Estimation of Reliability in Field Performance Studies

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ESTIMATION OF RELIABILITY IN FIELD PERFORMANCE STUDIES*

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ABSTRACT

Although manufactured products are typically subjected to an extensive reliability assessment during their development and sometimes during their manufacture, comprehensive analysis of product performance in the service of customers (i.e. in the 'field') is less common for various reasons: Scientific sampling of items in field use tends to be difficult and costly; warranty claims and other failure record data are usually in a form inconvenient for statistical analysis; and there has been a lack of interest on the part of many manufacturers in assessing quantitatively the performance of products in the field, except when major problems arise. Nonetheless, field performance data have the potential to be valuable in the systematic improvement of products, in the assessment and refinement of reliability predictions, in the comparison of products, in the design of warranty programs, and in planning the supply of replacement parts. The broad objectives of this paper are to draw attention to this area, and to discuss some of the statistical aspects. More specifically, we suggest procedures for the collection of field performance or reliability data, and propose some methods of analysis.

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CANADA

1. INTRODUCTION AND SUMMARY

Although manufactured products are typically subjected to an extensive reliability assessment during their development and sometimes during their manufacture, comprehensive analysis of product performance in the service of customers (i.e. in the 'field') is less common for various reasons: scientific sampling of items in field use tends to be difficult and costly; warranty claims and other failure record data are usually in a form inconvenient for statistical analysis; and there has been a lack of interest on the part of many manufacturers in assessing quantitatively the performance of products in the field, except when major problems arise. Nonetheless, field performance data have the potential to be valuable in the systematic improvement of products, in the assessment and refinement of reliability predictions, in the comparison of products, in the design of warranty programs, and in planning the supply of replacement parts. The broad objectives of this paper are to draw attention to this area, and to discuss some of the statistical aspects.

More specifically, we suggest procedures for the collection of field performance or reliability data, and propose some methods of analysis. Relatively little has been published about the collection of field reliability data, although there are a few notable exceptions (e.g. Amster et. al. 1982, who use the term "field tracking studies"). Similarly, there has not been much study of special statistical problems which can arise in such studies, although again there are exceptions (e.g. Suzuki 1985a, 1985b; Hahn and Meeker 1982).

Three aspects of data which can be collected in field performance studies are

A. Information on types and frequencies of 'problems' (e.g. failures, replacements,...) and on time patterns of problems (e.g. times to failure, performance degradation over time, life of the product,...).

B. Manufacturing characteristics of items in use (e.g. model, place or time of manufacture,...).

C. Environmental characteristics (e.g. personal characteristics of users, climatic conditions,...).

In this paper we assume that the objective is to examine data of Type A in relation to factors of Types B or C; in this regard we will think of a (Type A) response variable, with regressor variables of Types B and C. More specifically, we will consider situations where the response variable of interest is the time to some event, which for simplicity we will refer to as a failure. In many applications, the response is of a different type; categorical responses are, for example, common, or the response may involve several failures or failure types. We remark that the methods and many of the points discussed below apply quite generally, and not just for the specific models discussed here.

The framework we consider is as follows: suppose that the random variable T represents time to failure of a particular type for an item under study and that $\mathbf{x}$ is a vector of regressor variables which may affect T. We wish to learn about the distribution of T, given $\mathbf{x}$. This is a familiar problem, about which much has been written (e.g. Kalbfleisch and Prentice 1980, Lawless 1982, Nelson 1982). The novelty in what we discuss below arises from the fact that simple random samples of individual items in field use are often not available. Instead, what we have is either failure record data alone, or a combination of this and other selectively obtained data. In this article, we study this problem, propose statistical methods, and study related design issues.

Specifically we consider situations in which the time of failure and the corresponding regressor variables are observed only for items that fail in some specified follow-up or warranty period $(0,T^0]$. It is noted that, for satisfactory inference about baseline failure rates or regression effects, it is usually necessary to supplement these "failure record data" with prior information about the regressor variables in the whole population, or by taking a supplementary sample of items that survive to T^0. The general methods we propose can be used to combine these two types of information to make inferences about the effects of the regressor variables on reliability. It is shown that, in the context of exponential or Weibull failure time models, the methods proposed are simple to implement and are highly efficient. Several extensions are also considered to allow more complicated sampling plans and warranties that depend on both calendar time and operating time.

Section 2 deals with the estimation of failure time distributions and regression effects from failure record data. Section 3 discusses the use of additional supplementary data and methods of estimation. Section 4 provides an example and some checks on the adequacy of asymptotic approximations used to get confidence intervals. Section 5 deals with efficiency and design considerations, related to the amount of information in supplementary data. Section 6 discusses methods for some other follow-up scenarios. Section 7 summarizes our conclusions and indicates some areas for further research.

KEYWORDS: Warranty data, reliability follow-up, pseudo likelihood, likelihood methods, Weibull data, field performance data, regression analysis.

2. ESTIMATION FROM FAILURE RECORD DATA

Suppose that N items are in field use and that associated with the ith item is a time to failure t_i and a vector of regressor variables $\mathbf{x}_i$. We suppose further that $(t_i;\mathbf{x}_i)$, $i = 1, \ldots, N$ arise as a random sample from a distribution with joint probability density function (pdf)

$$f(t|\mathbf{x};\boldsymbol{\theta})g(\mathbf{x}),$$

where the conditional pdf of T given $\mathbf{x}$, $f(t|\mathbf{x};\boldsymbol{\theta})$, is completely specified up to a vector of parameters $\boldsymbol{\theta}$ to be estimated and $g(\mathbf{x})$ is the pdf of $\mathbf{X}$. It is convenient to let $F(t|\mathbf{x};\boldsymbol{\theta}) = P\{T \leq t|\mathbf{x};\boldsymbol{\theta}\}$ be the cumulative distribution function (cdf) of T given $\mathbf{x}$ and $\bar{F}(t|\mathbf{x};\boldsymbol{\theta}) = 1-F(t|\mathbf{x};\boldsymbol{\theta})$ be the survivor function (sf). Our main interest is in estimating $\boldsymbol{\theta}$, and thus the conditional distribution of failure time, given $\mathbf{x}$. In doing this we prefer, as is common in regression modelling, to avoid making assumptions about the distribution of $\mathbf{x}$, i.e. about $g(\mathbf{x})$.

Failure record data arise when the ith item is sampled if and only if $T_i \leq T^\circ$ for some prespecified T°. For these items, the time of failure t_i and the corresponding $\mathbf{x}_i$ are observed; for all other items, we know only that $T_i > T^\circ$. In particular, the $\mathbf{x}_i$'s for these items are not observed. Data of this type arise, for example, if a warranty period $(0,T^\circ]$ is in effect with all failures under warranty being reported. Two remarks seem important. First, it is assumed that T represents a relevant time variable in terms of the item's use, and that T° is expressed in the same units. For some manufactured items, there may be a warranty period which depends on more than one time scale (e.g. mileage and calendar time, in the case of automobiles). We discuss this further in Section 6. Second, it is assumed that all failures in $(0,T^\circ]$ are reported. Methods for relaxing this assumption can be developed, but are beyond the scope of the present paper; see Section 7 for a comment.

If only the failure record data up to time T° are available, inferences about $\boldsymbol{\theta}$ can be based on the 'truncated' conditional likelihood function

$$L_T(\boldsymbol{\theta}) = \prod_{i:t_i \leq T^\circ} \frac{f(t_i|\mathbf{x}_i;\boldsymbol{\theta})}{F(T^\circ|\mathbf{x}_i;\boldsymbol{\theta})} \tag{2.1}$$

which arises from the conditional distribution of the failure time T_i given $T_i \leq T^\circ$. As exemplified below, however, (2.1) can be quite uninformative about θ unless a high proportion of items fail by time $T°$. Note that (2.1) does not depend on N and would seem suitable for inference when N is unknown. When, as we assume here, N is known, however, it does not use the information that items not included in (2.1) did not fail by time $T°$. If the values of $\mathbf{x}_i$ were known for all N items in the population, then we could use the familiar censored data likelihood

$$L_F(\theta) = \prod_{i:t_i \le T°} f(t_i|\mathbf{x}_i;\theta) \prod_{i:t_i > T°} \bar{F}(T°|\mathbf{x}_i;\theta). \tag{2.2}$$

Very occasionally such information may be available: for example, if the regressor variables are categorical and refer only to simple manufacturing characteristics of an item, then the manufacturer may know how many items in the population have each possible combination of values. In general, however, we suppose that the $\mathbf{x}_i$'s are known only for individuals chosen for observation; this is what motivates subsequent developments in the paper.

Although the $\mathbf{x}$'s for items that do not fail in $(0,T°)$ are typically not known, occasionally the pdf $g(\mathbf{x})$ is known, or can be specified up to a few unknown parameters. The marginal probability of surviving past $T°$ is then

$$Pr\{T_i > T°\} = \int \bar{F}(T°|\mathbf{x};\theta) g(\mathbf{x}) d\mathbf{x} \tag{2.3}$$

and inference can be based on the likelihood arising from the full data $\{(t_i,\mathbf{x}_i)$ for $i:t_i \le T°$, and $t_i > T°$ for all other items$\}$: this gives

$$L_D(\theta) = \prod_{i:t_i \le T°} \{f(t_i|\mathbf{x}_i;\theta) g(\mathbf{x}_i)\} \prod_{i:t_i > T°} Pr(T_i > T°). \tag{2.4}$$

In most applications, and specifically those we consider here, the distribution of $\mathbf{x}$ is unknown, and cannot be represented adequately by a parsimonious parametric model. An alternative approach to that taken in this paper and which also would avoid modelling $g(\mathbf{x})$ would be to attempt non-parametric estimation of $g(\mathbf{x})$ jointly with θ, via the likelihood (2.4). We have not explored this approach, but for general application it appears that it would be computationally very complex, and there would be difficulties in obtaining interval estimates; this is the case for a simpler situation involving only categorical response and regressor variables discussed by Cosslett (1981).

Some detailed comparisons of (2.1), (2.2), (2.4) and alternative likelihoods are made in Section 5. To provide an example of the lack of information in failure record data, and additional motivation for the approach taken in the next section, however, (2.1) and (2.2) are compared here when failure times are exponentially distributed and no covariates are present. In this case, (2.2) and (2.4) are identical and with $f(t;\theta) = \theta e^{-\theta t}$ $(t>0)$, (2.1) and (2.2) respectively give

$$L_T(\theta) = \prod_{i:t_i \le T^\circ} \{\theta e^{-\theta t_i} / (1 - e^{-\theta T^\circ})\} \tag{2.5}$$

$$L_F(\theta) = \prod_{i:t_i \le T^\circ} \theta e^{-\theta t_i} \cdot \prod_{i:t_i > T^\circ} e^{-\theta T^\circ}. \tag{2.6}$$

Since there are no covariates, the censored data likelihood $L_F(\theta)$ is available for inference and there is no need to use the truncated data likelihood $L_T(\theta)$, but our purpose is to show that very little of the information about θ is contained in $L_T(\theta)$. Table 1 gives the asymptotic relative efficiency of $L_T(\theta)$ versus $L_F(\theta)$, defined as the ratio of the asymptotic variance of the maximum likelihood estimator (mle) $\hat{\theta}_F$ from $L_F(\theta)$ to that of the mle $\hat{\theta}_T$ from $L_T(\theta)$. (Appendix A gives relevant formulas.) The relative efficiency depends upon $F(T^\circ;\theta)$, the expected proportion of items failing by time T°; unless this proportion is high, the relative efficiency of $L_T(\theta)$ is very low.

Table 1. Asymptotic Relative Efficiency of $L_T(\theta)$ to $L_F(\theta)$ in the Exponential Distribution

$F(T^\circ;\theta) = 1 - e^{-\theta T^\circ}$	*Relative Efficiency*[a]
.01	.00001
.10	.0009
.20	.0041
.50	.0391
.90	.3454

[a] *Relative efficiency* = *as* $var\{\sqrt{N}(\hat{\theta}_F - \theta)\} \div$ *as* $var\{\sqrt{N}(\hat{\theta}_T - \theta)\}$

These calculations illustrate that although it is possible to estimate failure time distributions from failure record data only, much more precise estimation is possible if information on unfailed items can be incorporated. Since the information needed to employ (2.2) is not available, we consider supplementing the failure record data with a sample of the items which did not fail.

3. SUPPLEMENTING FAILURE RECORD DATA

The utility of failure record data can be increased greatly by collecting a supplementary sample on items which do not fail. This general approach is widely used in retrospective or case-control studies (e.g. Breslow and Day, 1980). For the present, we consider the following scheme: the failure record data are supplemented by selecting a sample of those items that do not fail by time T° and, for each sampled item, determining the corresponding $\mathbf{x}$. This could be implemented under various sampling schemes. The one most often used, and discussed here, is selection of a simple random sample without replacement of $n_2 = p_2 N_2$ items from the N_2 items surviving at T°, where p_2 is prespecified and typically small. We denote by D_1 the set of items failing by T°, and by D_2 the supplementary sample. For items in D_2, we observe $\mathbf{x}_i$ and know that $t_i > T^\circ$. It should be noted that this sampling scheme is **response-selective**; if $T_i \leq T^\circ$ an item is sampled with probability one, and if $T_i > T^\circ$ it is sampled with probability p_2. For examples of response selective sampling in different contexts see Hausman and Wise (1983), Holt et al. (1980), Jewell (1985), Scott and Wild (1986) and Kalbfleisch and Lawless (1988). Some authors refer to the sampling scheme described above as "standard stratified sampling".

In this section, we construct a pseudo likelihood that may be used for inference with the sampling procedure just described. First, however, we mention another supplementary sampling plan and "ordinary" likelihood estimation for it and other plans.

3.1 Exact Likelihoods

As above, let D_1 represent the set of items that fail and define $R_{1i} = I(i \epsilon D_1)$, where $I(A)$ is the indicator for event A. Suppose that a supplementary sample of those who do not fail is chosen according to the following "Bernoulli" scheme: each item in $\overline{D_1}$ is chosen for inclusion in the supplementary sample with probability p_2 independently. Let D_2 be the set of items in the supplementary sample and $R_{2i} = I(i \epsilon D_2)$. Finally, let $R_i = R_{1i} + R_{2i}$ be the indicator of the event, "the ith item is sampled." For the ith item, we could then consider the distribution of the observed data given R_{1i} and $\mathbf{x}_i$ to obtain the exact conditional likelihood

$$L_C = \prod_{i=1}^{N} \frac{f(t_i|\mathbf{x}_i;\boldsymbol{\theta})^{R_{1i}}[p_2\bar{F}(T^\circ|\mathbf{x}_i;\boldsymbol{\theta})]^{R_{2i}}}{[F(T^\circ|\mathbf{x}_i;\boldsymbol{\theta}) + p_2\bar{F}(T^\circ|\mathbf{x}_i;\boldsymbol{\theta})]^{R_i}} \tag{3.1}$$

where the ith term is 1 if $R_i = 0$ and otherwise gives the conditional probability density of $R_{1i} = 1$ and t_i, or $R_{2i} = 1$ and $t_i > T^\circ$, given $R_i = 1$ and $\mathbf{x}_i$. The contributions to (3.1) from different individuals are independent, and standard maximum likelihood procedures can be applied to the estimation of $\boldsymbol{\theta}$.

Unfortunately, Bernoulli sampling is not often used. For other sampling plans, and in particular the simple random sample of size $n_2 = p_2N_2$ referred to above, it is still possible in principle to obtain an exact likelihood from the conditional distribution of the data $\{(R_{1i},R_{2i},t_i^*), i = 1, \ldots, N\}$ given $R_1, \ldots, R_N$ and $\mathbf{x}_1, \ldots, \mathbf{x}_N$, where t_i^* represents t_i if $R_{1i} = 1$ and is otherwise null. However, with this "standard stratified sampling" the data for different individuals are not independent, and the likelihood is so complicated as to be intractable. In addition, it is not clear how to generalize it to handle varying "entry" times as discussed in Section 6.1, or other more complicated observational plans. Consequently, we introduce in Section 3.2 a pseudo likelihood which is easy to use for estimation of $\boldsymbol{\theta}$ with standard stratified sampling and other types of supplementary observational plans.

3.2 A Pseudo Likelihood

We present here a pseudo likelihood for $\boldsymbol{\theta}$, by which we mean a function of $\boldsymbol{\theta}$ which, when maximized, yields an estimator $\tilde{\boldsymbol{\theta}}$ with properties like those of an ordinary mle. For similar uses of the phrase pseudo likelihood see, for example, Suzuki (1985a) and Prentice (1986).

We propose to use for the case of standard stratified sampling the pseudo log likelihood

$$\log L_P(\boldsymbol{\theta}) = \sum_{i \in D_1} \log f(t_i|\mathbf{x}_i;\boldsymbol{\theta}) + \frac{1}{p_2}\sum_{i \in D_2} \log \bar{F}(T^\circ|\mathbf{x}_i;\boldsymbol{\theta}). \tag{3.2}$$

This can be thought of as an estimate of the logarithm of the likelihood function (2.2) that arises if the $\mathbf{x}_i$'s for all N items in the population are known; note that $p_2 = n_2/N_2$ is the probability that any individual unfailed item is included in the supplementary sample D_2. The use of this pseudo likelihood is similar in spirit to the use of weighted least squares estimators in regression analysis of data obtained from complex sample surveys (cf. Holt et al. 1980, Holt and Scott 1981). Godambe and Thompson (1986) also discuss a similar idea in the context of estimating equations.

Under mild conditions, the estimator $\tilde{\boldsymbol{\theta}}$ obtained by maximizing (3.2) is consistent as $N \rightarrow \infty$ with p_2 fixed, and $\sqrt{N}(\tilde{\boldsymbol{\theta}}-\boldsymbol{\theta})$ has a limiting normal distribution with mean vector $\mathbf{0}$ and covariance matrix $V(\boldsymbol{\theta}) = A(\boldsymbol{\theta})^{-1} + A(\boldsymbol{\theta})^{-1}C(\boldsymbol{\theta})A(\boldsymbol{\theta})^{-1}$ where

$$A(\boldsymbol{\theta})_{r,s} = \lim_{N\rightarrow\infty} \frac{1}{N} E\{-\frac{\partial^2 \log L_P}{\partial \theta_r \partial \theta_s}\}, \tag{3.3}$$

and

$$C(\boldsymbol{\theta})_{r,s} = \frac{1-p_2}{p_2} \lim_{N\rightarrow\infty} \frac{1}{N} E\{\frac{N_2}{n_2-1} \sum_{i \epsilon D_2} (m_{ir}-\bar{m}_r)(m_{is}-\bar{m}_s)\}, \tag{3.4}$$

$m_{ir} = \partial \log \bar{F}(T^\circ|\mathbf{x}_i;\boldsymbol{\theta})/\partial\theta_r$ and $\bar{m}_r = \sum_{i\epsilon D_2} m_{ir}/n_2$. $V(\boldsymbol{\theta})$ is consistently estimated by replacing $A(\boldsymbol{\theta})$ and $C(\boldsymbol{\theta})$ with $A_N(\tilde{\boldsymbol{\theta}})$ and $C_N(\tilde{\boldsymbol{\theta}})$ where

$$\begin{aligned} A_N(\boldsymbol{\theta})_{r,s} &= -\frac{1}{N}\frac{\partial^2 \log L_P}{\partial\theta_r\partial\theta_s} \\ &= -\frac{1}{N}\sum_{i\epsilon D_1}\frac{\partial^2 \log f(y_i|\mathbf{x}_i;\boldsymbol{\theta})}{\partial\theta_r\partial\theta_s} - \frac{1}{Np_2}\sum_{i\epsilon D_2}\frac{\partial \log \bar{F}(T^\circ|\mathbf{x}_i;\boldsymbol{\theta})}{\partial\theta_r\partial\theta_s} \end{aligned} \tag{3.5}$$

and

$$C_N(\boldsymbol{\theta})_{r,s} = \frac{N_2(1-p_2)}{Np_2(n_2-1)} \sum_{i\epsilon D_2} (m_{ir}-\bar{m}_r)(m_{is}-\bar{m}_s). \tag{3.6}$$

Some notes and references on these results are given in Appendix B.

The pseudo log likelihood (3.2) is equivalent to the log likelihood from a censored sample except that censored items have case weights p_2^{-1} and uncensored ones have case weights one. If software that allows case weights is available, it can be used to maximize (3.2) and will also give (3.5); (3.6) would need to be computed separately, but this is easy since (3.6) is essentially the sample covariance matrix of $\mathbf{m}_i = \partial \log \bar{F}(T^\circ|\mathbf{x}_i,\boldsymbol{\theta})/\partial \boldsymbol{\theta}$, $i \epsilon D_2$. Our experience indicates that estimates from (3.2) often have high efficiency when compared with $L_F(\boldsymbol{\theta})$ and $L_D(\boldsymbol{\theta})$ in (2.2) and (2.4). Examples are provided in Sections 4 and 5.

We remark that the pseudo likelihood (3.2) can also be used with Bernoulli supplementary sampling as described in Section 3.1. It is not really needed then, since (3.1) is available, but in situations where (3.2) is computationally much easier to handle than (3.1) it might be worth investigating its efficiency. For Bernoulli sampling one needs to replace $C(\theta)$ and $C_N(\theta)$ in (3.4) and (3.6) with (see Kalbfleisch and Lawless 1988)

$$C^*(\boldsymbol{\theta})_{r,s} = \frac{1-p_2}{p_2} E\{\sum_{i \epsilon \bar{D}_1} m_{ir} m_{is}\}$$

$$C^*_N(\boldsymbol{\theta})_{r,s} = \frac{1-p_2}{p_2^2} \sum_{i \epsilon D_2} m_{ir} m_{is}.$$

3.3 Formulas for Weibull and Exponential Models

The Weibull proportional hazards model is perhaps the most widely used parametric lifetime regression model; it has pdf and survivor function

$$f(t|\mathbf{x}_i) = \delta t^{\delta-1} e^{\mathbf{x}_i'\boldsymbol{\beta}} \exp(-t^\delta e^{\mathbf{x}_i'\boldsymbol{\beta}}), \quad \bar{F}(t|\mathbf{x}_i) = \exp(-t^\delta e^{\mathbf{x}_i'\boldsymbol{\beta}}), \tag{3.7}$$

respectively, where $\delta > 0$ is a shape parameter, $\mathbf{x}_i = (1, x_{i1}, \ldots, x_{ik})$ is the vector of regressor variables, and $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_k)'$ the vector of regression coefficients. The proportional hazards exponential regression model has the same form but with $\delta = 1$. This model is discussed at length, for example, in Kalbfleisch and Prentice (1980, pp. 55-62) and Lawless (1982, pp. 298-313). We give here the expressions needed to employ the pseudo likelihood of Section 3.2. These will be used in an example in Section 4 and in efficiency calculations in Section 5.

To make expressions more compact, we write $S_i = \bar{F}(T^\circ|\mathbf{x}_i)$, $H_i = \log S_i$ and $W_{ij} = (\log t_i)^j t_i^\delta e^{\mathbf{x}_i'\boldsymbol{\beta}}$, $j = 0,1,2$.

For the pseudo likelihood (3.2) the pseudo score vector has components

$$\frac{\partial \log L_P}{\partial \beta_r} = \sum_{i \epsilon D_1} x_{ir}(1-W_{i0}) + \frac{1}{p_2}\sum_{i \epsilon D_2} x_{ir} H_i$$

$$\frac{\partial \log L_P}{\partial \delta} = \sum_{i \epsilon D_1} (\delta^{-1} + \log t_i - W_{i1}) + \frac{1}{p_2}\sum_{i \epsilon D_2} (\log T^\circ) H_i$$

and, corresponding to (3.5), we find

$$NA_N(\beta,\delta)_{r,s} = \frac{-\partial^2 \log L_P}{\partial \beta_r \partial \beta_s} = \sum_{i \epsilon D_1} x_{ir} x_{is} W_{i0} - \frac{1}{p_2}\sum_{i \epsilon D_2} x_{ir} x_{is} H_i$$

$$NA_N(\beta,\delta)_{r,k+1} = \frac{-\partial^2 \log L_P}{\partial \beta_r \partial \delta} = \sum_{i \epsilon D_1} x_{ir} W_{i1} - \frac{1}{p_2}\sum_{i \epsilon D_2} x_{ir} (\log T^\circ) H_i$$

$$NA_N(\beta,\delta)_{k+1,k+1} = \frac{-\partial^2 \log L_P}{\partial \delta^2} = \sum_{i \epsilon D_1} (\delta^{-2} + W_{i2}) - \frac{1}{p_2}\sum_{i \epsilon D_2} (\log T^\circ)^2 H_i .$$

To obtain asymptotic variance estimates with L_P, we need also the matrix $C_N(\beta,\delta)$, with entries given by (3.6):

$$NC_N(\beta,\delta)_{r,s} = \frac{N_2(1-p_2)}{(n_2-1)p_2}\sum_{i \epsilon D_2} (x_{ir}H_i - \overline{H}_r)(x_{is}H_i - \overline{H}_s)$$

$$NC_N(\beta,\delta)_{r,k+1} = \frac{N_2(1-p_2)}{(n_2-1)p_2}\sum_{i \epsilon D_2} \log T^\circ (x_{ir}H_i - \overline{H}_r)(H_i - \overline{H})$$

$$NC_N(\beta,\delta)_{k+1,k+1} = \frac{N_2(1-p_2)}{(n_2-1)p_2}\sum_{i \epsilon D_2} (\log T^\circ)^2 (H_i - \overline{H})^2$$

where $\overline{H} = \sum_{i \epsilon D_2} H_i / n_2$ and $\overline{H}_r = \sum_{i \epsilon D_2} x_{ir} H_i / n_2$.

4. AN EXAMPLE

Hahn and Meeker (1982) present field data on the reliability of a population of $N = 5370$ electro-mechanical devices. In their example, inspections to determine failure were held at various times up to 38 months, by which time 270 items had failed. Hahn and Meeker fitted Weibull and lognormal distributions (no covariates) and discussed dangers of extrapolation beyond 38 months.

For purposes of illustration, we generated regression data based on this example with a single binary regressor variable, x. Such a covariate might, for example, indicate two different environments in which the devices are used. Half the x values were 0 and half 1, and failure times t_i were supposed to have arisen from a Weibull model with survivor function

$$\bar{F}(t\,|x) = \exp(-t^{\delta} e^{\beta_0+\beta_1 x}), \quad t>0.$$

The follow-up interval was taken to be $(0,T^{\circ}] = (0,38]$ and t_i's were generated so that there were 270 failures by 38 months, 205 with $x = 1$ and 65 with $x = 0$. More specifically, we chose $\beta_0 = -23.7$, $\beta_1 = 1.16$ and $\delta = 5.5$ to obtain $\bar{F}(38|0) = .0246 \doteq 65/2685$ and $\bar{F}(38|1) = .0766 \doteq 205/2685$. The δ value was chosen to be close to that estimated by Hahn and Meeker. The t_i values were obtained as the expected values of the first 65 and 205 order statistics from the respective Weibull distributions for $x = 0$ and $x = 1$, giving $\{\sum_{l=1}^{j}(2686-l)^{-1}\}^{1/\delta} e^{-\beta_0/\delta}$, $j = 1, \ldots, 65$ for $x = 0$ and $\{\sum_{l=1}^{j}(2686-l)^{-1}\}^{1/\delta} e^{-(\beta_0+\beta_1)/\delta}$, $j = 1, \ldots, 205$ for $x = 1$. The truncated likelihood L_T, based only on this failure record data, is very uninformative with regard to β_0 and β_1 and it leads to no useful estimation of these parameters.

In order to illustrate the utility of supplementary sampling along with the pseudo likelihood, we considered two cases. In the first a 5% sample was taken with the result that, of the 255 items selected from those surviving 38 months, 131 items had $x = 0$ and 124 had $x = 1$. In the second a 10% sample was taken with the result that, of 510 items selected, 262 were observed to have $x = 0$ and 248 to have $x = 1$. Table 2 summarizes the results of estimation based on L_P. The estimates are compared with those arising from L_F where complete knowledge of the covariate values for the unfailed items is assumed and with those based on L_D where it is assumed known that the covariate values were generated independently from the distribution $P(X=0) = P(X=1) = .5$ and no supplementary sample is available. Standard deviations in Table 2 are computed from the appropriate formulas in section 3.3 or Appendix C.

To examine the effect of length of follow-up, Table 2 reports also the results when $T^\circ = 28$ months. In this case, 51 items failed, of which 12 had $x = 0$ and 39 had $x = 1$. Supplementary samples had 266 items (134 with $x = 0$ and 132 with $x = 1$) for $p_2 = .05$ and 532 items (267 with $x = 0$ and 265 with $x = 1$) for $p_2 = .10$.

All methods give close to the same estimates and SD's. Increasing the sampling rate from $p_2 = .05$ to $p_2 = .10$ has a relatively small effect on the standard deviations; the standard deviations for L_P compare favourably with those for L_D. This rather modest degree of supplementary sampling is sufficient to overcome lack of information about the distribution from which covariate values in the population arise. We note also that SD's under L_D are only marginally greater than those under L_F.

An area which requires further study is the adequacy of large sample confidence interval estimation methods used with pseudo likelihood estimates. An extensive investigation is beyond the scope of this article, but we have run a few simulations to assess how close to standard normal the three approximate pivotals $Z_0 = (\hat{\beta}_0 - \beta_0)/\hat{s}_0$, $Z_1 = (\hat{\beta}_1 - \beta_1)/\hat{s}_1$, $Z_2 = (\hat{\delta} - \delta)/\hat{s}_2$ used in this example are. Here $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\delta}$ are the estimates obtained from L_P and $\hat{s}_0$, $\hat{s}_1$, and $\hat{s}_2$ are the corresponding estimated asymptotic standard deviations obtained from (3.5) and (3.6). We used the parameter values $\beta_0 = -23.7$, $\beta_1 = 1.16$, $\delta = 5.5$ and the supplementary sampling procedures described above to generate 500 samples for each of the four combinations of $T = 28,38$ and $p = .05,.10$. For these situations, the distributions of Z_0, Z_1 and Z_2 appeared close enough to standard normal so as to give reliable confidence intervals. The distribution of Z_2 appeared to depart to some extent from standard normal in the extreme left tail; a parameterization with δ^{-1} or $\log \delta$ instead of δ may be better. Table 3 shows the proportion of samples for which each of Z_0, Z_1, and Z_2 were less than selected $N(0,1)$ quantiles, for two of the cases considered; the other two cases gave very similar results.

Table 2. Estimates and Estimated Standard Deviations (SD) of Weibull Model Parameters

Follow-up to $T^\circ = 38$ *months* ($N = 5370$; 270 *failures)*

Method	$\hat{\beta}_0$	SD	$\hat{\beta}_1$	SD	$\hat{\delta}$	SD
$L_P(p_2 = .05)$	-24.13	1.241	1.176	.186	5.615	.339
$L_P(p_2 = .10)$	-24.13	1.240	1.176	.163	5.615	.339
L_D	-24.07	1.240	1.179	.144	5.625	.339
L_F	-24.13	1.239	1.176	.142	5.615	.339

Follow-up to $T^\circ = 28$ *months* ($N = 5370$; 51 *failures)*

Method	$\hat{\beta}_0$	SD	$\hat{\beta}_1$	SD	$\hat{\delta}$	SD
$L_P(p_2 = .05)$	-25.31	2.797	1.189	.350	5.971	.835
$L_P(p_2 = .10)$	-25.30	2.797	1.181	.339	5.971	.835
L_D	-25.21	2.797	1.185	.331	5.973	.835
L_F	-25.31	2.797	1.189	.330	5.971	.835

Table 3. Proportions of Values of Z_0, Z_1 *and* Z_2 *in* 500 *Samples that Fall Below Selected Standard Normal Quantiles*

Standard Normal Quantile

		$Z_{.005}$	$Z_{.025}$	$Z_{.05}$	$Z_{.95}$	$Z_{.975}$	$Z_{.995}$
$T = 28$	Z_0	.004	.020	.036	.946	.962	.986
$p = .05$	Z_1	.002	.028	.048	.960	.986	.996
	Z_2	.014	.042	.056	.962	.982	.996
$T = 38$	Z_0	.008	.020	.030	.962	.978	.986
$p = .05$	Z_1	.002	.026	.060	.958	.982	.996
	Z_2	.014	.028	.038	.966	.980	.994

5. SOME EFFICIENCY AND DESIGN CALCULATIONS

As the example in section 4 and earlier discussion indicate, data on failure records only are often uninformative about baseline failure rates and covariate effects. On account of this, we have identified two ways to supplement these data: first, by inserting knowledge about the distribution of covariates across the population and second, by taking a supplementary sample of those items which have not experienced failures. In this section, we evaluate the relative information in these approaches. Although only two specific examples are considered, the qualitative aspects of the results are likely rather general.

5.1 An example involving exponential failure times

We consider items with exponentially distributed time to failure and a single binary covariate x which equals 0 or 1. The p.d.f. of the time to failure is

$$f(t_i|\mathbf{x}_i) = \lambda_i e^{-\lambda_i t_i} \qquad (t_i > 0)$$

where $\lambda_i = e^{\beta_0 + \beta_1 x_i}$. For illustration we assume that half of the N items in field use have $x_i = 0$ and half have $x_i = 1$. This may or may not be known to the statistician. We will consider situations where e^{β_1}, which equals $E(T|x{=}0)/E(T|x{=}1)$, is 1 and 2, respectively, and suppose that the initial follow-up period $(0, T^\circ]$ is such that the expected proportion of items failing by T° is .10. (This implies that $.5\exp\{-T^\circ \exp(\beta_0)\} + .5\exp\{-T^\circ \exp(\beta_0+\beta_1)\} = .1$.) Without loss of generality we take $T^\circ = 1$.

We consider various possibilities; in each of the following, the failure record data (the data t_i, $\mathbf{x}_i$ for items failing in $(0, T^\circ]$) are available, but may be supplemented with additional data or knowledge. In each case, the purpose is to estimate β_0 and β_1.

I. Only the failure record data are available; β_0 and β_1 are estimated from (2.1).

II. The x_i's are known to be generated independently with $P(X_i = 0) = P(X_i = 1) = .50$; β_0 and β_1 are estimated from (2.4).

III. A supplementary simple random sample of units with $T_i > T^\circ$ is drawn, the probability of selection for each item being $p_2 = .11$ (In this case, the total expected fraction of items sampled is $.1 + .9(.11) \doteq .20$); β_0 and β_1 are estimated from the pseudo likelihood (3.2).

In addition, we also consider

IV. A randomly selected sample of $.2N$ items is followed from 0 to T° and estimation of β_0, β_1 is based on the likelihood

$$L(\beta_0,\beta_1) = \prod_{i \epsilon S} \{\lambda_i e^{-\lambda_i t_i}\}^{\delta_i} \{e^{-\lambda_i T^{\circ}}\}^{1-\delta_i}$$

where S is the set of items sampled and $\delta_i = 1$ if the ith item fails and 0 otherwise.

To compare the possibilities we compute asymptotic covariance matrices for $\sqrt{N}(\hat{\beta}_0-\beta_0, \hat{\beta}_1-\beta_1)$ and report asymptotic standard deviations in Table 4. In the cases of I and II, this involves computing the expectations of $-\partial^2 \log L_T / \partial \beta_r \partial \beta_s$ and of $-\partial^2 \log L_D / \partial \beta_r \partial \beta_s$, respectively. For III we need to evaluate (3.3) and (3.4). For IV a very straightforward computation applies. Expressions for all of these are given in Appendix C. Table 4 reports asymptotic standard deviations for $\sqrt{N}(\hat{\beta}_0-\beta_0)$ and $\sqrt{N}(\hat{\beta}_1-\beta_1)$ for each of cases I to IV.

Method I is indeed very uninformative relative to II or III. Of course, II requires knowledge about the distribution of the x_i's and III involves the additional cost of obtaining the supplementary sample. Both however, result in greatly increased precision. Note also that III gives considerably greater precision than IV. Thus, we are better to observe the failures in $(0,T^{\circ}]$ and supplement this with a secondary sample from the non-failed items, than to follow up a random sample of items over $(0,T^{\circ}]$ of equivalent total size. This fact was noted by Prentice (1986) in the context of relative risk estimation in medical follow-up studies and is important for reliability follow-up as well.

Table 4. Asymptotic Standard Deviations for $\hat{\beta}_0$ and $\hat{\beta}_1$ Under Four Approaches and Two Models

		$\beta_0 = -2.25$ $\beta_1 = 0$		$\beta_0 = -2.65$ $\beta_1 = .693$	
Method		$SD\{\sqrt{N}(\hat{\beta}_0-\beta_0)\}$	$SD\{\sqrt{N}(\hat{\beta}_1-\beta_1)\}$	$SD\{\sqrt{N}(\hat{\beta}_0-\beta_0)\}$	$SD\{\sqrt{N}(\hat{\beta}_1-\beta_1)\}$
I	L_T	147.8	209.0	246.2	264.1
II	L_D	4.59	6.67	5.51	7.00
III	L_P	5.30	8.50	6.09	8.76
IV		10.0	14.1	12.1	14.9

5.2 An example with Weibull failure times

Tables 5 and 6 report asymptotic standard deviations similar to those in Table 4, for the case of Weibull regression models of the form (3.7). Two sets of models are represented:

A: $\mathbf{x}_i'\boldsymbol{\beta} = \beta_0+\beta_1 x_i$, with half of the items in the population having each of $x_i = 0$ and $x_i = 1$. The follow-up interval has $T^\circ = 1$, without loss of generality. Results are shown in Table 5 for $\delta = 2.5$, $\beta_1 = 0$ or $\beta_1 = \log 2 = .693$, and β_0 chosen to make $Ave(S_i) = .5\exp(-T^{\circ\delta}e^{\beta_0}) + .5\exp(-T^{\circ\delta}e^{\beta_0+\beta_1}) = .95,.90$ or $.80$. Note that $Ave(S_i)$ is the expected proportion of items surviving past T°.

B: $\mathbf{x}_i'\boldsymbol{\beta} = \beta_0+\beta_1 x_{1i}+\beta_2 x_{2i}+\beta_3 x_{3i}$, with $x_{1i} = 0$ or 1, $x_{2i} = 0$ or 1, $x_{3i} = -1$, 0 or 1, and one-twelfth of the items in the population having each of the 12 possible combinations (x_{1i}, x_{2i}, x_{3i}). The follow-up interval has $T^\circ = 1$. Results are shown in Table 6 for $\delta = 2.5$, $\beta_2 = \beta_3 = 0$, $\beta_1 = \log 2 = .693$ or $\beta_1 = \log 3 = 1.099$, and β_0 chosen to make $Ave(S_i) = .90$ or $.80$.

Asymptotic standard deviations are shown for estimators based on the following four likelihoods:

L_T. (2.1), based on the failure record data only.

L_D. (2.4), which requires knowledge of the distribution from which the covariates $\mathbf{x}_i$ in the population are generated.

L_P. (3.2), which utilizes a supplementary sample of non-failed items; values $p_2 = .01, .05, .10$ and $.20$ are considered.

L_F. (2.2), which requires that we know exactly the covariates $\mathbf{x}_i$ for all items in the population.

The asymptotic covariance matrices upon which the calculations are based are given in Appendix C.

We examined a range of values for δ, $Ave(S_i)$ and p_2, but to conserve space present results only for the combinations shown in Tables 5 and 6; qualitative features of the results persisted across other models. The tables show that, as with the exponential models, methods L_D, L_P and L_F are much more informative than L_T although the difference is much greater with regard to the regression coefficients β_i than the shape parameter δ. We observe that a modest amount of supplementary sampling ($p_2 \geq .05$) yields most of the information that would be available if, as in method L_F, exact covariates for all items were known. Even a one percent supplementary sample brings about very large gains compared to L_T, and may well be adequate if N is large. Note that values in the tables are divided by $N^{1/2}$ to obtain approximate standard deviations of the estimators.

Table 5. Asymptotic Standard Deviations for Four Methods: Weibull Models A

	Method		$Ave(S_i) = .95$ $\hat\beta_0$	$\hat\beta_1$	$\hat\delta$	$Ave(S_i) = .90$ $\hat\beta_0$	$\hat\beta_1$	$\hat\delta$	$Ave(S_i) = .80$ $\hat\beta_0$	$\hat\beta_1$	$\hat\delta$
	L_T		670.	604.	22.0	229.	208.	15.3	75.1	69.5	10.4
	L_D		6.58	9.18	11.1	4.88	6.67	7.80	3.92	5.00	5.45
	L_P	$(p_2 = .01)$	11.8	21.8	11.1	10.9	20.9	7.80	10.4	20.4	5.44
$\beta_1 = 0$		$(p_2 = .05)$	7.68	12.5	11.1	6.24	10.8	7.80	5.38	9.78	5.44
$\delta = 2.5$		$(p_2 = .10)$	7.00	10.8	11.1	5.39	8.72	7.80	4.36	7.47	5.44
		$(p_2 = .20)$	6.63	9.80	11.1	4.90	7.48	7.80	3.74	5.99	5.44
	L_F		6.32	8.94	11.1	4.47	6.32	7.80	3.16	4.47	5.44
	L_T		1094.	911.	21.9	371.	310.	15.2	120.	101.	10.3
	L_D		7.91	9.68	11.1	5.74	7.00	7.79	4.39	5.20	5.44
	L_P	$(p_2 = .01)$	12.5	22.0	11.1	11.2	21.0	7.81	10.3	20.4	5.50
$\beta_1 = .693$		$(p_2 = .05)$	8.83	12.9	11.1	6.90	11.0	7.80	5.64	9.88	5.44
$\delta = 2.5$		$(p_2 = .10)$	8.26	11.2	11.1	6.16	8.97	7.80	4.75	7.60	5.43
		$(p_2 = .20)$	7.95	10.3	11.1	5.76	7.78	7.79	4.24	6.16	5.43
	L_F		7.30	9.46	11.1	5.41	6.67	7.79	3.78	4.69	5.43

Table 6. Asymptotic Standard Deviations for Four Methods: Weibull Models B

	Method		$Ave(S_i) = .90$ $\hat\beta_0$	$\hat\beta_1$	$\hat\beta_2$	$\hat\beta_3$	$\hat\delta$	$Ave(S_i) = .80$ $\hat\beta_0$	$\hat\beta_1$	$\hat\beta_2$	$\hat\beta_3$	$\hat\delta$
	L_T		382.	310.	180.	110.	15.2	124	101.	60.0	36.7	10.3
	L_D		6.65	7.00	6.70	4.11	7.79	5.07	5.20	5.05	3.09	5.44
	L_P	$(p_2 = .01)$	15.6	21.0	21.8	13.4	7.81	14.8	20.4	21.2	13.0	5.51
$\beta_1 = .693$		$(p_2 = .05)$	8.86	11.0	11.1	6.81	7.80	7.58	9.88	10.1	6.20	5.44
$\delta = 2.5$		$(p_2 = .10)$	7.61	8.97	8.92	5.46	7.80	6.11	7.60	7.68	4.70	5.43
		$(p_2 = .20)$	6.90	7.78	7.59	4.65	7.79	5.23	6.16	6.11	3.74	5.43
	L_F		6.27	6.67	6.32	3.87	7.79	4.39	4.69	4.47	2.74	5.43
	L_T		533.	469.	156.	95.6	15.1	170.	150.	51.8	31.7	10.2
	L_D		7.31	7.51	6.75	4.14	7.78	5.46	5.51	5.13	3.14	5.42
	L_P	$(p_2 = .01)$	16.2	21.2	22.9	14.1	7.81	15.1	20.5	22.2	13.6	5.58
$\beta_1 = 1.099$		$(p_2 = .05)$	9.47	11.3	11.6	7.07	7.78	7.93	10.0	10.5	6.45	5.44
$\delta = 2.5$		$(p_2 = .10)$	8.25	9.38	9.18	5.62	7.78	6.49	7.82	7.95	4.87	5.44
		$(p_2 = .20)$	7.56	8.25	7.72	4.73	7.78	5.64	6.42	6.26	3.83	5.42
	L_F		6.96	7.21	6.32	3.87	7.78	4.84	5.03	4.47	2.74	5.41

6. OTHER OBSERVATIONAL PLANS

There are many ways in which field performance data might be collected. The discussion to this point has focussed on a very specific sampling plan, but the ideas and methods apply much more generally. We consider briefly some other schemes and generalizations of the methods given.

6.1 Variations in Follow-up Period

The pseudo likelihood extends in a natural way to incorporate independent censoring. Suppose, for example, that items are placed in service over the calendar period $(0,\tau]$ and that an assessment is made at time τ. Let D_1 be the set of items placed in service prior to time τ that fail during their warranty periods and prior to τ. For these items, the time of failure t_i and the covariates $\mathbf{x}_i$ are observed. These failure record data are supplemented by a sample of items from $\bar{D}_1$, which are items placed in service over $(0,\tau]$ but not in D_1. We suppose the i'th item in $\bar{D}_1$ is sampled with probability p_{2i}, and let D_2 be the elements obtained in the supplementary sample. For elements in D_2 we observe $\mathbf{x}_i$, δ_i (= 1 if the ith item fails in $(0,\tau]$ and 0 otherwise), and t_i, the time to failure if $\delta_i = 1$ (which must exceed the warranty time $T°$), or the time to censoring if $\delta_i = 0$. The pseudo log likelihood is then

$$\log L_P = \sum_{i\epsilon D_1} \log f(t_i|\mathbf{x}_i;\boldsymbol{\theta}) + \sum_{i\epsilon D_2} [\delta_i \log f(t_i|\mathbf{x}_i;\boldsymbol{\theta}) + (1-\delta_i)\log \bar{F}(t_i|\mathbf{x}_i;\boldsymbol{\theta})]/p_{2i} \tag{6.1}$$

which provides an unbiased estimate of the likelihood that would arise if full information were available on all items placed in use in $(0,\tau]$.

Several questions of design arise. For example, the times that items entered service may be known, and sampling could be conditional upon these times. If time of entering service is not prognostic of subsequent failure experience given $\mathbf{x}$, then there would be advantage to sampling with higher probability those items that had begun exposure (entered service) early on. If, however, interest centred on the dependence of reliability on calendar time of entry, or possible associations between $\mathbf{x}$ and calendar time of entry, more uniform sampling would seem appropriate. Often we may expect the first situation but want to guard against the second, so a compromise is in order.

Estimators based on the pseudo likelihood (6.1) can be handled in the same way as those based on (3.2), and the approach outlined in Appendix B yields asymptotic covariance matrices. We note that withdrawals among items during the follow-up period, for example because of accidents, can also be handled. Provided the withdrawals arise through an independent censoring mechanism, the pseudo likelihood is unchanged.

6.2 'Case-Cohort' Designs

Prentice (1986) and Suzuki (1985a) describe follow-up studies in which a random sample of say n items out of the N in the population are followed up over a period of time, and in addition observations are taken on any items among the remaining $N-n$ that fail. Prentice calls this a case-cohort design, in line with common terminology in medicine and epidemiology. Such plans are similar to those discussed in Section 3, and can be handled in the same way with regard to L_P. We remark that Prentice shows how to carry out a semi-parametric analysis for proportional hazards and more general Cox models; his methods do not require the assumption of a specific failure time distribution but depend heavily on proportional hazards (e.g. Cox 1972) assumptions. Moreover, they only allow estimation of relative regression effects and not of the full distribution of failure time. We emphasize that our methods, based on the pseudo likelihood L_P, apply quite generally to arbitrary parametric models and so apply beyond the proportional hazards family.

For the case of proportional hazards models, and the case cohort design, Kalbfleisch and Lawless (1988) explore the relationships and make comparisons between L_P and the pseudo likelihood given by Prentice (1986). In particular, asymptotic variances of estimates of regression coefficients obtained by Prentice are compared with those obtained from L_P. A semi-parametric pseudo likelihood is also constructed; this gives rise to an estimating equation that is not exactly unbiased, but is a Fisher consistent estimate of zero.

6.3 Incorporating Additional Follow-up Information

Sometimes a decision may be made to carry out supplementary sampling at some point after the initial warranty or failure record period $(0,T°]$. Suppose for example that at time $T^1 > T°$ a simple random sample of items who had not failed by T° is taken, any individual item being selected with probability p_2. Define

$$R_{1i} = I(T_i \le T^\circ)$$

$$R_{2i} = I\{T_i = t_i \epsilon (T^\circ, T^1] \text{ and item } i \text{ is sampled}\}$$

$$R_{3i} = I\{T_i > T^1 \text{ and item } i \text{ is sampled}\},$$

where $I(A)$ is the indicator function for event A. Then a pseudo log likelihood is

$$\log L_P = \sum_{i=1}^{N} R_{1i} \log f(t_i|\mathbf{x}_i;\boldsymbol{\theta}) + \frac{R_{2i}}{p_2} \log f(t_i|\mathbf{x}_i;\boldsymbol{\theta}) + \frac{R_{3i}}{p_2} \log \bar{F}(T^1|\mathbf{x}_i;\boldsymbol{\theta}).$$

6.4 Follow-up when warranties have calendar time limits

Suppose that the warranty covers a fixed period of calendar time but that the distribution of the time of failure depends on operating time. To be more specific, suppose that each item has an associated usage process $Y_i(t)$, $0 \le t \le T^\circ$ where $Y_i(t)$ takes values 1 or 0 depending on whether the item would or would not be in service at calendar time t. The probabilistic structure by which $\{Y_i(t), i = 1, \ldots, N\}$ is generated conditional on $\mathbf{x}_1, \ldots, \mathbf{x}_N$ is left entirely arbitrary. Note however that it would often be expected that the $Y_i(t)$'s would be dependent (due to common environmental factors). Let $\mathbf{Y}_i = \{Y_i(t)\colon 0 \le t \le T^\circ\}$, $i = 1, \ldots, N$. To complete the specification, we work conditionally upon $\mathbf{Y}_1, \ldots, \mathbf{Y}_N$ and suppose that the p.d.f. of the time of failure T_i is

$$f(t;\mathbf{Y}_1, \ldots, \mathbf{Y}_N, \mathbf{x}_i) = \begin{cases} f^\circ(s|\mathbf{x}_i;\boldsymbol{\theta}), & Y_i(t) = 1 \\ 0, & Y_i(t) = 0 \end{cases} \tag{6.2}$$

where $s = \int_0^t Y(u)du$ is the operating time logged up to time t. Let $F^\circ(s|\mathbf{x};\theta) = \int_0^s f^\circ(u|\mathbf{x}_i;\boldsymbol{\theta})du$ and $\bar{F}^\circ(s|\mathbf{x};\theta) = 1 - F^\circ(s|\mathbf{x};\boldsymbol{\theta})$. Note that we have assumed that items are at risk of failure only when in use and that the probability of failure, conditional on $\mathbf{Y}_1, \ldots, \mathbf{Y}_N$ and $\mathbf{x}$, depends only on the operating time s.

Suppose that all items that fail prior to T° are observed so that, for these items, we know $\mathbf{x}_i$, the chronological time of failure $t_i \le T^\circ$ and the operating time at failure $s_i = \int_0^{t_i} Y(u)du$. In addition, suppose that a supplementary simple random sample of items that have not failed by time T° is taken, each such item being sampled with probability p_2. For these items we observe $\mathbf{x}_i$ and the operating time $s_i^\circ = \int_0^{T^\circ} Y(u)du$ at the end of the warranty period. Let T_i represent the chronological time to failure and let

$$R_{1i} = I(T_i \le T^\circ)$$

$$R_{2i} = I(T_i > T^\circ \text{ and item } i \text{ is sampled})$$

The pseudo log likelihood analogous to (3.2) is

$$\log L_P = \sum_1^N \{R_{1i} \log f^\circ(s_i | \mathbf{x}_i; \theta) + \frac{R_{2i}}{p_2} \log \bar{F}^\circ(s_i^\circ | \mathbf{x}_i; \theta)\}. \tag{6.3}$$

Suzuki (1985a), under a slightly different scenario (see Section 6.2), derives a likelihood that is equivalent to (6.3) for the case in which there are no covariates present. He formulates the problem in terms of a joint distribution of s_i and s_i° (i.e. the operating time at failure, and the total operating time which accrues over the warranty period; the latter then acts as a censoring time with regard to the former), however and restricts attention to the case in which s_i and s_i° are independent. Formulation in terms of the processes $\{Y_i(t)\}$ avoids these problems.

6.5 Warranties with calendar and operating time limits

Sometimes, for example with automobiles, a warranty may extend for a fixed calendar period or a fixed operating time, whichever comes first. The example of the previous section can easily be extended to this case. Let T° be the fixed calendar time as before and $Y(t)$ represent the usage process. If the chronological time to failure $T_i \le T^\circ$ and in addition $S_i = \int_0^{T_i} Y(u)du \le S^\circ$, the fixed operating time limit, then $(t_i, s_i, \mathbf{x}_i)$ are observed for the ith item. If either $T_i > T^\circ$ or $S_i > S^\circ$, the item is sampled with probability p_2 and $S_i^* = \min(S^\circ, \int_0^{T^\circ} Y_i(u)du)$ the operating time until expiration of warranty, and $\mathbf{x}_i$ are observed. Then under the same conditions and definitions as in Section 6.3, the pseudo log likelihood is

$$\log L_P = \sum_{i=1}^{N} R_{1i} \log f^\circ(s_i|\mathbf{x}_i) + \frac{R_{2i}}{p_2} \log \bar{F}^\circ(s_i^*|\mathbf{x}_i).$$

7. DISCUSSION

The likelihood methods proposed here need additional study to evaluate their robustness to incorrect assumptions, and to examine the appropriateness of the asymptotic approximations for inference. It seems likely that, in assessing the covariate effects on failure over the interval $(0,T^\circ]$, the methods have good robustness properties, but care must clearly be taken in extrapolation of any inference about survival patterns or effects past T°. The demonstrated benefits of employing supplementary sampling of unfailed items or information about the distribution of covariate values for the items in field use are, however, quite general, and with proper choice of model, the methods presented in the paper are widely applicable.

The models and sampling schemes considered here were fairly simple and chosen in order to study efficiency and other properties of the likelihood methods proposed. In a broader context, reliability studies may involve many modes of failure, continuous placement of items in the field, continuous observation of reliability problems, problems with data quality, and so on. It will also be common for there to be information on some covariates but not others, and sampling may be stratified with respect to both the response and the covariates. In addition, interesting and challenging problems exist when the two time scales of operating and calendar time are relevant. The ideas put forward here can be extended in various directions. The pseudo likelihood, in particular, allows relatively simple adjustment for complex sampling schemes. We are currently studying its uses further.

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APPENDIX A

The asymptotic variances of the estimators associated with (2.5) and (2.6) can be obtained by straightforward procedures involving direct computation of the Fisher information. We find, for (2.5)

$$as\ var(\sqrt{N}(\hat{\theta}_T-\theta)) = \frac{\theta^2(1-e^{-\theta T^\circ})}{(1-e^{-\theta T^\circ})^2-(\theta T^\circ)^2 e^{-\theta T^\circ}} \tag{A.1}$$

and for (2.6),

$$as\ var(\sqrt{N}(\hat{\theta}_F-\theta)) = \frac{\theta^2}{1-e^{-\theta T^\circ}}. \tag{A.2}$$

The asymptotic relative efficiencies in Table 1 are the ratios of (A.2) to (A.1).

APPENDIX B

Asymptotic Properties of the Pseudo Likelihood Function (3.2)

We show that the pseudo score vector $\mathbf{s}_p(\boldsymbol{\theta}) = \partial \log L_P / \partial \boldsymbol{\theta}$ has expectation zero and apply results of Inagaki (1973) and Crowder (1986) which show that under regularity conditions on $f(t|\mathbf{x};\boldsymbol{\theta})$ the estimator $\tilde{\boldsymbol{\theta}}$ from an unbiased estimating equation $\mathbf{s}(\boldsymbol{\theta}) = \mathbf{0}$ is consistent and asymptotically normal. In particular, $\sqrt{N}(\tilde{\boldsymbol{\theta}}-\boldsymbol{\theta})$ has a limiting multivariate normal distribution with mean $\mathbf{0}$ and covariance matrix $A(\boldsymbol{\theta})^{-1}B(\boldsymbol{\theta})A(\boldsymbol{\theta})^{-1}$, where

$$A(\boldsymbol{\theta}) = \lim_{N\to\infty}\frac{1}{N}E\left(-\frac{\partial \mathbf{s}}{\partial \boldsymbol{\theta}}\right) \text{ and } B(\boldsymbol{\theta}) = \lim_{N\to\infty}\frac{1}{N}E(\mathbf{s}(\boldsymbol{\theta})\mathbf{s}(\boldsymbol{\theta})').$$

If complete data on the whole cohort of N individuals over the period $(0,T^\circ]$ are available, the log likelihood is

$$l = \sum_{i\epsilon D_1}\log f(t_i|\mathbf{x}_i;\boldsymbol{\theta}) + \sum_{i\epsilon \bar{D}_1}\log \bar{F}(T^\circ|\mathbf{x}_i;\boldsymbol{\theta}) \tag{B.1}$$

with corresponding score function

$$\mathbf{s} = \sum_{i \epsilon D_1} \frac{\partial}{\partial \boldsymbol{\theta}} \log f(t_i | \mathbf{x}_i, \boldsymbol{\theta}) + \sum_{i \epsilon \bar{D}_1} \mathbf{m}_i(\boldsymbol{\theta}) \quad \text{(B.2)}$$

where, as before, $\mathbf{m}_i(\boldsymbol{\theta}) = \partial \log \bar{F}(T^\circ | \mathbf{x}_i ; \boldsymbol{\theta}) / \partial \boldsymbol{\theta}$. The pseudo log likelihood (3.2) can be written

$$l_P = \sum_{i \epsilon D_1} \log f(t_i | \mathbf{x}_i ; \boldsymbol{\theta}) + \sum_{i \epsilon D_2} \frac{1}{p_2} \log \bar{F}(T^\circ | \mathbf{x}_i ; \boldsymbol{\theta})$$

$$= l + \sum_{i \epsilon \bar{D}_1} (\frac{R_{2i}}{p_2} - 1) \log \bar{F}(T^\circ | \mathbf{x}_i ; \boldsymbol{\theta}) \quad \text{(B.3)}$$

where, as before, $R_{2i} = I(t_i > T^\circ$ and i is sampled) and $p_2 = P(R_{2i} = 1 | i \epsilon \bar{D}_1)$. The corresponding pseudo score function is

$$\mathbf{s}_P = \mathbf{s} + \sum_{i \epsilon \bar{D}_1} (\frac{R_{2i}}{p_2} - 1) \mathbf{m}_i(\boldsymbol{\theta}). \quad \text{(B.4)}$$

Since, for $i \epsilon \bar{D}_1$, $P\{R_{2i} = 1 | (t_i, \mathbf{x}_i),\ i = 1, \ldots, N\} = p_2$, it follows that $E\{\mathbf{s}_P | (t_i, \mathbf{x}_i)\ i = 1, \ldots, N\} = \mathbf{s}$ and so $E(\mathbf{s}_P) = 0$. It can also be seen that the two terms in (B.4) are uncorrelated. Since $\mathbf{s}$ is the full data score function,

$$cov(\mathbf{s}) = -E \left[\sum_{i \epsilon D_1} \frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \log f(t_i | \mathbf{x}_i, \boldsymbol{\theta}) + \sum_{i \epsilon \bar{D}_1} \frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \log \bar{F}(t_i | \mathbf{x}_i, \boldsymbol{\theta}) \right] = V_N(\boldsymbol{\theta}).$$

The covariance matrix of the second term in (B.4) requires calculation. As in Section 3, we assume that a simple random sample of size $n_2 = p_2 N_2$ of the N_2 items in $\bar{D}_1$ is selected. It can be shown that

$$U_N(\boldsymbol{\theta}) = cov\{\sum_{i \epsilon \bar{D}_1} (\frac{R_{2i}}{p_2} - 1) \mathbf{m}_i(\boldsymbol{\theta})\} = \frac{1 - p_2}{p_2} E \frac{N_2}{N_2 - 1} \sum_{i \epsilon \bar{D}_1} (\mathbf{m}_i(\boldsymbol{\theta}) - \bar{\mathbf{m}}(\boldsymbol{\theta}))(\mathbf{m}_i(\boldsymbol{\theta}) - \bar{\mathbf{m}}(\boldsymbol{\theta}))'$$

$$= \frac{1 - p_2}{p_2} E \frac{N_2}{n_2 - 1} \sum_{i \epsilon D_2} (\mathbf{m}_i(\boldsymbol{\theta}) - \bar{\mathbf{m}}(\boldsymbol{\theta}))(\mathbf{m}_i(\boldsymbol{\theta}) - \bar{\mathbf{m}}(\boldsymbol{\theta}))' \quad \text{(B.5)}$$

where $\mathbf{m}_i(\boldsymbol{\theta}) = \partial \log \bar{F}(T^\circ | x_i ; \boldsymbol{\theta}) / \partial \boldsymbol{\theta}$, $\bar{\mathbf{m}}(\boldsymbol{\theta}) = \sum_{i \epsilon D_2} \mathbf{m}_i(\boldsymbol{\theta}) / n_2$ and $\bar{\mathbf{m}}(\boldsymbol{\theta}) = \sum_{i \epsilon \bar{D}_1} \mathbf{m}_i(\boldsymbol{\theta}) / N_2$. Other sampling schemes could also be considered that, for example, involve stratification on some manufacturing characteristic that is known for each item.

Applying results of Inagaki (1973) and Crowder (1986), suitably specialized, shows that under regularity conditions

$$\sqrt{N}(\tilde{\theta}-\theta) \xrightarrow{L} N(0,A(\theta)^{-1}+A(\theta)^{-1}C(\theta)A(\theta)^{-1})$$

where $A(\theta) = \lim_{N\to\infty} V_N(\theta)/N$ and $C(\theta) = \lim_{N\to\infty} U_N(\theta)/N$. It is assumed here that the central limit theorem applies to the pseudo score $s_P(\theta)$ as $N \to \infty$ and that

$$A(\theta) = \underset{N\to\infty}{plim}\frac{1}{N}\{-\frac{\partial^2 l_P}{\partial\theta\partial\theta'}\}. \tag{B.6}$$

The estimates $A_N(\tilde{\theta})$ and $B_N(\tilde{\theta})$ for $A(\theta)$ and $B(\theta)$ follow from expressions (B.5) and (B.6).

APPENDIX C

Asymptotic Variance Calculations for Exponential and Weibull Examples of Section 5

The exponential results are a special case of the Weibull results with $\delta = 1$. For the exponential, only differentiation with respect to β is considered.

C.1. The Truncated Likelihood L_T

For method I, the truncated log likelihood for (2.1) is

$$\log L_T(\beta,\delta) = \sum_{i=1}^{N} R_{1i}\{\log\delta+(\delta-1)\log t_i+\mathbf{x}_i'\beta-W_{i0}-\log(1-S_i)\}$$

where $R_{1i} = I(i\epsilon D_1)$ and W_{ij} and S_i are defined as in Section 3.3. Straightforward calculation now gives

$$-\frac{\partial^2\log L_T}{\partial\beta_r\partial\beta_s} = \sum_{i=1}^{N} R_{1i}x_{is}x_{ir}\{W_{i0}-C_i/(1-S_i)\} \tag{C.1}$$

$$-\frac{\partial^2\log L_T}{\partial\beta_r\partial\delta} = \sum_{i=1}^{N} R_{1i}x_{ir}\{W_{i1}-(\log T^{\circ})C_i/(1-S_i)\} \tag{C.2}$$

$$-\frac{\partial^2 \log L_T}{\partial \delta^2} = \sum_{i=1}^{N} R_{1i}\{\frac{1}{\delta^2} + W_{i2} - (\log T^{\circ})^2 C_i/(1-S_i)\} \quad \text{(C.3)}$$

where $C_i = S_i \log S_i\{1+\log S_i/(1-S_i)\}$.

Expectations of these quantities involve the integrals

$$E_j(\mathbf{x}_i) = \int_0^{-\log S_i} (\log y)^j y e^{-y} dy, \quad j = 0,1,2. \quad \text{(C.4)}$$

It is easily verified that

$$E(R_{1i}W_{0i}) = E_0(\mathbf{x}_i) = S_i \log S_i + 1-S_i \quad \text{(C.5)}$$

while the quantities $E_1(\mathbf{x}_i)$, $E_2(\mathbf{x}_i)$ require numerical computation. Further,

$$\delta E(R_{1i}W_{1i}) = E_1(\mathbf{x}_i) - \mathbf{x}_i'\beta E_0(\mathbf{x}_i) \quad \text{(C.6)}$$

$$\delta^2 E(R_{1i}W_{2i}) = E_2(\mathbf{x}_i) - 2(\mathbf{x}_i'\beta)E_1(\mathbf{x}_i) + (\mathbf{x}_i'\beta)^2 E_0(\mathbf{x}_i). \quad \text{(C.7)}$$

For method I, the Fisher information matrix I_T can be obtained by taking expectations in (C.1), (C.2) and (C.3) to give

$$(I_T)_{r,s} = \sum_{i=1}^{N} x_{is}x_{ir}\{E_0(\mathbf{x}_i) - C_i\}$$

$$= \sum_{i=1}^{N} x_{is}x_{ir}\{1-S_i-S_i(\log S_i)^2/(1-S_i)\}$$

$$(I_T)_{r,k+1} = \frac{1}{\delta}\sum_{i=1}^{N} x_{ir}[E_1(\mathbf{x}_i) - \mathbf{x}_i'\beta E_0(\mathbf{x}_i) - \log(T^{\circ\delta})C_i]$$

$$(I_T)_{k+1,k+1} = \frac{1}{\delta^2}\sum_{i=1}^{N} 1-S_i+[E_2(\mathbf{x}_i) - 2\mathbf{x}_i'\beta E_1(\mathbf{x}_i) + (\mathbf{x}_i'\beta)^2 E_0(\mathbf{x}_i) - (\log T^{\circ\delta})^2 C_i].$$

Asymptotic variances are obtained as the appropriate entries in the inverse of I_T.

C.2. The Pseudo Likelihood L_P

The asymptotic covariance matrix of $\tilde{\theta}$ is the probability limit of

$$A_N^{-1} + A_N^{-1} C_N A_N^{-1}$$

where A_N and C_N are given by (3.5) and (3.6). We find

$$E(NA_N)_{r,s} = \sum_{i=1}^{N} x_{ir}x_{is}(1-S_i)$$

$$E(NA_N)_{r,k+1} = \frac{1}{\delta}\sum_{i=1}^{N} x_{ir}\{E_1(\mathbf{x}_i)-\mathbf{x}_i'\beta E_0(\mathbf{x}_i)-(log\ T^{\circ\delta})(\log S_i)S_i\}$$

$$E(NA_N)_{k+1,k+1} = \frac{1}{\delta^2}\sum_{i=1}^{N}\{1-S_i+E_2(\mathbf{x}_i)-2(\mathbf{x}_i'\beta)E_1(\mathbf{x}_i)+(\mathbf{x}_i'\beta)^2E_0(\mathbf{x}_i)-(log\ T^{\circ\delta})^2(log\ S_i)S_i\}.$$

In addition

$$\underset{N\to\infty}{plim}(C_N)_{r,s} = \frac{1-p_2}{p_2}\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}S_i(x_{ir}H_i-\overline{H}_r)(x_{is}H_i-\overline{H}_s)$$

$$\underset{N\to\infty}{plim}(C_N)_{r,k+1} = \frac{1-p_2}{p_2}\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}S_i\log T^{\circ}(x_{ir}H_i-\overline{H}_r)(H_i-\overline{H}_0)$$

$$\underset{N\to\infty}{plim}(C_N)_{k+1,k+1} = \frac{1-p_2}{p_2}\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}S_i(\log T^{\circ})^2(H_i-\overline{H}_0)^2$$

where $H_i = \log S_i$, $\overline{H}_r = \sum_{i=1}^{N} x_{ir}H_i/N$, and $\overline{H}_0 = \sum_{i=1}^{N} H_i/N$.

C.3. The Likelihood L_D

For method II, suppose that the covariate vector $\mathbf{x}$ can take L distinct values $x_1^*, \ldots, x_L^*$ with known probabilities $q_1, \ldots, q_L$ independently for each individual in the cohort. (In the specific example of Section 4, $\mathbf{x}_i' = (x_{i0}, x_{i1})$, $L = 2$, $q_1 = q_2 = .5$, $x_1^{*\prime} = (1,0)$ and $\mathbf{x}_2^{*\prime} = (1,1)$.) In keeping with the previous notation, let

$$S_l^* = \exp\{-T^{\circ\delta}e^{\mathbf{x}_l^*\beta}\}.$$

The likelihood (2.4) is then written as

$$\log L_D(\beta,\delta) = \sum_{i=1}^{N} R_{1i}[\log\delta+(\delta-1)\log t_i+\mathbf{x}_i'\beta-W_{0i}] + (1-R_{1i})\log\sum_{l=1}^{L} q_l S_l^*.$$

It is convenient to define

$$U_l^* = q_l^* S_l^* \log S_l^* / \sum_{l=1}^{L} q_l S_l^* \quad \text{and} \quad V_l^* = U_l^*(1+\log S_l^*).$$

Straightforward calculation now gives the second derivatives:

$$-\frac{\partial^2 \log L_D}{\partial \beta_r \partial \beta_s} = \sum_{i=1}^{N} R_{1i} x_{ir} x_{is} W_{0i} - (1-R_{1i})\{\sum_{l=1}^{L} x_{lr}^* x_{ls}^* V_l^* - \sum_{l=1}^{L} x_{lr}^* U_l^* \sum_{j=1}^{L} x_{js}^* U_j^*\}$$

$$-\frac{\partial^2 \log L_D}{\partial \beta_r \partial \delta} = \sum_{i=1}^{N} R_{1i} x_{ir} W_{1i} - (1-R_{1i})(\log T^\circ)\{\sum_{l=1}^{L} x_{lr}^* V_l^* - \sum_{l=1}^{L} x_{lr}^* U_l^* \sum_{j=1}^{L} U_j^*\}$$

$$-\frac{\partial^2 \log L_D}{\partial \delta^2} = \sum_{i=1}^{N} R_{1i}(\frac{1}{\delta^2} + W_{2i}) - (1-R_{1i})(\log T^\circ)^2\{\sum_{l=1}^{L} V_l^* - (\sum_{l=1}^{L} U_l^*)^2\}.$$

Finally, taking expectations, we obtain the entries of the Fisher information matrix I_D as

$$\frac{1}{N}(I_D)_{r,s} = \sum_{l=1}^{L} q_l x_{lr}^* x_{ls}^* E_0(\mathbf{x}_l^*) - (\sum_{l=1}^{L} q_l S_l^*)\{\sum_{l=1}^{L} x_{lr}^* x_{ls}^* V_l^* - \sum_{l=1}^{L} x_{lr}^* U_l^* \sum_{j=1}^{L} x_{js}^* U_j^*\}$$

$$\frac{1}{N}(I_D)_{r,k+1} = \frac{1}{\delta} \sum_{l=1}^{L} q_l x_{lr}^* [E_1(\mathbf{x}_l^*) - \mathbf{x}_l^{*\prime} \beta E_0(\mathbf{x}_l^*)] - \frac{1}{\delta}(\sum_{l=1}^{L} q_l S_l^*)(\log T^{\circ\delta})\{\sum_{l=1}^{L} x_{lr}^* V_l^* - \sum_{l=1}^{L} x_{lr}^* U_l^* \sum_{j=1}^{L} U_j^*\}$$

$$\frac{1}{N}(I_D)_{k+1,k+1} = \frac{1}{\delta^2} \sum_{l=1}^{N} q_l \{1 - S_l^* + E_2(\mathbf{x}_l^*) - 2(\mathbf{x}_l^{*\prime}\beta) E_1(\mathbf{x}_l^*) + (\mathbf{x}_l^{*\prime}\beta)^2 E_0(\mathbf{x}_l^*)\}$$

$$- \frac{1}{\delta^2}(\sum_{l=1}^{L} q_l S_l^*)(\log T^{\circ\delta})^2\{\sum_{l=1}^{L} V_l^* - (\sum_{l=1}^{L} U_l^*)^2\}.$$

Asymptotic variances are obtained as the appropriate entries in I_D^{-1}.

C.4. Full information on population covariates available

The calculations here are straightforward but can be obtained as those arising with pseudo likelihood when $p_2 = 1$. The resulting information is then $I_F = E(NA_N)$. For a random sample of size n followed on $(0,T^\circ]$, the information is nI_F/N.

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