \bar{\theta}^2$*,*$\gamma_4 = \lambda_{min}(Q)$*, then systems* (2.5.1) *and* (2.5.2) *synchronize.*

Proof. Consider the Lyapunov function candidate:

$$V = e^T P e + \frac{1}{2}\tilde{\theta}^T \gamma_\theta^{\quad 1}\tilde{\theta} \tag{2.5.7}$$

where $\tilde{\theta} = \hat{\theta} \quad \theta$. Then take the derivative of V:

$$\begin{aligned} \dot{V} =& \dot{e}^T P e + e^T P \dot{e} + \dot{\tilde{\theta}}^T \gamma_\theta {}^1 \tilde{\theta} \\ =& \quad e^T Q e + e^T K(u_r + d_s) \quad \gamma_2 \|e\| \tilde{\theta}^T \hat{\theta} \end{aligned} \tag{2.5.8}$$

Letting $\tilde{\theta}^T \hat{\theta} = \frac{1}{2}(\tilde{\theta}^2 + \hat{\theta}^2 \quad \theta^2)$ implies:

$$\begin{aligned} \dot{V} \le& \quad \gamma_4 \|e\|^2 \quad \frac{\gamma_2}{2} \|\tilde{\theta}\|^2 \|e\| + \frac{\gamma_3}{2} \|e\| \quad \frac{\gamma 3 \|e\|^2}{\|e\| + \gamma_1 e^{\quad \gamma_0 t}} \\ \le& \quad \|e\| (\gamma_4 \|e\| + \frac{\gamma_2}{2} \|\tilde{\theta}\|^2 \quad \frac{3\gamma_3}{2}) \end{aligned} \tag{2.5.9}$$

Hence $\dot{V} \le 0$ as long as $\|e\| \ge \frac{3\gamma_3}{2\gamma_4} \equiv \quad _e$ or $\|\tilde{\theta} \ge \frac{3\gamma_3}{\gamma_2}^{0.5} \equiv \quad _{\tilde{\theta}}$. Thus, e(t) and $\tilde{\theta}(t)$ are uniformly bounded. Moreover, define a region $\Omega = \{e | \|e\| \le \gamma_1 e^{\quad \gamma_0 t}\}$, then in case $\|e\| > \gamma_1 e^{\quad \gamma_0 t}$, we have $\dot{V} \le \quad \gamma_4 \|e\|^2$. Since V is bounded from below and non-increasing with time, we have:

$$\lim_{t \to \infty} \int_0^t \|e(\tau)\|^2 d\tau \le \frac{V(0) \quad V_\infty}{\gamma_4} < \infty \tag{2.5.10}$$

where $\lim_{t \to \infty} V(t) = V_\infty < \infty$. In addition, with the bounds on e, $\tilde{\theta}$,d and u_r, $\dot{e}$ is also bounded, thus $\dot{V}$ is uniformly continuous. By Barbalat's Lemma, $\lim_{t \to \infty} e(t) = 0$ for all $e \in \Omega^c$. In general, $\lim_{t \to \infty} e(t) = 0$ holds in the large with expanding Ω set.

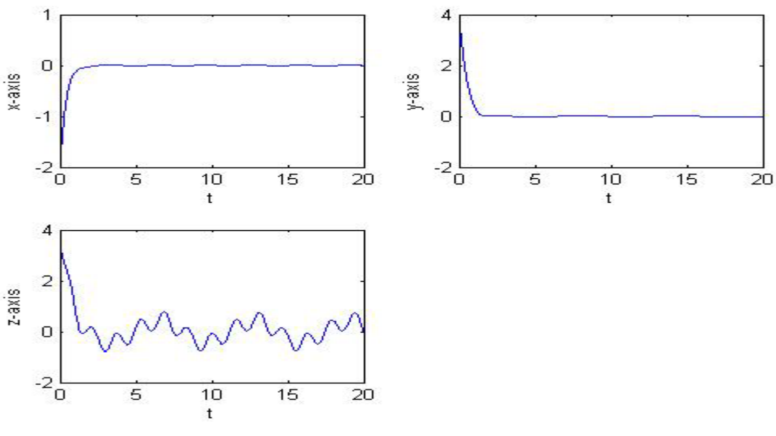

Figure 2.10: Error system with robust adaptive controller

Here, we apply the disturbance to the response New Chua's system, the error system converges to zero solution as expected.

2.6 Global Synchronization with Nonlinear Control

Consider the following three-dimensional autonomous system which displays two chaotic attractors simultaneously:

$$\begin{aligned} \dot{x} &= \quad \frac{ab}{a+b}x \quad yz + c \\ \dot{y} &= ay + xz \\ \dot{z} &= xy + bz \end{aligned} \tag{2.6.1}$$

where a,b,c are real constants, $a + b \neq 0$, and x,y,z are state variables. Also the system is chaotic for the parameters $a = \quad 10$,$b = \quad 4$ and $|c| < 19.2$.

Then we can construct the master and slave systems respectively:

$$\begin{aligned} \dot{x}_m &= \quad \frac{ab}{a+b}x_m \quad y_m z_m + c \\ \dot{y}_m &= ay_m + x_m z_m \\ \dot{z}_m &= x_m y_m + bz_m \end{aligned} \tag{2.6.2}$$

$$\begin{aligned} \dot{x}_s &= \quad \frac{ab}{a+b}x_s \quad y_s z_s + c + u_1 \\ \dot{y}_s &= ay_s + x_s z_s + u_2 \\ \dot{z}_s &= x_s y_s + bz_s + u_3 \end{aligned} \tag{2.6.3}$$

where $u_i, i = 1, 2, 3$ are nonlinear controllers which are designed to make sure above two systems can be synchronized.

Moreover, the error system can be derived as the following:

$$\begin{aligned} \dot{e}_1 &= \quad \beta e_1 \quad y_s e_3 \quad z_m e_2 + u_1 \\ \dot{e}_2 &= z_m e_1 + ae_2 + x_s e_3 + u_2 \\ \dot{e}_3 &= be_3 + x_s e_2 + y_m e_1 + u_3 \end{aligned} \tag{2.6.4}$$

where $\beta = \frac{ab}{a+b}$;$e_1 = x_s \quad x_m$,$e_2 = y_s \quad y_m$,$e_3 = z_s \quad z_m$

Theorem 2.6.1. *Systems* (2.6.2) *and* (2.6.3) *will approach global and exponential asymptotical synchronization for any initial condition with the following control law:*
$1.u_1 = (\beta \quad 1)e_1, u_2 = e_1e_3 \quad 2x_se_3, u_3 = 0$
$2.u_1 = (\beta \quad 1)e_1, u_3 = e_1e_2 \quad 2x_se_2, u_2 = 0$

Proof. For both cases, we select the Lyapunov function as $V(t) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2)$, then we obtain the derivative of the Lyapunoc function as $\dot{V}(t) = \quad e_1^2 + ae_2^2 + be_3^2$ for both conditions. Choosing $\quad = min(1, \quad a, \quad b) > 0$, then $\dot{V}(t) \leq \quad (e_1^2 + e_2^2 + e_3^2) = \quad 2 \quad V(t)$, which yields $V(t) \leq V(0)e^{\quad 2 \quad t}$. Therefore, the result implies that $lim_{t\to\infty}e_i = 0, i = 1, 2, 3$; in other words, systems (2.6.2) and (2.6.3) achieve global and exponential asymptotical synchronization.

In the numerical simulation, we set (3,4,2) and (5,-5,1) as the initial conditions of master and slave systems respectively; also we choose a=-10,b=-4 and c=15 to maintain the chaotic behaviors. Based on the graphs, the error system converges to zero solution after a short period of time, independent of initial conditions of master and slave systems. Thus, the nonlinear controllers are valid for chaos synchronization.

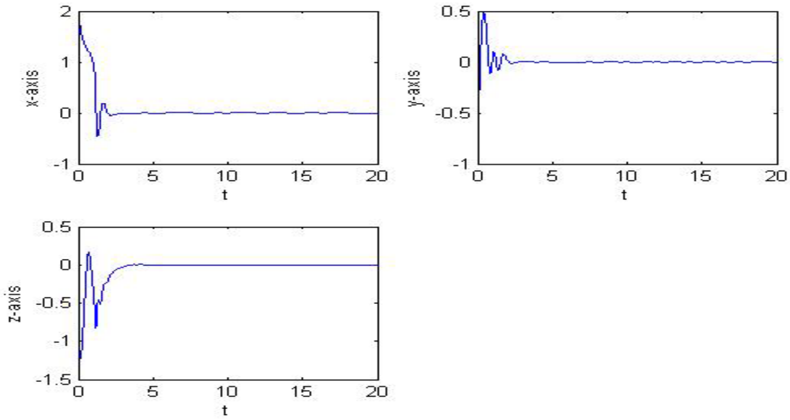

Figure 2.11: Error systems with type 1 controller

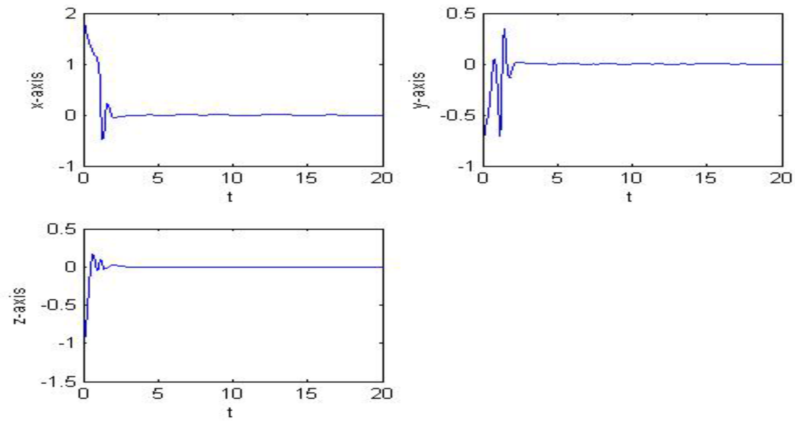

Figure 2.12: Error systems with type 2 controller

2.7 Global Synchronization with Sliding Mode Control

Consider the unified chaotic system:

$$\begin{aligned} \dot{x} &= (25 \quad + 10)(y \quad x) \\ \dot{y} &= (28 \quad 35 \quad)x + (29 \qquad 1)y \quad xz \\ \dot{z} &= xy \quad (\frac{8+}{3})z \end{aligned} \tag{2.7.1}$$

where $\in [0,1]$. If $\in [0,0.8)$, above system is called the general Lorenz system; if $= 0.8$, above system is called the general *Lü* system; if $\in [0.8,1]$, above system is called the general Chen system.

Thus we obtain the master and slave systems respectively:

$$\begin{aligned} \dot{x}_m &= (25 \quad + 10)(y_m \quad x_m) \\ \dot{y}_m &= (28 \quad 35 \quad)x_m + (29 \qquad 1)y_m \quad x_m z_m \\ \dot{z}_m &= x_m y_m \quad (\frac{8+}{3})z_m \end{aligned} \tag{2.7.2}$$

$$\begin{aligned} \dot{x}_s &= (25 \quad + 10)(y_s \quad x_s) \\ \dot{y}_s &= (28 \quad 35 \quad)x_s + (29 \qquad 1)y_s \quad x_s z_s + u_1 + \Delta f_1(x_s, y_s, z_s, p) \\ \dot{z}_s &= x_s y_s \quad (\frac{8+}{3})z_s + \Delta f_2(x_s, y_s, z_s, p) \end{aligned} \tag{2.7.3}$$

where $p \in R$ is the external perturbation; Δf_1 and Δf_2 are the uncertainties including parameter uncertainty and external perturbation applied to the slave system. In general,

$$\|\Delta f\| = \| \begin{pmatrix} \Delta f_1 \\ \Delta f_2 \end{pmatrix} \| \leq \beta_1 \| \begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix} \| + \beta_2 \tag{2.7.4}$$

where β_1 and β_2 are positive.

As usual, we have the error system:

$$\begin{aligned} \dot{e}_1 &= (25 \quad + 10)(e_2 \quad e_1) \\ \dot{e}_2 &= (28 \quad 35 \quad)e_1 + (29 \qquad 1)e_2 \quad x_s e_3 \quad z_m e_1 + u_1 + \Delta f_1 \\ \dot{e}_3 &= \quad (\frac{8+}{3})e_3 + x_s e_2 + y_m e_1 + u_2 + \Delta f_2 \end{aligned} \tag{2.7.5}$$

Alternatively, we can rewrite the error system as the form:

$$\dot{e} = Ae + Bf + Bu + B\Delta f \tag{2.7.6}$$

where $e = (e_1, e_2, e_3)^T$,$u = (0, u_1, u_2)^T$,$\Delta f = (0, \Delta f_1, \Delta f_2)^T$, and

$$A = \begin{pmatrix} (25 \quad +10) & (25 \quad +10) & 0 \\ (28 \quad 35 \quad) & (29 \qquad 1) & 0 \\ 0 & 0 & \frac{8+}{3} \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, f = \begin{pmatrix} 0 \\ x_s e_3 \quad z_m e_1 \\ x_s e_2 + y_m e_1 \end{pmatrix}$$

In order to use a sliding mode control method for synchronization successfully, we require two basic steps:

1. Selecting an appropriate switching surface so that the sliding motion is stable and ensures the error system converges to zero as time increasing.
2. Establishing a robust control law which guarantees the existence of the sliding manifold $s(t) = 0$ within the consideration of parameter uncertainty and external perturbation. (i.e., p and Δf)

First of all, the proportional-integral switching surface is defined as:

$$s = Ce \qquad \int_0^t (CA + CBK)e(\tau)d\tau \tag{2.7.7}$$

where $s \in R^{2\times1}$,$C \in R^{2\times3}$ and $K \in R^{2\times3}$; C is chosen such that $CB \neq 0$ (CB is singular) and K is chosen such that $\lambda_{max}(A + BK) < 0$ (A+BK is stable).

According to [16,17], then we can obtain the equivalent control $u_{eq}(t)$ in the sliding manifold by differentiating (2.7.7) with respect to time and substituting form (2.7.6):

$$\begin{aligned} \dot{s} = C\dot{e} \quad (CA + CBK)e = CAe + CBf + CBu_{eq} + CB\Delta f \quad CAe \quad CBKe \\ = CB(u_{eq} + f + \Delta f \quad Ke) = 0 \end{aligned} \tag{2.7.8}$$

Since CB is nonsingular, the equivalent control u_{eq} in the sliding mode is $u_{eq} = Ke \quad f \quad \Delta f$; substitute it back to (2.7.6), we obtain:

$$\dot{e} = Ae + Bf + BKe \quad Bf \quad B\Delta f + B\Delta f = (A + BK)e \tag{2.7.9}$$

From the above result, notice that the system is insensitive to parameter uncertainty and external perturbation (i.e., the controlled system is robust).

Next, introduce Lemma 1.7.1 which indicates the reaching condition of the sliding mode:

Lemma 2.7.1. *The motion of the sliding mode is asymptotically stable, if the following reaching condition is held:* $s^T(t)\dot{s}(t) < 0$.

To achieve the reaching condition indicated in Lemma 1.7.1, a control law is proposed as:

$$u = Ke \quad \gamma(CB)^{\ 1}[\|CB\|(\|f\| + \beta_1\|X_s\| + \beta_2)]sign(s) \tag{2.7.10}$$

where γ is an arbitrarily constant lager than 1, and $X_s = (x_s, y_s, z_s)^T$. In order to ensure this control law can derive the error dynamics (2.7.6) onto the sliding manifold $s(t) = 0$, the following theorem is introduced:

Theorem 2.7.2. *The reaching condition $s^T(t)\dot{s}(t) < 0$ of the sliding mode is satisfied, if the control u(t) is given by* (2.7.10).

Proof. Substituting (2.7.6) and (2.7.10) into $s^T(t)\dot{s}(t)$, we have:

$$\begin{aligned} s^T\dot{s} &= s^T[CBf + CBu + CB\Delta f \quad CBKe] \\ &= s^T[CBf + CBKe \quad \gamma[\|CB\|(\|f\| + \beta_1\|X_s\| + \beta_2)]sign(s) \\ &\qquad + CB\Delta f \quad CBKe] \\ &\leq \quad \gamma[\|CB\|(\|f\| + \beta_1\|X_s\| + \beta_2)]s^T sign(s) \\ &\qquad + \|CB\|(\|f\| + \|\Delta f\|)\|s\| \end{aligned} \tag{2.7.11}$$

Moreover, since $\gamma > 1$ and $s^T sign(s) = |s_1| + |s_2| \geq \|s\| = \sqrt{s_1^2 + s_2^2}$, we have $s^T\dot{s} \leq (1 \quad \gamma)[\|CB\|(\|f\| + \beta_1\|X_s\| + \beta_2)]\|s\|$ and $s^T(t)\dot{s}(t) < 0$ is always satisfied.

In order to verify the analytic results of sliding mode, by using the RK4 method, we simulate the case when the system operates in the sliding mode with the e-quivalent controller $u_{eq}(t)$. We choose $\quad = 0.8$ and $K == \begin{pmatrix} 32.2 & 4 & 0 \\ 0 & 0 & 0.6667 \end{pmatrix}$ so that $\lambda_{max}(A + BK) = \quad 2 < 0$; also we use the same initial conditions, (1.5,2,1) and (-1,-5,-10) for master and slave system respectively. As a result, based on the graphs above, observe that the error system converges to zero as time increasing, which guarantees the synchronization of the unified chaotic system. Furthermore, the controller u_{eq} involves both the consideration of the parameter uncertainty and external perturbation, it is efficient with the occurrence of the sliding motion.

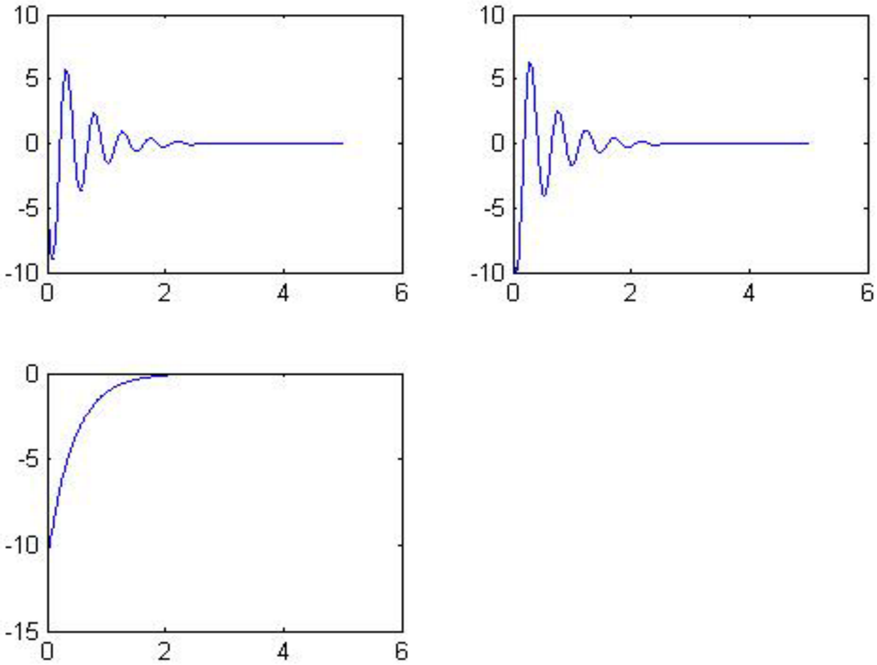

Figure 2.13: Synchronization errors between network (2.7.2) and (2.7.3)

2.8 Global Synchronization with Impulsive Control

Notice that the adaptive controller is activated all the time during the whole synchronization process, now we consider an impulsive controller [8] which allows itself to activate when it is needed instead of running all the time. We continue using the unified chaotic system to demonstrate.

Based on the unified chaotic system, there exists three equilibrium points: $E_1 = (0,0,0)^T$,$E_2 = (\beta, \beta, \eta)^T$,$E_3 = (\quad \beta, \beta, \eta)^T$, where $\beta = \sqrt{(8+a)(9 \quad 2 \quad)}$ and $\eta = 27 \quad 6 \quad$. If we denote the equilibrium points as $(y_1^0, y_2^0, y_3^0)^T$, then we can apply the transformation $x_i = y_i \quad y_i^0$, i=1,2,3. Therefore, we can rewrite the unified chaotic system as the form: $\dot{x} = Ax + \Phi(x)$, where

$$A = \begin{pmatrix} & (25 \quad +10) & 25 \quad +10 & 0 \\ 28 & 35 \quad y_3^0 & 29 \quad 1 & y_1^0 \\ & y_2^0 & y_1^0 & \frac{+8}{3} \end{pmatrix}$$

and

$$\Phi(x) = \begin{pmatrix} 0 \\ x_1x_3 \\ x_1x_2 \end{pmatrix}$$

From here, we introduce an impulsive controller to stabilize the equilibrium points, which described by:

$$\dot{x} = Ax(t) + \Phi(x(t)) + \sum_{k=1}^{\infty} \delta(t \quad \tau_k) B_k x(t) \tag{2.8.1}$$

where $\delta(.)$ is the Dirac delta function, the time sequence τ_k satisfies $0 = \tau_0 < ... < \tau_k < \tau_{k+1}...$, $lim_{k\to\infty}\tau_k = \infty$, and B_k is a 3×3 constant matrix for each $k \in Z^+ = 1, 2, ..., n, n+1, ...$

According to the above construction, we can derive the driving system: $\dot{y} = Ay + \Phi(y)$ where

$$A = \begin{pmatrix} & (25 \quad +10) & 25 \quad +10 & 0 \\ 28 & 35 \quad y_3^0 & 29 \quad 1 & y_1^0 \\ & 0 & 0 & \frac{+8}{3} \end{pmatrix}$$

and

$$\Phi(x) = \begin{pmatrix} 0 \\ y_1y_3 \\ y_1y_2 \end{pmatrix}$$

and the driven system which the impulsive controller involved:

$$\dot{\tilde{y}} = A\tilde{y}(t) + \Phi(\tilde{y}(t)) + \sum_{k=1}^{\infty} \delta(t - \tau_k) B_k(\tilde{y}(t) - y(t)) \tag{2.8.2}$$

where all the parameters remain the same as defined previously.

If we apply integration from $\tau_k - h$ to $\tau_k + h$ for both sides of $\dot{x} = Ax(t) + \Phi(x(t)) + \sum_{k=1}^{\infty} \delta(t - \tau_k) B_k x(t)$ then as $h \to 0^+$, by the properties of Dirac delta function, we have the following $x(\tau_k^+) - x(\tau_k^-) = B_k x(\tau_k)$. Moreover, by the assumption of $x_i(t) = lim_{t \to \tau_k^-} x_i(t)$ and the definition of the Dirac delta function, $\dot{x}(t)$ can be rewritten as:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \Phi(x(t)), t \neq \tau_k \\ x(t^+) - x(t) &= B_k x(t), t = \tau_k \end{aligned} \tag{2.8.3}$$

Thus, similar technique can be applied to the driven system $\tilde{y}$:

$$\begin{aligned} \dot{\tilde{y}} &= A\tilde{y} + \Phi(\tilde{y}), t \neq \tau_k \\ \Delta\tilde{y} &= B_k(\tilde{y}(t) - y(t)), t = \tau_k \end{aligned} \tag{2.8.4}$$

Finally, the error system can be expressed as follows:

$$\begin{aligned} \dot{e} &= Ae + \Psi(y, \tilde{y}), t \neq \tau_k \\ \Delta\tilde{y} &= B_k e, t = \tau_k \end{aligned} \tag{2.8.5}$$

where the synchronization error is $e = (\tilde{y}_1 - y_1, \tilde{y}_2 - y_2, \tilde{y}_3 - y_3)^T$ and $\Psi(y, \tilde{y}) = \Phi(\tilde{y}) - \Phi(y) = \begin{pmatrix} 0 \\ y_1 y_3 - \tilde{y}_1 \tilde{y}_3 \\ \tilde{y}_1 \tilde{y}_2 - y_1 y_2 \end{pmatrix}$.

Remark.

According to Theorem 3 and Theorem 10 in [8], they ensure the globally asymptotical stability under the construction of impulsive controller, and their proof can be illustrated by solving $D^+V(t), t \in (\tau_{k-1}, \tau_k], k = 1, 2, ...$ where the piecewise continuous auxiliary function $V(t, x) = e^{b(\tau_{k-1} - t)} x^T(t) P x(t)$ for $\dot{x}(t)$ system and $V(t, x) = e^{b(\tau_{k-1} - t)} e^T(t) e(t)$ for the error system respectively. Moreover, P is a 3×3 symmetric and positive definite matrix, B can be determined by the two conditions in Theorem 3; in addition, according to Corollary 8 in [8], we can obtain the impulsive interval $\delta \leq \frac{1}{q-b} In \frac{1}{\mu d}$, where q is the largest eigenvalue of $(PA + A^T P)$, d is the largest eigenvalue of $(I + B)^T (I + B)$,$\mu > 1$, and $b \geq 0$.

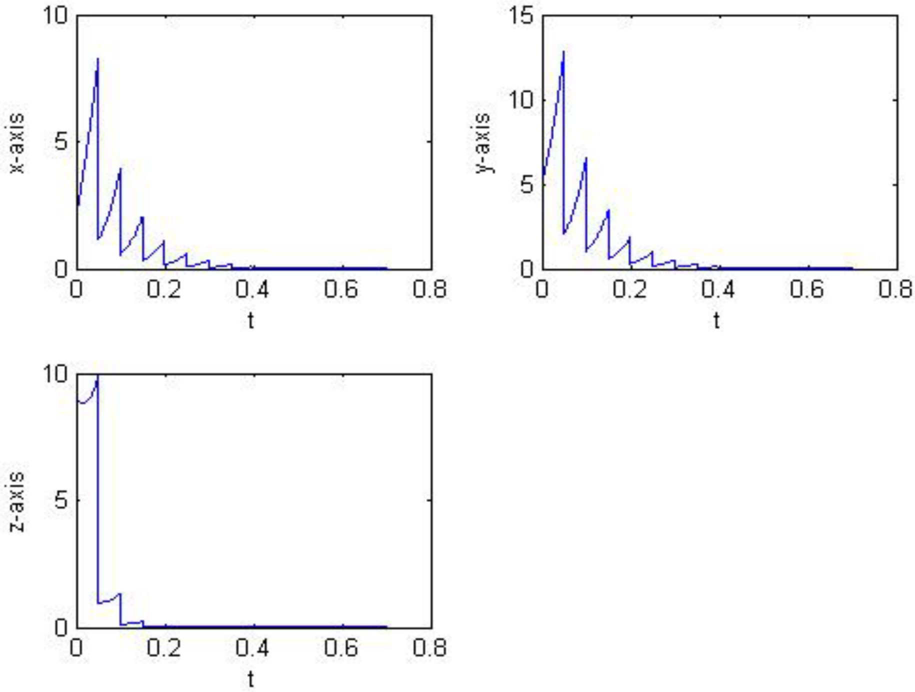

Figure 2.14: Error system (2.8.5)

We assume the initial position at (2,5,9), $\quad = 1, B = diag(\quad 0.86, \quad 0.84, \quad 0.9)$ which is obtained by the conditions in Theorem 3 in [8]. Based on the graphs, the impulsive controller takes effect only at the vertical "jump" ($t = \tau_k$); then the error system converges to zero solution as expected. Therefore, the impulsive controller is valid for the chaos synchronization.

2.9 Impulsive Synchronization With Time Delay

Consider the chaos-based communication system which consists of two chaotic systems at the transmitter and the receiver ends, respectively. At the transmitter end, we have:

$$\dot{x}(t) = Ax(t) + \Phi_1(x(t)) \tag{2.9.1}$$

and at the receiver end, we have:

$$\begin{aligned} \dot{u}(t) &= Au(t) + \Phi_2(u(t), x(t \quad r), u(t \quad r)), t \neq \tau_k \\ \Delta u(t) &= \quad B_k e(t), t = \tau_k, k = 1, 2, ... \end{aligned} \tag{2.9.2}$$

where matrices $A \in R^{n \times n}$ and $B_k \in R^{n \times n}$; functions Φ_1,Φ_2 are continuous functions in their respective domain of definition. $\Delta u(t_k) = u(t_k^+) \quad u(t_k^-)$, $u(t_k^+ = \lim_{t \to t_k} u(t)$ for k=1,2,... A typical form of the function Φ_2 is:

$$\Phi_2(u(t), x(t \quad r), u(t \quad r)) = \Phi_1(u(t)) + FK(x(t \quad r) \quad u(t \quad r)) \tag{2.9.3}$$

where $F \in R^n \times R^m$,$K \in R^m \times R^n$ are constant matrices. The second term in $\dot{u}(t)$ is known as the delayed feedback control and $B_k e(t)$ is called the impulsive control. Moreover, let e=x-u be the synchronization error, it follows the error dynamics is given by:

$$\begin{aligned} \dot{e}(t) &= Ae(t) + \Psi(x(t), u(t), x(t \quad r), u(t \quad r)), t \neq \tau_k \\ \Delta e(t) &= B_k e(t), t = \tau_k, k = 1, 2, ... \end{aligned} \tag{2.9.4}$$

where $\Psi(x(t), u(t), x(t \quad r), u(t \quad r)) = \Phi_1(x(t)) \quad \Phi_2(u(t), x(t \quad r), u(t \quad r))$. Notice that e=0 is the trivial solution of system (2.9.4), thus the global asymptotical stability of this trivial solution implies the global synchronization of systems (2.9.1) and (2.9.2). In addition, we assume that Ψ satisfies the following assumption:

Assumption 2.9.1. *For a positive definite matrix P, there exist constant matrices $D_i \in R^n \times R^m$, i.1,2 such that*

$$(x \quad y)^T P\Psi(x, y, u, v) \leq (x \quad y)^T PD_1(x \quad y) + (x \quad y)^T PD_2(u \quad v) \tag{2.9.5}$$

and introduce the following Lemma:

Lemma 2.9.1. *Let $\gamma > 0$ and $m \in C^1[J, R_+]$, where $J = [a \quad \gamma, b)$,$0 < b \quad a \leq \Delta_i$. Assume that there exist constants $l > 0$ and $\beta \in (0, 1)$ such that $m'(t) \leq lm(t)$ whenever $m(t) \geq \beta m(t + s)$, $s \in [\quad \gamma, 0]$; which is in the spirit of the Razumikhin technique. Also there exists a constant $\eta > 0$ such that $m(s) < \eta$,$s \in [a \quad \gamma, a)$,$m(a) \leq \beta\eta$, and*

$$l\Delta_i + ln\beta < 0 \tag{2.9.6}$$

Then there exists $d = d(\eta)$, $0 < d < \eta$ such that $m(t) < \eta \quad d$,$t \geq a$.

Theorem 2.9.2. *Assume that:*
(i) there exists a positive definite matrix P and constats $\ _i$*,i=1,2, with* $\ _2 \in R_+$ *such that*

$$\begin{pmatrix} A^TP + PA + PD_1 \quad {}_1P & PD_2 \\ D_2^TP & {}_2P \end{pmatrix} \leq 0 \tag{2.9.7}$$

(ii) there exists a real number $\beta \in (0,1)$ *such that for each k=1,2,...,*

$$(I + B_k^T)P(I + B_k) \quad \beta P \leq 0 \tag{2.9.8}$$

(iii) there exists a positive number τ *such that*

$$\tau_k \quad \tau_{k\ 1} \leq \tau, k = 2, 3, .., and \frac{\tau}{\beta}(\beta \ _1 + \ _2) + ln\beta \leq 0 \tag{2.9.9}$$

Then the trivial solution of system (2.9.4) *is globally asymptotically stable, and systems* (2.9.1) *and* (2.9.2) *realize global synchronization.*

Proof. Let $\lambda_1 = \lambda_{min}(P)$ and $\lambda_2 = \lambda_{max}(P)$. Then $0 < \lambda_1 \leq \lambda_2$. For and $\varepsilon > 0$, choose $\delta = \delta(\varepsilon) > 0$ so that $\delta < \sqrt{\frac{\lambda_1\beta}{\lambda_2}}\varepsilon$. Let $t_0 \in [\tau_{l\ 1}, \tau_l]$,$\phi \in PC[\ r, 0]$ with $\|\phi\| < \delta$. Let $e(t) = e(t, t_0, \phi)$ be any solution of (2.9.4) such that $e_{t_0} = \phi$. By the choice of δ, $|e(t_0)| = |\phi_0| < \varepsilon$, then $|e(t)| < \varepsilon$,$t \geq t_0$. Define $m(t) = e^T(t)Pe(t)$,$t \geq t_0 \quad \gamma$; then for $t \in [t_0 \quad r, t_0]$ we have $m(t) \leq \lambda_2\delta^2 < \lambda_1\beta\varepsilon^2$. For $t \neq \tau_k$, we obtain

$$\begin{aligned} m^{'}(t) &= e^{'T}(t)Pe(t) + e^T(t)Pe^{'}(t) \\ &= e^T(t)[A^TP + PA]e(t) \\ &+ 2e^T(t)P\Psi(x(t), u(t), x(t \quad r), u(t \quad r)) \end{aligned} \tag{2.9.10}$$

Then by inequality (2.9.5), we obtain

$$\begin{aligned} m^{'}(t) &\leq e^T(t)[A^tP + PA + PD_1]e(t) \\ &+ e^TPD_2e(t \quad r) + e^T(t \quad r)D_2^TPe(t) \\ &= [x^T(t)X^T(t \quad r)] \begin{pmatrix} A^TP + PA + PD_1 & PD_2 \\ D_2^TP & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ x(t \quad r) \end{pmatrix} \end{aligned} \tag{2.9.11}$$

By condition (i), $m^{'}(t) \leq \ _1m(t) + \ _2m(t \quad r), t \neq \tau_k$, which implies if $m(t) \geq \beta m(t + s), s \in [\ r, 0]$, then

$$m^{'}(t) \leq \frac{1}{\beta}(\beta \ _1 + \ _2)m(t), t \neq \tau_k \tag{2.9.12}$$

When $t = \tau_k$, by condition (ii)

$$m(\tau_k) = e^T(\tau_k \ (I + B_k^T)P(I + B_k)e(\tau_k \ ) \leq \beta m(\tau_k \ ) \tag{2.9.13}$$

Based on the Lemma, condition (ii) and (iii) in the Theorem, we have $m(t) < \lambda\varepsilon^2, t \in [t_0, \tau_l)$. By condition (iii) we have $m(\tau l) \leq \beta m(\tau_l \ ) < \beta\lambda_1\varepsilon^2$. By the Lemma again gives $m(t) < \lambda_1\varepsilon, t \in [\tau_l, \tau_{l+1})$. By repeating this process, $|e(t)| < \varepsilon, t \geq t_0$ holds, thus the trivial solution of system (2.9.4) is stable. Moreover, we need to prove it is also globally attractive.

Let $\sigma > 0$ be given. Let $e(t) = e(t, t_0, \phi)$ be any solution of inequality (2.9.7) such that $\|\phi\| < \sigma$. Let $\varepsilon > 0$ be sufficiently small so that $\lambda_1\varepsilon^2 < \frac{\lambda_2\sigma^2}{\beta}$. Then we need to show that there exists a $T = T(\varepsilon) > 0$ such that $|e(t)| < \varepsilon, t \geq t_0 + T$. Also define $m(t) = e^T Pe(t), t \geq t_0 \quad r$, we have $m(t) \leq \lambda_2\sigma^2 < \frac{\lambda_2\sigma^2}{\beta}, t \in [t_0 \quad r, t_0]$, which by applying the Lemma yields $m(t) < \frac{\lambda_2\sigma^2}{\beta}, t \geq t_0$ and there exists a $d = d(\sigma) > 0$ such that $m(t) < \frac{\lambda_2\sigma^2}{\beta} \quad d, t \in [t_0, \tau_l)$.

Let N be the smallest integer for which $\frac{\lambda_2\sigma^2}{\beta} < \lambda_1\varepsilon^2 + Nd$. Define $T = \tau + (r + \tau)(N \quad 1)$. Since $m(t) \leq \frac{\lambda_2\sigma^2}{\beta}, t \in [\tau_l \quad \gamma, \tau_l)$ and $m(\tau_l) \leq \beta m(\tau_l \ ) \leq \lambda_2\sigma^2$, it follows from the Lemma that $m(t) \leq \frac{\lambda_2\sigma^2}{\beta} \quad d, t \in [\tau_l, \tau_{l+1})$; thus we have $m(t) \leq \frac{\lambda_2\sigma^2}{\beta} \quad d, t \in [t_0, \tau_{l+1})$.

Suppose $m(t) \leq \frac{\lambda_2\sigma^2}{\beta}, t \in [t_0, t_k)$; then $m(\tau_k) \leq \beta m(\tau_k^1) \leq \beta(\frac{\lambda_2\sigma^2}{\beta} \quad d) < \lambda_2\sigma^2$ and $m(t) \leq \frac{\lambda_2\sigma^2}{\beta} \quad d, t \in [\tau_k \quad \gamma, \tau_k]$; hence by the Lemma, $m(t) \leq \frac{\lambda_2\sigma^2}{\beta} \quad d, t \in [\tau_k, \tau_{k+1}]$ which shows that $m(t) \leq \frac{\lambda_2\sigma^2}{\beta} \quad d, t \in [t_0, \tau_{k+1}]$. Therefore by induction, $m(t) \leq \frac{\lambda_2\sigma^2}{\beta}, t \geq t_0$.

Let $\tau_{k_1} = \inf_k\{\tau_k \in [t_0 + \gamma + I, \infty)\}$, then $\tau_{k_1} \quad r \geq t_0$, $m(t) \leq \frac{\lambda_2\sigma^2}{\beta} \quad d, t \in [\tau_{k_1} \quad r, \tau_{k_1})$, and $m(\tau_{k_1}) \leq \beta m(\tau_{k_1}) \leq \beta(\frac{\lambda_2\sigma^2}{\beta} \quad d)$. Then by the Lemma we have $m(t) < \frac{\lambda_2\sigma^2}{\beta} \quad 2d, t \in [t_{k_1}, \infty)$.

By the definition of τ_{k_1}, notice that $\tau_{k_1} \leq t_0 + r + 2\tau$. Let $\tau_{k_2} = \inf_k\{t_0 + 2(r + \tau), \infty)\}$, then $m(t) \leq \frac{\lambda_2\sigma^2}{\beta} \quad 2d, t \in [\tau_{k_2} \quad r, \tau_{k_2})$ and $m(\tau_{k_2}) \leq \beta m(\tau_{k_2}) \leq \beta(\frac{\lambda_2\sigma^2}{\beta} \quad d)$. By continuing this process with the Lemma, we obtain $m(t) \leq \frac{\lambda_2\sigma^2}{\beta} \quad Nd, t \in [\tau_{k_1}, \infty)$, where $\tau_{k_{N \ 1}} \leq t_0 + \tau + (r + \tau)(N \quad 1)$. This shows that $m(t) \leq \lambda_1\varepsilon^2, t \geq t_0 + T$ which yields $|e(t)| < \varepsilon, t \geq t_0 + T$. Thus the trivial solution of system (2.9.4) is also globally asymptotically stable, which indicates the globally synchronization between systems (2.9.1) and (2.9.2).

Remark. In condition (i) of the Theorem, the constant ${}_1 > 0$ measures the degree of instability of the delay-free system and is determined by the eigenvalues of the matrix $A + PD_1$, while ${}_2$ is determined by the matrices D_2. In condition (ii) of the Theorem, the constant β measures the amplitude of the control impulses, the smaller the β the larger the amplitude. And the condition (iii) of the Theorem characterizes the relationship among the interval length of consecutive impulses and the other parameters ${}_i,\beta$ and the delay length r. If the delay-free system is stable, then ${}_1$ can be a negative constant; in that case, a larger interval length of consecutive impulses is allowed.

Consider the following Chua circuit

$$\begin{aligned} \dot{x}_1 &= a_1[x_2 \quad h(x_1)] \\ \dot{x}_2 &= x_1 \quad x_2 + x_3 \\ \dot{x}_3 &= \quad a_2 x_2 \end{aligned} \tag{2.9.14}$$

where $h(x_1) = m_1 x_1 + \frac{1}{2}(m_0 \quad m_1)(|x_1 + a_3| \quad |x_1 \quad a_3|)$.

When $a_1 = 9, a_2 = 14.286, a_3 = 1, m_0 = \quad \frac{1}{7}, m_1 = \frac{1.5}{7}$, we obtain the double scroll attractor. Chua's circuit can be rewritten in the form of $\dot{x}(t) = Ax(t) + \Phi_1(x(t))$ where $A = \begin{pmatrix} a_1 m_1 & a_1 & 0 \\ 1 & \quad 1 & 1 \\ 0 & \quad a_2 & 0 \end{pmatrix}$,

$\Phi_1 = \begin{pmatrix} \frac{1}{2}(m_0 \quad m_1)(|x_1 + a_3| \quad |x_1 \quad a_3|) \\ 0 \\ 0 \end{pmatrix}$.

Moreover, we choose Φ_2 the same as that defined by (2.9.3) with FK=I. Set the delay r=2, and the impulse matrices $B_k = 0.5I$ for all k=1,2,.... Then the inequality (2.9.5) and all conditions of the Theorem can be hold with $D_1 = \begin{pmatrix} a_1(m_1 \quad m_0) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $D_2 = \quad FK = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$P = \begin{pmatrix} 4.825 & 1.2816 & 3.5975 \\ 1.2816 & 9.4887 & 1.2610 \\ 3.5975 & 1.2610 & 2.8618 \end{pmatrix}$

${}_1 = 4.2, {}_2 = 0.9, \beta = 0.25$ and $\tau_{i+1} \quad \tau_i = \tau \leq 0.1777$.

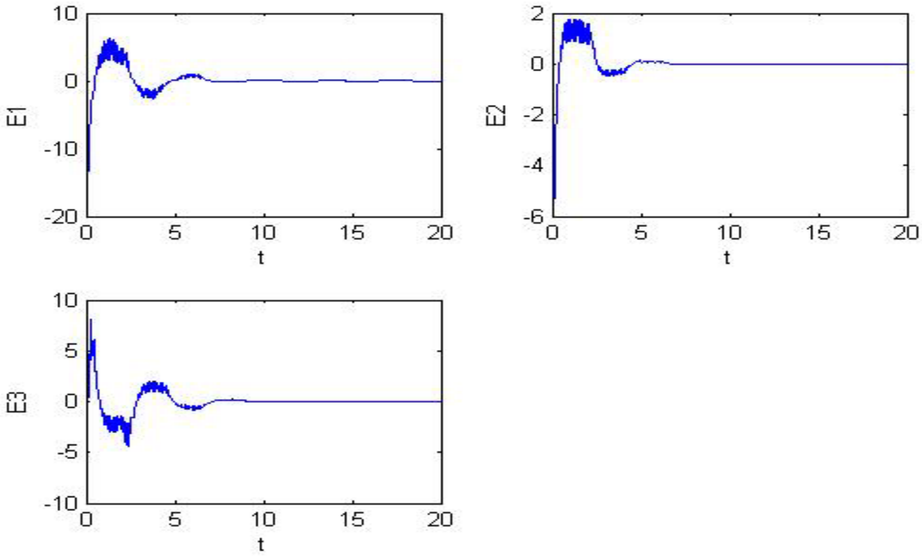

Figure 2.15: Impulsive synchronization with $\tau = 0.15$

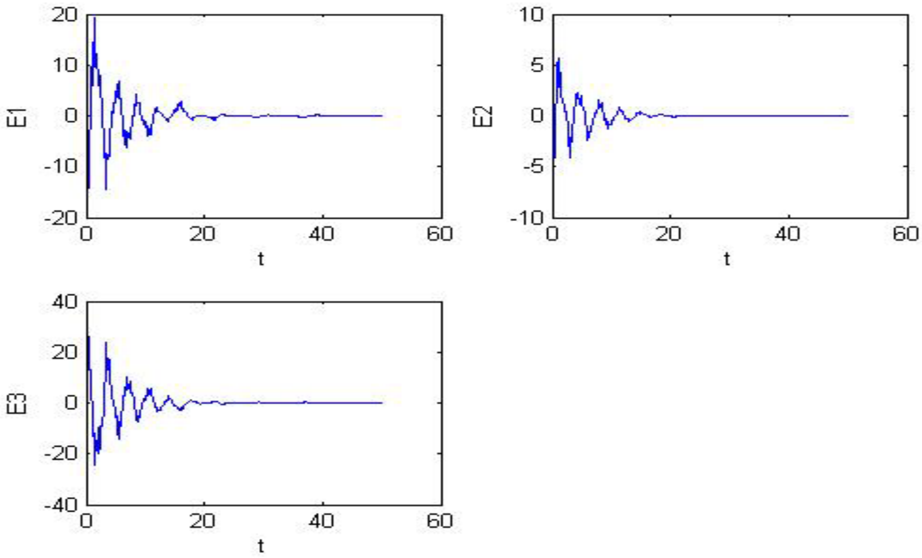

Figure 2.16: Impulsive synchronization with $\tau = 0.5$

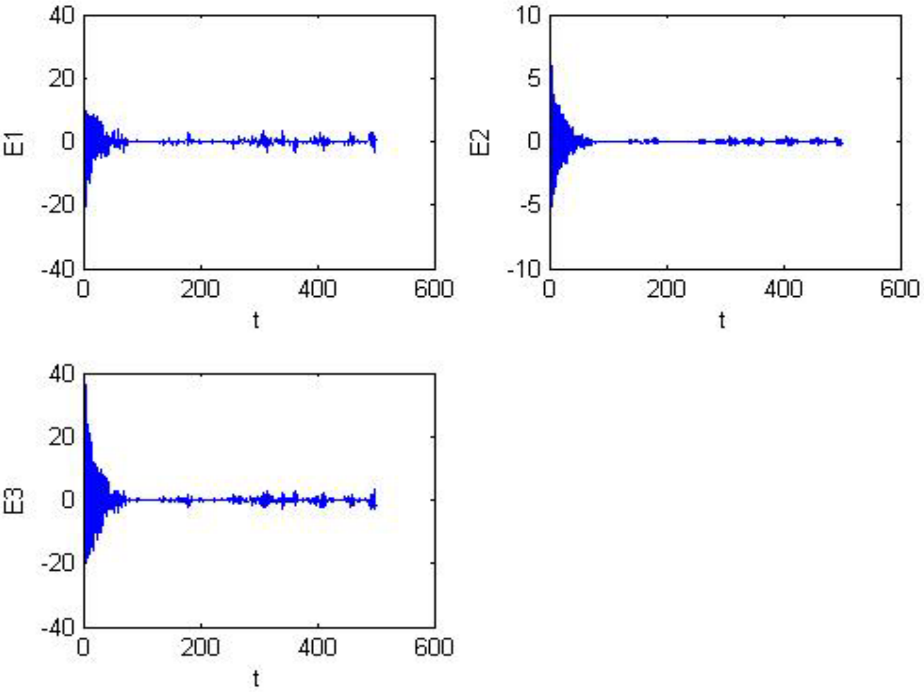

Figure 2.17: Impulsive synchronization with $\tau = 0.7$

The above graphs indicate the behavior of the error systems with impulsive interval τ=0.15,0.5,0.7 respectively. Notice all conditions in the Theorem are hold when $\tau = 0.15$, thus its error system converges to zero in a short period of time as expected and synchronization is realized. Interestingly, conditions (ii) and (iii) fail to hold when $\tau = 0.5 > 0.177$, but the corresponding error system converges to zero as well; it suggests the synchronization conditions in the Theorem are sufficient but not necessary. Moreover, the synchronization fails when $\tau = 0.7$.

2.10 Projective Synchronization

Chaos in the economy had made a tremendous impact on market economy, then the Synchronization of the financial system is one of the related issues, such as the synchronous development in different countries. Moreover, the economic system is a high-dimensional nonlinear system, whose chaos is mostly super chaos; therefore, we construct the projective synchronization of the hyperchaotic financial systems, which have two or more positive Lyapunov exponents.

Consider the following financial system:

$$\begin{cases} \dot{x} = z + (y \quad a)x \\ \dot{y} = 1 \quad by \quad x^2 \\ \dot{z} = \quad x \quad cz \end{cases} \tag{2.10.1}$$

where x,y,z represent the interest rate,investment demand,and price index, respectively. The positive parameter a,b,c are the saving,the per-investment cost,and the elasticity of demands of commercials.

Furthermore, based on the chaotic finance of system (2.10.1), since the factors affecting the interest rate x also related to the average profit margin ω; therefore, we consider the improved system which presents hyperchaotic behavior as follows:

$$\begin{cases} \dot{x} = z + (y \quad a)x + \omega \\ \dot{y} = 1 \quad by \quad x^2 \\ \dot{z} = \quad x \quad cz \\ \dot{\omega} = \quad dxy \quad k\omega \end{cases} \tag{2.10.2}$$

where a,b,c,d,k are all positive. We focus on the projective synchronization which two relative chaotic dynamical systems can be synchronous with a desired scaling factor, thus we rewrite the system (2.10.2) as:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} \ a & 0 & 1 & 1 \\ 0 & \ b & 0 & 0 \\ \ 1 & 0 & \ c & 0 \\ 0 & 0 & 0 & \ k \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ \omega \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} xy \\ \ x^2 \\ 0 \\ dxy \end{pmatrix} \tag{2.10.3}$$

Then we can construct the response system with controller designed for system (2.10.3) as follows:

$$\begin{pmatrix} \dot{x}_s \\ \dot{y}_s \\ \dot{z}_s \\ \dot{\omega}_s \end{pmatrix} = \begin{pmatrix} \ a & 0 & 1 & 1 \\ 0 & \ b & 0 & 0 \\ \ 1 & 0 & \ c & 0 \\ 0 & 0 & 0 & \ k \end{pmatrix} \begin{pmatrix} x_s \\ y_s \\ z_s \\ \omega_s \end{pmatrix} + \ [\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} xy \\ \ x^2 \\ 0 \\ dxy \end{pmatrix}] + \mathbf{u} \tag{2.10.4}$$

where is a scaling factor and $\mathbf{u}$ is the external input control vector.

Denote $\mathbf{x} = (x, y, z, \omega)^T$, $\mathbf{y} = (x_s, y_s, z_s, \omega_s)^T$,and $\mathbf{e} = (e_1, e_2, e_3, e_4)^T$. Then define $\mathbf{e} = \mathbf{y} \quad \mathbf{x}$, the error system can be obtained:

$$\dot{\mathbf{e}} = A\mathbf{e} + \mathbf{u} \tag{2.10.5}$$

where

$$A = \begin{pmatrix} \ a & 0 & 1 & 1 \\ 0 & \ b & 0 & 0 \\ \ 1 & 0 & \ c & 0 \\ 0 & 0 & 0 & \ k \end{pmatrix} \tag{2.10.6}$$

In order to determine the particular controller, the following theorem is introduced:

Theorem 2.10.1. *If we design the controller* $\mathbf{u}$=$B\mathbf{e}$ *for the error system* (2.10.5)*, the they system* (2.10.5) *is asymptotically stable at the origin, where*

$$B = \begin{pmatrix} k_1(\frac{b_{11}}{k_1}) & k_1(\ \frac{a_1}{k_2}) & k_1(\frac{1}{k_3} \ \frac{1}{k_1} \ \frac{a_2}{k_3}) & k_1(\frac{1}{k_1} \ \frac{a_3}{k_4}) \\ k_2(\frac{a_1}{k_2}) & k_2(\frac{b_{22}}{k_2}) & k_2(\ \frac{a_4}{k_3}) & k_2(\ \frac{a_5}{k_4}) \\ k_3(\frac{a_2}{k_3}) & k_3(\frac{a_4}{k_3}) & k_3(\frac{b_{33}}{k_3}) & k_3(\ \frac{a_6}{k_4}) \\ k_4(\frac{a_3}{k_4}) & k_4(\frac{a_5}{k_4}) & k_4(\frac{a_6}{k_4}) & k_4(\frac{b_{44}}{k_4}) \end{pmatrix} \tag{2.10.7}$$

where $b_{11} \quad a$,$b_{22} \quad a$,$b_{33} \quad a$,$b_{44} \quad a$;$a_i \in R(i = 1, 2, 3, 4, 5, 6)$,$k_i \in R^+(i = 1, 2, 3, 4)$.

In order to prove Theorem 1, we require a Lemma:

Lemma 2.10.2. *Suppose a dynamic system ca be written as:*

$$\begin{pmatrix} \dot{x_1} \\ \dot{y_2} \\ \vdots \\ \dot{x_n} \end{pmatrix} = \begin{pmatrix} k_1a_{11} & k_1a_{12} & \cdots & k_1a_{1n} \\ k_2a_{21} & k_2a_{22} & \cdots & k_2a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ k_na_{n1} & k_na_{n2} & \cdots & k_na_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{2.10.8}$$

If the system (2.10.8) *satisfies the following conditions:*
(1)$\forall a_{ij} \in R$,
(2)$a_{ij} = \quad a_{ji}(i \neq j)$,
(3)$a_{ii} \leq 0(notalla_{ii}areequaltozero)$,
(4)$\forall k_i > 0$
then the states of system (2.10.8) *will decrease to zero gradually.*

Proof of Theorem 1.10.1.

Let $\mathbf{u} = B\mathbf{e}$, where $B = (b_{ij})_{4x4}$ is a 4x4 order constant matrix to be determined; $\mathbf{e} = (e_1, e_2, e_3, e_4)^T$. We can rewrite error system (2.10.5) as

$$\dot{\mathbf{e}} = (A + B)\mathbf{e} \tag{2.10.9}$$

where

$$A = (a_{ij})_{4x4} = \begin{pmatrix} k_1(\frac{a}{k_1}) & 0 & k_1(\frac{1}{k_1}) & k_1(\frac{1}{k_1}) \\ 0 & k_2(\frac{b}{k_2}) & 0 & 0 \\ k_3(\frac{1}{k_3}) & 0 & k_3(\frac{c}{k_3}) & 0 \\ 0 & 0 & 0 & k_4(\frac{k}{k_4}) \end{pmatrix} \tag{2.10.10}$$

Based on conditions from Lemma 2, design matrix B to satisfy conditions (2.10.11) and (2.10.12) as follows:

$$\frac{a}{k_1} + \frac{b_{11}}{k_1} \leq 0, \frac{b}{k_2} + \frac{b_{22}}{k_2} \leq 0, \frac{c}{k_3} + \frac{b_{33}}{k_3} \leq 0, \frac{k}{k_4} + \frac{b_{44}}{k_4} \leq 0, \tag{2.10.11}$$

$$\begin{aligned} \frac{b_{12}}{k_1} &= \frac{b_{21}}{k_2}, \\ \frac{1}{k_1} + \frac{b_{13}}{k_1} &= (\frac{1}{k_3} + \frac{b_{31}}{k_3}), \\ \frac{1}{k_1} + \frac{b_{14}}{k_1} &= \frac{b_{14}}{k_4}, \\ \frac{b_{23}}{k_2} &= \frac{b_{32}}{k_3}, \\ \frac{b_{24}}{k_2} &= \frac{b_{42}}{k_4}, \\ \frac{b_{34}}{k_3} &= \frac{b_{43}}{k_4}, \end{aligned} \tag{2.10.12}$$

From (2.10.11), we can see that $b_{11} \leq a$,$b_{22} \leq b$,$b_{33} \leq c$,and $b_{44} \leq k$. Moreover, let the coefficient matrix of (2.10.12) be H, and let its augmented matrix be $\bar{H}$, we can get $r(H) = r(\bar{H})$, and the numbers of unknowns are greater than the numbers of (2.10.12), so (2.10.12) has infinitely many solutions as follows:

$$\begin{aligned} \eta = (&\frac{a_1 k_1}{k_2}, a_1, \frac{k_1}{k_3} \quad 1 \quad \frac{a_2 k_1}{k_3}, a_2, \quad 1 \quad \frac{a_3 k_1}{k_4}, a_3, \\ &\frac{a_4 k_2}{k_3}, a_4, \frac{a_5 k_2}{k_4}, a_5, \frac{a_6 k_3}{k_4}, a_6) \\ = (&b_{12}, b_{21}, b_{13}, b_{31}, b_{14}, b_{41}, b_{23}, b_{32}, b_{24}, b_{42}, b_{34}, b_{43}) \end{aligned} \tag{2.10.13}$$

where $a_i \in R$ (i=1,2,3,4,5,6); $k_i \in R^+$ (i=1,2,3,4).

Therefore, combine (2.10.11) ,(2.10.12) and (2.10.13) , we can obtain the matrix B from Theorem 1.10.1:

$$B = \begin{pmatrix} k_1(\frac{b_{11}}{k_1}) & k_1(\frac{-a_1}{k_2}) & k_1(\frac{1}{k_3} \quad \frac{1}{k_1} \quad \frac{a_2}{k_3}) & k_1(\frac{1}{k_1} \quad \frac{a_3}{k_4}) \\ k_2(\frac{a_1}{k_2}) & k_2(\frac{b_{22}}{k_2}) & k_2(\frac{-a_4}{k_3}) & k_2(\frac{-a_5}{k_4}) \\ k_3(\frac{a_2}{k_3}) & k_3(\frac{a_4}{k_3}) & k_3(\frac{b_{33}}{k_3}) & k_3(\frac{-a_6}{k_4}) \\ k_4(\frac{a_3}{k_4}) & k_4(\frac{a_5}{k_4}) & k_4(\frac{a_6}{k_4}) & k_4(\frac{b_{44}}{k_4}) \end{pmatrix} \tag{2.10.14}$$

Then we can write system (32) as:

$$\begin{pmatrix} \dot{e_1} \\ \dot{e_2} \\ \dot{e_3} \\ \dot{e_4} \end{pmatrix} = \begin{pmatrix} k_1(\frac{a}{k_1} + \frac{b_{11}}{k_1}) & k_1(\frac{-a_1}{k_2}) & k_1(\frac{1}{k_3} \quad \frac{a_2}{k_3}) & k_1(\quad \frac{a_3}{k_4}) \\ k_2(\frac{a_1}{k_2}) & k_2(\frac{b}{k_2} + \frac{b_{22}}{k_2}) & k_2(\frac{-a_4}{k_3}) & k_2(\frac{-a_5}{k_4}) \\ k_3(\frac{1}{k_3} + \frac{a_2}{k_3}) & k_3(\frac{a_4}{k_3}) & k_3(\frac{c}{k_3} + \frac{b_{33}}{k_3}) & k_3(\frac{-a_6}{k_4}) \\ k_4(\frac{a_3}{k_4}) & k_4(\frac{a_5}{k_4}) & k_4(\frac{a_6}{k_4}) & k_4(\frac{k}{k_4} + \frac{b_{44}}{k_4}) \end{pmatrix} * \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \tag{2.10.15}$$

According to Lemma 2, the system (2.10.5) and (2.10.15) are asymptotically stable at the origin, which means the projective synchronization of system (2.10.3) and (2.10.4) achieved.

Remark. There are several particular scenarios occur by setting specific values to the variables:
(1) Set $b_{ii} = 0$,i=1,2,3,4;$a_i = 0$,i=1,2,3,4,5,6;and $k_i = 1$,i=1,2,3,4; then we can construct the simplest controller, where the control matrix as follows:

$$A = \begin{pmatrix} 0 & 0 & k_1(\frac{1}{k_3} \quad \frac{1}{k_1}) & k_1(\frac{1}{k_1}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.10.16}$$

where $k_i \in R^+$,i=1,2,3,4
(2) When $\quad = 1$,the system (2.10.3) and (2.10.4) are of complete synchronization; when $\quad = \quad 1$, the system (2.10.3) and (2.10.4) are anaesthetization.
(3) When $k_i = 1$ for i=1,2,3,4, then the coefficient matrix of system (2.10.15) is the antisymmetric matrix in [20], which means they have the same control scheme.

Instead of synchronizing two distinct financial systems, in general, we often encounter two different hyperchaotic financial systems and we apply projective synchronization on them. Here, we consider a new hyperchaotic financial system with a state feedback controller ω_t:

$$\begin{cases} \dot{x} = \quad a^{'}(x_t + y_t) + \omega_t \\ \qquad \dot{y} = \quad y_t \quad a^{'}x_tz_t \\ \qquad\quad \dot{z} = b^{'} + a^{'}x_ty_t \\ \quad \dot{\omega} = \quad c^{'}x_tz_t \quad d^{'}\omega_t \end{cases} \tag{2.10.17}$$

where $a^{'}$ and $b^{'}$ are parameters of the system (2.10.17), $c^{'} = 0.2$ is a constant, and $d^{'}$ is a control parameter. When $a^{'} = 3$,$b^{'} = 15$,$c^{'} = 0.2$,and $d^{'}0.12$, system (2.10.17) has hyperchaotic behavior.

Let the system (2.10.3) be the drive system and system (2.10.17) be the response system, we obtain from system (2.10.17):

$$\begin{pmatrix} \dot{x}_t \\ \dot{y}_t \\ \dot{z}_t \\ \dot{\omega}_t \end{pmatrix} = \begin{pmatrix} \quad a^{'} & \quad a^{'} & 0 & 1 \\ 0 & \quad 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \quad d^{'} \end{pmatrix} \begin{pmatrix} x_t \\ y_t \\ z_t \\ \omega_t \end{pmatrix} + \begin{pmatrix} 0 \\ \quad a^{'}x_tz_t \\ a^{'}x_ty_t \\ \quad c^{'}x_tz_t \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ b^{'} \\ 0 \end{pmatrix} + \mathbf{u} \tag{2.10.18}$$

then, we denote

$$A = \begin{pmatrix} \quad a & 0 & 1 & 1 \\ 0 & \quad b & 0 & 0 \\ \quad 1 & 0 & \quad c & 0 \\ 0 & 0 & 0 & \quad k \end{pmatrix}, \mathbf{h}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

,

$$f(\mathbf{x}) = \begin{pmatrix} xy \\ \quad x^2 \\ 0 \\ dxy \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ \omega \end{pmatrix}$$

,

$$D = \begin{pmatrix} \quad a^{'} & \quad a^{'} & 0 & 1 \\ 0 & \quad 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \quad d^{'} \end{pmatrix}, \mathbf{h}_2 = \begin{pmatrix} 0 \\ 0 \\ b^{'} \\ 0 \end{pmatrix}$$

,

$$g(\mathbf{y}) = \begin{pmatrix} 0 \\ a' x_t z_t \\ a' x_t y_t \\ c' x_t z_t \end{pmatrix}, \mathbf{y} = \begin{pmatrix} x_t \\ y_t \\ z_t \\ \omega_t \end{pmatrix}$$

.

Let $\mathbf{u} = (A \quad D)\mathbf{y} + \quad (\mathbf{h}_1 + f(\mathbf{x}) \quad \mathbf{h}_2 + g(\mathbf{y}) + E\mathbf{e}$, where $E = (e_{ij})_{4x4}$ is a 4x4 order constant matrix to be determined, and $\mathbf{e} = (e_1, e_2, e_3, e_4)^T$. So the response system (41) becomes:

$$\begin{pmatrix} \dot{x}_t \\ \dot{y}_t \\ \dot{z}_t \\ \dot{\omega}_t \end{pmatrix} = \begin{pmatrix} \ a & 0 & 1 & 1 \\ 0 & \ b & 0 & 0 \\ \ 1 & 0 & \ c & 0 \\ 0 & 0 & 0 & \ k \end{pmatrix} \begin{pmatrix} x_t \\ y_t \\ z_t \\ \omega_t \end{pmatrix} + \quad [\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} xy \\ x^2 \\ 0 \\ dxy \end{pmatrix}] + E\mathbf{e} \tag{2.10.19}$$

If we continue define the error system as $\mathbf{e} = \mathbf{y} \quad \mathbf{x}$, then we have:

$$\begin{pmatrix} \dot{e_1} \\ \dot{e_2} \\ \dot{e_3} \\ \dot{e_4} \end{pmatrix} = \begin{pmatrix} \ a & 0 & 1 & 1 \\ 0 & \ b & 0 & 0 \\ \ 1 & 0 & \ c & 0 \\ 0 & 0 & 0 & \ k \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} + E \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \tag{2.10.20}$$

Theorem 2.10.3. *For the error system* (2.10.20)*, let the controller be*

$$\mathbf{u} = (A \quad D)\mathbf{y} + \quad (\mathbf{h}_1 + f(\mathbf{x}) \quad \mathbf{h}_2 + g(\mathbf{y}) + E\mathbf{e} \tag{2.10.21}$$

where

$$E = \begin{pmatrix} k_1(\frac{b_{11}}{k_1}) & k_1(\frac{\ a_1}{k_2}) & k_1(\frac{1}{k_3} \quad \frac{1}{k_1} \quad \frac{a_2}{k_3}) & k_1(\frac{\ 1}{k_1} \quad \frac{a_3}{k_4}) \\ k_2(\frac{a_1}{k_2}) & k_2(\frac{b_{22}}{k_2}) & k_2(\frac{\ a_4}{k_3}) & k_2(\frac{\ a_5}{k_4}) \\ k_3(\frac{a_2}{k_3}) & k_3(\frac{a_4}{k_3}) & k_3(\frac{b_{33}}{k_3}) & k_3(\frac{\ a_6}{k_4}) \\ k_4(\frac{a_3}{k_4}) & k_4(\frac{a_5}{k_4}) & k_4(\frac{a_6}{k_4}) & k_4(\frac{b_{44}}{k_4}) \end{pmatrix} \tag{2.10.22}$$

$e_{11} \quad a$,$e_{22} \quad a$,$e_{33} \quad a$,$e_{44} \quad a$;$a_i \in R(i = 1, 2, 3, 4, 5, 6)$,$k_i \in R^+(i = 1, 2, 3, 4)$. *Then the system* (2.10.20) *is asymptotically stable at the origin.*

Proof of Theorem 1.10.3. The proof here is similar to the previous proof, since $\mathbf{u} = (A \quad D)\mathbf{y} + \quad (\mathbf{h}_1 + f(\mathbf{x}) \quad \mathbf{h}_2 + g(\mathbf{y}) + E\mathbf{e}$, and we can rewrite the error system (2.10.20) of the driven system (2.10.3) and response system (2.10.18) as:

$$\dot{\mathbf{e}} = (A + E)\mathbf{e} \tag{2.10.23}$$

Notice that system (2.10.23) has the same structure as the system (2.10.9), apply Lemma 1.10.2 again, we will have E=B; therefore, $\dot{\mathbf{e}} = (A + E)\mathbf{e} = (A + B)\mathbf{e}$. According to Lemma 1.10.2, the system (2.10.23) tends to zero gradually; namely, the system (2.10.20) is driven to the origin gradually. Therefore, the system (2.10.3) and (2.10.18) achieve the projective synchronization.

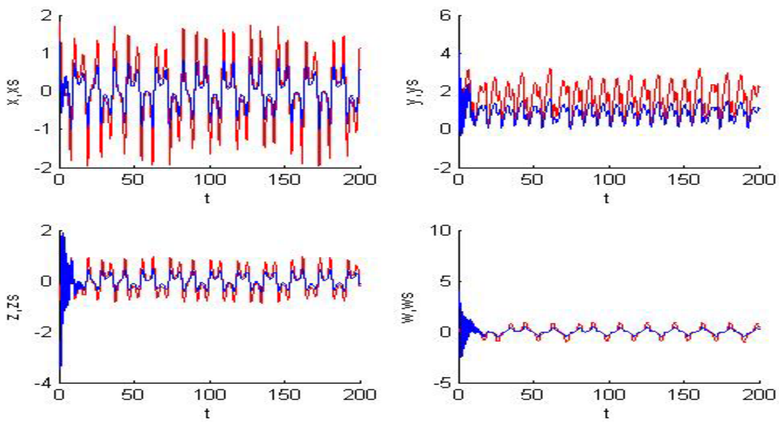

Figure 2.18: Time evolutions of the state variables of identical drive and response systems)

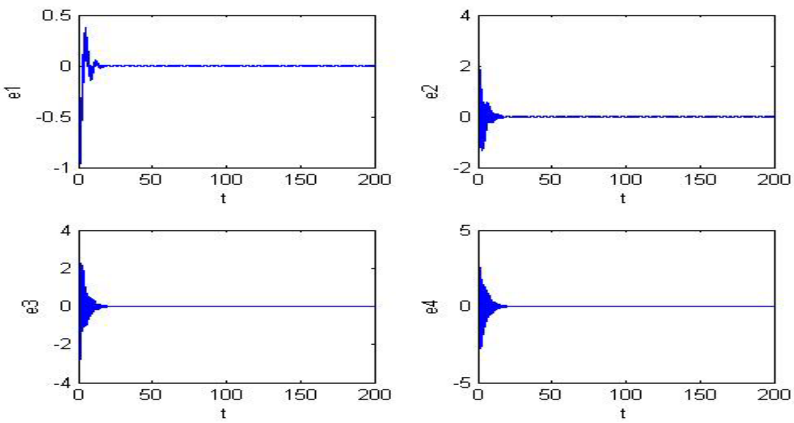

Figure 2.19: Time evolutions of error system

In the above numerical simulation of two distinct hyperchaotic financial systems, we observe the hyperchaotic behaviors from the phase portraits of system (2.10.2), and the corresponding state variables between driven and response system are converging to each other as time increasing (i.e. x x_s,y y_s,z z_s,ω ω_s), since the setting of $= 0.5$ produces nearly complete synchronization. Moreover, the error system converges to zero as time increasing. Thus, the system (2.10.3) and (2.10.4) achieve the projective synchronization as expected.

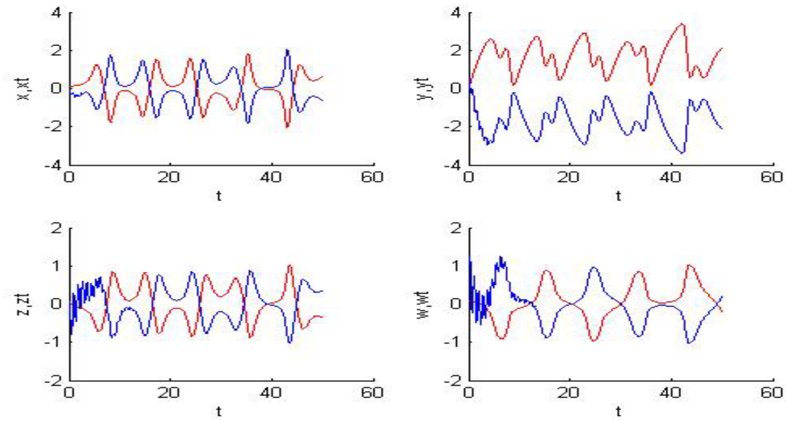

Figure 2.20: Time evolutions of the state variables of nonidentical drive and response systems

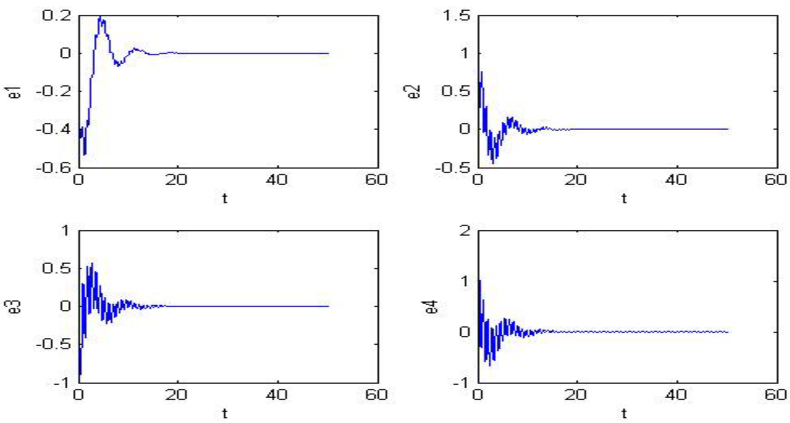

Figure 2.21: Time evolutions of error system

In the above numerical simulation of two different hyperchaotic financial systems, we observe the hyperchaotic behaviors from the phase portraits of system (2.10.17), and the corresponding state variables between driven and response system behave symmetrically to each other along zero (i.e. $x \quad x_t, y \quad y_t, z \quad z_t, \omega \quad \omega_t$), since the setting of $\quad = \quad 1$ produces the antisynchronization between system (2.10.3) and (2.10.19). Moreover, the error system still converges to zero as time increasing. Thus, the system (2.10.3) and (2.10.19) achieve the projective synchronization as expected.

Chapter 3

System Generalized Synchronization Techniques

3.1 Generalized Synchronization

Generalized synchronization means that states of the response system synchronize that of the drive system through a nonlinear smooth functional mapping.Therefore, generalized synchronization has more applications than complete synchronization.Specifically, here we consider a generalized synchronization algorithm based on nonlinear control; and we focus on the generalized synchronization of two arbitrary chaotic systems without the limitation of dimension and the invertible of Jacobian matrix.

Consider the following generalized systems:

$$\dot{x} = f(x) \tag{3.1.1}$$

$$\dot{y} = g(y) + u(x, y) \tag{3.1.2}$$

where $x = (x_1, x_2, ..., x_n)^T$ and $y = (y_1, y_2, ..., y_m)^T$ are the state vectors, and $f(x) : R^n \to R^n$,and $g(y) : R^m \to R^m$ are two continuous vector functions. The above systems are called drive and response system with control input u(x,y) respectively.

De nition 3.1.1. *Systems* (3.1.1) *and* (3.1.2) *are said to achieve generalized synchronization (GS), if there exist a controller u(x,y) and a given map* $\varphi = (\varphi_1, \varphi_2, ..., \varphi n)^T : R^m \to R^n$ *such that the following equation:* $\lim_{t\to\infty} \|x \quad \varphi(y)\| = 0$ *holds for any initial conditions* x_0 *and* y_0.

Moreover, we define the generalized synchronization error between the systems (3.1.1) and (3.1.2) as $e = x \quad \varphi(y)$, and the error system is given by:

$$\dot{e} = \dot{x} \quad \dot{\varphi}(y) = \dot{x} \quad D\varphi y\dot{y} \tag{3.1.3}$$

where $D\varphi(y)$ is the Jacobian matrix of the map $\varphi(y)$:

$$D\varphi(y) = \begin{pmatrix} \partial\varphi_1(y)\partial y_1 & \partial\varphi_1(y)\partial y_2 & \cdots & \partial\varphi_1(y)\partial y_m \\ \partial\varphi_2(y)\partial y_2 & \partial\varphi_2(y)\partial y_2 & \cdots & \partial\varphi_2(y)\partial y_m \\ \vdots & \vdots & \vdots & \vdots \\ \partial\varphi_n(y)\partial y_1 & \partial\varphi_n(y)\partial y_2 & \cdots & \partial\varphi_n(y)\partial y_m \end{pmatrix} \tag{3.1.4}$$

Remark. It is difficult to determine the suitable controller u(x,y) to achieve generalized synchronization since we are unable to ensure the reversibility of $D\varphi(y)$. Thus, we need to construct a new response system which contains the controller u(x,y) to synchronize system (3.1.1), and a new controller to obtain the generalized synchronization.

1. Construct a new response system

1.1. $n \neq m$ (For the convenience, we assume $n < m$)

Assumption 3.1.1. *The elements of each line of $D\varphi(y)$ are not all zero.*

Remark. Suppose there exist a i_0 line with all zero elements in $D\varphi(y)$,thus the value of $\varphi(y)$ on this line will give a constant,namely $\varphi_{i_0} = c$, which cannot evolve with time. In this case, we are unable to proceed the generalized synchronization between x_i and c. Also, according to matrix theory, we are able to determine a invertible matrix Q and a ladder-type matrix M from a series of elementary column transformation to $D\varphi(y)$, which has the following form:

$$M = \begin{pmatrix} a_{11} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & \cdots & 0 \end{pmatrix} \tag{3.1.5}$$

where $a_{ij}(1 \leq i \leq n)$ can be constant or function, and $a_{ii}(2 \leq i \leq n)$ can also be zero, to satisfy the condition: $D\varphi(y)Q = M$.

As a result, system (3.1.3) can be rewritten as: $\dot{e} = \dot{x} \quad \dot{\varphi}(y) = \dot{x} \quad D\varphi(y)QQ \;^{1}\dot{y}$, and if we let

$$\dot{y} = Q\dot{z} \tag{3.1.6}$$

where $z = (z_1, z_2, ..., z_m)^T$ is and m-dimensional column vector, then

$$\dot{e} = \dot{x} \quad M\dot{z} = \begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \vdots \\ \dot{e}_n \end{pmatrix} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} \quad \begin{pmatrix} a_{11} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & \cdots & 0 \end{pmatrix}$$
$$\times \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_n \end{pmatrix} = \begin{pmatrix} \dot{x}_1 \quad a_{11}\dot{z}_1 \\ \dot{x}_2 \quad a_{21}\dot{z}_1 \quad a_{22}\dot{z}_2 \\ \vdots \\ \dot{x}_n \quad a_{n1}\dot{z}_1 \quad a_{n2}\dot{z}_2 \quad \cdots \quad a_{nn}\dot{z}_n \end{pmatrix} \tag{3.1.7}$$

and let

$$\dot{e} = \dot{x} \quad M\dot{z} = \begin{pmatrix} \dot{x}_1 \quad a_{11}\dot{z}_1 \\ \dot{x}_2 \quad a_{21}\dot{z}_1 \quad a_{22}\dot{z}_2 \\ \vdots \\ \dot{x}_n \quad a_{n1}\dot{z}_1 \quad a_{n2}\dot{z}_2 \quad \cdots \quad a_{nn}\dot{z}_n \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} \tag{3.1.8}$$

Based on the Assumption 1, we obtain $a_{11} \neq 0$,which produces the following cases:

(a) All $a_{ii} \neq 0 (2 \leq i \leq n)$ in (3.1.5)

Since, the value of a_{11} is known, then we can figure out $\dot{z}_2$ by substituting $\dot{z}_1$, repeatedly, we can obtain the values up to $\dot{z}_n$ term by term and use these values into (3.1.6) to form the new response system of (3.1.2). However, $\dot{z}_{n+1}$ up to $\dot{z}_m$ cannot be obtained directly by (3.1.8), thus we can set their values as needed.

(b) Every element in all columns after $a_{i_0 i_0} (2 \leq i \leq n)$ in (3.1.5) is zero, which has the form:

$$\begin{pmatrix} a_1 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ a_2 1 & a_2 2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ a_{i_0} 1 & a_{i_0} 2 & \cdots & a_{i_0 i_0} & 0 & \cdots & 0 \\ a_{i_0+1,1} & a_{i_0+1,2} & \cdots & a_{i_0+1,i_0} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n i_0} & 0 & \cdots & 0 \end{pmatrix} \tag{3.1.9}$$

By using the same procedure in (a), we obtain all $\dot{z}_j(j < i_0)$. For $\dot{z}_{i_0}$, we sum up all of the lines between i_0 and n in (3.1.6); and for $\dot{z}_k(i_0 < k < n)$, we can still set those values as needed.

Finally, we have the final form of (3.1.5) as follows:

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 & 0 & \cdots & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i_0 1} & a_{i_0 2} & \cdots & a_{i_0 i_0} & 0 & \cdots & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i_0+1,1} & a_{i_0+1,2} & \cdots & a_{i_0+1,i_0} & 0 & \cdots & \cdots & 0 & \cdots & 0 \\ a_{i_0+2,1} & a_{i_0+2,2} & \cdots & a_{i_0+2,i_0} & a_{i_0+2,i_0+1} & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni_0} & a_{n,i_0+1} & \cdots & a_{n,i_0+t} & 0 & \cdots & 0 \end{pmatrix} \quad (3.1.10)$$

where $t < n \quad i_0$, then we can use method in (a) and (b) to complete all $\dot{z}_i$.

1.2 n=m

When n=m, the Jacobian matrix $D\varphi y$ will be either a reversible or irreversible square matrix. If $D\varphi(y)$ is reversible, we can follow the generalized synchronization method in (a) in 2.1.1; if $D\varphi(y)$ is irreversible, generalized synchronization method in (b) and final form in 2.1.1 can be referred.

2 Realization of generalized synchronization

Based on all the contents above, we realize that generalized synchronization between two systems can be transformed in to the asymptotical stability analysis of zero solution of error system, then a theorem is introduced:

Theorem 3.1.1. *Let controller* $u(x, y) = Q\dot{z} \quad g(y)$, *and condition* (3.1.8) *is satisfied; then the generalized synchronization between master systems* (3.1.1) *and slave system* (3.1.2) *is achieved.*

Proof. Construct a Lyapunov function $V = \frac{1}{2}(e_1^2 + ... + e_n^2) \geq 0$, thus V=0 if and only if $e_1 = e_2 = ... = e_n$. Also the derivative of V will be $\dot{v} = e_1\dot{e}_1 + ... + e_n\dot{e}_n = \quad e_1^2 \quad ... \quad e_n^2 < 0$. Therefore, from the Lyapunov function theory, the error system is asymptotically stable at the origin, which means the generalized synchronization between (3.1.1) and (3.1.2) is achieved.

We use the Hyperchaotic Chen system, Lorenz system and Lu system as specific examples to verify the effectiveness of the generalized synchronization; and they are defined as following respectively:

$$\begin{aligned} \dot{x}_1 &= a(x_2 \quad x_1) + x_4 \\ \dot{x}_2 &= dx_1 \quad x_1x_3 + cx_2 \\ \dot{x}_3 &= x_1x_2 \quad bx_3 \\ \dot{x}_4 &= x_2x_3 + rx_4 \end{aligned} \tag{3.1.11}$$

where $a = 35, b = 3, c = 12, d = 7, r = 0.5$, system is chaotic.

$$\begin{aligned} \dot{x}_1 &= a_1(x_2 \quad x_1) \\ \dot{x}_2 &= c_1x_1 \quad x_2 \quad x_1x_3 \\ \dot{x}_3 &= x_1x_2 \quad b_1x_3 \end{aligned} \tag{3.1.12}$$

where $a_1 = 10, b_1 = 28, c_1 = \frac{8}{3}$, system is chaotic.

$$\begin{aligned} \dot{x}_1 &= a_2(x_2 \quad x_1) \\ \dot{x}_2 &= \quad x_1x_3 + c_2x_2 \\ \dot{x}_3 &= x_1x_2 \quad b_2x_3 \end{aligned} \tag{3.1.13}$$

where a_2, b_2andc_2 are all positive; if $a_2 = 36, b_2 = 20, c_2 = 3$, then system is chaotic.

Example 1. Consider the Hyperchaotic Chen system as drive system and Lorenz system as response system, define $\varphi(y) = (y_1 + y_3, y_1 + y_2, y_2^2, y_1 \quad y_2 + 2y_3)^T$, then we obtain

$$D\varphi(y) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 2y_2 & 0 \\ 1 & \;\;1 & 2 \end{pmatrix}$$

Next we apply a series of column transformations to $D\varphi(y)$ which forms matrix Q to obtain:

$$M = D\varphi(y)Q = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 2y_2 & 0 \\ 1 & \;\;1 & 2 \end{pmatrix} * \begin{pmatrix} 0 & \;\;1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \;\;1 & 0 \\ 0 & 0 & 2y_2 \\ 2 & 1 & 0 \end{pmatrix} \tag{3.1.14}$$

Then we have:

$$\dot{y} = Q\dot{z} = \begin{pmatrix} 0 & \;\;1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \dot{z} = \begin{pmatrix} \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_1 + \dot{z}_2 + \dot{z}_3 \end{pmatrix} \tag{3.1.15}$$

and also we obtain the error system:

$$\dot{e} = \dot{x} \quad M\dot{z} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} \quad \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ 2y_2\dot{z}_3 \\ 2\dot{z}_1 + \dot{z}_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \tag{3.1.16}$$

Finally, based on equation the method in case (a), we obtain:
$\dot{z}_1 = \dot{x}_1 + e_1$
$\dot{z}_2 = \frac{1}{2}(\dot{x}_4 \quad 2\dot{x}_1 \quad \dot{x}_2 \quad 2e_1 + e_4 \quad e_2)$
$\dot{z}_3 = \frac{1}{2y_2}(\dot{x}_3 + e_3)$
therefore, we can determine the controller u for this example:

$$u(x, y) = Q\dot{z} \quad g(y) = \begin{pmatrix} \frac{1}{2}(\dot{x}_4 \quad 2\dot{x}_1 \quad \dot{x}_2 \quad 2e_1 + e_4 \quad e_2) \\ \frac{1}{2y_2}(\dot{x}_3 + e_3) \\ \frac{1}{2}(\dot{x}_4 \quad \dot{x}_2 + e_4 \quad e_2) + \frac{1}{2y_2}(\dot{x}_3 + e_3) \end{pmatrix}$$
$$\begin{pmatrix} a_1(y_2 \quad y1) \\ c_1y_1 \quad y_2 \quad y_1y_3 \\ y_1y_2 \quad b_1y_3 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2}(\dot{x}_4 \quad 2\dot{x}_1 \quad \dot{x}_2 \quad 2e_1 + e_4 \quad e_2) \quad a_1(y_2 \quad y1) \\ \frac{1}{2y_2}(\dot{x}_3 + e_3) \quad c_1y_1 + y_2 + y_1y_3 \\ \frac{1}{2}(\dot{x}_4 \quad \dot{x}_2 + e_4 \quad e_2) + \frac{1}{2y_2}(\dot{x}_3 + e_3) \quad y_1y_2 + b_1y_3 \end{pmatrix}$$

Example 2. Oppositely,consider the Lorenz system as drive system and Hyperchaotic Chen system as response system, define $\varphi(y) = (y_1 + y_3 + y_4, y_2 + y_3, 2y_2 + 2y_3 + y_4)^T$, then we have

$$D\varphi(y) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

Next we apply a series of column transformations to $D\varphi(y)$ which forms matrix Q to obtain:

$$M = D\varphi(y)Q = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

This time, we follow the method of case (b) and substitute M and Q into $u(x, y) = Q\dot{z} \quad g(y)$, we can obtain:

$\dot{z}_1 = \dot{x}_1 + e_1$
$\dot{z}_2 = \dot{x}_2 + e_2$
$\dot{z}_3 = \dot{x}_3 + e_3$
$\dot{z}_4 = \quad e_3$
and the controller u(x,y) will be:

$$u(x,y) = \begin{pmatrix} \dot{x}_1 + e_1 \quad \dot{x}_3 \quad e_3 + e_3 \quad ay_2 + ay_1 \quad y_4 \\ \dot{x}_2 + e_2 + e_3 \quad dy_1 + y_1 y_3 \quad cy_2 \\ e_3 \quad y_1 y_2 + by_3 \\ 2\dot{x}_2 \quad 2e_2 + \dot{x}_3 + e_3 \quad y_2 y_3 \quad ry_4 \end{pmatrix}$$

Example 3. Consider the Lorenz system as drive and Lu system as response system, where $D\varphi(y)$ is irreversible; define $\varphi(y) = (y_1 + y_3, y_2, y_1 + y_2 + y_3)^T$, then

$$D\varphi(y) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

where its rank is 2, thus irreversible.

After carrying out a series of column transformation to $D\varphi(y)$ which forms matrix Q to obtain

$$M = D\varphi(y)Q = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & \ 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Also, we have

$$\dot{y} = Q\dot{z} = \begin{pmatrix} 1 & 0 & \ 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{z} = \begin{pmatrix} \dot{z}_1 \quad \dot{z}_3 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix}$$

and also we obtain the error system:

$$\dot{e} = \dot{x} \quad M\dot{z} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} \quad \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_1 + \dot{z}_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

In this irreversible example, we apply the method of case (b) again and substitute M and Q into $u(x,y) = Q\dot{z} \quad g(y)$, we can obtain:
$\dot{z}_1 = \dot{x}_1 + e_1$
$\dot{z}_2 = \frac{1}{2}(\dot{x}_2 + \dot{x}_3 \quad \dot{x}_1 + e_2 + e_3 \quad e_1)$

$\dot{z}_3 = \dot{x}_3 + e_1$
and the controller u(x,y) will be:

$$u(x,y) = \begin{pmatrix} \dot{x}_1 + e_1 \quad \dot{x}_3 \quad e_1 \quad a_2y_2 + a_2y_1 \\ \frac{1}{2}(\dot{x}_2 + \dot{x}_3 \quad \dot{x}_1 + e_2 + e_3 \quad e_1) \quad c_2y_2 + y_1y_3 \\ \dot{x}_3 + e_1 + b_2y_2 + y_1y_2 \end{pmatrix}$$

Example 4. In this last example, we consider the Lorenz system as the drive system and Lu system as response system, but $D\varphi(y)$ is reversible; define the functional map is $D\varphi(y) = (y_1, y_1 + y_2, y_1 + y_2 + y_3)^T$, we have:

$$D\varphi(y) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

where its rank is 3, thus reversible. Then we apply same steps as previous examples, we are able to get the matrix Q, matrix M, system $\dot{y}$, error system, column vector $\dot{z}$ (apply the method in case (a)), and finally the controller u respectively:

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$M = D\varphi(y)Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\dot{y} = Q\dot{z} = \begin{pmatrix} \dot{z}_1 \\ \dot{z}_1 + \dot{z}_2 \\ \dot{z}_2 + \dot{z}_3 \end{pmatrix}$$

$$\dot{e} = \dot{x} \quad M\dot{z} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} \quad \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

$$\dot{z} = \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} \dot{x}_1 + e_1 \\ \dot{x}_2 + e_2 \\ \dot{x}_3 + e_3 \end{pmatrix}$$

and finally,

$$u(x,y) = \begin{pmatrix} \dot{x}_1 + e_1 \quad \dot{x}_3 \quad e_3 \quad a_2y_2 + a_2y_1 \\ \dot{x}_2 + e_2 \quad c_2y_2 + y_1y_3 \\ \dot{x}_3 + e_3 + b_2y_2 \quad y_1y_2 \end{pmatrix}$$

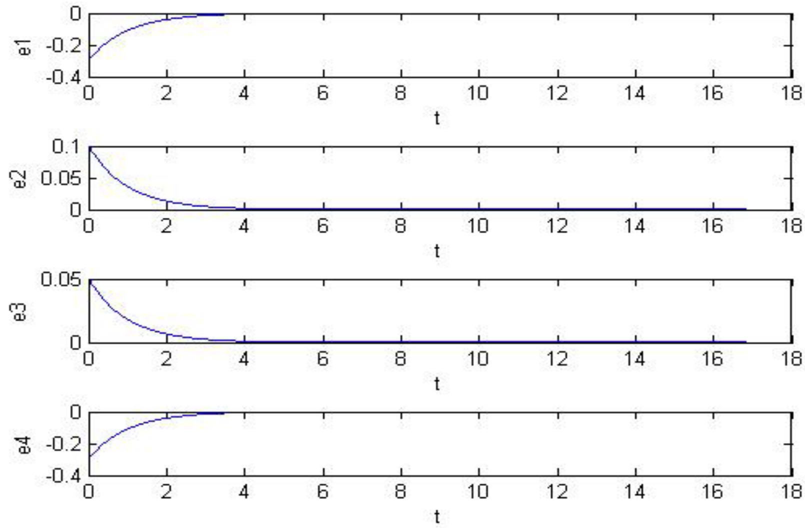

Figure 3.1: Error system of Hyperchaotic Chen and Lorenz system

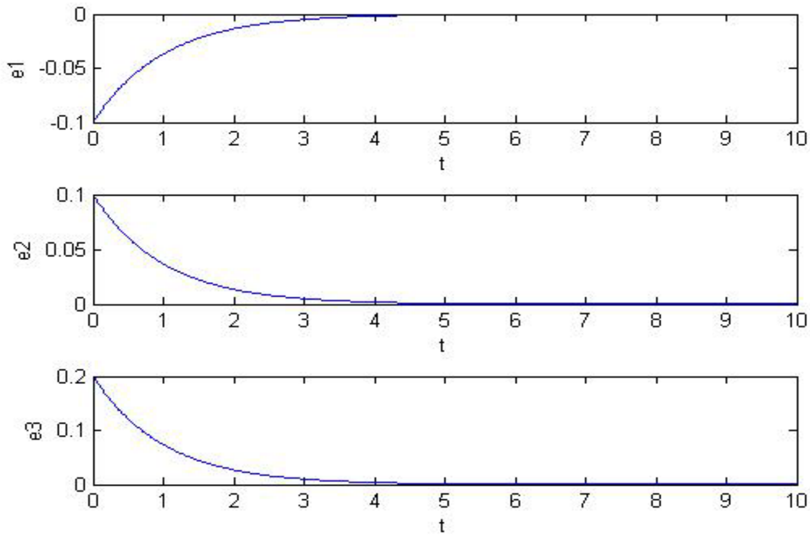

Figure 3.2: Error system of Lorenz and Hyperchaotic Chen system

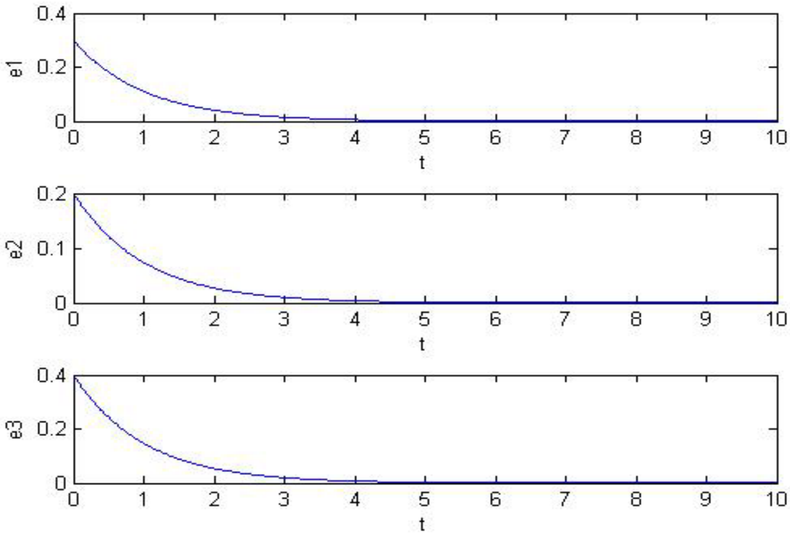

Figure 3.3: Error system of Lorenz and Lu system with $D\varphi(y)$ irreversible

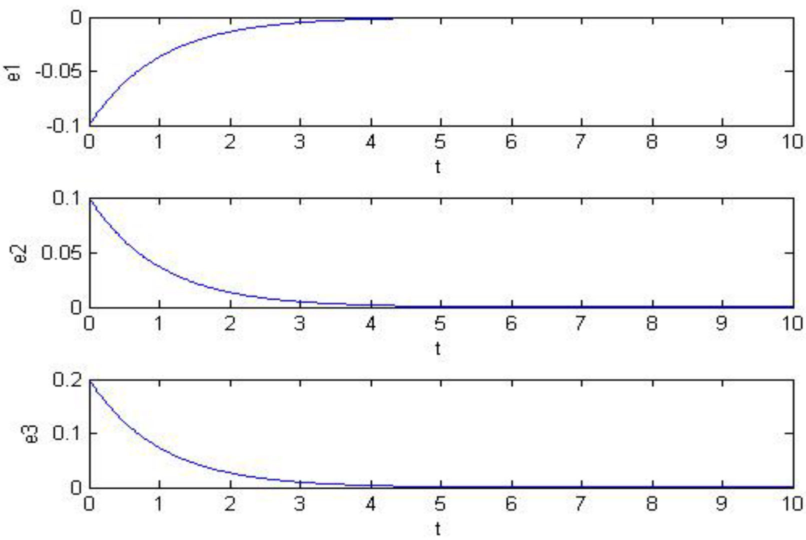

Figure 3.4: Error system of Lorenz and Lu system with $D\varphi(y)$ reversible

Based on the graphs above, the error systems of all four examples converge to zero; thus the generalized synchronization can be achieved.

3.2 Generalized Synchronization in discrete time

1. Classical problem of GS in discrete time

First, we construct the following chaotic drive system:

$$X(k+1) = f_1(X(k)) \tag{3.2.1}$$

where $X(k) = (x_1(k), ..., x_n(k))^T \in R^n$ is the state vector and $f_1 : R^m \to R^n$ contains the linear and the nonlinear parts of this drive system.

Second, we construct the following chaotic response system with controller U:

$$Y(k+1) = BY(k) + g_1(Y(k)) + U \tag{3.2.2}$$

where $Y(k) = (y_1(k), ..., y_m(k))^T \in R^m$, B is an $m \times m$ matrix which represents the linear part of the system dynamics, $g_1 : R^m \to R^m$ is the nonlinear part of the response system, and $U = (u_i)_{1 \leq i \leq m} \in R^m$ is the vector controller to be determined.

In order to understand and achieve the classical generalized synchronization between system (3.2.1) and (3.2.2), we introduce the following definition and theorem:

Definition 3.2.1. *The drive system and the response system* (3.2.2) *are said to be generalized synchronized with respect to the vector map ϕ if there exists a controller $U = (u_i)_{1 \leq i \leq m} \in R^m$ and a given map $\phi : R^n \to R^m$ such that the synchronization error:*

$$e(k) = Y(k) - \phi(X(k)) \tag{3.2.3}$$

satisfies that $lim_{k \to +\infty} \|e(k)\| = 0$.

Based on this definition, we need to design the controller U such that the solutions of the error system (3.2.3) go to zero as k goes to infinity. Since the error dynamics between (3.2.1) and (3.2.2) can be derived as:

$$e(k+1) = BY(k) + g_1(Y(k)) - \phi(f_1(X(k))) + U \tag{3.2.4}$$

then we can choose U to be:

$$U = -L_1 Y(k) - g_1(Y(k)) + \phi(f_1(X(k))) + (L_1 - B)\phi(X(k)) \tag{3.2.5}$$

where $L_1 \in R^{m \times m}$ is an undetermined control matrix. If we substitute controller (3.2.5) into error system (3.2.4), we obtain:

$$e(k+1) = (B - L_1)e(k) \tag{3.2.6}$$

Theorem 3.2.1. *If we select the control matrix* L_1 *such that* $P_1 = I \quad (B \ L_1)^T(B \quad L_1)$ *is a positive definite matrix, then system* (3.2.1) *and* (3.2.2) *are globally generalized synchronized with respect to* ϕ*, under the controller* (3.2.5).

Proof. Choose the Lyapunov candidate as follows:

$$V(e(k)) = e^T(k)e(k) \tag{3.2.7}$$

then we have the derivative of the Lyapunov candidate:

$$\begin{aligned}
&\Delta V(e(k)) = e^T(k+1)e(k+1) \quad e^T(k)e(k) \\
&= e^T(k)(B \quad L_1)^T(B \quad L_1)e(k) \quad e^T(k)e(k) \\
&= e^T(k)[(B \quad L_1)^T(B \quad L_1) \quad I]e(k) \\
&= \quad e^T(k)P_1e(k) < 0
\end{aligned} \tag{3.2.8}$$

Therefore, since the derivative of the Lyapunov candidate is always less than zero; by Lyapunov stability theory, we can obtain:

$$\lim_{k\to\infty} e_i(k) = 0, (i = 1, 2, ..., m) \tag{3.2.9}$$

which means system (3.2.1) and (3.2.2) are globally generalized synchronized and the error system (3.2.6) is globally asymptotically stable.

2. Inverse problem of GS in discrete time

Consider another type drive and response chaotic system:

$$X(k+1) = AX(k) + f_2(X(k)) \tag{3.2.10}$$

$$Y(k+1) = g_2(Y(k)) + U \tag{3.2.11}$$

where $X(k) = (x_1(k), ..., x_n(k))^T \in R^n$ and $Y(k) = (y_1(k), ..., y_m(k))^T \in R^m$ are the state vectors of drive and response system respectively; $A \in R^{n\times n}$,$f_2 : R^n \to R^n$ is the nonlinear part of the drive system (3.2.10) and $g_2 : R^m \to R^m$ contains the linear and nonlinear part of response system (3.2.11); $U = (u_i)_{1\le i\le m} \in R^m$ is the vector controller to be determined.

Similarly, in order to understand and achieve the inverse generalized synchronization between system (3.2.10) and (3.2.11), we introduce another likely definition and theorem:

De nition 3.2.2. *The drive system* (3.2.10) *and the response system* (3.2.11) *are said to be inverse generalized synchronized with respect to the vector map* ϕ *if there exists a controller* $U = (u_i)_{1 \leq i \leq m} \in R^m$ *and a given map* $\varphi : R^m \to R^n$ *such that the synchronization error:*

$$e(k) = X(k) - \varphi(Y(k)) \tag{3.2.12}$$

satisfies that $lim_{k \to \infty} \|e(k)\| = 0$.

According to the definition, we need to design the controller U such that the solutions of the error system (3.2.12) go to zero as k goes to infinity. Since the error dynamics between (3.2.10) and (3.2.11) can be derived as:

$$e(k+1) = AX(k) + f_2(X(k)) - \varphi(g_2(Y(k)) + U) \tag{3.2.13}$$

Then, in order to achieve inverse generalized synchronization, we choose U as follows:

$$U = -g_2(Y(k)) + \varphi^{-1}[f_2(X(k)) + L_2 X(k) + (A - L_2)\psi(Y(k))] \tag{3.2.14}$$

where $\varphi^{-1} : R^n \to R^m$ is the inverse of the map φ and $L_2 \in R^{n \times n}$ is an undetermined control matrix. If we substitute controller U in (3.2.14) into error system (3.2.13), we obtain:

$$e(k+1) = (A - L_2)e(k) \tag{3.2.15}$$

Theorem 3.2.2. *If we select the control matrix* L_2 *such that* $P_2 = I - (A - L_2)^T (A - L_2)$ *is a positive definite matrix, then the drive system* (3.2.10) *and the response system* (3.2.11) *are globally inverse generalized synchronized with respect to* φ*, under the controller law* (3.2.14).

Proof. Similarly, we choose the Lyapunov candidate to be:

$$V(e(k)) = e^T(k)e(k) \tag{3.2.16}$$

then we calculate the derivative of the Lyapunov candidate:

$$\begin{aligned} \Delta V(e(k)) &= e^T(k+1)e(k+1) - e^T(k)e(k) \\ &= e^T(k)(A - L_2)^T(A - L_2)e(k) - e^T(k)e(k) \\ &= e^T(k)[(A - L_2)^T(A - L_2) - I]e(k) \\ &= -e^T(k)P_2 e(k) < 0 \end{aligned} \tag{3.2.17}$$

Therefore, based on the Lyapunov stability theory, we have

$$\lim_{k \to \infty} e_i(k) = 0, (i = 1, 2, ..., n) \tag{3.2.18}$$

the zero solution of error system (3.2.15) is globally asymptotically stable as well as system (3.2.10) and (3.2.11) achieve inverse generalized synchronization globally.

In order to check the effectiveness of theoretical generalized synchronization with discrete time, we discuss the following two numerical examples:

Example 1: Classical GS 3D Henon-like system and Lorenz system

In this example, we construct the hyperchaotic 3D Henom-like map as the drive system and the controlled Lorenz discrete time system as the response system respectively as follows:

$$\begin{aligned} x_1(k+1) &= 1 + x_3(k) \qquad x_2^2(k) \\ x_2(k+1) &= 1 + \beta x_2(k) \qquad x_1^2(k) \\ x_3(k+1) &= \beta x_1(k) \end{aligned} \tag{3.2.19}$$

where we choose $(\quad , \beta) = (1.4, 0.2)$ to obtain the chaotic attractor.

$$\begin{aligned} y_1(k+1) &= (1+ab)y_1(k) \quad by_1(k)y_2(k) + u_1 \\ y_2(k+1) &= (1 \quad b)y_2(k) + by_1^2(k) + u_2 \end{aligned} \tag{3.2.20}$$

where we choose $(a, b) = (1.25, 0.75)$ to obtain the chaotic attractor.

Then we define $\phi : R^3 \to R^2$ as:

$$\phi(x_1(k), x_2(k), x_3(k)) = (x_1(k) + x_3(k), x_2(k) + x_3(k)) \tag{3.2.21}$$

and the control matrix L_1 as:

$$L_1 = \left. \begin{matrix} 1.4 + ab & 0 \\ 1 \quad b & 0.82 \end{matrix} \right) \tag{3.2.22}$$

Moreover, based on the structure of system (3.2.20), we can obtain

$$B = \left. \begin{matrix} 1 + ab & 0 \\ 1 \quad b & 0 \end{matrix} \right) \tag{3.2.23}$$

and

$$g_1(Y(k)) = \left. \begin{matrix} by_1y_2 \\ by_1^2 \end{matrix} \right) \tag{3.2.24}$$

Finally, based on theorem 1, system (3.2.19) and (3.2.20) are theocratically generalized synchronized.

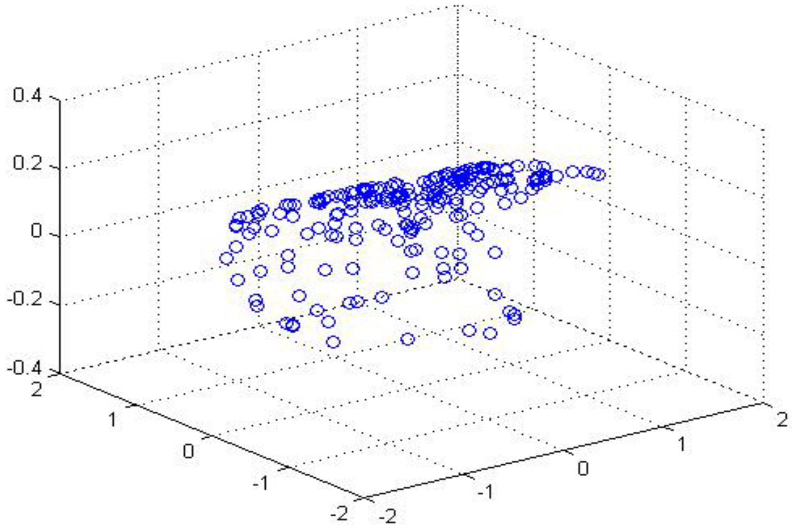

Figure 3.5: Phase portraits of hyperchaotic 3D Henom-like map

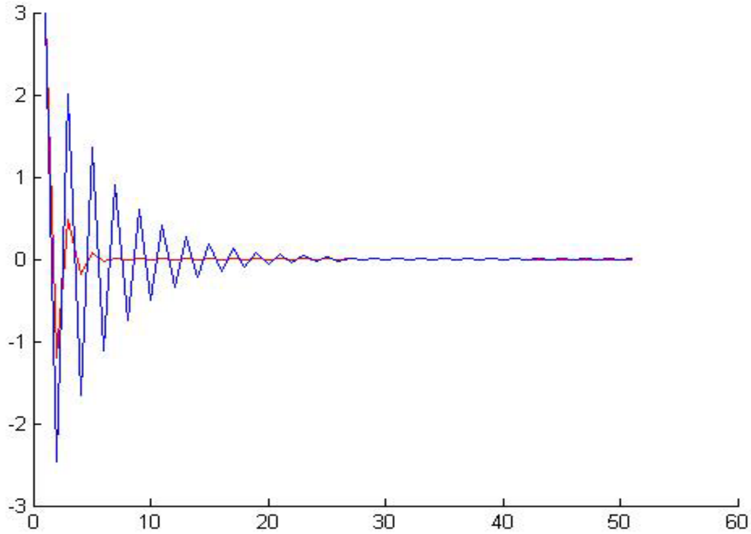

Figure 3.6: Error system of the hyperchaotic 3D Henom-like map and the controlled Lorenz discrete time system

The error system converges to zero as time increasing based on graph as well, which verify the generalized synchronization between system (3.2.19) and (3.2.20) numerically.

Example 2. Inverse GS between the 3D generalized Henon system and the Fold system

In this example, we construct the hyperchaotic 3D Henom-like map as the drive system and the controlled Fold discrete time system as the response system respectively as follows:

$$\begin{aligned} x_1(k+1) &= \quad \beta x_2(k) \\ x_2(k+1) &= 1 + x_3(k) \qquad x_2^2(k) \\ x_3(k+1) &= \beta x_s(k) + x_1(k) \end{aligned} \tag{3.2.25}$$

where we choose $(\quad , \beta) = (1.07, 0.3)$ to obtain the chaotic attractor.

$$\begin{aligned} y_1(k+1) &= y_2(k) + a y_1(k) + u_1 \\ y_2(k+1) &= b + y_1^2(k) + u_2 \end{aligned} \tag{3.2.26}$$

where we choose $(a, b) = (\quad 0.1, \quad 1.7)$ to obtain the chaotic attractor.

Then we define $\phi : R^2 \to R^3$ as:

$$\phi(y_1(k), y_2) = (y_1(k), y_2(k), 2y_1(k)) \tag{3.2.27}$$

and the control matrix L_2 as:

$$L_2 = \begin{pmatrix} 0.1 & \quad \beta & 0 \\ 0 & 0 & 1 \\ 1 & \beta & 0.25 \end{pmatrix} \tag{3.2.28}$$

Moreover, based on the structure of system (3.2.25), we can obtain

$$A = \begin{pmatrix} 0 & \quad \beta & 0 \\ 0 & 0 & 1 \\ 1 & \beta & 0 \end{pmatrix} \tag{3.2.29}$$

and

$$f_2(X(k)) = \begin{pmatrix} 0 \\ 1 \qquad x_2^2 \\ 0 \end{pmatrix} \tag{3.2.30}$$

Finally, based on theorem 2, system (3.2.25) and (3.2.26) are theocratically inverse generalized synchronized.

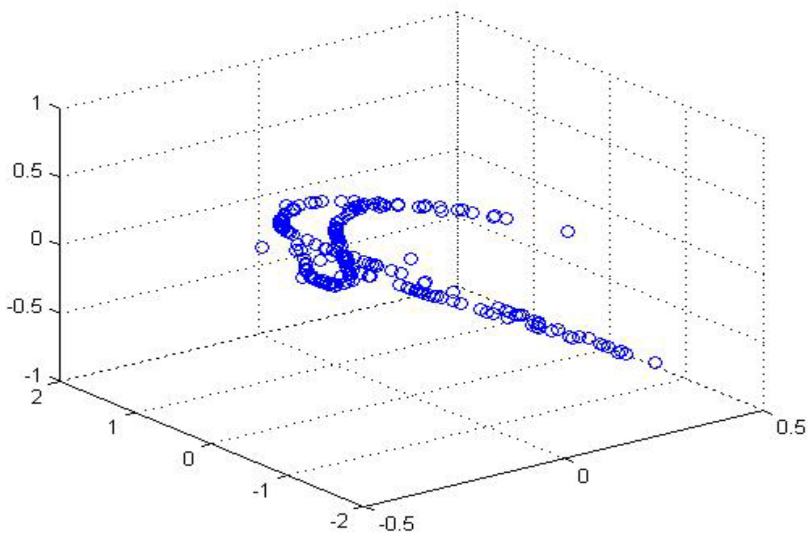

Figure 3.7: Phase portraits of hyperchaotic 3D Henom-like map

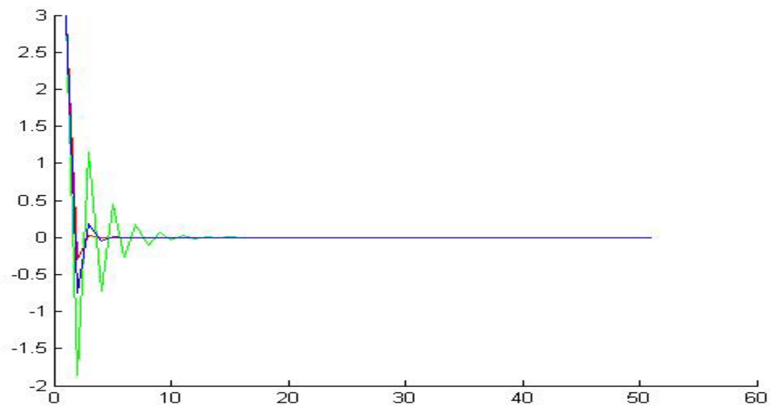

Figure 3.8: Error system of the hyperchaotic 3D Henom-like map and the controlled Fold discrete time system

The error system converges to zero as time increasing based on graph as well, which verify the generalized synchronization between system (3.2.25) and (3.2.26) numerically.

Chapter 4

Network Complete Synchronization Techniques

4.1 Synchronization of Identical Networks

1.Model Construction

A general complex dynamical network consisting of N identical linearly and d-iffusively coupled nodes, with each node being an n-dimensional dynamical system and be described by:

$$\dot{x}_i = f(x_i) + \sum_{j=1,j\neq i}^{N} c_{ij}(t)A(t)x_j \quad x_i, i = 1, 2, ..., N \tag{4.1.1}$$

where $x_i(t) = (x_{i1}(t), ..., x_{in}(t))^T \in R^n$ is the state vector of the ith node, $f : R \times R^n \to R^n$ is a smooth nonlinear vector-valued function, the constant matrix $A(t)$ is the coupling link matrix between node i and node j ($i \neq j$) for all $1 \leq i, j \leq N$ at time t; the coupling matrix $C = (c_{ij})_{N\times N}$ represent the coupling configuration of the network,if there is a connection between node i and j ($i \neq j$), then $c_{ij} > 0$; $c_{ij} = 0$ otherwise; also, C is set to be a diffusively coupled matrix where $c_{ii} = \sum_{j=1,j\neq i}^{N} c_{ij}, i = 1, 2, ..., N$.

In addition, define the diagonal elements of C(t) by:

$$c_{ii}(t) = \sum_{j=1,j\neq i}^{N} c_{ij}(t), i = 1, 2, ..., N \tag{4.1.2}$$

then the time-varying network (104) can be rewritten in a compact form as:

$$\dot{x}_i = f(x_i) + \sum_{j=1,j\neq i}^{N} c_{ij}(t)A(t)x_j, i = 1, 2, ..., N \tag{4.1.3}$$

Therefore, if network (4.1.3) is connected without isolate clusters, then C(t) is an irreducible matrix at any time t and network is time-varying. Meanwhile, we do not make assumption of the symmetry and all the off-diagonal elements are nonnegative of C(t).

2. Mathematical preliminaries

In this section, necessary definitions are presented in order to provide basic understanding of chaotic network synchronization.

De nition 4.1.1. *For a given real matrix $C = (c_{ij})_{N\times N}$, if sum of all elements in each row is equal to zero, then C is called a di usively coupled matrix. In addition, if all the o -diagonal elements of C are nonnegative, then C is called a nonnegative di usively coupled matrix. The set of all nonnegative di usively coupled matrices is denoted T.*

De nition 4.1.2. *Let $x_i(t; t_0, x_1^0, ..., x_N^0) i = 1, 2, ..., N$ be a solution of the dynamical network:*

$$\dot{x}_i = f(x_i) + g_i(x_1, ..., x_N), i = 1, 2, ..., N \qquad (4.1.4)$$

where $f : D \rightarrow R^n$ and $g_i : D \times ... \times D \rightarrow R^n$ are continuously di erentiable, $D \subseteq R^n$. If there is a nonempty open subset $D^0(t_0) \subseteq D$, with $x_i^0 \subseteq D^0(t_0) i = 1, 2, ..., N$, such that $x_i \in D$ for all $t \geq t_0, i = 1, 2, ..., N$ and $lim_{t\rightarrow\infty}\|x_i(t; t_0, x_1^0, ..., x_N^0) \quad x_j(t; t_0, x_1^0, ..., x_N^0)\|_2 = 0$ for $1 \leq i, j \leq N$, then the dynamical network (4.1.3) *is said to realize synchronization and $D^0(t_0) \times ... \times D^0(t_0)$ is called the region of synchrony of* (4.1.2).

De nition 4.1.3. *From above previous definition, if the system $\dot{s}(t) = f(s(t))$ where s(t) is the synchronous solution: $x_1(t; t_0, x_1^0, ..., x_N^0) = ... = x_N(t; t_0, x_1^0, ..., x_N^0)$ $(x_1^0 = ... = x_N^0 \in D)$ of an individual node $lim_{t\rightarrow\infty}\|x_i(t; t_0, x_1^0, ..., x_N^0) \quad x_1(t; t_0, x_1^0, ..., x_N^0)\|_2 = 0$ for $2 \leq i \leq N$, then network* (4.1.3) *achieves synchronization, where the errors: $\dot{\eta}(t) = x_i(t; t_0, x_1^0, ..., x_N^0) \quad x_1(t; t_0, x_1^0, ..., x_N^0)$ for $2 \leq i \leq N$ are called the transverse errors of the synchronous manifold $x_1(t) = ... = x_N(t)$, where $x_1(t)$ is called the reference direction of the synchronous manifold. When all the transverse errors exponentially (uniformly exponentially stable) tend to zero, the chaotic synchronous state:$x_1(t) = ... = x_N(t)$ is called exponentially stable (uniformly exponentially stable). Moreover, if $\|x(t) \quad s(t)\|_2$ approaches zero uniformly and exponentially as t approaches infinity, then x(t)-s(t) is said to be uniformly exponentially stable.*

Remark:

Based on Definition 3.1.1 and 3.1.2 above, the network (4.1.3) contains the diffusive coupling matrix C; thus it ensures the synchronous solution: $x_1(t; t_0, x_1^0, ..., x_N^0) = ... = x_N(t; t_0, x_1^0, ..., x_N^0)(x_1^0 = ... = x_N^0 \in D)$ to be a solution of system $\dot{s}(t) = f(s(t))$, where s(t) assumes to be chaotic.

According to Definition 3.1.3, the uniformly exponential stability of a chaotic synchronous state is equivalent to the uniformly exponential stability of the zero transverse errors of the synchronous manifold for network (4.1.3). However, from the diffusive coupling condition, if s(t) is not uniformly exponentially stable; then it is impossible for the synchronous solution to be uniformly exponentially stable.

Therefore, by choosing $x_1(t) = s(t)$ as the reference direction of the synchronous manifold $x_1(t) = ... = x_N(t)$; we have $\eta_1(t) = x_1(t) \quad s(t) \equiv 0$ and $x_i(t) = s(t) + \eta_i(t), i = 1, 2, ..., N$, which yields network (4.1.3) as:

$$\dot{\eta}_i(t) = f(s(t) + \eta_i(t)) \quad f(s(t)) + \sum_{j=2}^{N} c_{ij}(t)A(t)\eta_i(t), 2 \leq i \leq N \tag{4.1.5}$$

Also, we can rewrite it as:

$$\dot{\bar{\eta}}(t) = F(t, \bar{\eta}) \tag{4.1.6}$$

where $\bar{\eta} = (\eta_2(t), ..., \eta_N(t))^T$, $\eta_i(t) = (\eta_{i1}(t), ..., \eta_{in}(t))^T$, $\bar{S}(t) = (s(t), ..., s(t))^T \in R^{n(N \quad 1)}$, and the Jacobian matrix of $F(t, \bar{\eta})$ at $\bar{\eta} = 0$ is $DF(t, 0)$:

$$\begin{pmatrix} Df(s(t)) + c_{22}(t)A(t) & c_{23}(t)A(t) & \cdots & c_{2N}(t)A(t) \\ c_{32}(t)A(t) & Df(s(t)) + c_{33}(t)A(t) & \cdots & c_{3N}(t)A(t) \\ \vdots & \vdots & Df(s(t)) + c_{22}(t)A(t) & \vdots \\ c_{N2}(t)A(t) & c_{N3}(t)A(t) & \cdots & Df(s(t)) + c_{NN}(t)A(t) \end{pmatrix} \tag{4.1.7}$$

3. Theorem analysis

In this section, a specific chaos synchronization theorem [2] is discussed which allows us to obtain the synchronization by verifying linear time-varying systems, rather than directly synchronize the dynamical network.

Theorem 4.1.1. *If the following two assumptions holds, then the chaotic synchronous state $x_1(t) = ... = x_N(t) = s(t)$ is exponentially stable for dynamical network (4.1.3) if and only if the linear time-varying systems $\dot{w} = [Df(s(t)) + \lambda_k(t)A(t) \quad \beta_k(t)I_n]w, k = 2, ..., N$ are exponentially stable about the zero solution.*

Assumption 4.1.1. *$F : \Omega \to R^{n(N\ 1)}$ is continuously di erentiable on $\Omega = \{x \in \frac{R^{n\ N\ 1)}}{\|x\|_2} < r\}$ with F(t,0)=0 for all t, and the Jacobian DF(t,x) is bounded and Lipschitz on Ω, uniformly in t.*

Assumption 4.1.2. *There exists a bounded non-singular real matrix $\Phi(t)$, such that $\Phi^{\ 1}(t)(C^T(t))\Phi(t) = diag\{\lambda_1(t), ..., \lambda_N(t)\}$, there exists $t_0 \geq 0$, for any $\lambda_i(t), 1 \leq i \leq N$, either $\lambda_i(t) \neq 0$ for all $t > t_0$, or $\lambda_i(t) \equiv 0$ for all $t > t_0$, and $\dot{\Phi}^{\ 1}\Phi(t) = diag\{\beta_1(t), .., \beta_N(t)\}$*

Remark:

Based on definition 3.1.3, we are able to investigate the proof of above theorem by verifying the exponential stability of the zero transverse errors of the synchronous manifold. According to the assumption 1 and Lyapunov converse theorem [7], we can implement equation (4.1.5) to be $\dot{\eta}(t) = Df(s(t))\eta(t) + A(t)\eta(t)(C(t))^T$, where $\eta(t) = (\eta_1(t), ..., \eta_N(t))^T$ and construct the nonsingular linear transformation $\eta(t) = v(t)\Phi^{\ 1}(t)$, which leads to the equation

$$\dot{v}_k(t) = [Df(s(t)) + \lambda_k(t)A(t) \quad \beta_k(t)I_n]v_k(t), k = 1, 2, ..., N \tag{4.1.8}$$

and it is N independent n-dimensional linear time-varying system. Moreover, according to Lemma 3 in [2], $\phi'_{11}(t) = 1$, hence $v_1(t) = \quad \sum_{k=2}^{N} \phi'_{k1}(t)v_k(t)$; if $v_k(t) \to 0$ for k=2,..,N, then $v_1(t) \to 0$ as well. Therefore, we can finally transferred the exponential stability of the synchronous state to the exponential stability of the N-1 independent n-dimensional linear time-varying systems.

Based on the results above, this chaos synchronization theorem proposes that the synchronization of time-varying chaotic network (4.1.3) is determined by means of A(t), the eigenvalues $\lambda_k k = 2, .., N$ and the corresponding eigenvectors $\phi_k(t)k = 2, ..., N$ of C(t). Overall, it provides a necessary and sufficient condition for synchronization of network.

4. Specify conditions of constructing A(t) and C(t)

Notice that the theorem discussed above does not provide the specific conditions of constructing the coupled matrix A(t) and the coupled configuration matrix C(t) in network (4.1.3). Thus in this section, specific conditions are introduced in order to obtain A(t) and C(t), which allows the network (4.1.3) more applicable for the further numerical investigations.

Consider the following hypothesis:

(H1) Assume $F : \Omega \to R^{n(N\ 1)}$ is continuously differentiable on $\Omega = \{x \in \frac{R^{n\ N\ 1)}}{\|x\|_2} < r\}$ with F(t,0)=0 for all t, and the Jacobian DF(t,x) is bounded and

Lipschitz on Ω, uniformly in t; and C(t) can be diagonalized by using a constant nonsingular matrix $\Phi(t)$, there exists $t_0 \geq 0$, for any $\lambda_i(t), 1 \leq i \leq N$, either $\lambda_i(t) \neq 0$ for all $t > t_0$, or $\lambda_i(t) \equiv 0$ for all $t > t_0$.

(H2) Assume that C(t) is a nonnegative diffusively coupled matrix and its eigenvalues are real numbers satisfying $\lambda_1(t) \geq ... \geq \lambda_N(t)$ for all $t > t_0 \geq 0$.

Then either H1 holds or both H1 and H2 hold, the following conditions are valid for constructing specific matrices A(t) and C(t):
1. H1 holds, if the linear time-varying systems $\dot{w} = [Df(s(t)) + \lambda_k(t)A(t)]wk = 2, .., N$ are exponentially stable about zero solution.
2. H1 and H2 hold, if $(A(t) + (A(t))^T)$ is a semi-positive-definite matrix, and the maximum eigenvalue $\mu[Df(s(t)) + \lambda_2(t)A(t)] \leq a < 0$ for all $t \geq 0$.
3. H1 and H2 hold, if $\int_{t_0}^{\infty} |\lambda_N(t)A(t)|dt < \infty$ and the linear time-varying system $\dot{w} = [Df(s(t)) + \lambda_k(t)A(t)]w$ is exponentially stable about the zero solution.
4. H1 and H2 hold, if there exists and $n \times n$ positive-definite matrix B, such that $[Df(s(t)) + dA(t)]^T B + B[Df(s(t)) + dA(t)] \leq \quad I_n$ for all $d \leq \lambda_2^0$ and $t \geq t_0$.
5. H1 and H2 hold, if there exists and $n \times n$ positive-definite matrix B, such that for all $t \geq t_0$, $(A(t))^T B + BA(t)$ is a semi-positive-definite matrix, and $[Df(s(t)) + \lambda_2^0 A(t)]^T B + B[Df(s(t)) + \lambda_2^0 A(t)] \leq \quad I_n$

Consequently, if either one of the above conditions is satisfied upon constructing particular A(t) and C(t), then the chaotic synchronous state $x_1(t) = ... = x_N(t) = s(t)$ of network (106) will be exponentially stable.

Based on the above discussion, thus A(t) and C(t) with respect to unified chaotic system can be described as follows:

$$A(t) = \begin{pmatrix} (25 \quad + 38 \quad 35 \quad)^2(1 + e \quad ^t) & 0 & 0 \\ 0 & (\frac{8+}{3})^2(1 + e \quad ^{3t}) & 0 \\ 0 & 0 & (\frac{8+}{3})(1 + e \quad ^{2t}) \end{pmatrix}$$

$$C(t) = (\frac{1}{2e^2 \quad e \quad 1}) \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$

where
$C_{11}(t) = (e^2 \quad 1)th(t) + (e)arctan(t)$, $C_{12}(t) = (1 \quad e)th(t) \quad (2e)arctan(t)$, $C_{13}(t) = (e \quad e^2)th(t) + (e)arctan(t)$, $C_{21}(t) = 2(e^2 \quad 1)th(t) + (e^2)arctan(t)$, $C_{22}(t) = 2(1 \quad e)th(t) \quad (2e^2)arctan(t)$, $C_{23}(t) = 2(e \quad e^2)th(t) + (e^2)arctan(t)$, $C_{31}(t) = 3(e^2 \quad 1)th(t) + arctan(t)$, $C_{32}(t) = 3(1 \quad e)th(t) \quad 2arctan(t)$, $C_{33}(t) = 3(e \quad e^2)th(t) + arctan(t)$, with $th(t) = \frac{(e^t \quad e \quad ^t)}{(e^t + e \quad ^t)}$

In order to construct the time-varying systems, we require a nonsingular real matrix Φ. In this case, we choose

$$\Phi(t) = (\frac{1}{2e^2 \quad e \quad 1}) \begin{pmatrix} 3e^2 \quad 2 & (1 \quad e^2)e^t & e^{1+sin(t)} \\ 1 \quad 3e & (e \quad 1)e^t & 2e^{1+sin(t)} \\ 2e \quad e^2 & (e^2 \quad 1)e^t & e^{1+sin(t)} \end{pmatrix}$$

then $\Phi \;\, ^1(t)(C(t))^T\Phi(t) = diag\{0, \quad 1, \quad cos(t)\}$, thus we can determine the linear time-varying systems: $\dot{w} = [Df(s(t)) + \lambda_k(t)A(t) \quad \beta_k(t)I_n]w, k = 2, 3$.

5. Numerical Simulation

The numerical simulations illustrate the results for both directly simulate the unified chaotic network and simulate the transformed linear time-varying systems. In order to improve the effectiveness of the simulations, we apply ode45 method to those ODE systems, and apply the interpolation method to deal with the time-dependent coefficients in the matrix A(t), C(t) and the matrix $[Df(s(t)) + \lambda_k(t)A(t) \quad \beta_k(t)I_n]$. According to the time series plots, it can be clearly viewed that each system of the network will converge to zero solutions after certain period of time, which means those systems are synchronized, no matter what the initial condition for each system will be assigned to and which   value is chosen from [0,1]. Especially for the linear time-varying systems, there exists an obvious exponentially stable behavior when converging to the zero solution. Therefore, the numerical simulations indicates the validity of the chaotic synchronization theorem.

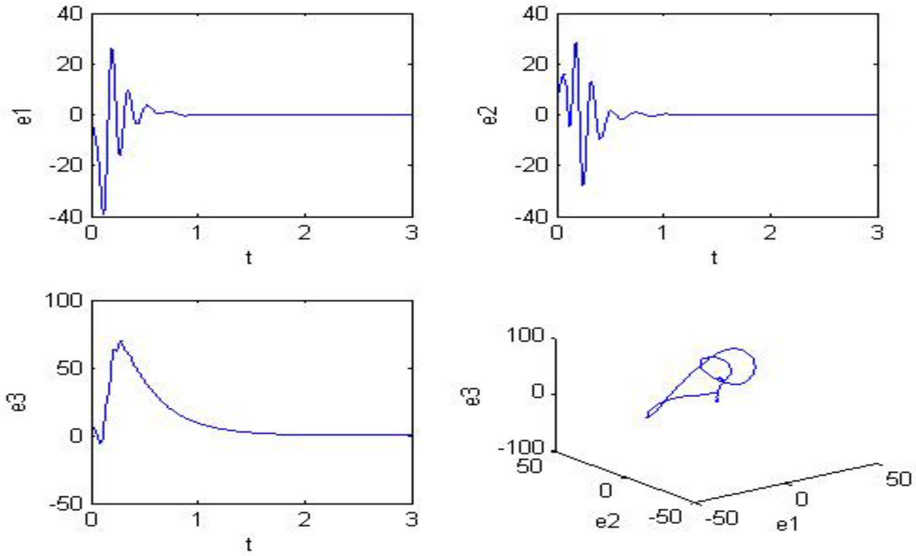

Figure 4.1: Single system node with directly control

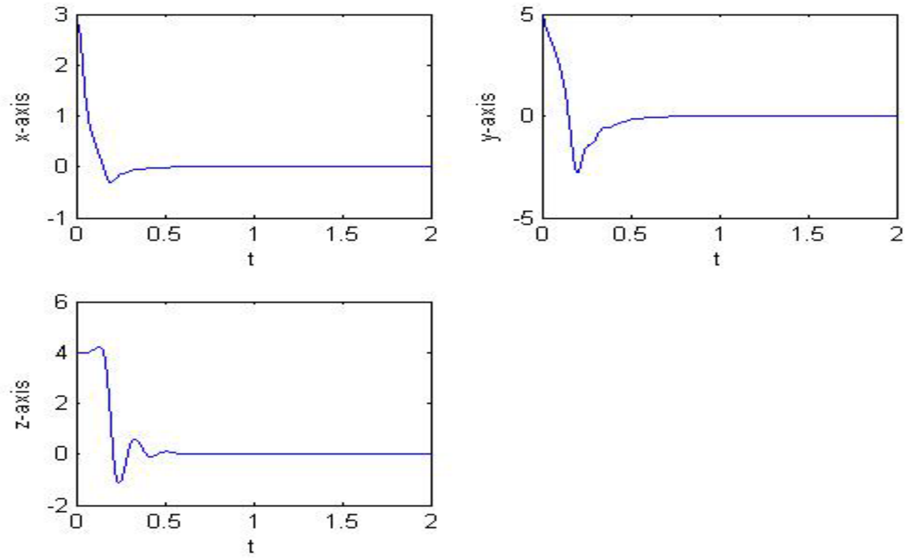

Figure 4.2: Corresponding linear time-varying system

4.2 Synchronization of Nonidentical Networks via Adaptive Linear Generalized Control

Besides the generalized synchronization between two systems, we can also apply the generalized synchronization on two networks for more broad application. Previously, we have discussed the synchronization between two identical complex networks.. Therefore, now we study the generalized synchronization between two complex networks with nonidentical topological structures.

1. Adaptive Linear generalized synchronization

First of all, recall the general structure of the complex network:

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{N} c_{ij} \quad x_j, i = 1, 2, ..., N \tag{4.2.1}$$

where $x_i(t) = (x_{i1}(t), ..., x_{in}(t))^T \in R^n$ is the state vector of the ith node, $f : R \times R^n \to R^n$ is a smooth nonlinear vector-valued function, the constant matrix is the inner-coupling matrix between the elements of the node itself, $\| \quad \| \leq \gamma$, the coupling matrix $C = (c_{ij})_{N\times N}$ represent the coupling configuration of the network,if there is a connection between node i and j $(i \neq j)$, then $c_{ij} > 0$; $c_{ij} = 0$ otherwise; also, C is set to be a diffusively coupled matrix where $c_{ii} = \sum_{j=1,j\neq i}^{N} c_{ij}, i = 1, 2, ..., N$.

Then we obtain the controlled response network as:

$$\dot{y}_i = Ay_i + Bg(y_i) + \sum_{j=1}^{N} c_{ij} \quad y_j + u_i, i = 1, 2, ..., N \tag{4.2.2}$$

where $y_i(t) = (y_{i1}(t), ..., y_{in}(t))^T \in R^n$ is the state vector of the ith node, A and B are system matrices with proper dimensions, satisfied $\|A\| \leq \quad ,\|B\| \leq \beta$, g is a continuous vector function, u_i is the controller designed for node i.

According previous construction, we also implement the following definition:

De nition 4.2.1. *If system* (4.2.1) *and* (4.2.2) *satisfy the following property:*

$$\lim t \to \infty \|y_i(t) \quad Px_i(t) \quad Q\| = 0, i = 1, 2, ..., N \tag{4.2.3}$$

then system (4.2.1) *achieves linear generalized synchronization with system* (4.2.2), *where* P, Q *are constant matrices. Moreover, we designed the controller as:*

$$u_i = Pf(x_i) \quad \varepsilon e_i \quad Bg(Px_i + Q) \quad A(Px_i + Q) \tag{4.2.4}$$

where $e_i = y_i \quad Px_i \quad Q, \varepsilon = diag(\varepsilon_1, ...\varepsilon_n)$.

Now, compare with the construction above, we can construct the drive-response network with nonidentical topological structures as follows:

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{N} c_{ij} \quad x_j, i = 1, 2, ..., N \tag{4.2.5}$$

$$\dot{y}_i = Ay_i + Bg(y_i) + \sum_{j=1}^{N} d_{ij} \quad y_j + u_i, i = 1, 2, ..., N \tag{4.2.6}$$

where the only difference is that in our new non-identical response network,$D = (d_{ij})_{N \times N}$ is the distinct configuration matrix than C, then u_i is the controller for node i to be designed based on the specify network structures C and D.

Continuously, we need some mathematic preliminaries to theocratically construct the linear generalized synchronization.

Assumption 4.2.1. *For function g(z), there exists constant* $L > 0$*, such that*

$$\|g(z_1) \quad g(z_2)\| \leq L\|z_1 \quad z_2\| \tag{4.2.7}$$

holds for any z_1, z_2*. If this assumption satisfied, we have the Lipschitz chaotic systems.*

Theorem 4.2.1. *Suppose that Assumption 3.2.1 holds, under the condition of nonidentical configurations, i.e* $C \neq D$*, the driving network* (4.2.5) *and response network* (4.2.6) *can realize linear generalized synchronization by using the following adaptive control scheme:*

$$u_i = Pf(x_i) + \sum_{j=1}^{N} b_{ij} \quad y_j \quad \varepsilon_i e_i \quad Bg(Px_i + Q) \quad A(Px_i + Q) \tag{4.2.8}$$

$$\dot{\varepsilon}_i = k_i\|e_i\|^2, \dot{b}_{ij} = \quad e_i^T \quad y_j \tag{4.2.9}$$

where $e_i = y_i \quad Px_i \quad Q$ *and* k_i *is any positive constant for* $i = 1, 2, ..., N$.

Proof. The proof of the theorem is straightforward, first we substitute (4.2.5),(4.2.6),(4.2.8) and (4.2.9) into $e_i = y_i \quad Px_i \quad Q$ and k_i. Notice that since the configuration matrices satisfying $C \neq D$, also $\sum_{j=1}^{N} c_{ij} = 0$ leads to $\sum_{j=1}^{N} c_{ij} \quad Q = 0$, thus we add this term to then end of the equation:

$$
\begin{aligned}
\dot{e}_i &= \dot{y}_i \quad P\dot{x}_i \\
&= Ay_i + Bg(y_i) + \sum_{j=1}^{N} d_{ij} \; y_j + u_i \quad Pf(x_i) \quad P\sum_{j=1}^{N} c_{ij} \; x_j \\
&= Ay_i + Bg(y_i) + \sum_{j=1}^{N} d_{ij} \; y_j \\
&+ Pf(x_i) + \sum_{j=1}^{N} b_{ij} \; y_j \quad \varepsilon_i e_i \quad Bg(Px_i + Q) \quad A(Px_i + Q) \\
&\quad Pf(x_i) \quad P\sum_{j=1}^{N} c_{ij} \; x_j \\
&= Ae_i \quad \varepsilon_i e_i + B(g(y_i) \quad g(Px_i + Q)) \\
&+ \sum_{j=1}^{N} (d_{ij} + b_{ij}) \; y_j \quad \sum_{j=1}^{N} c_{ij} P \; x_j \quad \sum_{j=1}^{N} c_{ij} \; Q
\end{aligned}
\tag{4.2.10}
$$

Secondly, define the non-negative Lyapunov candidate as:

$$
\begin{aligned}
V &= \frac{1}{2}\sum_{i=1}^{N} e_i^T e_i + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} (d_{ij} + b_{ij} \quad c_{ij})^2 \\
&+ \frac{1}{2}\sum_{i=1}^{N} \frac{1}{k_i}(\varepsilon_i \quad \bar{\varepsilon})^2
\end{aligned}
\tag{4.2.11}
$$

where $\bar{\varepsilon}$ is a sufficiently large positive constant to be determined. Then, we can derive the derivative of V as follows:

$$
\begin{aligned}
\dot{V} &= \sum_{i=1}^{N} e_i^T [Ae_i \quad \varepsilon_i e_i + B(g(y_i) \quad g(Px_i + Q))] \\
&+ \sum_{i=1}^{N} e_i^T \sum_{i=1}^{N} \sum_{j=1}^{N} (d_{ij} + b_{ij}) \quad y_j \\
&\quad \sum_{i=1}^{N} e_i^T \sum_{j=1}^{N} c_{ij} \quad (Px_j + Q) \\
&\quad \sum_{i=1}^{N} \sum_{j=1}^{N} (d_{ij} + b_{ij} \quad c_{ij}) e_i^T \quad y_j \\
&+ \sum_{i=1}^{N} (\varepsilon_i \quad \bar{\varepsilon}) \|e_i\|^2 \\
&= \sum_{i=1}^{N} e_i^T [Ae_i + B(g(y_i) \quad g(Px_i + Q))] \\
&+ \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} e_i^T \quad y_j \quad \bar{\varepsilon} \sum_{i=1}^{N} \|e_i\|^2 \\
&\leq \sum_{i=1}^{N} (\quad + \beta L \quad \bar{\varepsilon}) e_i^T e_i + \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma |c_{ij}| \|e_i\| \|e_j\|
\end{aligned}
\tag{4.2.12}
$$

As a result, define $e = (\|e_1\|, ..., \|e_N\|)^T$,$\hat{C} = (|c_{ij}|)_{N \times N}$, then we can rewrite (4.2.12) as $\dot{V} \leq e^T[(\quad + \beta L \quad \bar{\varepsilon}) I_N + \quad \hat{C}]e$. In order to ensure network (4.2.5) and (4.2.6) are synchronized, we need $(e_1^T, ...e_N^T)^T$ converges to zero as time goes to infinity, thus select proper value of $\bar{\varepsilon}$ as long as $(\quad + \beta L \quad \bar{\varepsilon}) I_N + \quad \hat{C}$ is negative definite.

In the following, based on the theorem, two more alternative adaptive controllers can be directly obtained as long as they satisfy corresponding conditions:

Corollary 4.2.1. *Suppose that assumption 3.2.1 holds, the driving network* (4.2.5) *and the response network* (4.2.6) *can realize Linear GS by using the following adaptive control scheme:*

$$u_i = Pf(x_i) + \sum_{j=1}^{N} b_{ij} \quad y_j \quad \varepsilon_i e_i \quad Bg(Px_i + Q) \quad A(Px_i + Q) \tag{4.2.13}$$

$$\dot{\varepsilon}_i = k_i \|e_i\|^2 \tag{4.2.14}$$

$$\dot{b}_{ij} = \quad e_i^T \quad y_j, c_{ij} \neq d_{ij} \qquad b_{ij} = 0, c_{ij} = d_{ij} \tag{4.2.15}$$

where $e_i = y_i \quad Px_i \quad Q$ *and* k_i *is any positive constant for* $i = 1, 2, ..., N$.

Corollary 4.2.2. *Suppose that assumption 3.2.1 holds. If the two networks have the same configuration matrices, i.e., C=D, network* (4.2.5) *and* (4.2.6) *can achieve Linear GS by using the following adaptive controller:*

$$u_i = Pf(x_i) \quad \varepsilon_i e_i \quad Bg(Px_i + Q) \quad A(Px_i + Q), \dot{\varepsilon}_i = k_i \|e_i\|^2, \tag{4.2.16}$$

where $e_i = y_i \quad Px_i \quad Q$ *and* k_i *is any positive constant for* $i = 1, 2, ..., N$.

Remark. Overall, although the theorem proposed with a very weak condition, i.e., $C \neq D$, it can be applied to handle the nonidentical topological structures, non-diffusive structures and two networks having different dynamics in particular.

2. Numerical simulations

Consider the *Lü* system and *Rössler* (satisfy Assumption 1 by [19]) system as the nodes dynamical of the drive and response networks to verify the effectiveness of the controller construction to achieve linear generalized synchronization:

$$\dot{x} = f(x) = \begin{pmatrix} \rho(x_2 \quad x_1) \\ x_1 x_3 + \upsilon x_2 \\ x_1 x_2 \quad \mu x_3 \end{pmatrix} \tag{4.2.17}$$

where $\rho = 36$,$\mu = 3$,$\upsilon = 20$.

$$\dot{y} = Ay + Bg(y) = \begin{pmatrix} y_2 \quad y_3 \\ y_1 + \quad y_2 \\ \beta + y_3(y_1 \quad \gamma) \end{pmatrix} \tag{4.2.18}$$

where $\quad = 0.2, \beta = 0.2, \gamma = 5.7$. Then we have:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0.2 & 0 \\ 0 & 0 & 5.7 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, g(y) = \begin{pmatrix} 0 \\ 0 \\ y_1 y_3 + \beta \end{pmatrix} \tag{4.2.19}$$

where $\|A\| = 5.7898, \|B\| = 1$.

Consider the drive network composed of three *Lü* nodes coupled via following configuration matrix C:

$$C = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

and the response network composed of three *Rössler* nodes coupled via following configuration matrix D:

$$D = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

Moreover, we choose $P = diag(1, \quad 1, 1)$,$Q = [0, 0, 0]^T$, $\quad = diag(1, 1.2, 1)$,$b_{ij} = \begin{pmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{pmatrix}$,where each entry$\in$ $(0, 1)$; and initial values of state vectors X,Y, control variable ε_i randomly.

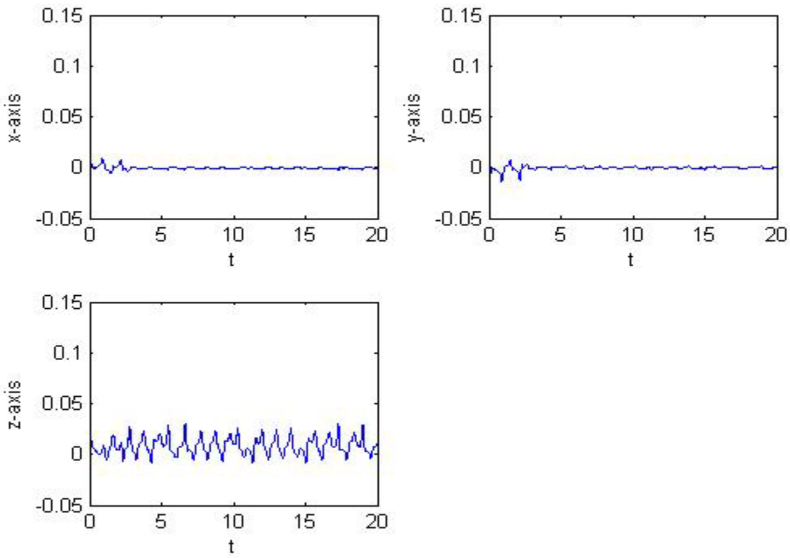

Figure 4.3: Synchronization errors between network (4.2.5) and (4.2.6)

Based on the graph above, when suitable ε_i values are selected, all of the states error tend to zero; although in this particular example, the z-state error tends to zero slower than the other two, which requires larger ε value. Thus, two nonidentical networks achieve synchronization.

Chapter 5

Network Cluster Synchronization Techniques

5.1 Cluster Synchronization For Delayed complex Networks Via Periodically Intermittent Pinning Control

1. Introduction

As discussed previously, in general, N linearly coupled identical systems with external control can be described as:

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{N} a_{ij} \quad x_j + u_i, i = 1, 2, ..., N \tag{5.1.1}$$

where $x_i(t) = (x_{i1}(t), ..., x_{in}(t))^T \in R^n$ is the state vector of the ith node, $f : R \times R^n \to R^n$ is a smooth nonlinear vector-valued function, which denotes the dynamical behaviour of each uncoupled node; $\sum_{j=1}^{N} a_{ij} \quad x_j$ represents the linear and local coupling among pairs of connected nodes, where the positive definite matrix $\quad = diag(\gamma_1, ..., \gamma_n)$ denotes the inner coupling matrix, and the network topology is represented by the outer coupling matrix $A = (a_{ij})_{N \times N}$ represent the coupling configuration of the network,if there is a connection between node i and j $(i \neq j)$, then $a_{ij} > 0$; $a_{ij} = 0$ otherwise. u_i is the external control, if controllers are added on only a small fraction of network nodes, then it is called the pinning control problem.

On the one hand, if $\|x_i \quad y_i\| \to 0$ as $t \to \infty$, $i, j = 1, ..., N$, we say that the complete synchronization is realized. On the other hand, if the set of nodes can be divided into m nonempty clusters as:

$$\{1, 2, ..., N\} = C_1 \cup C_2 \cup ... \cup C_m \tag{5.1.2}$$

where $C_1 = \{1, ..., r_1\}, C_2 = \{r_1 + 1, ..., r_2\}, ..., C_m = \{r_{m-1} + 1, ...N\}$ such that coupled nodes in the same cluster can be synchronized, but there is no synchronization among different clusters, then the network is said to realize the cluster synchronization. In particular, when m=1, the cluster synchronization is the complete synchronization. In order to avoid undesirable dynamical behaviours such as oscillation and instability, we consider the cluster synchronization of linearly coupled systems with time delay by pinning periodically intermittent control:

$$\dot{x}_i = f(x_i(t), x_i(t-\tau)) + \sum_{j=1}^{N} a_{ij} x_j + u_i, i = 1, 2, ..., N \tag{5.1.3}$$

Moreover, we will investigate if we can apply the adaptive approach of theocratical resulted on periodically intermittent control with delay.

2. Preliminaries

Definition 5.1.1. *Matrix $A = (a_{ij})_{i,j=1}^N$ is said to belong to class A1, if the following conditions are satisfied:*
1. $a_{ij} > 0, i \neq j, a_{ii} = -\sum_{j=1, j\neq i}^N a_{ij}, i = 1, ..., N$;
2. A is irreducible.
If $A \in A1$ is symmetric, then A belongs to class A2.

Definition 5.1.2. *Matrix $A = (a_{ij}) \in R^{N_1 \times N_2}$ is said to belong to class A3 if its each row-sum is zero, i.e., $\sum_{j=1}^{N_2} a_{ij} = 0, i = 1, ..., N_1$.*

Definition 5.1.3. *Suppose $A \in R^{N\times N}$, the indexes 1,...,N can be divided into m clusters defined by* (5.1.2)*, and the following form holds:*

$$A = \begin{pmatrix} A_{11} & A_{12} & ... & A_{1m} \\ A_{21} & A_{22} & ... & A_{2m} \\ ... & ... & ... & ... \\ A_{m1} & A_{m2} & ... & A_{mm} \end{pmatrix} \tag{5.1.4}$$

where $A_{ij} \in R^{(r_i - r_{i-1})\times(r_j - r_{j-1})}, r_0 = 0$, r_i is defined by (5.1.2)*, $A_{ii} \in A1$ and $A_{ij} \in A3, i \neq j \in 1, ..., m$ Then A is said to belong to class A4.*

Definition 5.1.4. *Network* (5.1.3) *with N nodes is said to realize cluster synchronization, if N nodes can be divided into m clusters as defined by* (5.1.2)*, such that, for and node $i \in C_k, k = 1, ..., m$, $\lim_{t\to+\infty} \|x_i(t) - s_k(t)\| = 0$ and $\lim_{t\to+\infty} \|s_k(t) - s_l(t)\| = 0, l \neq k$ where $s_k(t)$ is a trajectory defined by*

$$\dot{s_k}(t) = f(s_k(t), s_k(t-\tau)) \tag{5.1.5}$$

Assumption 5.1.1. *The function f(.,.) is said to satisfy the QUAD condition, denoted as* $f(.,.) \in QUAD(P, \Delta_1, \Delta_2)$*, if there exist three positive definite diagonal matrices* $P = diag(p_1, ..., p_n)$,$\Delta_1 = diag(\delta_1, ..., \delta_n)$ *and* $\Delta_2 = diag(\delta_1^{'}, ..., \delta_n^{'})$*, such that for any* $x(t), y(t) \in R^n$*, the following condition holds:*

$$\begin{aligned} &[x(t) \quad y(t)]^T P[f(x(t), x(t \quad \tau)) \quad f(y(t), y(t \quad \tau))] \\ &\leq [x(t) \quad y(t)]^T P\Delta_1[x(t) \quad y(t)] + \\ &[x(t \quad \tau) \quad y(t \quad \tau)]^T P\Delta_2[x(t \quad \tau) \quad y(t \quad \tau)] \end{aligned} \tag{5.1.6}$$

Lemma 5.1.1. *Suppose* $A \in A1$ *and* $\varepsilon < 0$*. Then, there exists a positive definite diagonal matrix* $\Phi = diag(\phi_1, ..., \phi_N)$ *such that* $A_\varepsilon = A + diag(\varepsilon, ..., 0)$ *is Lyapunov stable, i.e.,*

$$\Phi A_\varepsilon + A_\varepsilon^T \Phi < 0 \tag{5.1.7}$$

Lemma 5.1.2. *Let w(t) be a nonnegative function defined on the interval* $[t_0 \quad \tau, \infty]$*, and be continuous on the subinterval* $[t_0, \infty]$*. Assume that the following inequality holds for* $t \geq t_0$*,*

$$\dot{w}(t) \leq \quad aw(t) + bw(t \quad \tau), b > 0 \tag{5.1.8}$$

1. If $a > b$*, then*

$$w(t) \leq \bar{w}_{t_0} exp\{ \quad \gamma(t \quad t_0)\}, t \geq t_0 \tag{5.1.9}$$

2. If $a < b$*, then*

$$w(t) \leq \bar{w}_{t_0} exp\{\eta(t \quad t_0 + \tau)\}, t \geq t_0 \tag{5.1.10}$$

where $\bar{w}_{t_0} = sup_{t_0 \quad \tau \leq \kappa \leq t_0} w(\kappa), \gamma > 0$ *is the smallest real root of the equation*

$$a \quad \gamma \quad bexp\{\gamma\tau\} = 0 \tag{5.1.11}$$

and $\eta > 0$ *is the unique root of the equation*

$$a + bexp\{ \quad \eta\tau\} = \eta \tag{5.1.12}$$

Notation. If all eigenvalues of a matrix $A \in R^{N \times N}$ are real, then we sort them as $\lambda_1(A) \leq \lambda_2(A) \leq ... \leq \lambda_N(A)$. Also the symmetrical part of matrix A is defined as $A^s = (A + A^T)/2$. A symmetric real matrix A is positive definite (semi-definite) if $x^T Ax > 0(\geq 0)$ for all nonzero x, denote as $A > 0(A \geq 0)$. Then Kronecker product of and N by M matrix $A = (a_{ij}$ and a p by q matrix B is the Np by Mq matrix $A \otimes B$, defined as $A \otimes B = (a_{ij}B)$, and the Kronecker product has the property $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

Remark. In general, $a_{ij} > 0(or < 0), i \neq j$ is regarded as the cooperative (or competitive) relationship between node i and node j. Thus, $A \in A4$ means that nodes in the same cluster only have cooperative relationship, while nodes belonging to different clusters can have both cooperative and competitive relationship. For the delayed differential inequality (5.1.8), when $b \leq 0$, the second term can be just enlarged to zero, so $b > 0$ is natural. As for parameter a, when $a > b$, it becomes the Halanay inequality.

3. Cluster synchronization with static intermittent control

Consider the cluster synchronization in complex dynamical networks with delay is realized by periodical intermittent pinning control as follows:

$$
\begin{aligned}
&\dot{x}_{r_{k\ \ 1}+1} = f(x_{r_{k\ \ 1}+1}(t), x_{r_{k\ \ 1}+1}(t\quad \tau)) \\
&+\sum_{j=1}^{N} a_{r_{k\ \ 1}+1,j}\ \ x_j(t) + g\ \ (s_k(t)\quad x_{r_{k\ \ 1}+1}(t)) \\
&\dot{x}_i = f(x_i(t), x_i(t\quad \tau)) + \sum_{j=1}^{N} a_{ij}\ \ x_j(t), i = r_{k\ \ 1} + 2, ..., r_k \\
&t \in [l\omega, l\omega + \theta], l = 0, 1, 2, ... \\
&\dot{x}_i = f(x_i(t), x_i(t\quad \tau)) + \sum_{j=1}^{N} a_{ij}\ \ x_j(t), i = 1, ..., N \\
&t \in [l\omega + \theta, (l + 1)\theta], l = 0, 1, 2, ...
\end{aligned}
\tag{5.1.13}
$$

where assume that only the first node in each cluster is pinned with the same control gain; constant scalar $g > 0$ is control gain, which means the zoom coefficient of control strength; $\omega > 0$ is the control period, and $\theta > 0$ is called the control width (control duration). In addition, time delay is less than the control width, $\tau < \theta$.

Moreover, if node $i \in C_k$, denote $e_i(t) = x_i(t) \quad s_k(t)$,$i = r_{k \quad 1} + 2, ..., r_k$, $E_k(t) = (e_{r_{k \quad 1}+1}(t)^T, ..., e_{r_K}(t)^T)^T$,
$E(t) = (E_1(t)^T, ..., E_m(t)^T)^T$ and $F(E_k(t), E_k(t \quad \tau)$
$= ((f(x_{r_{k \quad 1}+1}(t), x_{r_{k \quad 1}+1}(t \quad \tau)) \quad f(s_k(t), s_k(t \quad \tau)))^T, ...,$
$(f(x_{r_k}(t), x_{r_k}(t \quad \tau)) \quad f(s_k(t), s_k(t \quad \tau)))^T)^T$.
Then for $1 \leq k \leq m$, l=0,1,2,... we obtain:

$$\begin{aligned} \dot{E}_k &= F(E_k(t), E_k(t \quad \tau)) + \sum_{j=1}^{m} (\hat{A}_{kj} \otimes \quad) E_j(t), t \in [l\omega, l\omega + \theta] \\ \dot{E}_k &= F(E_k(t), E_k(t \quad \tau)) + \sum_{j=1}^{m} (A_{kj} \otimes \quad) E_j(t), t \in [l\omega + \theta, (l+1)\theta] \end{aligned} \tag{5.1.14}$$

where sub-matrices $\hat{A}_{kj}$ are defined as

$$\begin{aligned} \hat{A}_{kj} &= A_{kj}, k \neq j \\ \hat{A}_{kj} &= A_{kk} \quad diag(g, 0, ..., 0), k = j \end{aligned} \tag{5.1.15}$$

Furthermore, based on (5.1.15) and Lemma 4.1.1, there exist positive definite diagonal matrices

$$\Phi_k = diag(\phi_{r_{k \quad 1}+1}, ..., \phi_{r_k}), k = 1, ..., m \tag{5.1.16}$$

such that

$$\Phi_k \hat{A}_{kk} + \hat{A}_{kk}^T \Phi_k < 0 \tag{5.1.17}$$

Without loss of generality, we always assume that $max_{i=1}^N \phi_i = 1$. Denote $a_1 = max_{i=1}^n \frac{\delta}{\gamma}$,

$$\bar{A} = \begin{pmatrix} (\Phi_1 \hat{A}_{11})^s + a_1 \Phi_1 & \cdots & \frac{\Phi_1 A_{1m} + A_{m1}^T \Phi_m}{2} \\ \vdots & \ddots & \vdots \\ \frac{\Phi_m A_{m1} + A_{1m}^T \Phi_1}{2} & \cdots & (\Phi_m \hat{A}_{mm})^s + a_1 \Phi_m \end{pmatrix} \tag{5.1.18}$$

$$\underline{A} = \begin{pmatrix} (\Phi_1 A_{11})^s + a_1 \Phi_1 & \cdots & \frac{\Phi_1 A_{1m} + A_{m1}^T \Phi_m}{2} \\ \vdots & \ddots & \vdots \\ \frac{\Phi_m A_{m1} + A_{1m}^T \Phi_1}{2} & \cdots & (\Phi_m A_{mm})^s + a_1 \Phi_m \end{pmatrix} \tag{5.1.19}$$

Theorem 5.1.3. *Suppose* $f(.,.) \in QUAD(P, \Delta_1, \Delta_2)$ *and* $\tau < \theta$ *holds. Then linearly coupled networks* (5.1.13) *with periodically intermittent pinning control can realize cluster synchronization under the following conditions:*
1. $\quad a_3 < b < a_2$
2. $\gamma(\theta \quad \tau) \quad \eta(\omega \quad \theta + \tau) > 0$
where $a_2 = \quad 2min_{i=1}^{n}\gamma_i\lambda_N(\bar{A})$, $b = 2max_{i=1}^{n}\delta_i^{'} > 0$, *and*

$$\begin{aligned} a_3 &= 2min_{i=1}^{n}\gamma_i\lambda_N(\underline{A}), \lambda_N(\underline{A}) < 0 \\ a_3 &= 2max_{i=1}^{n}\gamma_i\lambda_N(\underline{A}), \lambda_N(\underline{A}) \geq 0 \end{aligned} \tag{5.1.20}$$

and $\eta > 0$ *is the unique root of the equation:*

$$a_3 + bexp \quad \eta\tau = \eta \tag{5.1.21}$$

and $\gamma > 0$ *is the smallest real root of the equation:*

$$a_2 \quad \gamma \quad bexp\gamma\tau = 0 \tag{5.1.22}$$

Proof. From the definition of Δ_1 and , one can get $\delta_i = \gamma_i \frac{\delta_i}{\gamma_i}$ for any i=1,...,n; thus $\delta_1 \leq a_1$, where $a_1 = max_{i=1}^{n}\frac{\delta_i}{\gamma}$. Define the Lyapunov function as

$$V(t) = \frac{1}{2}\sum_{k=1}^{m} E_k^T(t)(\Phi_k \otimes P)E_k(t) \tag{5.1.23}$$

Then when $t \in [l\omega, l\omega + \theta]$, l=0,1,2,...

$$
\begin{aligned}
\dot{V}(t) &= \sum_{k=1}^{m} E_k^T(t)(\Phi_k \otimes P)\dot{E}_k(t) \\
&= \sum_{k=1}^{m} E_k^T(t)(\Phi_k \otimes P)[F(E_k(t), E_k(t \;\; \tau)) + \sum_{j=1}^{m}(\hat{A}_{kj} \otimes \;\;)E_j(t)] \\
&= \sum_{k=1}^{m} \sum_{i=r_k \;\; 1+1}^{r_k} \phi_i(x_i(t) \;\; s_k(t))^T P[f(x_i, x_i(t \;\; \tau)) \;\; f(s_k(t), s_k(t \;\; \tau))] \\
&+ \sum_{k=1}^{m} \sum_{j=1}^{m} E_k^T[(\Phi_k \hat{A}_{kj}) \otimes (P \;\;)]E_j(t) \\
&\leq \sum_{k=1}^{m} E_k^T[\Phi_k \otimes (P\Delta_1)]E_k(t) + \sum_{k=1}^{m} E_k^T(t \;\; \tau)[\Phi_k \otimes (P\Delta_2)]E_k(t \;\; \tau) \\
&+ \sum_{k=1}^{m} \sum_{j=1}^{m} E_k^T(t)[(\Phi_k \hat{A}_{kj}) \otimes (P \;\;)]E_j(t) \\
&\leq \sum_{k=1}^{m} E_k^T[a_1\Phi_k \otimes (P \;\;)]E_k(t) + \sum_{k=1}^{m} \sum_{j=1}^{m} E_k^T(t)[(\Phi_k \hat{A}_{kj}) \otimes (P \;\;)]E_j(t) \\
&+ \sum_{k=1}^{m} E_k^T(t \;\; \tau)[\Phi_k \otimes (P\Delta_2)]E_k(t \;\; \tau) \\
&\leq E^T(t)[\bar{A} \otimes (P \;\;)]E(t) \\
&+ max_{1\leq i\leq n}\gamma_i^{'} \sum_{k=1}^{m} E_k^T(t \;\; \tau)[\Phi_k \otimes P]E_k(t \;\; \tau) \\
&\leq 2min_{1\leq i\leq n}\gamma_i\lambda_N(\bar{A})V(t) + 2max_{1\leq i\leq n}\gamma_i^{'}V(t \;\; \tau) \\
&= \;\; a_2V(t) + bV(t \;\; \tau)
\end{aligned}
\tag{5.1.24}
$$

Since $a_2 > b > 0$, from Lemma 4.1.2 we obtain

$$V(t) \leq \bar{V}(l\omega)exp\{ \;\; \gamma(t \;\; l\omega)\} \tag{5.1.25}$$

where $\bar{V}(l\omega) = sup_{l\omega \;\; \tau\leq\kappa\leq l\omega}V(\kappa)$,$\gamma > 0$ is defined by (5.1.22)

Similarly for $t \in [l\omega + \theta, (l + 1)\theta], l = 0, 1, 2, ...$, we have

$$
\begin{aligned}
\dot{V}(t) &\leq E^T(t)[\underline{A} \otimes (P \;\;)]E(t) + 2max_{1\leq i\leq n}\gamma_i^{'}V(t \;\; \tau) \\
&\leq a_3V(t) + bV(t \;\; \tau)
\end{aligned}
\tag{5.1.26}
$$

Since $b > \;\; a_3$, from Lemma 4.1.2 we obtain

$$V(t) \leq \bar{V}(l\omega + \theta)exp\{\eta(t \;\; l\omega \;\; \theta + \tau)\} \tag{5.1.27}$$

where $\eta > 0$ is defined by (5.1.21). Furthermore, estimate V(t) based on (5.1.25),(5.1.26) and $\tau < \theta$:

1.$V(t) \leq \bar{V}(0)exp\{ \quad \gamma t\}$ for $0 \leq t \leq theta$
2.$V(t) \leq \bar{V}(\theta)exp\{\eta(t \quad \theta + \tau)\} \leq \bar{V}(0)exp\{\eta(t \quad \theta + \tau) \quad \gamma(\theta \quad \tau)\}$

By induction, we can derive the estimations of V(t) for any integer l:

a. for any time t, if $l\omega \leq t \leq l\omega + \theta, l = 0, 1, ...,$ then $V(t) \leq \bar{V}(0)exp\{ \quad \gamma[t \quad l(\omega \quad \theta + \tau)] + l\eta(\omega$ and we obtain $V(t) \leq \bar{V}(0)exp\{ \quad [\gamma(\theta \quad tau) \quad \eta(\omega \quad \theta + \tau)]\frac{t}{\omega}\}$ in the case of $l\omega \leq t$.
b. If $l\omega + \theta < t < (l+1)\omega, l = 0, 1, ...,$ then $V(t) \leq \bar{V}(0)exp\{\eta(t \quad (l + 1)(\theta \quad \tau)) \quad \gamma(l + 1)(\theta \quad \tau)\}$ and we obtain $V(t) \leq \bar{V}(0)exp\{ \quad [\gamma(\theta \quad tau) \quad \eta(\omega \quad \theta + \tau)]\frac{t}{\omega}\}$ in the case of $t \leq (l + 1)\omega$.

Therefore, for any $t \geq 0$, we have:
$V(t) \leq \bar{V}(0)exp\{ \quad [\gamma(\theta \quad tau) \quad \eta(\omega \quad \theta + \tau)]\frac{t}{\omega}\}$; which means as $\gamma(\theta \quad \tau) \quad \eta(\omega \quad \theta + \tau) > 0$, $e_i(t) \to 0$ when $t \to \infty$. Hence the cluster synchronization is realized.

Remark. Based on (5.1.22), γ increases as a_2 increases; hence in order to make $\gamma(\theta \quad \tau) \quad \eta(\omega \quad \theta + \tau) > 0$, we have to make a_2 large enough as well as $\lambda_N(\bar{A})$ is negative enough. Meanwhile, since $(\Phi_k \bar{A}_{kk}), k = 1, ..., m$ are negative definite, thus making it large enough is the effective way to achieve cluster synchronization.

4. Cluster synchronization with adaptive intermittent control

The adaptive method allows the intermittent control gain g to be time varying like $g \cdot H(t)$, then introduce the following lemma to support the theoretical result.

Lemma 5.1.4. *Suppose*

$$\dot{x}_t = M(x(t), x(t \quad \tau)) \quad H(t)x(t) \tag{5.1.28}$$

where $x \in R$,$M(.,.) : R \times R \to R$ *is continuous and satisfies* $xM(x(t), x(t \quad \tau)) \leq L_M x^2(t) + \tilde{L}_M x^2(t \quad \tau)$, *where* $L_M > 0, \tilde{L}_M > 0$, *and* $M(0, 0) = 0$. *Function* $H(t) : \{0 \cup R_+ \to R$ *is the adaptive intermittent feedback control gain defined as:*

$$\begin{aligned} &H(t) = 0, t = 0 \\ &H(t) = H(l\omega + \theta), t = (l + 1)\omega \\ &H(t) = 0, l\omega + \theta < t < (l + 1)\omega \end{aligned} \tag{5.1.29}$$

and

$$\dot{H}(t) = \max_{t \quad \tau \leq \kappa \leq t} x^2(\kappa), l\omega \leq t \leq l\omega + \theta \tag{5.1.30}$$

where l=0,1,..., then $\lim_{t\to\infty} x(t) = 0$.

Based on system (5.1.13), the dynamical behavior with adaptive intermittent pinning control is described as follows:

$$
\begin{aligned}
&\dot{x}_{r_{k \ \ 1}+1} = f(x_{r_{k \ \ 1}+1}(t), x_{r_{k \ \ 1}+1}(t \quad \tau)) \\
&+ \sum_{j \ \ C_k}^{N} a_{r_{k \ \ 1}+1,j} \ \ x_j(t) + H(t) \sum_{j\in C_k}^{N} a_{r_{k \ \ 1}+1,j} \ \ x_j(t) \\
&+ gH(t) \ \ (s_k(t) \quad x_{r_{k \ \ 1}+1}(t)) \\
&\dot{x}_i = f(x_i(t), x_i(t \quad \tau)) + \sum_{j \ \ C_k}^{N} a_{ij} \ \ x_j(t) + H(t) \sum_{j\in C_k}^{N} a_{ij} \ \ x_j(t) \\
&i = r_{k \ \ 1} + 2, ..., r_k, t \in [l\omega, l\omega + \theta], l = 0, 1, 2, ... \\
&\dot{x}_i = f(x_i(t), x_i(t \quad \tau)) + \sum_{j=1}^{N} a_{ij} \ \ x_j(t), i = 1, ..., N \\
&t \in [l\omega + \theta, (l + 1)\theta], l = 0, 1, 2, ...
\end{aligned}
\tag{5.1.31}
$$

With the same notations as those in Theorem 4.1.3, for $1 \le k \le m$, we have

$$
\begin{aligned}
&\dot{E}_k = F(E_k(t), E_k(t \quad \tau)) + \sum_{j=1, j\neq k}^{m} (\hat{A}_{kj} \otimes \quad)E_j(t) \\
&+ H(t) \sum_{j=1, j\neq k}^{m} (\hat{A}_{kk} \otimes \quad)E_k(t) \\
&t \in [l\omega, l\omega + \theta], l = 0, 1, 2, ... \\
&\dot{E}_k = F(E_k(t), E_k(t \quad \tau)) + \sum_{j=1}^{m} (A_{kj} \otimes \quad)E_j(t) \\
&t \in [l\omega + \theta, (l + 1)\theta], l = 0, 1, 2, ...
\end{aligned}
\tag{5.1.32}
$$

where the adaptive control gain H(t) is defined by (5.1.29) with the adaptive rule:

$$
\dot{H}(t) = h \sum_{k=1}^{m} \sum_{r_{k \ \ 1}+1}^{r_k} \max_{t \ \ \tau \le \kappa \le t} \|x_i(\kappa) \quad s_k(\kappa)\|^2 \tag{5.1.33}
$$

where $h > 0$ and $l\omega \le t \le l\omega + \theta$, l=0,1,2,...

Furthermore, apply Lemma 4.1.4 to system (5.1.31), the following theorem of the cluster synchronization with adaptive intermittent control is introduced:

Theorem 5.1.5. *If function $f(.,.)$ satisfied the QUAD condition and $\tau < \theta$, then linearly coupled networks* (5.1.32) *with adaptive rules* (5.1.29) *and* (5.1.33) *can realize cluster synchronization.*

Proof. We can first rewrite (5.1.32) as:

$$\begin{aligned} &\dot{E}(t) = F(E(t), E(t \quad \tau)) + (\tilde{A} \otimes \quad)(t) + H(t)(\breve{A} \otimes \quad)E(t) \\ &t \in [l\omega, l\omega + \theta], l = 0, 1, 2, ... \\ &\dot{E}(t) = F(E(t), E(t \quad \tau)) + (A \otimes \quad)E(t) \\ &t \in [l\omega + \theta, (l+1)\theta], l = 0, 1, 2, ... \end{aligned} \tag{5.1.34}$$

where $\tilde{A} = \hat{A} \quad \breve{A}$ and $\breve{A} = diag(\hat{A}_{11}, ..., \hat{A}_{mm})$. Then define a function:

$$\begin{aligned} &\bar{F}(E(t), E(t \quad \tau)) = F(E(t), E(t \quad \tau)) + (\tilde{A} \otimes \quad)E(t) \\ &t \in [l\omega, l\omega + \theta] \\ &\bar{F}(E(t), E(t \quad \tau)) = F(E(t), E(t \quad \tau)) + (A \otimes \quad)E(t) \\ &t \in [l\omega + \theta, (l+1)\theta] \end{aligned} \tag{5.1.35}$$

which (5.1.34) can be stated as:

$$\dot{E}(t) = \bar{F}(E(t), E(t \quad \tau)) + H(t)(\breve{A} \otimes \quad)E(t) \tag{5.1.36}$$

Moreover, define
$L_1 = \max_{1 \le i \le n} \delta_i + \max_{1 \le n} \gamma_i \cdot \max\{\|A\|_2, \|\tilde{A}\|_2\}$,
$L_2 = \max_{1 \le i \le n} \delta_i^{'}$. Then:

$$\begin{aligned} &E(t)^T (I \otimes P) \bar{F}(E(t), E(t \quad \tau)) \\ &\le E(t)^T (I \otimes P) F(E(t), E(t \quad \tau)) \\ &+ \max_{1 \le n} \gamma_i \cdot \max\{\|A\|_2, \|\tilde{A}\|_2\} E(t)^T (I \otimes P) E(t) \\ &\le \max_{1 \le n} \delta_i E(t)^T (I \otimes P) E(t) + \max_{1 \le n} \delta_i^{'} E(t \quad \tau)^T (I \otimes P) E(t \quad \tau) \\ &+ \max_{1 \le n} \gamma_i \cdot \max\{\|A\|_2, \|\tilde{A}\|_2\} E(t)^T (I \otimes P) E(t) \end{aligned} \tag{5.1.37}$$

Lastly, we choose the Lyapunov candidate as:
$V(t) = \frac{1}{2}E(t)^T(\Phi \otimes P)E(t)$, where Φ is defined in (5.1.16), then

$$\begin{aligned}
&\dot{V}(t) = 2E(t)^T(\Phi \otimes P)[\bar{F}(E(t), E(t \quad \tau)) + H(t)(\check{A} \otimes \quad)E(t)] \\
&= 2E(t)^T(\Phi \otimes P)\bar{F}(E(t), E(t \quad \tau)) \\
&+ 2H(t)E(t)^T(\Phi \otimes P)(\check{A} \otimes \quad)E(t) \\
&\leq E(t)^T(I \otimes P)\bar{F}(E(t), E(t \quad \tau)) \quad 2|\lambda^*| \min_{1 \leq i \leq n} \gamma_i H(t)V(t)
\end{aligned} \tag{5.1.38}$$

where $\lambda^* = \max_{k=1,...,m} \lambda_{r_k}\{\Phi_k \hat{A}_{kk}\}^s < 0$.

Consequently, from the results of Lemma 4.1.4, we are able to obtain $\lim_{t \to \infty} V(t) = 0$, which means $\lim_{t\to\infty} E(t) = 0$. Thus, cluster synchronization is realized.

Remark.
1. When m=1, the coupled network (5.1.32) with adaptive rules (5.1.29) and (5.1.33) leads to complete synchronization with adaptive intermittent control.
2. If f=0, then coupled network (5.1.32) with adaptive rules (5.1.29) and (5.1.33) can realize cluster consensus.

5. Numerical Simulation

Consider a linearly coupled with delay, and suppose the network can be divided into two clusters: $C_1 = \{1, 2\}$, $C_2 = \{3, 4, 5\}$. Suppose the intermittent controllers with constant control gain g=1 are added to nodes 1 and 3 only, then the network can be described as follows:

$$\begin{aligned}
&\dot{x}_1(t) = f(x_1(t), x_1(t \quad \tau)) + \sum_{j=1}^{5} a_{1j} \quad x_j(t) + \quad (s_1(t) \quad x_1(t)) \\
&\dot{x}_3(t) = f(x_3(t), x_3(t \quad \tau)) + \sum_{j=1}^{5} a_{3j} \quad x_j(t) + \quad (s_2(t) \quad x_3(t)) \\
&\dot{x}_i(t) = f(x_i(t), x_i(t \quad \tau)) + \sum_{j=1}^{5} a_{ij} \quad x_j(t), i = 2, 4, 5 \\
&t \in [l\omega, l\omega + \theta], l = 0, 1, 2, ... \\
&\dot{x}_i(t) = f(x_i(t), x_i(t \quad \tau)) + \sum_{j=1}^{5} a_{ij} \quad x_j(t), i = 1, 2, 3, 4, 5 \\
&t \in [l\omega + \theta, (l + 1)\omega], l = 0, 1, 2, ...
\end{aligned} \tag{5.1.39}$$

where trajectories $s_i(t)$ are defined by $\dot{s}_i(t) = f(s_i(t), s_i(t))$. Then the error network can be obtained as:

$$\begin{aligned}
&\dot{E}_1(t) = F(E_1(t), E_1(t \quad \tau)) + (\hat{A}_{11} \otimes \quad)E_1(t) + (\hat{A}_{12} \otimes \quad)E_2(t) \\
&\dot{E}_2(t) = F(E_2(t), E_2(t \quad \tau)) + (\hat{A}_{21} \otimes \quad)E_1(t) + (\hat{A}_{22} \otimes \quad)E_2(t) \\
&if t \in [l\omega, l\omega + \theta], l = 0, 1, 2, ... \\
&\dot{E}_1(t) = F(E_1(t), E_1(t \quad \tau)) + (A_{11} \otimes \quad)E_1(t) + (A_{12} \otimes \quad)E_2(t) \\
&\dot{E}_2(t) = F(E_2(t), E_2(t \quad \tau)) + (A_{21} \otimes \quad)E_1(t) + (A_{22} \otimes \quad)E_2(t) \\
&if t \in [l\omega + \theta, (l+1)\omega], l = 0, 1, 2, ...
\end{aligned} \tag{5.1.40}$$

Moreover, we set up values of series of parameters needed as follows: $\hat{A} = A_{11} \quad diag(g, 0) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$, $\hat{A}_{12} = A_{12} = \begin{pmatrix} 2 & 3 & 5 \\ 0 & 4 & 4 \end{pmatrix}$, $\hat{A}_{21} = A_{21} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \\ 3 & 3 \end{pmatrix}$, $\hat{A}_{22} = A_{22} \quad diag(g, 0, 0) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$ and the dynamics of uncoupled system is described by Chua oscillator:

$$\dot{x}(t) = f(x(t), x(t \quad \tau)) = Cx(t) + g_1(x(t)) + g_2(x(t \quad \tau)) \tag{5.1.41}$$

where $x(t) = (x^1(t), x^2(t), x^3(t))^T \in R^3$,$C = \begin{pmatrix} \delta(1+b) & \delta & 0 \\ 1 & 1 & 1 \\ 0 & \eta & \varrho \end{pmatrix}$, $g_1(x(t)) = (\quad \frac{1}{2}\delta(a \quad b)(|x^1(t)+1| \quad |x^1(t) \quad 1|), 0, 0)^T \in R^3$, $g_2(x(t \quad \tau)) = (0, 0, \quad \eta\epsilon sin(vx^1(t \quad \tau)))^T \in R^3$, and $\delta = 10$,$\eta = 19.53$,$\varrho = 0.1636$,$a = \quad 1.4325$,$b = \quad 0.7831$,$v = 0.5$,$\epsilon = 0.2$,$\tau = 0.02$. Moreover, assume P=I and $e(t) = x(t) \quad s(t) = (e^1(t), e^2(t), e^3(t))^T$ to get appropriate δ_1,δ_2 satisfying QUAD condition:

$$\begin{aligned}
&[x(t) \quad s(t)]^T[f(x(t), x(t \quad \tau)) \quad f(s(t), s(t \quad \tau))] \\
&\leq 11.3931e(t)^T e(t) + 0.9765e(t \quad \tau)^T e(t \quad \tau)
\end{aligned} \tag{5.1.42}$$

and assume $\quad = I$, $\omega = 2$, $\theta = 1$; $\Phi_1 = I$, $\Phi_2 = 0.2244 * I$ by LMI tools.

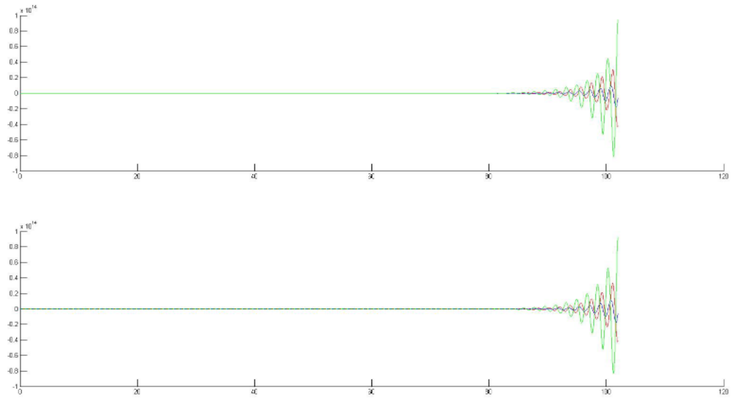

Figure 5.1: Error network with cluster 1

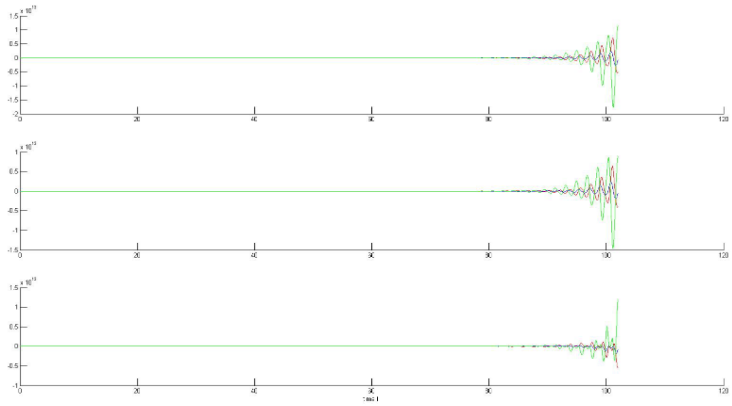

Figure 5.2: Error network with cluster 2

The figure 1 and 2 show the cluster synchronization cannot be realized. Based on Theorem 1, since here we have $a_3 = 25.8918$, $b = 1.953$, $a_2 = \quad 24.9836$, then $a_3 < b < a_2$ is false. Therefore, we increase the scale of $(\Phi_{kk}\hat{A}_{kk})^s$ by a positive constant c. (i.e., c=640)

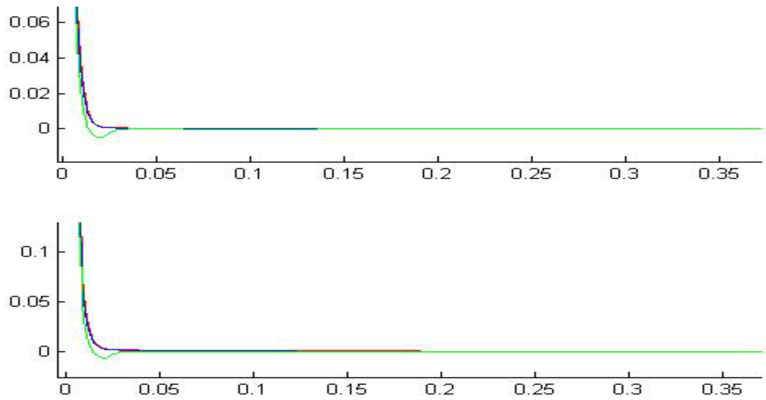

Figure 5.3: Error network with cluster 1

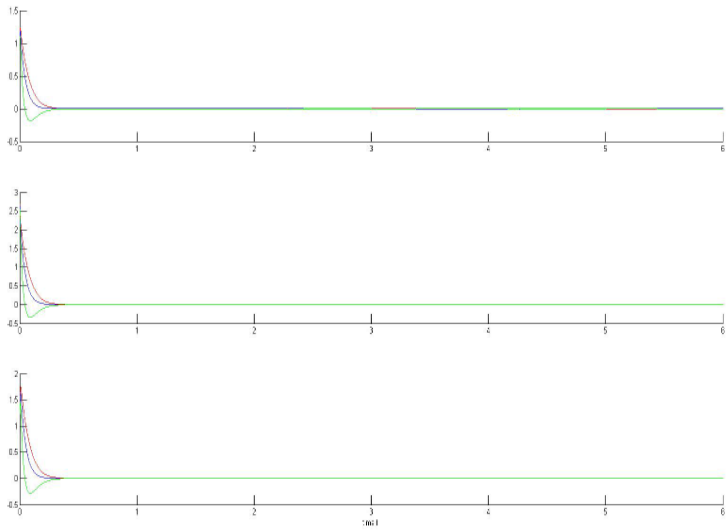

Figure 5.4: Error network with cluster 2

The figure 3 and 4 show the cluster synchronization is realized very fast with errors reduce to zero in a short time.

Remark. First of all, since we are dealing with linearly coupled identical systems, we can choose the "first" node in each cluster freely; the stability behaviors may vary by setting different nodes in each cluster otherwise; however, in this paper, no matter the dynamical behaviours among nodes are identical or not, they are all obtained to satisfy the QUAD condition and $\Phi_k \hat{A}_{kk} < 0$ negative enough, hence there will be no significant difference by assigning controller to different nodes. Secondly, since dynamics of all nodes are identical in the numerical example as well as the dynamics are chaotic (Chua oscillator), we can choose distinct initial dynamical conditions to ensure $s_i(t)$ are distinct even they are not particularly given in the paper. At last, even the cluster synchronization is realized, the value of c is much larger than we need in fact; thus we can apply the adaptive controller $gH(t)$ based on Theorem 4.1.5, in this case with adaptive rules (Matlab code omitted):

$$\begin{aligned}\dot{H}(t) = 0.1 * \max_{t \quad \tau \le \kappa \le t} (\sum_{i=1}^{2} \|x_i(\kappa) \quad s_1(\kappa)\| \\ + \sum_{i=3}^{5} \|x_i(\kappa) \quad s_1(\kappa)\|)\end{aligned} \tag{5.1.43}$$

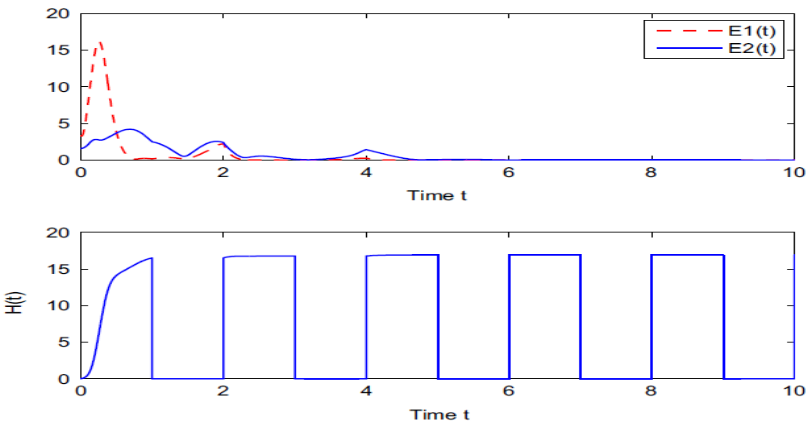

Figure 5.5: Error networks of both clusters with adaptive controller H(t)

From figure 5, it only requires control gain to be approximately 16.9228 to realize cluster synchronization, which is indeed far less then 640 with static intermittent control gain. In general, adaptive intermittent control is more practical and eco-friendly.

5.2 Impulsive cluster synchronization of chaotic systems subject to time delay

1. Introduction

As an extension of both cluster synchronization [24] and impulsive synchronization [25], we investigate the cluster synchronization via impulsive control. Due to the complication of deriving theoretical analysis, however, we will mainly take advantage of numerical simulation in order to exam the properties which lead to the synchronization.

2. Model construction

According to [24], consider a linearly coupled network with delay, and suppose the network can be divided into two clusters: $C_1 = \{1, 2\}$ and $C_2 = \{3, 4, 5\}$, where 1,2,3,4,5 represent five nodes. Meanwhile, the dynamical system for each node is chosen to be:

$$\begin{aligned} &\dot{u}(t) = u(t) + \Phi_2(u(t), x(t \quad \tau), u(t \quad r)), t \neq \tau_k \\ &\Delta u(t) = \quad B_k e(t) \end{aligned} \tag{5.2.1}$$

where it is the receiver end of Chaos-based communication system in [2]. And since it is dependent on the transmitter end:

$$\dot{x}(t) = Ax(t) + \Phi_1(x(t)) \tag{5.2.2}$$

and we have two clusters, hence we introduce two distinct transmitter systems: one connects to two receiver nodes, and the other connects three receiver nodes.

Remark. Since the controller already exists to be the impulsive controller on each single node, we will focus on determining the proper coupling matrix to realize cluster synchronization numerically; for simplicity, we will keep the dynamical systems for all nodes in the same cluster identical. By inspection, we expect the coupling matrix which satisfy either the conditions in Theorem 1 or Definition 3 in [25], could effect the impulsive cluster synchronization as well.

3. Numerical simulation

Particularly, we set up the following two transmitter systems, consider the Chua circuit:

$$\begin{aligned} \dot{x}_1 &= a_1[x_2 - h(x_1)] \\ \dot{x}_2 &= x_1 - x_2 + x_3 \\ \dot{x}_3 &= -a_2 x_2 \end{aligned} \tag{5.2.3}$$

where $h(x_1) = m_1 x_1 + \frac{1}{2}(m_0 - m_1)(|x_1 + a_3| - |x_1 - a_3|)$. The first system with $a_1 = 9$,$a_2 = 14.286$,$a_3 = 1$,$m_0 = -\frac{1}{7}$,$m_1 = \frac{1.5}{7}$; and the second one with $a_1 = 9$,$a_2 = 14.286$,$a_3 = 1$,$m_0 = -\frac{8}{7}$,$m_1 = \frac{5}{7}$.

Notice that according to [25] and following graphs, the impulsive system synchronization has been achieved for both transmitter systems with their corresponding receiver systems:

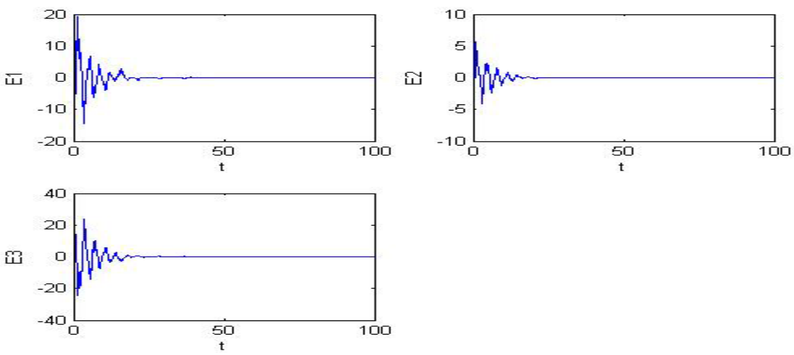

Figure 5.6: Impulsive synchronization with first transmitter

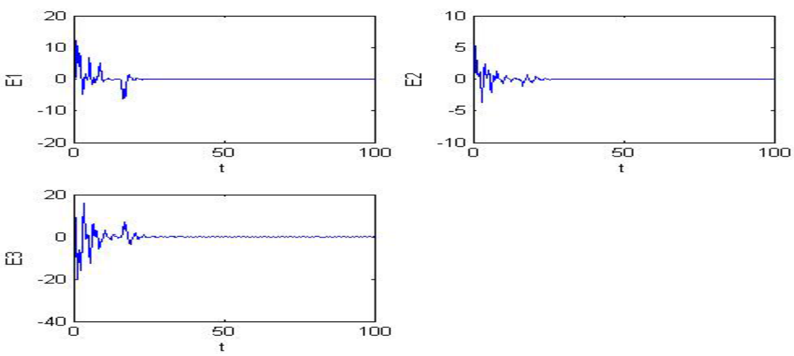

Figure 5.7: Impulsive synchronization with second transmitter

Case 1. In this case, the following coupling matrix which holds all conditions of Theorem 1 and Definition 3 in [24] for the five receiver nodes:

$$A = \begin{pmatrix} 143.616 & 143.616 & 2 & 3 & 5 \\ 287.232 & 287.232 & 0 & 4 & 4 \\ 2 & 2 & 640 & 640 & 0 \\ 2 & 2 & 0 & 640 & 640 \\ 3 & 3 & 640 & 640 & 1280 \end{pmatrix} \qquad (5.2.4)$$

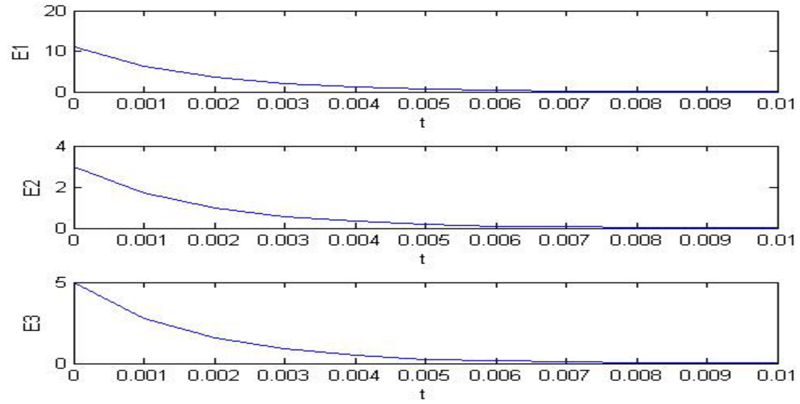

Figure 5.8: Errors between node 1 and 2

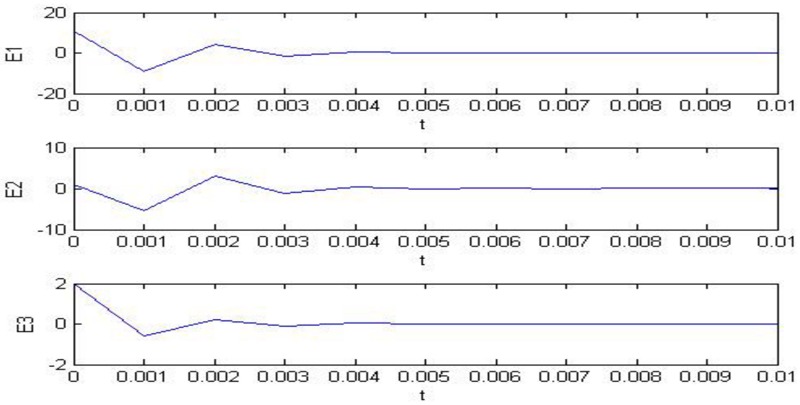

Figure 5.9: Errors between node 3 and 4

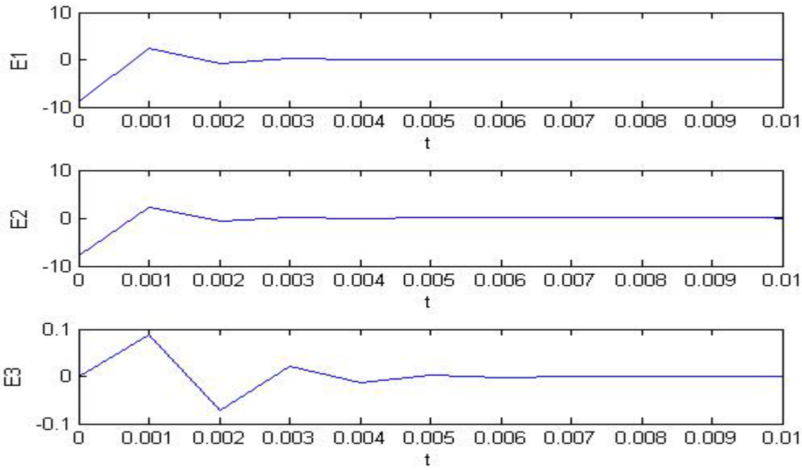

Figure 5.10: Errors between node 3 and 5

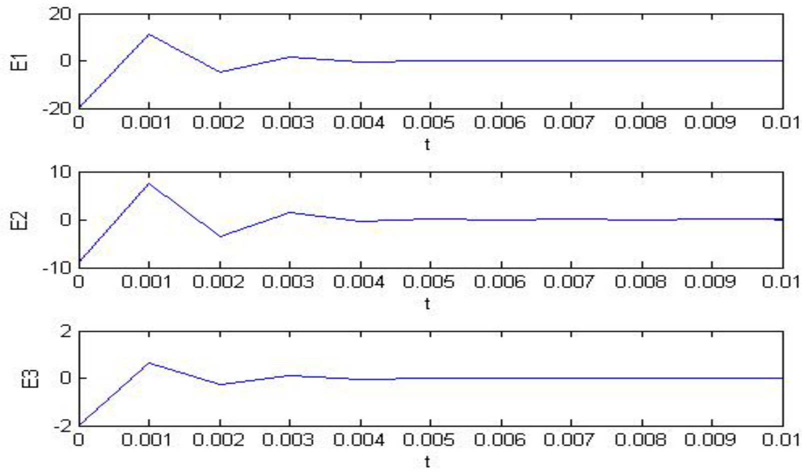

Figure 5.11: Errors between node 4 and 5

Based on the graphs above, the numerical simulations indicate that the coupling matrix which satisfy both the conditions in Theorem 1 and Definition 3 in [24] generates impulsive cluster synchronization. Due to the choice of matrix, the errors converge to zero rapidly.

Case 2. In this case, we consider the coupling matrices where Definition 3 holds but fails to hold all conditions in Theorem 1 in [24]. (i.e., $a_{ii} = \sum_{j=1,j\neq i}^{N} a_{ij}, i = 1, .., N$):

$$A1 = \begin{pmatrix} 1 & 1 & 2 & 3 & 5 \\ 2 & 2 & 0 & 4 & 4 \\ 2 & 2 & 1 & 1 & 0 \\ 2 & 2 & 0 & 1 & 1 \\ 3 & 3 & 1 & 1 & 2 \end{pmatrix} \tag{5.2.5}$$

$$A2 = \begin{pmatrix} 5 & 5 & 4 & 3 & 1 \\ 2 & 2 & 0 & 4 & 4 \\ 2 & 2 & 1 & 1 & 0 \\ 2 & 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix} \tag{5.2.6}$$

A1:

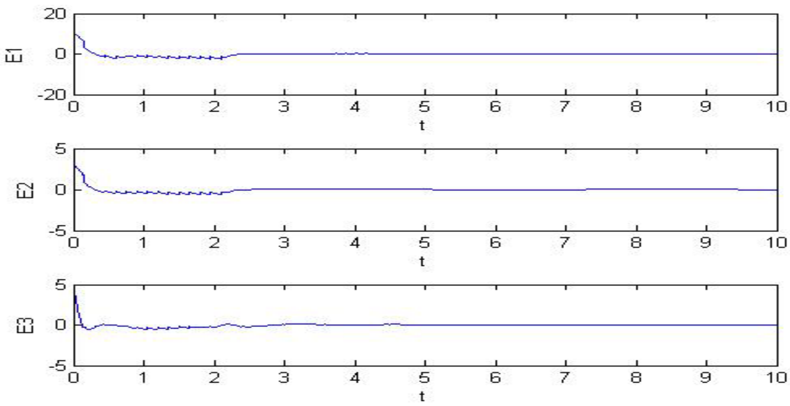

Figure 5.12: Errors between node 1 and 2

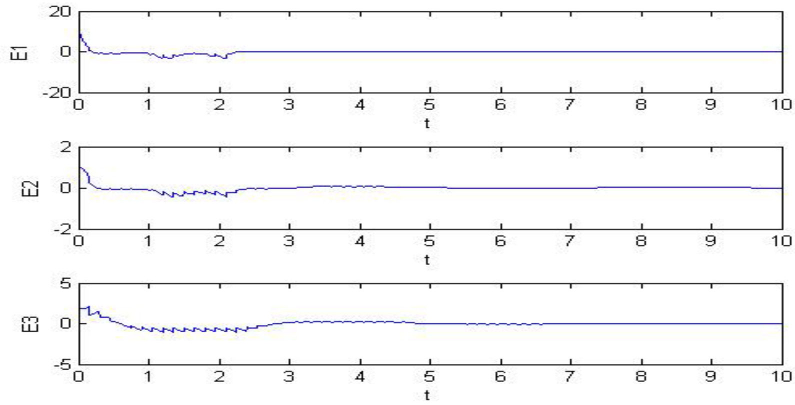

Figure 5.13: Errors between node 3 and 4

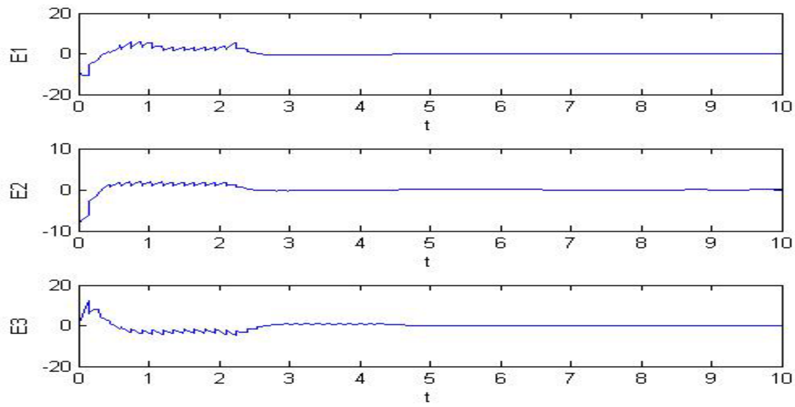

Figure 5.14: Errors between node 3 and 5

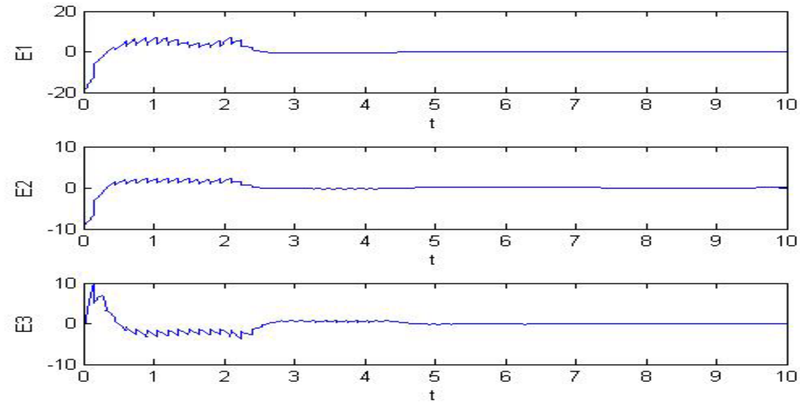

Figure 5.15: Errors between node 4 and 5

A2:

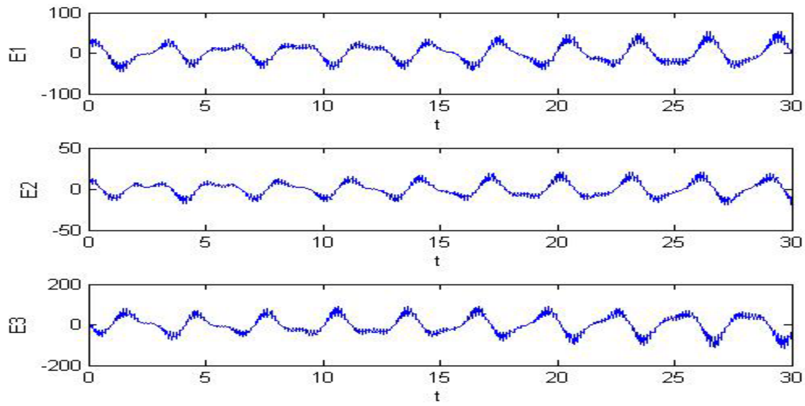

Figure 5.16: Errors between node 1 and 2

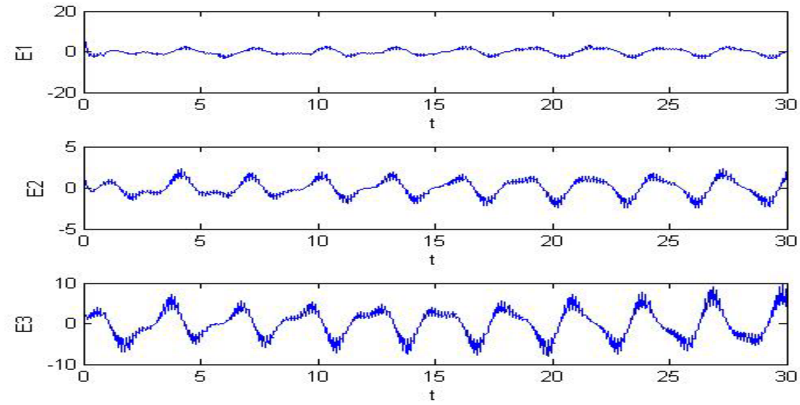

Figure 5.17: Errors between node 3 and 4

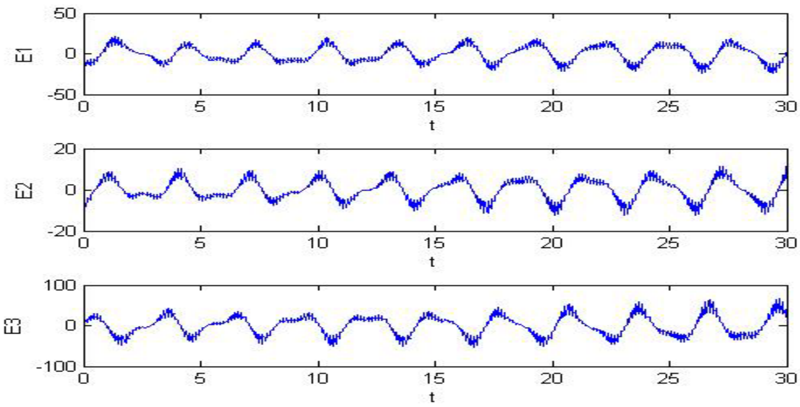

Figure 5.18: Errors between node 3 and 5

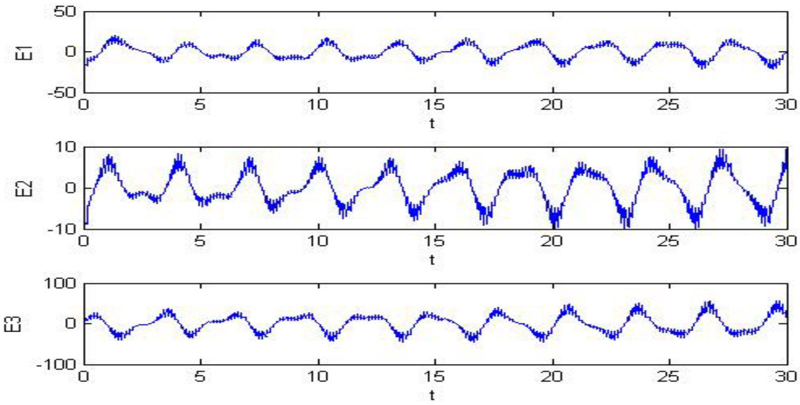

Figure 5.19: Errors between node 4 and 5

Based on the graphs above, the numerical simulations indicate if only the Definition 3 in [24] holds, two opposite error behaviours occur. This suggests there exists more factors to consider when designing the coupling matrix, and its theoretical analysis can be a potential research area to explore in the future.

Case 3. In this case, we consider the coupling matrix where both Definition 3 and all conditions in Theorem 1 in [24] fails:

$$A = \begin{pmatrix} 5 & 1 & 2 & 4 & 5 \\ 2 & 2 & 0 & 5 & 4 \\ 2 & 4 & 1 & 2 & 0 \\ 2 & 5 & 2 & 1 & 1 \\ 3 & 3 & 7 & 1 & 2 \end{pmatrix} \tag{5.2.7}$$

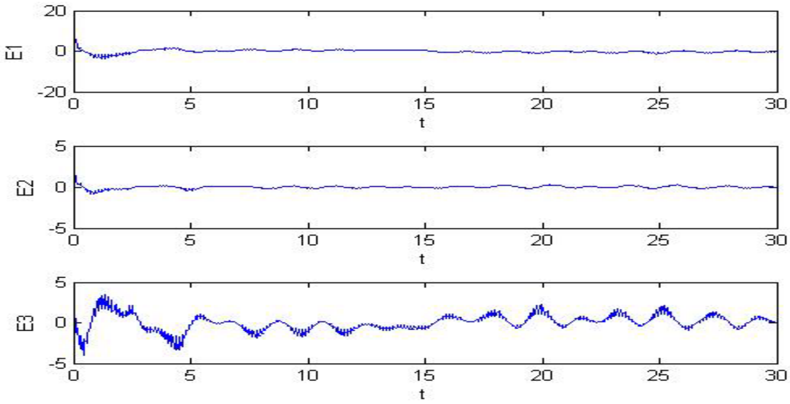

Figure 5.20: Errors between node 1 and 2

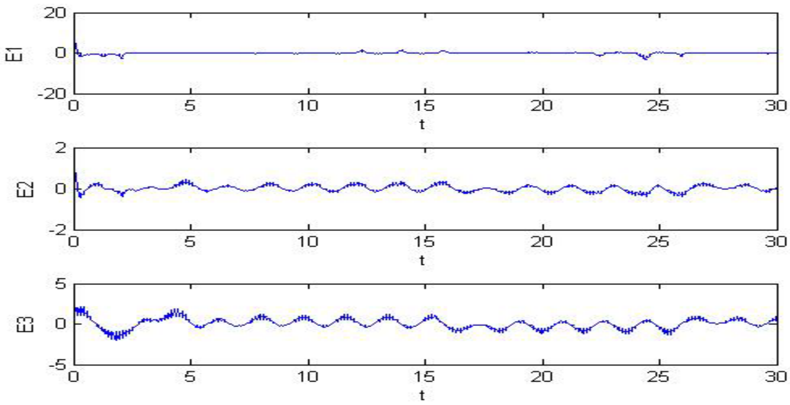

Figure 5.21: Errors between node 3 and 4

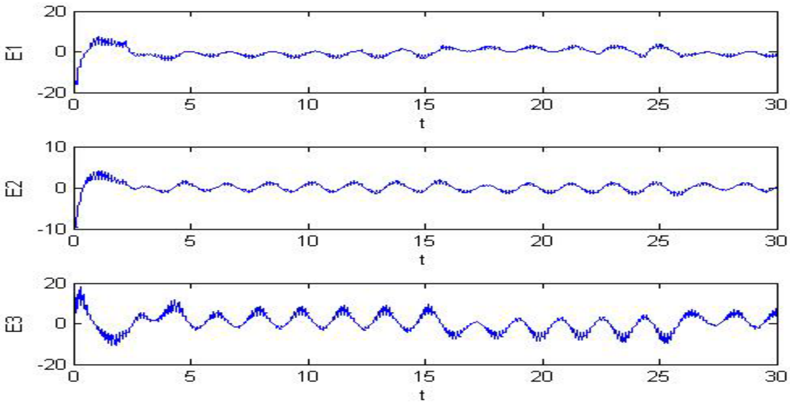

Figure 5.22: Errors between node 3 and 5

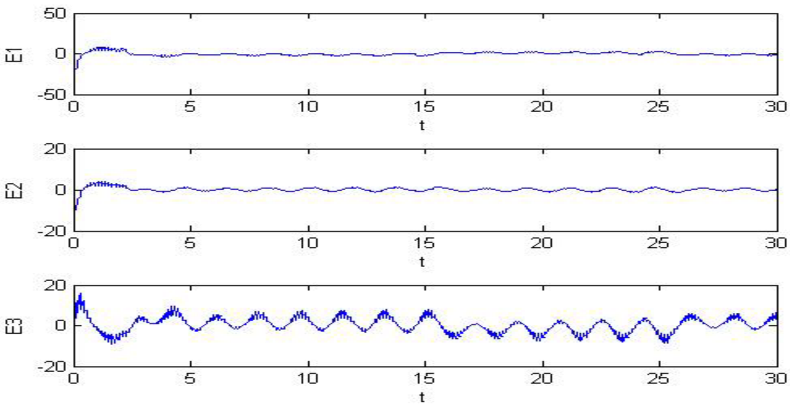

Figure 5.23: Errors between node 4 and 5

Based on the graphs above, the numerical simulations indicate the impulsive cluster synchronization cannot be realized if both requirements fail.

Remark. Even the pattern of numerical simulation is limited without combining the theocratical results, but it still reveals that Theorem 1 and Definition 3 in [24] are the sufficient conditions to obtain impulsive cluster synchronization as well. Moreover, such requirements could also be the potential requirements to other types of cluster synchronization, besides pinning intermittent and impulsive control.

Chapter 6

Conclusion

In this report, the synchronization of both chaos systems and networks have been summarized comprehensively. Specifically, by generating the system synchronization in terms of linear feedback control, decoupling feedback control, adaptive control, delay synchronization, robust control, nonlinear control, sliding mode control, impulsive control (with time delay) and projective synchronization; then the synchronization with identical or nonidentical networks as well as cluster synchronization.In general, we can import these synchronization methods to the broad range of ODE,DDE systems and complex networks in analyzing real applications. Conclusively, to achieve the synchronization in practice will be the matter of designing appropriate controllers with consideration of pros and cons such as timing, storage or prime cost etc.

Moreover, our numerical simulations on each category are also consistently applied to support the effectiveness of the theoretical approaches. The numerical experiments performed with cross using of Euler and Runge-Kutta methods in order to maintain high algorithm stability and accuracy. The aim of theses numerical simulations mainly focus on detecting the convergence behaviours of error systems (networks) generated by corresponding drive and response systems (networks).

Longitudinally, once we understand these fundamental process combine with numerical simulations, many potential researches can be considered as directions for further work, such as 3D chaotic cipher for encrypting two data streams simultaneously, exponential synchronization of fractional-order complex networks via pinning impulsive control etc. Thus, more and more complex scenarios can be involved into the chaos synchronization in many real world fields.

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