

Trapped wave solutions to the Dubreil-Jacotin-Long equation

N. Soontiens*, M. Stastna, M. Waite

Department of Applied Mathematics, University of Waterloo, Canada

*nksoonti@uwaterloo.ca

Abstract

The Dubreil-Jacotin-Long (DJL) equation is a scalar equation that is equivalent to the steady, stratified Euler equations of motion for an incompressible fluid. This study examines solutions to this equation for waves trapped over topography with a pycnocline stratification. Solutions are derived for flows with a shear background current and for different topographic profiles with and without the Boussinesq approximation. It is found that multiple states can occur for certain background currents and topographic profiles. An examination of the effects of making the Boussinesq approximation is carried out. Large trapped waves with amplitudes up to four times the topographic height are discovered for background speeds near a critical minimum speed. As the background speed increases the waves become smaller until the wave amplitude is close to the topographic height. Solutions under the Boussinesq approximation experience a very sharp transition from large to small amplitude waves when compared to their non-Boussinesq counterparts. Finally, asymmetric states across the topographic crest are considered.

Equations

The DJL equation can be expressed in terms of the isopycnal displacement η , which is the distance between an isopycnal and its far upstream height. Under the Boussinesq approximation with a constant background current U_0 and bottom topography $h(x)$ this equation with boundary conditions is given by:

$$(1) \quad \begin{aligned} \nabla^2 \eta + \frac{N_B^2(z-\eta)}{U_0^2} \eta &= 0 \\ \eta(x, H) &= 0 \\ \eta(x, h(x)) &= h(x) \\ \eta(\pm L, z) &= 0 \end{aligned}$$

The squared buoyancy frequency $N_B^2(z)$ is related to the density $\bar{\rho}(z)$ by:

$$N_B^2(z) = -\frac{g}{\rho_0} \frac{d\bar{\rho}}{dz}(z)$$

This equation is used to study the trapped waves of elevation and depression discussed on the right.

A different form of the DJL equation can be derived when the Boussinesq approximation is relaxed:

$$(2) \quad \nabla^2 \eta + \frac{N_{NB}^2(z-\eta)}{U_0^2} \eta + \frac{N_{NB}^2(z-\eta)}{2g} [\eta_x^2 + \eta_z(\eta_z - 2)] = 0$$

where the squared buoyancy frequency $N_{NB}^2(z)$ is related to the density $\bar{\rho}(z)$ by:

$$N_{NB}^2(z) = -\frac{g}{\bar{\rho}(z)} \frac{d\bar{\rho}}{dz}(z)$$

This equation is used to compare Boussinesq and non-Boussinesq results on the right.

With a shear background current, under the Boussinesq approximation the DJL equation is written as:

$$(3) \quad \nabla^2 \eta + \frac{U'(z-\eta)}{U(z-\eta)} \left[1 - (\eta_x^2 + (1-\eta_z)^2) \right] + \frac{N_B^2(z-\eta)}{U^2(z-\eta)} \eta = 0$$

Solutions to this equation are found using a background current:

$$U(z) = U_0 + \frac{\Delta u}{2} \tanh\left(\frac{z-z_j}{d_j}\right)$$

Finally, asymmetric states can be achieved by altering the boundary conditions to:

$$(4) \quad \eta(-L, z) = 0, \quad \eta_x(L, z) = 0$$

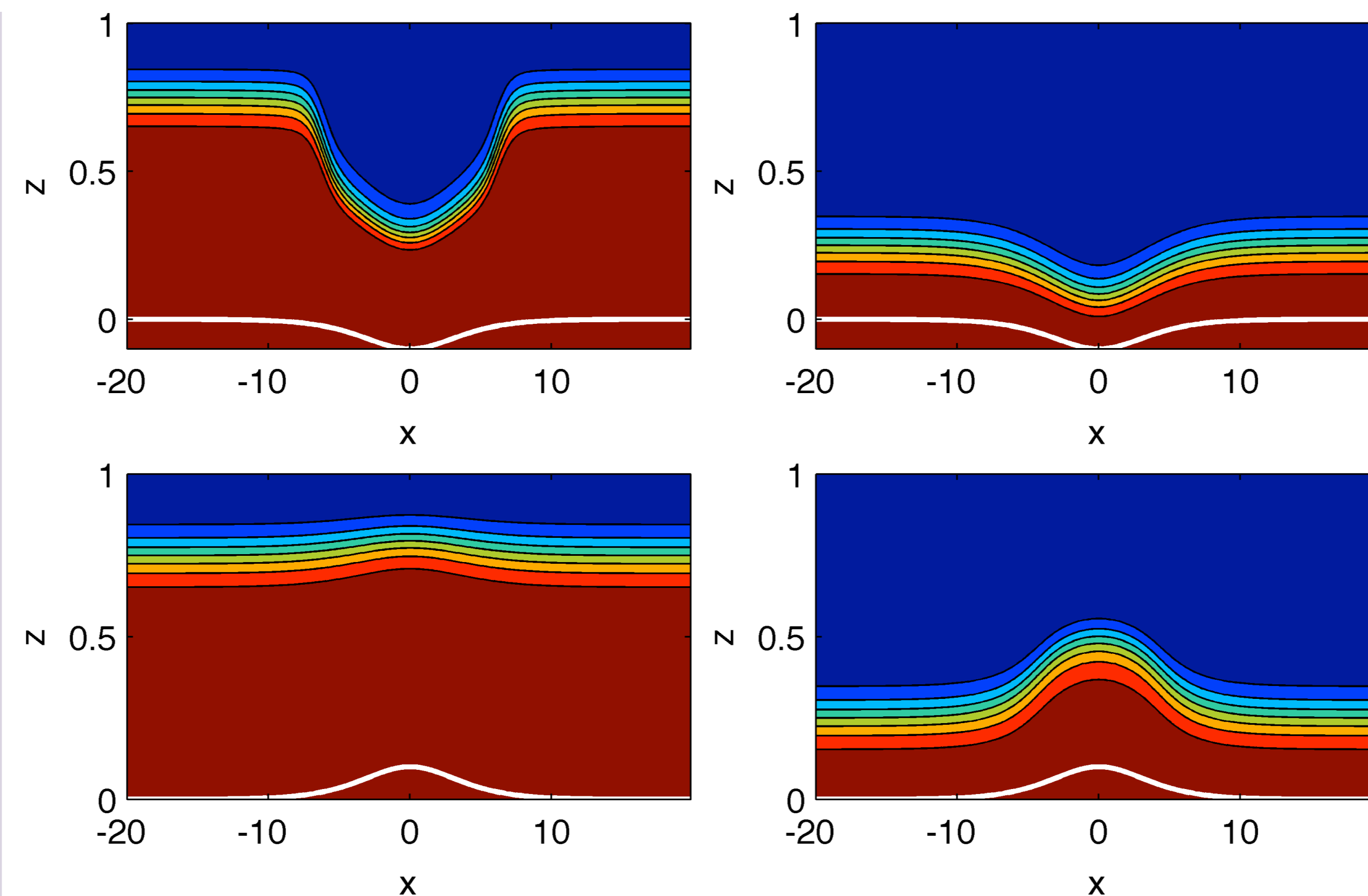
Solutions to equation (1) with these boundary conditions and slightly subcritical flow conditions are asymmetric about the topographic crest.

References

N. Soontiens, C. Subich, and M. Stastna, "Numerical simulation of supercritical trapped internal waves over topography," *Phys. Fluids* 22, 11605 (2010).

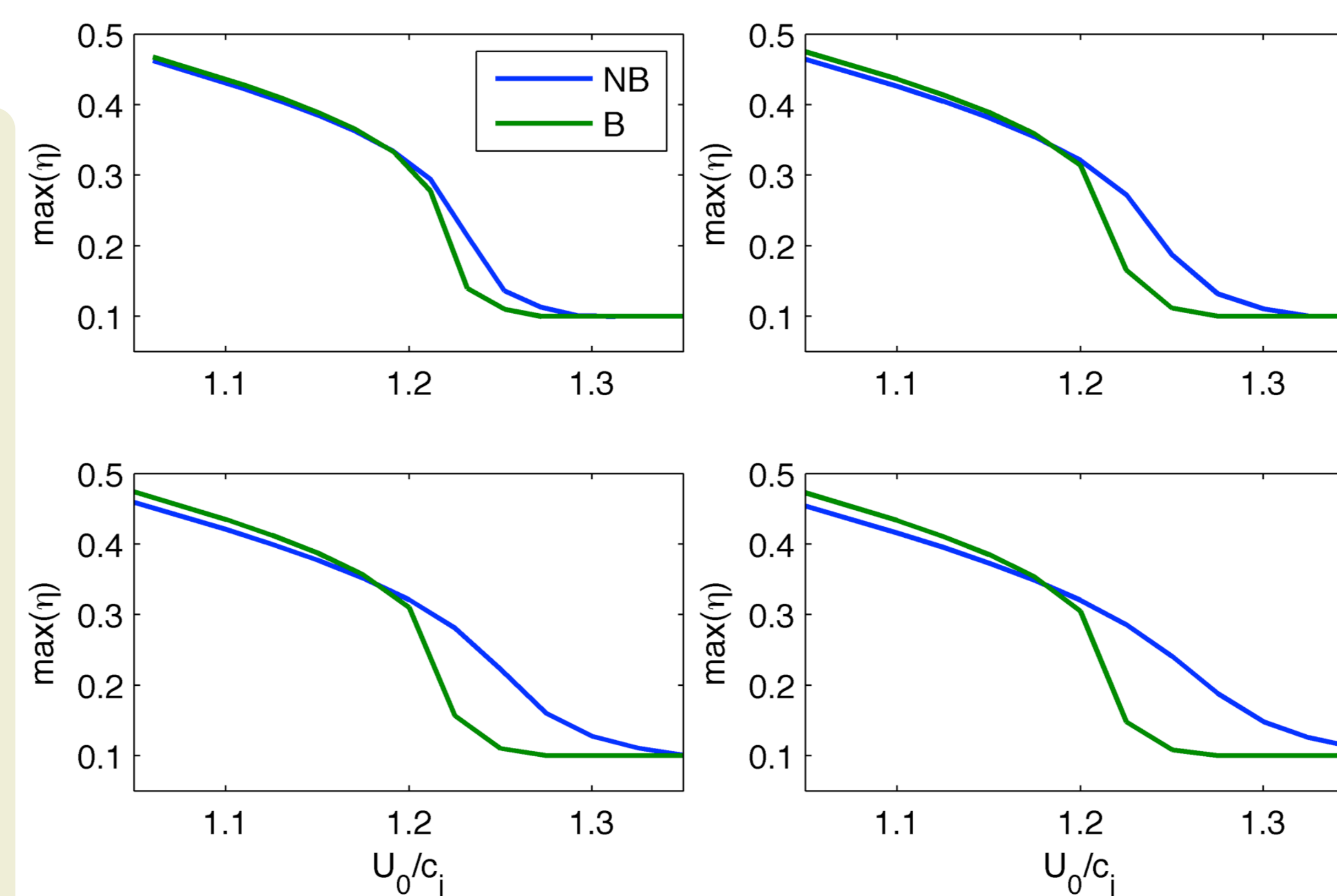
N. Soontiens, M. Stastna, and M.L. Waite, "Trapped internal waves over topography: Non-Boussinesq effects, symmetry breaking, and downstream recovery jumps," *Phys. Fluids* 25, 066602 (2013).

(1) Waves of elevation and depression



Freely propagating internal waves of elevation and depression exhibit a symmetry property when the profile $N_B^2(z)$ is reflected about the mid-depth. Trapped internal waves over topography do not exhibit such a symmetry property as demonstrated in the figure to the left where we have plotted density contours for several background stratifications and topography shapes. Over depression topography, very large trapped waves of depression can develop if the pycnocline is centered above the mid-depth. Over elevation topography large trapped waves exist if the pycnocline is centered below the mid-depth. However, the large trapped waves of elevation are not nearly as large as the corresponding waves of depression. The largest trapped waves occur when the background current U_0 is close to a minimal speed called the conjugate flow speed c_j .

(2) Boussinesq vs non-Boussinesq

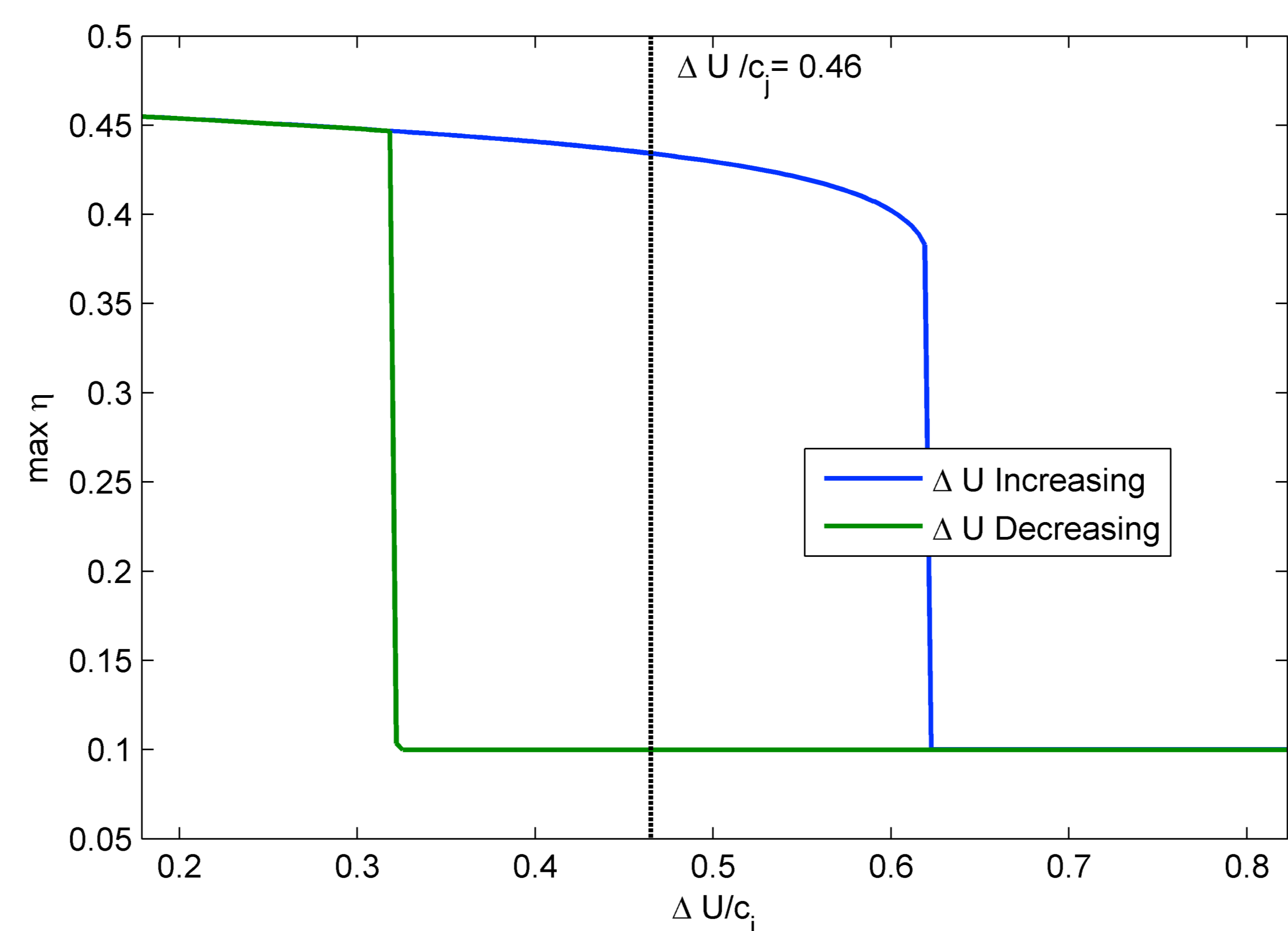


Differences between Boussinesq and non-Boussinesq waves are apparent for moderate values of U_0/c_j as demonstrated in the figure to the left. Here we plot the maximum isopycnal displacement for several values of U_0 and $\Delta\rho$ under both non-Boussinesq (blue) and Boussinesq (green) conditions. In each case the density stratification is given by

$$\bar{\rho}(z) = \rho_0 \left(1 - \Delta\rho \tanh\left(\frac{z-z_0}{d_0}\right) \right)$$

where $\Delta\rho = 0.05, 0.1, 0.15,$ or 0.2 from top left to bottom right. Both Boussinesq and non-Boussinesq curves exhibit a similar trend: the wave amplitude decreases as U_0 increases. However, the non-Boussinesq waves experience a more gradual transition to small amplitude. The Boussinesq approximation can significantly underestimate the wave amplitude for moderate U_0/c_j , even for small values of $\Delta\rho$.

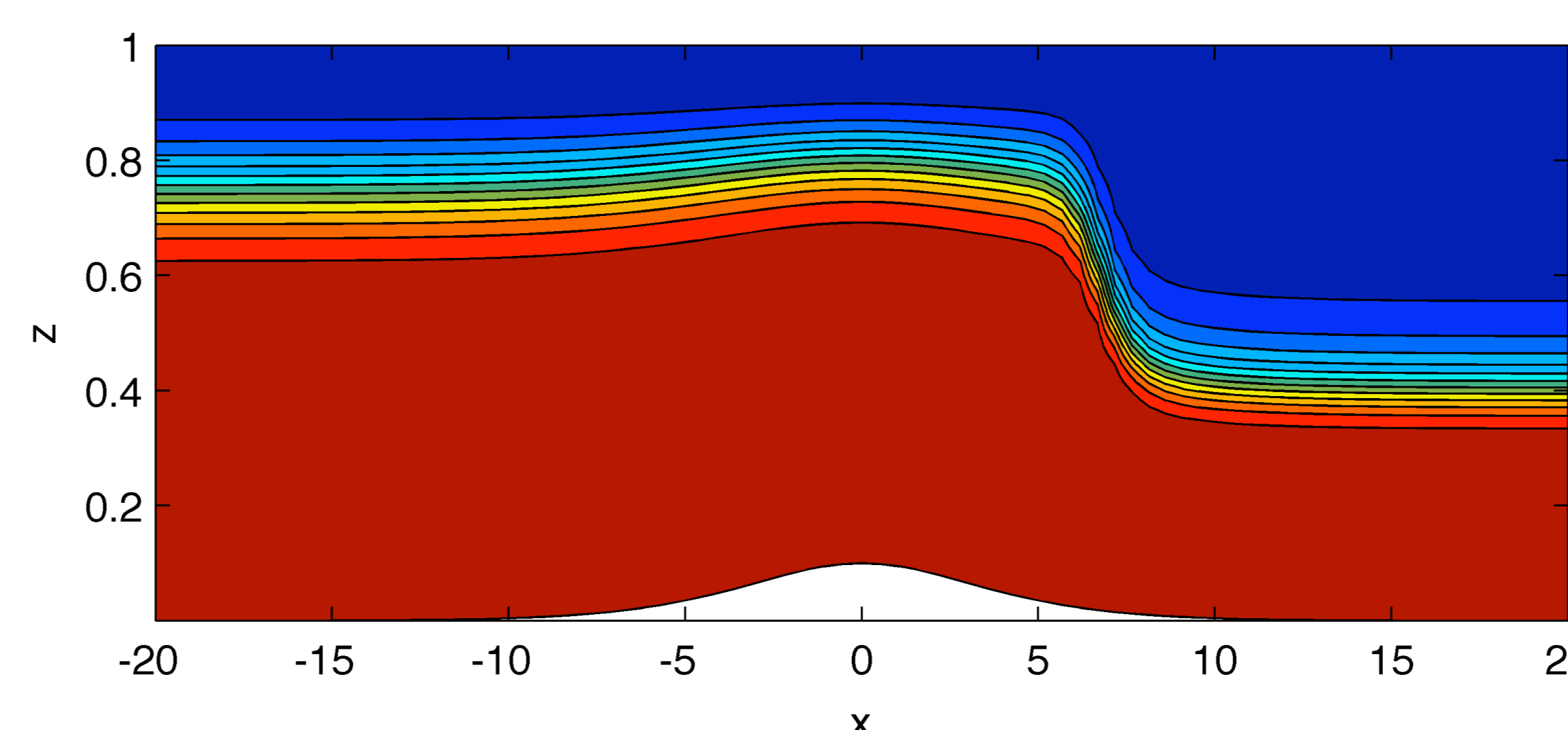
(3) Shear background currents and hysteresis



Multiple states can exist for steady flows with a shear background current. These solutions are produced numerically using an iterative solver which modifies the shear strength Δu slightly between successive solutions. On the left, the blue curve displays the maximum isopycnal displacement when Δu is increased from $0.18c_j$ to $0.82c_j$ in the iterative solver. The green curve displays the maximum isopycnal displacement when Δu is decreased from $0.82c_j$ to $0.18c_j$. Multiple solutions exist for a wide range of Δu , in this case $0.31 < \Delta u/c_j < 0.62$. The region of hysteresis changes with the strength of the background current U_0 .

Other examples of hysteresis have been observed in cases with a constant background current and very narrow topography.

(4) Asymmetric states



Asymmetric states have been computed under slightly subcritical flow conditions, $U_0/c_j < 1$, where c_j is the conjugate flow speed. A large wave that is asymmetric about the topographic crest is produced as displayed in the density contours to the left. These types of waves have also been discovered through weakly nonlinear theory and fully nonlinear, time-dependent numerical simulations.

Acknowledgements: We would like to thank The Fields Institute of Mathematical Sciences for providing travel support.