INTERNAL WAVE TURNING POINTS AND TUNNELING

MODEL EQUATIONS

$$\begin{aligned}
\rho_0\left(\vec{U}_t + \vec{U}\cdot\vec{\nabla}\vec{U}\right) &= -\vec{\nabla}p - \rho g\hat{k}, \\
\rho_t + \vec{U}\cdot\vec{\nabla}\rho &= 0, \\
\vec{\nabla}\cdot\vec{U} &= 0.
\end{aligned}$$

- 2D vertical x-z plane.
- Boussinesq approximation.
- Inviscid
- Earth's rotation ignored.

LINEARIZED EQUATION FOR THE VERTICAL VELOCITY w

Background State:

$$\rho = \bar{\rho}(z), \quad p = \bar{p}(z), \quad \vec{U} = (0, 0), \tag{1}$$

with $\bar{p}'(z) = -\rho g$.

Add perturbation:

$$\begin{aligned} \rho &= \bar{\rho}(z) + \epsilon \rho', \\ p &= \bar{p}(z) + \epsilon p', \\ \vec{U} &= \epsilon(u, w). \end{aligned} \tag{2}$$

At $O(\epsilon)$ a single equation for the vertical velocity can be derived:

$$\frac{\partial^2}{\partial t^2} \nabla^2 w + N^2(z) w_{xx} = 0, \tag{3}$$

where

$$N^2(z) = -\frac{g}{\rho_0} \frac{d\bar{\rho}}{dz}. \tag{4}$$

$N(z)$ is the vertically varying buoyancy frequency.

PLANE WAVE SOLUTIONS: CONSTANT N

If N is constant, solutions of the form

$$w = e^{i(kx+mz-\omega t)}, \tag{5}$$

exist provided

$$\omega^2 = \frac{N^2 k^2}{k^2+m^2} = \frac{N^2}{1+(m/k)^2}. \tag{6}$$

The wave frequency depends only on the direction of the wave vector (k, m). The wave frequency is bounded by N.

GROUP VELOCITY

FACT: Energy propagates with the group velocity

$$\vec{c}_g = \vec{\nabla}_k \omega = (\omega_k, \omega_m) = \frac{\omega}{k^2 + m^2} \frac{m}{k} (m, -k), \quad (7)$$

which is perpendicular to the wave vector $\vec{k} = (k, m)$.

This means energy propagates along lines of constant phase.

Internal waves are an example of *dispersive waves* for which the group velocity depends on $\vec{k}$.

TURNING POINTS

Recall w is given by

$$\frac{\partial^2}{\partial t^2}\nabla^2 w + N^2(z) w_{xx} = 0. \qquad (8)$$

This time let $N = N(z)$ and look for solutions of the form

$$w = e^{i(kx-\omega t)}\Psi(z). \qquad (9)$$

Substituting into the equation for w gives

$$\Psi'' + \left(\frac{N^2 - \omega^2}{\omega^2}\right) k^2 \Psi = 0. \qquad (10)$$

or

$$\Psi'' + Q(z)\Psi = 0, \tag{11}$$

where

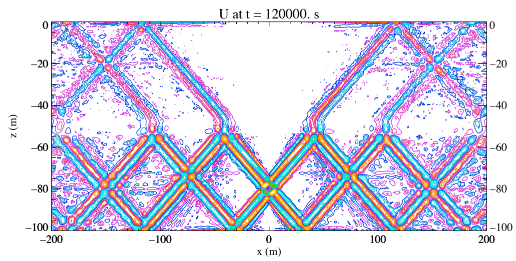
$$Q(z) = \frac{N^2 - \omega^2}{\omega^2} k^2. \tag{12}$$

$Q(z)$ is positive in regions where $|\omega| < N$ and is negative in regions where $|\omega| > N$.

Recall internal plane waves only exist if $|\omega| < N$.

ANIMATIONS

(1) Turning point. N decreases linearly with height.



(2) Tunneling: N decreases below ω at mid-depth and then increases to be larger than ω.