AMATH 732: Linear Damped Oscillator

November 4, 2010

Governing equation:

$$\begin{aligned} m\ddot{x} + 2\beta\dot{x} + kx &= 0, \\ x(0) &= x_0, \\ \dot{x} = v_0 &= 0, \end{aligned} \tag{1}$$

Three time scales:

- $T_c^{(1)} = \sqrt{m/k}$;
- $T_c^{(2)} = \beta/k$;
- $T_c^{(3)} = m/\beta$.

Defining

$$\epsilon = \frac{\beta}{\sqrt{mk}}, \tag{2}$$

we have

$$T_c^{(2)} = \epsilon T_c^{(1)} = \epsilon^2 T_c^{(3)}. \tag{3}$$

- If $\epsilon \ll 1$ then $T_c^{(2)} \ll T_c^{(1)} \ll T_c^{(3)}$.
- If $\epsilon \gg 1$ then $T_c^{(2)} \gg T_c^{(1)} \gg T_c^{(3)}$.

Case I: $\epsilon < 1$ (underdamped).

In dimensional time the exact solution is

$$x = e^{-\frac{t}{T_c^{(3)}}}\left(\cos\left(\sqrt{1-\epsilon^2}\frac{t}{T_c^{(1)}}\right) + \frac{\epsilon}{\sqrt{1-\epsilon^2}}\sin\left(\sqrt{1-\epsilon^2}\frac{t}{T_c^{(1)}}\right)\right)$$

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The solution consists of

- oscillations on the time scale $T_c^{(1)}$ if $\epsilon \ll 1$;
- plus an amplitude that decays on the time scale $T_c^{(3)}$.

Since $T_c^{(1)} = \epsilon T_c^{(3)}$, for $\epsilon \ll 1$ the time scale of the oscillations is much shorter than the time scale of the amplitude decay.

- Scaling with time scale $T_c^{(1)}$ we obtained

$$x_0 = \cos(t/T_c^{(1)}).$$

  - Does not capture exponential decay on slow time scale $T_c^{(3)} = \epsilon T_c^{(1)}$.
  - Does not capture $\mathcal{O}(\epsilon^2)$ change in frequency. The oscillatory behaviour is approximately

$$\begin{aligned}
&\cos\left(\sqrt{1-\epsilon^2}\frac{t}{T_c^{(1)}}\right) + \frac{\epsilon}{\sqrt{1-\epsilon^2}}\sin\left(\sqrt{1-\epsilon^2}\frac{t}{T_c^{(1)}}\right) \\
&\quad \approx \cos\left((1-\frac{1}{2}\epsilon^2)\frac{t}{T_c^{(1)}}\right) + (\epsilon + \frac{1}{2}\epsilon^3)\sin\left((1-\frac{1}{2}\epsilon^2)\frac{t}{T_c^{(1)}}\right) \\
&\quad = \cos\left(\frac{t}{T_c^{(1)}} - \frac{1}{2}\frac{\epsilon^2 t}{T_c^{(1)}}\right) + (\epsilon + \frac{1}{2}\epsilon^3)\sin\left(\frac{t}{T_c^{(1)}} - \frac{1}{2}\frac{\epsilon^2 t}{T_c^{(1)}}\right)
\end{aligned}$$

so this can also be interpreted as dependence on time scale $T_c^{(1)}$ and on the slow time scale $T_c^{(1)}/\epsilon^2$.

- Scaling with time scale $T_c^{(2)}$ we obtained

$$x_0 = 1 - \frac{1}{2}\epsilon^2\left(\frac{t}{T_c^{(2)}}\right)^2 = 1 - \frac{1}{2}\left(\frac{t}{T_c^{(1)}}\right)^2$$

  which is becomes disordered on times of order $T_c^{(1)}$.
  - Exact solution depends on time scales $T_c^{(1)}$ and $T_c^{(3)}$, not on time scale $T_c^{(2)}$.
- Scaling with time scale $T_c^{(3)}$ we could not find a solution.
  - Exact solution is dominated by behaviour on faster time scale $T_c^{(1)}$.

Regular Perturbation Theory runs into difficulty for two reasons.

1. The frequency of the oscillations depends on ϵ. This results in secular forcing terms and the solution breaks down after a time of $\mathcal{O}(\epsilon^{-1})$. We have seen this behaviour before when we consider the simple nonlinear pendulum.
2. The solution includes behaviour on two very different time scales: the amplitude decays on a slow time scale which significantly modifies the solution after times of $\mathcal{O}(T_c^{(3)})$.

Case III: $\epsilon > 1$ (overdamped).

The exact solution now has the form

$$x = \frac{1}{2}\left(1 + \frac{\epsilon}{\sqrt{\epsilon^2 - 1}}\right) e^{-(\epsilon - \sqrt{\epsilon^2 - 1})t/T_c^{(1)}} + \frac{1}{2}\left(1 - \frac{\epsilon}{\sqrt{\epsilon^2 - 1}}\right) e^{-(\epsilon + \sqrt{\epsilon^2 - 1})t/T_c^{(1)}}.$$

For $\epsilon \gg 1$, $\epsilon - \sqrt{\epsilon^2 - 1} = \epsilon - \epsilon\sqrt{1 - \frac{1}{\epsilon^2}} \approx \frac{1}{2\epsilon}$ and $\epsilon + \sqrt{\epsilon^2 - 1} \approx 2\epsilon$. Hence, for large ϵ

$$x \approx e^{-t/(2\epsilon T_c^{(1)})} - \frac{1}{4\epsilon^2} e^{-2\epsilon t/T_c^{(1)}}, \\ = e^{-t/(2T_c^{(2)})} - \frac{1}{4\epsilon^2} e^{-2t/T_c^{(3)}}. \qquad (4)$$

- $T_c^{(2)} = \epsilon^2 T_c^{(3)} \gg T_c^{(3)}$
- The first term decays on the slow time scale $T_c^{(2)}$
- The second term decays on the fast time scale $T_c^{(3)}$.

The RPT solution runs into difficulty because of the presence of two time scales.

- Scaling with time scale $T_c^{(1)}$ we could not find a solution.
  - The exact solution does not depend on time scale $T_c^{(1)}$.
- Scaling with time scale $T_c^{(2)}$ we could not find a solution.
  - At short times the solution decays on the faster time scale $T_c^{(3)}$. Need this time scale in solution to satisfy initial conditions.

- Scaling with time scale $T_c^{(3)}$ we obtained

$$x = 1 + \left(\frac{1}{4}(1 - e^{-\frac{2t}{T_c^{(3)}}}) - \frac{1}{2}\frac{t}{T_c^{(3)}} \right) \frac{1}{\epsilon^2} + \cdots ,$$

which becomes disordered after a time of $\mathcal{O}(\epsilon^2)$.

In terms of $\xi = t/T_c^{(3)}$, the exact solution is

$$x = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1 - \frac{1}{\epsilon^2}}}\right) e^{-\left(1-\sqrt{1-\frac{1}{\epsilon^2}}\right)\xi}$$
$$+ \frac{1}{2}\left(1 - \frac{1}{\sqrt{1 - \frac{1}{\epsilon^2}}}\right) e^{-\left(1+\sqrt{1-\frac{1}{\epsilon^2}}\right)\xi}.$$

Expanding $\sqrt{1-\frac{1}{\epsilon^2}}$ and $1/\sqrt{1-\frac{1}{\epsilon^2}}$ in powers of $\frac{1}{\epsilon^2}$,

$$y = \left(1 + \frac{1}{4\epsilon^2} + \cdots\right) e^{-\xi/(2\epsilon^2)+\cdots} + \left(-\frac{1}{4\epsilon^2} + \cdots\right) e^{-2\xi} e^{\left(\frac{1}{2\epsilon^2}+\cdots\right)\xi},$$
$$= 1 + \left(\frac{1}{4}\left(1 - e^{-2\xi}\right) - \frac{\xi}{2}\right)\frac{1}{\epsilon^2} + \mathcal{O}_F\left(\frac{1}{\epsilon^4}\right).$$

which recovers the Regular Perturbation Theory solution.

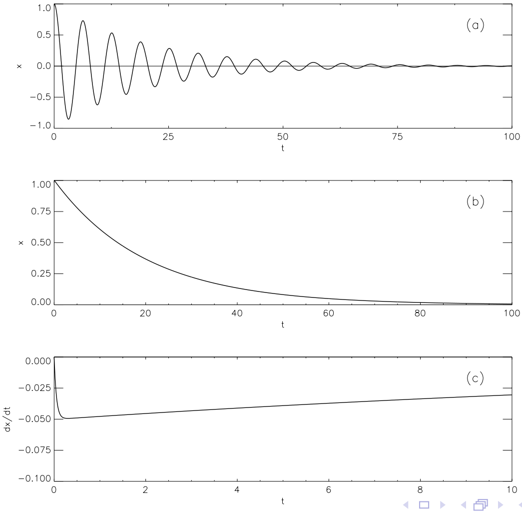
- The solution becomes disordered because the Taylor Series expansion

$$e^{-\frac{\xi}{2\epsilon^2}} = 1 - \frac{\xi}{2\epsilon^2} + \frac{1}{8}\frac{\xi^2}{\epsilon^4} + \cdots, \tag{5}$$

  which converges as the number of terms goes to ∞ for fixed ξ, is a disordered Asymptotic Expansion. It becomes disordered when ξ is $\mathcal{O}(\epsilon^2)$.

Solutions I

(a) Underdamped ($\epsilon = 0.05$). (b,c) Overdamped ($\epsilon = 10$).



Solutions II

(a) Underdamped ($\epsilon = 0.05$). (b,c) Overdamped ($\epsilon = 10$).