Testing Linear Hypotheses with a Generalized Multivariate Modified Bessel Error Variable

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TESTING LINEAR HYPOTHESES WITH A GENERALIZED MULTIVARIATE MODIFIED BESSEL ERROR VARIABLE

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ABSTRACT

Most of the classical literature on linear hypothesis testing assumes that the errors are normally distributed. In situations where the distribution of the error variable has heavier tails than the normal, several authors have studied the problem using other distributions such as the multivariate-t distribution. In this paper, we consider linear hypothesis testing problems in which the error variable follows the generalized multivariate modified Bessel distribution. This distribution is a much more general distribution that includes both the multivariate normal and multivariate-t as special cases.

Key words and phrases: Linear regression model; generalized multivariate modified Bessel distribution; hypothesis test; non-central F-Bessel distribution; power of the test.

1 INTRODUCTION

Consider the general linear regression model given by

$$\mathbf{Y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon} \tag{1}$$

where $\mathbf{Y}$ is the $n \times 1$ observed response vector, X is the $n \times p$ matrix of fixed predictor variables, $\boldsymbol{\beta}$ is the $p \times 1$ parameter vector, and $\boldsymbol{\epsilon}$ is the $n \times 1$ error vector. Most of the classical literature on this model assumes that $\boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2 I_n)$ where I_n denotes the $n \times n$ identity matrix. However, there exists numerous practical situations where this assumption may not be suitable. For instance, in quality control studies of gage and measurement system capability, the random error component (consisting of differences among instruments, differences among operators, instability over time, environmental changes, different setups, etc.) of the measurement error may have a distribution with heavier tails than the traditionally-used normal distribution. In these kinds of situations, it is more appropriate to use a thicker-tailed alternative like the multivariate-t distribution (cf. Blattberg and Gonedes [1], Zellner [2], Sutradhar and Ali [3], and Sutradhar [4] among others). In particular, Zellner [2] considered the renowned "market-model" in which the errors associated with the linear market model for stock data have a common variance with an inverted gamma density. Other authors have used the assumption of spherical errors (cf. Dawid [5], Jammalamadaka et al. [6], and Chib et al. [7] to mention a few). Box and Tiao [8] have explored the consequences of non-normality of the error distribution by assuming a class of symmetric power distributions such as the exponential power distributon.

Generally, a broader assumption of mixture distributions can be employed. One such choice is the *generalized multivariate modified Bessel distribution* considered by Thabane and Haq [9]. More specifically, we assume that the errors $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)'$ have joint probability density function (pdf) given by

$$\begin{aligned} p(\epsilon_1, \ldots, \epsilon_n) &= \frac{\left(\frac{\lambda}{\psi}\right)^{\frac{n}{4}}}{(2\pi\sigma^2)^{\frac{n}{2}} K_\nu\left(\sqrt{\lambda\psi}\right)} \left\{1 + \frac{1}{\psi}\sum_{j=1}^{n} \epsilon_j^2/\sigma^2\right\}^{\frac{2\nu-n}{4}} \\ &\times \; K_{\frac{2\nu-n}{2}}\left(\sqrt{\lambda\psi\left(1 + \frac{1}{\psi}\sum_{j=1}^{n}\epsilon_j^2/\sigma^2\right)}\right); \quad -\infty < \epsilon_j < \infty \quad \forall j, \quad (2) \end{aligned}$$

where $K_\nu(z)$ denotes the modified Bessel function of the third kind of order ν (cf. Gradshteyn and Ryzhik [10], p. 970) and the domain of the shape parameters (ψ, λ, ν) is given by:

$$\psi > 0, \; \lambda \geq 0 \qquad \text{for} \qquad \nu < 0,$$

$$\psi > 0,\ \lambda > 0 \qquad \text{for} \qquad \nu = 0,$$
$$\psi \geq 0,\ \lambda > 0 \qquad \text{for} \qquad \nu > 0. \tag{3}$$

It is to be noted that the pdf (2) is: (i) a member of the spherically symmetric class of distributions (cf. Fang et al. [11]), and (ii) a special case of the symmetric multivariate hyperbolic distributions of Barndorff-Nielsen [12]. Moreover, the distribution of the errors $(\epsilon_1, \ldots, \epsilon_n)$ can be expressed as a mixture of the multivariate normal distribution with the generalized inverse Gaussian pdf as follows (cf. Thabane and Haq [9] for further details):

$$p(\epsilon_1, \ldots, \epsilon_n) = \int_0^\infty p(\epsilon_1, \ldots, \epsilon_n | \tau) p(\tau) d\tau,$$

where

$$p(\epsilon_1, \ldots, \epsilon_n | \tau) = \frac{1}{(2\pi\tau\sigma^2)^{\frac{n}{2}}} \exp\left\{ -\frac{1}{2\tau} \sum_{j=1}^{n} \epsilon_j^2 / \sigma^2 \right\},$$

and

$$p(\tau) = \frac{\left(\frac{\lambda}{\psi}\right)^{\frac{\nu}{2}}}{2K_\nu\left(\sqrt{\lambda\psi}\right)} \tau^{\nu-1} \exp\left\{ -\frac{1}{2}\left(\frac{\psi}{\tau} + \lambda\tau\right)\right\}; \quad \tau > 0. \tag{4}$$

The model described by (2) would be applicable in various practical situations where the error distribution is symmetric, but with heavier tails than the normal distribution. This model also has added advantage because of its rich parametric structure, as different values of (ψ, λ, ν) correspond to different model choices. The great flexibility in the choice of these parameters allows us to study how our inferences are affected by changes in model assumptions. This is normally referred to as *inference robustness* (cf. Box and Tiao [8]). Table 1 provides some special cases of the model for different values of (ψ, λ, ν).

As (2) is a member of the spherically symmetric class of distributions, we would expect inferences concerning linear combinations of the regression parameters to remain robust under the null hypothesis (cf. Giri [13]). Sutradhar [4] made this observation for the multivariate-t model, which is a special case of (2). However, we would expect the non-null distribution of the corresponding test statistic to be different from that obtained under either the normal or t-models. In the following section, we derive the non-null distribution of the test statistic under the generalized multivariate modified Bessel distribution. The derived distribution is then used to calculate powers for several different choices of error distributions in the simple linear regression model. We also obtain a confidence region for the regression parameter estimates. Concluding remarks are given in Section 3.

2 THE NON-NULL DISTRIBUTION OF THE TEST STATISTIC

Returning to the linear model (1), consider the problem of testing

$$H_0 : C\boldsymbol{\beta} = \mathbf{0} \quad versus \quad H_a : C\boldsymbol{\beta} \neq \mathbf{0} \tag{5}$$

where C is an $m \times p$ matrix of known elements with rank(C)=q. Let $r = p - q$. It is well known that when $\boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2 I_n)$, one tests (5) by using the classical test statistic

$$T = \frac{S_2 - S_1}{S_1}$$

where

$$S_1 = Y'(I_n - X(X'X)^{-1}X')Y/\sigma^2 \tag{6}$$

and

$$S_2 = Y'(I_n - Z(Z'Z)^{-1}Z')Y/\sigma^2.$$

In the above, S_1 is the residual sum of squares of the full model (1), and S_2 is the residual sum of squares of the reduced model

$$E(\mathbf{Y}) = Z\boldsymbol{\gamma}, \tag{7}$$

which is obtained from (1) under the restriction $C\boldsymbol{\beta} = \mathbf{0}$. In (7), Z is an $n \times r$ design matrix and $\boldsymbol{\gamma}$ is an $r \times 1$ parameter vector. We now present the following result:

Theorem 1 *Consider the linear model* (1) *and the test defined by* (5). *Let*

$$\delta^2 = (X\beta)'(I_n - Z(Z'Z)^{-1}Z')X\beta/\sigma^2. \tag{8}$$

If the error vector ϵ *has pdf* (2), *then the pdf of the test statistic* T *for testing* H_0 *versus* H_a *is given by*

$$\begin{aligned} g(t) &= \frac{\left(\frac{\psi+\delta^2}{\psi}\right)^{\frac{\nu}{2}} t^{\frac{q-2}{2}}}{\Gamma\left(\frac{n-p}{2}\right) K_\nu\left(\sqrt{\lambda\psi}\right)(1+t)^{\frac{n-r}{2}}} \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{n-r}{2}+j\right) K_{\nu-j}\left(\sqrt{\lambda(\psi+\delta^2)}\right)}{j!\Gamma\left(\frac{q}{2}+j\right)} \\ &\times \left(\frac{\sqrt{\lambda}\delta^2 t}{2\sqrt{\psi+\delta^2}(1+t)}\right)^j; \quad t > 0. \end{aligned} \tag{9}$$

Proof: If ϵ has pdf (2), the conditional distribution of ϵ given τ, where τ has pdf (4), is normal with mean vector $\mathbf{0}$ and covariance matrix $\tau\sigma^2 I_n$. Therefore, given τ, it is well known that $F = (n-p)T/q$ has a non-central F-distribution with q and $n-p$ degrees of freedom and non-centrality parameter δ^2/τ. That is, the conditional pdf of F given τ has the form (cf. Anderson [14], p. 174):

$$h(f|\tau) = \frac{q^{\frac{q}{2}}(n-p)^{\frac{n-p}{2}} \exp\left\{-\frac{\delta^2}{2\tau}\right\} f^{\frac{q-2}{2}}}{\Gamma\left(\frac{n-p}{2}\right)(n-p+qf)^{\frac{n-r}{2}}} \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{n-r}{2}+j\right)}{j!\Gamma\left(\frac{q}{2}+j\right)} \left(\frac{\delta^2 qf}{2\tau(n-p+qf)}\right)^j; \quad f > 0. \tag{10}$$

Integration with respect to τ subsequently yields

$$\begin{aligned} h(f) &= \int_0^\infty h(f|\tau)p(\tau)d\tau \\ &= \frac{q^{\frac{q}{2}}(n-p)^{\frac{n-p}{2}}\left(\frac{\lambda}{\psi}\right)^{\frac{\nu}{2}} f^{\frac{q-2}{2}}}{2\Gamma\left(\frac{n-p}{2}\right) K_\nu\left(\sqrt{\lambda\psi}\right)(n-p+qf)^{\frac{n-r}{2}}} \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{n-r}{2}+j\right)}{j!\Gamma\left(\frac{q}{2}+j\right)} \left(\frac{\delta^2 qf}{2(n-p+qf)}\right)^j \\ &\times \int_0^\infty \tau^{\nu-j-1} \exp\left\{-\frac{1}{2}\left(\frac{(\psi+\delta^2)}{\tau}+\lambda\tau\right)\right\} d\tau. \end{aligned} \tag{11}$$

Applying formula 9 of Gradshteyn and Ryzhik [10], p. 340, we get

$$\int_0^\infty \tau^{\nu-j-1} \exp\left\{-\frac{1}{2}\left(\frac{(\psi+\delta^2)}{\tau}+\lambda\tau\right)\right\} d\tau = \frac{2K_{\nu-j}\left(\sqrt{\lambda(\psi+\delta^2)}\right)}{\left\{\frac{\lambda}{\psi+\delta^2}\right\}^{\frac{\nu-j}{2}}}, \tag{12}$$

and substituting (12) back into (11) gives rise to the marginal pdf of F:

$$
\begin{aligned}
h(f) &= \frac{q^{\frac{q}{2}}(n-p)^{\frac{n-p}{2}}\left(\frac{\psi+\delta^2}{\psi}\right)^{\frac{\nu}{2}} f^{\frac{q-2}{2}}}{\Gamma\left(\frac{n-p}{2}\right) K_\nu\left(\sqrt{\lambda\psi}\right)(n-p+qf)^{\frac{n-r}{2}}} \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{n-r}{2}+j\right) K_{\nu-j}\left(\sqrt{\lambda(\psi+\delta^2)}\right)}{j!\,\Gamma\left(\frac{q}{2}+j\right)} \\
&\times \left(\frac{\sqrt{\lambda}\delta^2 qf}{2\sqrt{\psi+\delta^2}(n-p+qf)}\right)^j; \quad f>0. \qquad (13)
\end{aligned}
$$

Finally, by making a simple transformation, one obtains the distribution of T as given in the theorem. □

We observe that (13) is the pdf of the *non-central F-Bessel distribution* with parameters $(q, n-p, \delta^2, \psi, \lambda, \nu)$. Thabane and Drekic [15] first introduced this distribution and have studied some of its statistical properties. Furthermore, if the null hypothesis is true, the full model (1) corresponds with the reduced model (7). This implies $\delta^2 = 0$ from (8), and it is easily verified via direct substitution into (13) that the distribution of F simplifies to the central F-distribution with q and $n-p$ degrees of freedom (cf. Evans et al. [16], p. 91). We point out that this is identical to the null distribution obtained under the classical multivariate normal model (cf. Draper and Smith [17]) as well as the multivariate t model (cf. Sutradhar [4]).

2.1 A Special Case

Consider the special case of $C = [\mathbf{0}^* \mid I_q]$, where $\mathbf{0}^*$ is a $q \times (p-q)$ matrix of zeros, $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_{p-1})'$, and design matrix given by

$$
X = \begin{bmatrix} 1 & x_{11} & \cdots & x_{(p-1)1} \\ 1 & x_{12} & \cdots & x_{(p-1)2} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & x_{1n} & \cdots & x_{(p-1)n} \end{bmatrix}.
$$

Testing the null hypothesis $H_0 : C\boldsymbol{\beta} = \mathbf{0}$ is equivalent to testing $H_0 : \beta_{p-q} = \cdots = \beta_{p-1} = 0$. Clearly, q=rank(C). Also, note that the reduced design matrix Z under this restriction becomes an $n \times (p-q)$ matrix. If we let $p = 2$ (i.e., the simple linear regression model), then $C = [0 \;\; 1]$ with q=rank(C)=1. For this particular case, the corresponding non-centrality parameter in Theorem 1 reduces to

$$\delta^2 = \beta_1^2 \sum_{j=1}^{n} (x_j - \bar{x})^2 / \sigma^2.$$

It is of great interest to investigate the power of this test under the generalized multivariate modified Bessel model. Tables 2 and 3 give power calculations for $p = 2$ at levels of significance $\alpha = 0.05$ and $\alpha = 0.01$ respectively, and various values of n and δ^2. The calculations are conducted for several different choices of (ψ, λ, ν), corresponding to varying model assumptions. Among the more heavily-tailed variants considered are the multivariate t (the (5,0,-2.5) column), Bessel (the (0,2,2) column), Pearson Type VII (the (2,0,-1) column), and Cauchy (the (1,0,-0.5) column) distributions. All calculations were carried out using the computational package *Mathematica.*

It is interesting to observe from the tables that in most cases: (i) the powers are generally lower than the powers obtained under the normal model, and (ii) differences become more pronounced for larger values of δ^2. This suggests that if the errors are truly non-normal, then power calculations of T based on the classical normal model would generally result in an overestimation of the true power, particularly if the non-centrality parameter δ^2 is large.

2.2 Confidence Regions For Regression Parameter Estimates

Suppose we wish to construct a confidence ellipsoid for $\boldsymbol{\beta}$. Let $\hat{\boldsymbol{\beta}} = (X'X)^{-1}X'\mathbf{Y}$ be the vector of least-squares estimators. Under the assumed model (2), it is readily verified from Thabane and Haq [9] that $\hat{\boldsymbol{\beta}} = (\hat{\beta}_1, \dots, \hat{\beta}_p)'$ has a generalized multivariate modified Bessel distribution with pdf

$$\begin{aligned} p(\hat{\boldsymbol{\beta}}) &= \frac{|X'X|^{\frac{1}{2}} \left(\frac{\lambda}{\psi}\right)^{\frac{p}{4}}}{(2\pi\sigma^2)^{\frac{p}{2}} K_\nu\left(\sqrt{\lambda\psi}\right)} \left\{1 + \frac{1}{\psi\sigma^2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'X'X(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})\right\}^{\frac{2\nu - p}{4}} \\ &\times \; K_{\frac{2\nu-p}{2}}\left(\sqrt{\lambda\psi\left[1 + \frac{1}{\psi\sigma^2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'X'X(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})\right]}\right); \quad -\infty < \hat{\beta}_j < \infty \quad \forall\, j. \end{aligned}$$

Furthermore, conditional on τ where τ has pdf (4), $\hat{\boldsymbol{\beta}}$ is normally distributed with mean vector $\boldsymbol{\beta}$ and covariance matrix $\tau\sigma^2(X'X)^{-1}$ (cf. Thabane and Haq [9] for details). Therefore,

$$Q_1 = \frac{\left(\hat{\beta} - \beta\right)' X'X \left(\hat{\beta} - \beta\right)}{\tau\sigma^2} \sim \chi^2_p.$$

That is, Q_1 has a chi-squared distribution with p degrees of freedom. Moreover, conditional on τ, $Q_2 = S_1/\tau\sigma^2$ has a chi-squared distribution with $n-p$ degrees of freedom where S_1 is given by (6). It then follows that the quantity

$$\frac{Q_1/p}{Q_2/(n-p)} = \frac{(n-p)\left(\hat{\beta} - \beta\right)' X'X \left(\hat{\beta} - \beta\right)}{pS_1} \sim F_{p,n-p}.$$

Note that this result is independent of τ. Thus, the standard $100(1-\alpha)\%$ confidence region for β given by

$$\left(\hat{\beta} - \beta\right)' X'X \left(\hat{\beta} - \beta\right) \leq \frac{pS_1}{n-p} F_{p,n-p,\alpha}$$

remains robust under this model. In the above, $F_{p,n-p,\alpha}$ is the upper α-point of a central F-distribution with p and $n-p$ degrees of freedom.

3 CONCLUDING REMARKS

In this paper, we have considered the linear model (1) and the general linear hypothesis testing problem defined by (5) under the assumption that the errors follow the generalized multivariate modified Bessel distribution. We have derived the non-null distribution of the test statistic for testing $H_0 : C\boldsymbol{\beta} = \mathbf{0}$ against the alternative $H_a : C\boldsymbol{\beta} \neq \mathbf{0}$. This distribution was then used to calculate powers for the special case of the simple linear regression model. Several different error distributions were considered, including the normal and t distributions. Although inferences under the generalized multivariate modified Bessel model remain robust, our analysis does indicate that an incorrect normal model assumption for the errors typically leads to an overestimation of the power.

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Table 1: Special Cases of the Generalized Multivariate Modified Bessel Distribution

Distribution	Values of (ψ, λ, ν)
Multivariate Modified Bessel	$\psi > 0,\ \lambda > 0,\ \nu > \frac{n}{2}$
Pearson Type VII	$\psi > 0,\ \lambda = 0,\ \nu = -(M - \frac{n}{2}),\ M > \frac{n}{2},$
Multivariate-t Type	$\psi > 0,\ \lambda = 0,\ \nu = -\frac{r}{2},\ r > 0$
Multivariate-t	$\psi = r,\ \lambda = 0,\ \nu = -\frac{r}{2},\ r > 0$
Mean-Variance Representation of Multivariate-t	$\psi = r,\ \lambda = 0,\ \nu = -\frac{r}{2},\ r > 2$
Multivariate Normal	$\psi = r,\ \lambda = 0,\ \nu = -\frac{r}{2},\ r \to \infty$
Multivariate Cauchy	$\psi = 1,\ \lambda = 0,\ \nu = -\frac{1}{2}$
Multivariate Bessel	$\psi = 0,\ \lambda > 0,\ \nu > 0$

Table 2: Powers of T for $p = 2$, $\alpha = 0.05$

n	δ^2	(ψ, λ, ν) Normal	$(5, 0, -2.5)$	$(2, 1, 0)$	$(0, 2, 2)$	$(2, 0, -1)$	$(1, 1, 1)$	$(1, 0, -0.5)$
5	0.5	0.0806	0.0806	0.0784	0.0798	0.0806	0.0714	0.0805
	1.0	0.1113	0.1111	0.1067	0.1080	0.1108	0.0926	0.1099
	2.0	0.1721	0.1707	0.1614	0.1604	0.1682	0.1339	0.1635
	4.0	0.2888	0.2803	0.2609	0.2515	0.2680	0.2097	0.2498
	8.0	0.4900	0.4553	0.4188	0.3943	0.4150	0.3353	0.3666
	16.0	0.7550	0.6702	0.6198	0.5841	0.5860	0.5118	0.4964
10	0.5	0.0960	0.0962	0.0929	0.0948	0.0964	0.0822	0.0965
	1.0	0.1433	0.1435	0.1366	0.1375	0.1432	0.1149	0.1416
	2.0	0.2390	0.2363	0.2211	0.2155	0.2305	0.1781	0.2198
	4.0	0.4211	0.3990	0.3667	0.3455	0.3702	0.2902	0.3326
	8.0	0.6985	0.6207	0.5688	0.5319	0.5448	0.4599	0.4642
	16.0	0.9367	0.8238	0.7751	0.7417	0.7100	0.6648	0.5913
15	0.5	0.1006	0.1009	0.0973	0.0992	0.1011	0.0855	0.1012
	1.0	0.1529	0.1531	0.1455	0.1459	0.1526	0.1215	0.1506
	2.0	0.2587	0.2550	0.2379	0.2306	0.2475	0.1907	0.2343
	4.0	0.4571	0.4294	0.3937	0.3696	0.3949	0.3115	0.3516
	8.0	0.7438	0.6555	0.6017	0.5634	0.5716	0.4894	0.4842
	16.0	0.9582	0.8485	0.8028	0.7719	0.7320	0.6959	0.6091
20	0.5	0.1028	0.1031	0.0994	0.1013	0.1034	0.0870	0.1034
	1.0	0.1575	0.1577	0.1496	0.1499	0.1571	0.1246	0.1547
	2.0	0.2679	0.2636	0.2457	0.2375	0.2553	0.1966	0.2408
	4.0	0.4734	0.4429	0.4058	0.3805	0.4057	0.3211	0.3599
	8.0	0.7627	0.6701	0.6157	0.5770	0.5830	0.5023	0.4927
	16.0	0.9655	0.8582	0.8141	0.7843	0.7411	0.7090	0.6166
25	0.5	0.1041	0.1044	0.1006	0.1025	0.1047	0.0879	0.1047
	1.0	0.1601	0.1603	0.1521	0.1521	0.1596	0.1263	0.1571
	2.0	0.2732	0.2686	0.2501	0.2415	0.2597	0.1999	0.2445
	4.0	0.4826	0.4505	0.4126	0.3866	0.4118	0.3266	0.3645
	8.0	0.7730	0.6781	0.6235	0.5846	0.5892	0.5095	0.4974
	16.0	0.9691	0.8633	0.8201	0.7911	0.7460	0.7161	0.6206
30	0.5	0.1049	0.1053	0.1013	0.1033	0.1055	0.0885	0.1056
	1.0	0.1618	0.1620	0.1536	0.1536	0.1613	0.1275	0.1586
	2.0	0.2767	0.2718	0.2530	0.2441	0.2626	0.2021	0.2469
	4.0	0.4886	0.4554	0.4169	0.3905	0.4157	0.3301	0.3675
	8.0	0.7794	0.6832	0.6284	0.5895	0.5932	0.5141	0.5003
	16.0	0.9712	0.8665	0.8239	0.7953	0.7490	0.7206	0.6232

Table 3: Powers of T for $p = 2$, $\alpha = 0.01$

n	δ^2	(ψ, λ, ν)						
		Normal	$(5, 0, -2.5)$	$(2, 1, 0)$	$(0, 2, 2)$	$(2, 0, -1)$	$(1, 1, 1)$	$(1, 0, -0.5)$
5	0.5	0.0172	0.0173	0.0168	0.0175	0.0174	0.0151	0.0175
	1.0	0.0248	0.0250	0.0240	0.0253	0.0253	0.0205	0.0258
	2.0	0.0408	0.0415	0.0394	0.0416	0.0424	0.0319	0.0433
	4.0	0.0757	0.0773	0.0725	0.0747	0.0787	0.0562	0.0795
	8.0	0.1519	0.1526	0.1411	0.1388	0.1514	0.1066	0.1467
	16.0	0.3077	0.2950	0.2684	0.2527	0.2771	0.2022	0.2516
10	0.5	0.0236	0.0240	0.0231	0.0250	0.0246	0.0198	0.0255
	1.0	0.0393	0.0408	0.0389	0.0424	0.0427	0.0314	0.0451
	2.0	0.0764	0.0806	0.0761	0.0798	0.0850	0.0582	0.0887
	4.0	0.1672	0.1736	0.1601	0.1561	0.1759	0.1180	0.1719
	8.0	0.3758	0.3584	0.3213	0.2960	0.3320	0.2363	0.2958
	16.0	0.7204	0.6164	0.5534	0.5081	0.5274	0.4269	0.4393
15	0.5	0.0261	0.0267	0.0257	0.0280	0.0275	0.0217	0.0287
	1.0	0.0452	0.0473	0.0450	0.0490	0.0498	0.0358	0.0529
	2.0	0.0913	0.0968	0.0910	0.0940	0.1020	0.0687	0.1056
	4.0	0.2053	0.2110	0.1930	0.1848	0.2102	0.1412	0.2010
	8.0	0.4583	0.4238	0.3779	0.3453	0.3828	0.2792	0.3337
	16.0	0.8167	0.6867	0.6209	0.5740	0.5803	0.4881	0.4780
20	0.5	0.0274	0.0281	0.0270	0.0295	0.0290	0.0226	0.0304
	1.0	0.0483	0.0506	0.0482	0.0523	0.0535	0.0381	0.0568
	2.0	0.0990	0.1051	0.0986	0.1011	0.1105	0.0741	0.1139
	4.0	0.2248	0.2294	0.2091	0.1986	0.2264	0.1527	0.2143
	8.0	0.4972	0.4532	0.4037	0.3681	0.4051	0.2993	0.3500
	16.0	0.8527	0.7148	0.6489	0.6023	0.6019	0.5149	0.4940
25	0.5	0.0282	0.0289	0.0278	0.0304	0.0299	0.0232	0.0314
	1.0	0.0502	0.0526	0.0501	0.0543	0.0557	0.0395	0.0592
	2.0	0.1037	0.1101	0.1032	0.1054	0.1156	0.0773	0.1187
	4.0	0.2364	0.2403	0.2185	0.2067	0.2359	0.1594	0.2219
	8.0	0.5195	0.4698	0.4183	0.3811	0.4176	0.3108	0.3591
	16.0	0.8709	0.7297	0.6642	0.6180	0.6136	0.5299	0.5027
30	0.5	0.0287	0.0295	0.0283	0.0310	0.0305	0.0236	0.0321
	1.0	0.0514	0.0540	0.0514	0.0556	0.0572	0.0404	0.0608
	2.0	0.1069	0.1134	0.1062	0.1082	0.1190	0.0795	0.1219
	4.0	0.2442	0.2474	0.2247	0.2120	0.2420	0.1639	0.2268
	8.0	0.5339	0.4804	0.4276	0.3895	0.4255	0.3183	0.3649
	16.0	0.8817	0.7389	0.6737	0.6278	0.6209	0.5394	0.5082