AMATH 732: Method of Multiple Scales

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The linear damped oscillator is an example of a physical system that varies on more than one time scale. There many other such examples - far too numerous to list. The simple nonlinear pendulum is one such example. We will now solve it using the method of multiple scales.

The method of multiple scales is an extremely powerful method with widespread applications.

The Simple Nonlinear Pendulum

Recall the scaled problem

$$\begin{aligned} \theta'' + \frac{\sin(a\theta)}{a} &= 0, \\ \theta(0) &= 1, \\ \theta'(0) &= 0, \end{aligned}$$

The nonlinearity in the system causes a slow drift (period not exactly 2π) so we can think of θ varying on two time scales: t and $\tau = a^2 t$.

With this in mind we try looking for a solution of the form

$$\theta = f(t, \epsilon t; a^2) = f(t, \tau; a^2)$$

The idea is to treat t and τ as two independent variables. The chain rule gives

$$\frac{d\theta}{dt} = f_t + a^2 f_\tau,$$
$$\frac{d^2\theta}{dt^2} = f_{tt} + 2a^2 f_{t\tau} + a^4 f_{\tau\tau},$$

So the governing equation becomes

$$
\begin{aligned}
f_{tt} + 2a^2 f_{t\tau} + a^4 f_{\tau\tau} + f - a^2\frac{f^3}{6} + \cdots &= 0, \\
f(0,0;a^2) &= 1, \\
f_t(0,0;a^2) + a^2 f_\tau(0,0;a^2) &= 0.
\end{aligned}
$$

As usual look for solutions of the form

$$
f = f_0(t,\tau) + a^2 f_1(t,\tau) + a^4 f_2(t,\tau) + \cdots .
$$

$\mathcal{O}(1)$ Problem:

$$f_{0_{tt}} + f_0 = 0,$$
$$f_0(0,0) = 1,$$
$$f_{0_t}(0,0) = 0,$$

which has the solution

$$f_0 = A(\tau)\cos(t) + B(\tau)\sin(t)$$

where

$$A(0) = 1 \qquad \text{and} \qquad B(0) = 0.$$

$\mathcal{O}(a^2)$ Problem:

$$f_{1_{tt}} + f_1 = -2f_{0_{t\tau}} + \frac{1}{6}f_0^3,$$

$$f_1(0,0) = 0,$$

$$f_{1_t}(0,0) + f_{0_\tau}(0,0) = 0.$$

After some algebra to evaluate the forcing terms we get

$$\begin{aligned} f_{1_{tt}} + f_1 = {} & \left[2A'(\tau) + \frac{1}{8}(A^2B + B^3)\right]\sin t \\ & + \left[-2B'(\tau) + \frac{1}{8}(AB^2 + A^3)\right]\cos t \\ & + \frac{1}{24}(A^3 - 3AB^2)\cos 3t - \frac{1}{24}(B^3 - 3A^2B)\sin 3t. \end{aligned}$$

We must now eliminate the secular terms. This gives

$$A'(\tau) = -\frac{1}{16}(A^2 + B^2)B,$$

$$B'(\tau) = \frac{1}{16}(A^2 + B^2)A,$$

We now have two coupled nonlinear ODEs to solve!

Multiplying the first by A, the second by B and adding gives

$$\frac{d}{d\tau}\left(\frac{1}{2}(A^2+B^2)\right)=0,$$

so A^2+B^2 is constant. From the initial conditions $A^2+B^2=1$ so

$$A'(\tau)=-\frac{1}{16}B,$$

$$B'(\tau)=\frac{1}{16}A,$$

We now have a couple set of linear ODEs which are easily solved. Eliminating B gives

$$A''(\tau)+\frac{1}{16^2}A=0.$$

The solution is

$$A(\tau) = \cos(\frac{\tau}{16}),$$

$$B(\tau) = \sin(\frac{\tau}{16}),$$

The $\mathcal{O}(1)$ solution is

$$\begin{aligned} f_0 &= \cos\left(\frac{\tau}{16}\right)\cos t + \sin\left(\frac{\tau}{16}\right)\sin t, \\ &= \cos\left(t - \frac{\tau}{16}\right), \\ &= \cos\left((1 - \frac{a^2}{16})t\right). \end{aligned}$$

Using the method of strained coordinates we obtained

$$\theta(t) = \cos\left((1 - \frac{a^2}{16} + \cdots)t\right) + \frac{a^2}{192}\left[\cos\left((1 - \frac{a^2}{16} + \cdots)t\right) - \cos\left(3\left(1 - \frac{a^2}{16} + \cdots\right)t\right)\right] + \mathcal{O}_F(a^4).$$

The method of multiple scales has recovered the same first term with identical frequencies to $\mathcal{O}(a^2)$

With the resonant forcing terms eliminated in the $\mathcal{O}(a^2)$ problem the problem for f_1 simplifies to

$$
\begin{aligned}
f_{1_{tt}} + f_1 &= \frac{1}{24}(A^3 - 3AB^2)\cos 3t - \frac{1}{24}(B^3 - 3A^2B)\sin 3t,\\
f_1(0,0) &= 0,\\
f_{1_t}(0,0) + f_{0_\tau}(0,0) &= 0.
\end{aligned}
$$

Using the known forms for A and B this simplifies to

$$
\begin{aligned}
f_{1_{tt}} + f_1 &= \frac{1}{24}\cos(3\tau)\cos 3t + \frac{1}{24}\sin(3\tau)\sin 3t,\\
f_1(0,0) &= 0,\\
f_{1_t}(0,0) &= 0.
\end{aligned}
$$

Solving gives

$$f_1(t,\tau) = -\frac{1}{192}\cos(3\tau)\cos 3t - \frac{1}{192}\sin 3\tau \sin 3t$$
$$+ A(\tau)\cos t + B(\tau)\sin t,$$

where A and B are new unknown functions.

Note that the first two terms combine to give

$$-\frac{1}{192}\cos(3\tau)\cos 3t - \frac{1}{192}\sin 3\tau \sin 3t = -\frac{1}{192}\cos\Big((1-\frac{a^2}{16})3t\Big)$$

which is again in agreement with our previous solution.

The new unknown function A and B can be determined by eliminating the resonant forcing in the $\mathcal{O}(a^4)$ problem. To go to higher order, however, will require the introduction of an even longer time scale $\tau_1 = a^4 t$, consistent with our previously obtained amplitude dependent frequency $\sigma = 1 - a^2/16 + \mathcal{O}(a^4)$.