AN APPROACH TO THE CONSTRUCTION OF ASYMMETRICAL ORTHOGONAL ARRAYS

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IIQP Research Report
RR-89-01

January 1989

AN APPROACH TO THE CONSTRUCTION OF ASYMMETRICAL ORTHOGONAL ARRAYS*

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ABSTRACT

Use of asymmetrical orthogonal arrays (OAs) for planning industrial experiments with mixed levels has become increasingly popular. In addition to applications to experimental investigations in many disciplines, they can also be used in the balanced repeated replications method of inference from general stratified survey samples. In this paper we propose a general approach to the construction of asymmetrical OAs with economic run size and flexibility in the choice of factor levels. As applications of this approach we construct several general classes of arrays which include numerous existing classes of arrays as special cases. An overwhelming majority of known asymmetrical OAs can be reproduced by our approach in a unified and simple manner. Many new arrays are obtained. A catalog of asymmetrical OAs with run size less than 100 is given. Our approach consists of three steps. We first construct a symmetrical OA as the Kronecker sum of an OA and a difference matrix. We then add, to the constructed OA, columns which are based on another OA. Since the choice of the second OA is quite flexible, the resulting array can take several forms. As the third step we use the replacement method to substitute p -level columns by p^r -level columns, $r > 1$. It greatly increases the variety of asymmetrical OAs. As an example, for 48 runs, we have found thirteen asymmetrical OAs with factor levels equal to 2, 3, 4, 6, 8, or 12. Out of these OAs, ten are apparently new.

*This research is supported by the US National Science Foundation Grant DMS-84-20968 and the GM/NSERC Chair in Quality and Productivity through the Natural Sciences and Engineering Research Council of Canada.

1 Introduction

Factorial experiments with mixed levels are often encountered in practice because the choice of factor levels may vary with the nature of the factor. Asymmetrical orthogonal arrays, to be defined below, are commonly used for planning such experiments. They have been used extensively by G. Taguchi (1987) and his colleagues in industrial experiments for quality improvement. Their use in agricultural experiments is also quite common. For inference from stratified samples in sample surveys, use of asymmetrical orthogonal arrays allows the balanced repeated replications method (McCarthy 1969) to be extended to general sampling designs with arbitrary numbers of primary selections per stratum (Gupta and Nigam 1987). In this paper we propose a method for constructing these arrays with economic run size and flexibility in the choice of factor levels. Many new arrays are found and several existing classes of arrays are reproduced in a unified and simple manner.

Formally an orthogonal array of strength d with k_i s_i-level columns, $i = 1, ..., r$, is an $N \times m$ matrix, $m = k_1 + \cdots + k_r$, in which all possible combinations of levels in any d columns appear the same number of times (Rao 1947, 1973). Our definition follows the convention in statistics of using rows to denote runs. An orthogonal array is symmetrical or asymmetrical if its columns have the same or different numbers of levels. Since only $d = 2$ is considered in the paper, we will use the notation

$$L_N(s_1^{k_1} \cdots s_r^{k_r})$$

to denote an orthogonal array of size N and strength 2 having k_i columns with s_i levels. An orthogonal array of strength 2 is saturated if $\sum_{i=1}^{r} k_i(s_i - 1) = N - 1$, i.e., if all the degrees of freedom are used for estimating the main effects of the m factors (columns).

Our approach consists of three steps. First we construct a symmetrical orthogonal array as the Kronecker sum of a symmetrical orthogonal array and a difference matrix. For this and the construction of difference matrices, see Section 2. Then we add to the constructed array columns which are based on another orthogonal array. Since the latter array can be asymmetrical and usually can be chosen in several ways, the resulting array from this step shares the same properties. The last step is optional. If some p-level columns of the array, p being prime, together with the column of zeros form a subgroup under addition modulus p, they can be replaced by a p^r-level column, $r > 1$. The most useful case is the replacement of three 2-level columns by a 4-level column. The resulting array is asymmetrical. For steps two and three, see Section 3. In Section 4 we construct several new classes of asymmetrical orthogonal arrays which include numerous existing methods as special cases. In Section 5 we give a list of orthogonal arrays, including many new ones, with run size less than 100. Since the replacement method is applied to the maximum number of disjoint sets of three 2-level columns, the number of 4-level columns in the constructed array is maximized.

2 Construction through Difference Matrices and Kronecker Sum

First we give a brief review of difference matrix. Let M be an additive group of p elements denoted by $\{0, 1, \ldots, p-1\}$. A $\lambda p \times r$ matrix with elements from M, denoted by $D_{\lambda p,r;p}$, is called a *difference matrix* if among the differences, modulus p, of the corresponding elements of any two columns, each element of M occurs exactly λ times. It is known that $r \leq \lambda p$ (Beth, Jungnickel and Lenz 1985, p. 362). If the transpose of a difference matrix is also a difference matrix, then we call it a *generalized Hadamard matrix*. For $p = 2$, $D_{k,k,2}$ is a *Hadamard matrix* of order k. Without loss of generality we can always put the column of zeros as the first column of $D_{\lambda p,r;p}$.

For prime p, a $D_{p,p;p}$ can be constructed as follows. Let $x = [0, ..., (p-1)]^T$ and 0_p be the $p \times 1$ vector of zeros. Then $D_{p,p;p} = [0_p, x, 2x, \cdots, (p-1)x]$ is a difference matrix. Several construction methods are reviewed as follows.

Bose and Bush (1952) provided a method of constructing $D_{p^{u+v},p^{u+v};p^v}$ for prime p

through Galois field. Masuyama (1957) gave a method of constructing $D_{2p,2p;p}$ for odd prime p using equimodular matrices. For two matrices $A = [a_{ij}]$ of order $n \times r$ and B of order $m \times s$ both with entries from M, define their *Kronecker sum* to be

$$A * B = [B^{a_{ij}}]_{1\leq i\leq n, 1\leq j\leq r}, \tag{1}$$

where

$$B^k = (B + k\mathrm{J}) \bmod p \tag{2}$$

is obtained from adding $k \bmod p$ to the elements of B and J is the $m \times s$ matrix of ones. Shrikhande (1964) showed that for two difference matrices $D_{\mu p,s;p}$ and $D_{\lambda p,r;p}$, their Kronecker sum is a difference matrix $D_{\lambda\mu p^2,rs;p}$. For example, $D_{6,6;3} * D_{3,3;3}$ gives a $D_{18,18;3}$. Other results on difference matrices can be found in Beth, Jungnickel and Lenz (1985), de Launey (1986) and Seberry (1979).

A collection of small difference matrices is given in the Appendix.

We now consider the construction of orthogonal arrays through difference matrices. It follows from their definitions that an orthogonal array $L_{\lambda p^2}(p^r)$ can be obtained from

$$\begin{bmatrix} D^0 \\ \vdots \\ D^{p-1} \end{bmatrix},$$

where $D_{\lambda p,r;p}$ is a difference matrix and $D^k = (D_{\lambda p,r;p} + k\mathrm{J}) \bmod p$ is defined analogously to (2). This method is due to Bose and Bush (1952). A more general construction method is given by the Kronecker sum

$$\mathbf{D} = L_{\mu p}(p^s) * D_{\lambda p,r;p} \tag{3}$$

of an orthogonal array $L_{\mu p}(p^s)$ and a difference matrix $D_{\lambda p,r;p}$. It can be shown that $\mathbf{D}$ is an orthogonal array $L_{\lambda\mu p^2}(p^{rs})$ (Beth, Jungnickel and Lenz 1985, p. 417). Note that the Bose-Bush method is a special case with $\mu = s = 1$.

3 Accommodation of Additional Columns and Method of Replacement

By counting the degrees of freedom in (3), we have $\mu p - 1 \geq s(p-1)$ for $L_{\mu p}(p^s)$ and $\lambda p \geq r$ for $D_{\lambda p,r;p}$ which imply that $\lambda p(\mu p - 1) \geq rs(p-1)$. Among the $\lambda\mu p^2 - 1$ degrees of freedom for the array $\mathbf{D}$ in (3), there are at least $\lambda p - 1$ $(= \lambda\mu p^2 - 1 - \lambda p(\mu p - 1))$ unused degrees of freedom. We can add additional columns to $\mathbf{D}$ to use up these degrees of freedom. For example, if an orthogonal array $L_{\lambda p}(q_1^{r_1} \cdots q_m^{r_m})$ exists, the matrix $[\mathbf{D}\ \mathbf{L}]$ constitutes an orthogonal array $L_{\lambda\mu p^2}(p^{rs} \cdot q_1^{r_1} \cdots q_m^{r_m})$, where $\mathbf{L} = 0_{\mu p} * L_{\lambda p}(q_1^{r_1} \cdots q_m^{r_m})$ is a matrix consisting of μp copies of $L_{\lambda p}(q_1^{r_1} \cdots q_m^{r_m})$ as its rows. The orthogonality between the columns in $\mathbf{D}$ and the columns in $\mathbf{L}$ is due to the fact that $\mathbf{L}$ has μp identical submatrices and $\mathbf{D}$ has μp submatrices which are the permutations of each other.

The constructed array $L_{\lambda\mu p^2}(p^{rs} \cdot q_1^{r_1} \cdots q_m^{r_m})$ is saturated if the three components in its construction satisfy the following conditions:

(i) $L_{\mu p}(p^s)$ and $L_{\lambda p}(q_1^{r_1} \cdots q_m^{r_m})$ are saturated, i.e.

$s(p-1) = \mu p - 1$ and $\sum_{i=1}^m r_i(q_i - 1) = \lambda p - 1$;

(ii) in $D_{\lambda p,r;p}$, r attains its maximum, i.e., $r = \lambda p$.

Regarding (ii), recall that $r \leq \lambda p$ in general (see Section 2).

To prove this, note that conditions (i) and (ii) imply

$rs(p-1) + \sum_{i=1}^m r_i(q_i - 1) = \lambda p(\mu p - 1) + \lambda p - 1 = \lambda\mu p^2 - 1.$

Special cases of this method for adding columns were considered before, e.g., $L_{\lambda p}(p^1)$ in Bose and Bush (1952), $L_{\lambda p}(q^r)$ in Taguchi (1987) and Dey (1985).

Besides exploiting unused degrees of freedom, the method of adding columns provides flexibility in the choice of factor levels because usually $L_{\lambda p}(q_1^{r_1} \cdots q_m^{r_m})$ is available for a variety of q_i and r_i. This flexibility can also be achieved by the following method of replacement.

Suppose there is an orthogonal array containing three 2-level columns with one column being the sum (mod 2) of the other two. These three columns can be replaced by a 4-level column according to the following scheme:

$$\begin{array}{cccccc} \multicolumn{3}{c}{2-\text{level columns}} & & 4-\text{level column} \\ 0 & 0 & 0 & \longrightarrow & 0 \\ 0 & 1 & 1 & \longrightarrow & 1 \\ 1 & 0 & 1 & \longrightarrow & 2 \\ 1 & 1 & 0 & \longrightarrow & 3 \end{array} \tag{4}$$

The 4-level column is balanced and is orthogonal to any other columns that are orthogonal to the three 2-level columns (Addelman 1962). It will be used in Section 4 to construct mixed 2- and 4-level arrays from 2-level arrays. Similarly if an array contains four 3-level columns of the form a, b, $a+b \pmod 3$ and $a+2b \pmod 3$, these four columns can be replaced by a 9-level column. Extension to other prime p is obvious.

4 Construction of Some Classes of Asymmetrical Orthogonal Arrays

In this section we construct several general classes of asymmetrical orthogonal arrays by using the methods in Sections 2 and 3. Some existing classes of arrays are easily obtained as special cases.

Denote a Hadamard matrix of order n by H_n. A positive integer n is called a *Hadamard number* if H_n exists. Without loss of generality, we rewrite

$$H_n = [0_n,\ L_n(2^{n-1})],$$

where 0_n is the $n \times 1$ vector of zeros. The following classes of arrays will be referred as Class A, B, C, A1, B1, etc.

A. Construction of $L_{nt}(4^s \cdot 2^{n(t-1)-2s} \cdot q_1^{r_1} \cdots q_m^{r_m})$, where n and t are Hadamard numbers,

$s < \min(n,t)$ and an $L_n(2^s \cdot q_1^{r_1} \cdots q_m^{r_m})$ exists with its s 2-level columns coinciding with the first s nonzero columns of a Hadamard matrix of order n.

An $L_t(2^{t-1})$ exists since t is a Hadamard number. Therefore we can construct

$$L_{nt}(2^{n(t-1)+s} \cdot q_1^{r_1} \cdots q_m^{r_m}) = [L_t(2^{t-1}) * H_n,\ 0_t * L_n(2^s \cdot q_1^{r_1} \cdots q_m^{r_m})],$$

where the second part of the matrix is obtained by the method of adding columns. Let $L_t(2^{t-1}) = [a_1, \cdots, a_s, \mathbf{A}]$, $L_n(2^{n-1}) = [b_1, \cdots, b_s, \mathbf{B}]$ and $L_n(2^s \cdot q_1^{r_1} \cdots q_m^{r_m}) = [b_1, \cdots, b_s, \mathbf{C}]$, where **A**, **B** and **C** are the matrices of the remaining columns in the corresponding arrays. Then for $1 \leq i \leq s$,

$$a_i * 0_n + a_i * b_i \equiv 0_t * b_i \mod 2,$$

which is the ith column of $0_t * L_n(2^s \cdot q_1^{r_1} \cdots q_m^{r_m})$. This implies that for each i a 4-level column can be obtained from the 2-level columns $a_i * 0_n$, $a_i * b_i$, and $0_t * b_i$ through the method of replacement. Repeating this for all i gives $L_{nt}(4^s \cdot 2^{n(t-1)-2s} \cdot q_1^{r_1} \cdots q_m^{r_m})$. The constructed array is saturated if $L_n(2^s \cdot q_1^{r_1} \cdots q_m^{r_m})$ is saturated.

By taking $n = 4k$ and $t = 2$ in Class A, we have the following class of arrays which will be used later.

A1. $L_{8k}(4 \cdot 2^{4k-2} \cdot q_1^{r_1} \cdots q_m^{r_m})$, where an $L_{4k}(2 \cdot q_1^{r_1} \cdots q_m^{r_m})$ exists.

If $n \geq t$ and an $L_n(2^{n-1})$ is used in the construction, i.e. $q_i = 2$ for all i, then Class A gives $L_{nt}(4^{t-1} \cdot 2^{nt-3t+2})$ (Cheng 1987). In particular if an $L_{4k}(2^{4k-1})$ is used in the construction, Class A1 gives $L_{8k}(4 \cdot 2^{8k-4})$ (Dey and Ramakrishna 1977). Using the array

$$L_{4k}((2k)^1 \cdot 2^2) = [L_2(2) * D_{2k,2;2},\ 0_2 * L_{2k}((2k)^1)]$$

in Class A1, we obtain $L_{8k}((2k)^1 \cdot 4 \cdot 2^{4k-1})$ for odd k (Agrawal and Dey 1982). For even k, a better array than $L_{8k}((2k)^1 \cdot 4 \cdot 2^{4k-1})$ will be obtained as a special case of Class B1.

B. Construction of $L_{nt}(4^s \cdot 2^{n(t-1)-3s} \cdot q_1^{r_1} \cdots q_m^{r_m})$, where $s = \min(n-1, t-1)$, n and $t/2$ are Hadamard numbers and an $L_n(q_1^{r_1} \cdots q_m^{r_m})$ exists.

Since $t/2$ is a Hadamard number, we can construct

$$\begin{aligned} L_t(2^{t-1}) &= [L_2(2^1) * H_{t/2},\ 0_2 * L_{t/2}(2^{t/2-1})] \\ &= [L_2(2^1) * 0_{t/2},\ L_2(2^1) * L_{t/2}(2^{t/2}),\ 0_2 * L_{t/2}(2^{t/2-1})]. \end{aligned}$$

Let $L_t(2^{t-1}) = [a_1, \cdots, a_{t-1}]$. From the construction,

$$a_1 + a_{i+1} \equiv a_{t/2+i} \mod 2,\ 1 \le i \le t/2 - 1. \tag{5}$$

Let $L_n(2^{n-1}) = [b_1, \cdots, b_s, \mathbf{B}]$. Then we construct

$$\begin{aligned} L_{nt}(2^{n(t-1)} \cdot q_1^{r_1} \cdots q_m^{r_m}) &= [L_t(2^{t-1}) * H_n,\ 0_t * L_n(q_1^{r_1} \cdots q_m^{r_m})] \\ &= [[a_1, \cdots, a_{t-1}] * [0_n, b_1, \cdots, b_s, \mathbf{B}], 0_t * L_n(q_1^{r_1} \cdots q_m^{r_m})]. \end{aligned} \tag{6}$$

From (5), we have

$$\begin{aligned} a_1 * 0_n + a_2 * b_1 &\equiv a_{t/2+1} * b_1 \mod 2, \\ a_1 * b_{i+1} + a_{i+1} * 0_n &\equiv a_{t/2+i} * b_{i+1} \mod 2,\ 1 \le i \le \min(s-1,\ t/2-1),\ \text{if } s > 1, \\ a_1 * b_{t/2+i} + a_{i+1} * b_{t/2+i} &\equiv a_{t/2+i} * 0_n \mod 2,\ 1 \le i \le s - t/2,\ \text{if } s > t/2. \end{aligned}$$

From these relations, we can group s triplets of 2-level columns in (6) to obtain s 4-level columns through replacement, thus obtaining $L_{nt}(4^s \cdot 2^{n(t-1)-3s} \cdot q_1^{r_1} \cdots q_m^{r_m})$. If $L_n(q_1^{r_1} \cdots q_m^{r_m})$ is saturated, then the constructed array is saturated.

The following special cases of the constructed arrays will be used later:
provided the existence of an $L_{4k}(q_1^{r_1} \cdots q_m^{r_m})$, we have

B1. $L_{16k}(4^3 \cdot 2^{12k-9} \cdot q_1^{r_1} \cdots q_m^{r_m})$ and
B2. $L_{32k}(4^7 \cdot 2^{28k-21} \cdot q_1^{r_1} \cdots q_m^{r_m})$ for $k \ge 2$.

If an $L_n(n)$ is used in the construction, i.e. $m = r_1 = 1$ and $q_1 = n$, then Class B gives $L_{nt}(n \cdot 4^s \cdot 2^{n(t-1)-3s})$. Cheng (1987) gave separate constructions of these arrays according to $n \geq t$ and $t > n$ in a complicated manner. If an $L_{4k}(2^{4k-1})$ is used in the construction, i.e. $q_i = 2$ for all i, then B1 gives $L_{16k}(4^3 \cdot 2^{16k-10})$ (Chacko, Dey and Ramakrishna 1979). Writing an even v as $v = 2k$ and using $L_{2v}((2v)^1)$ for $L_{4k}(q_1^{r_1} \cdots q_m^{r_m})$, Class B1 gives $L_{8v}(4^3 \cdot 2^{6v-9} \cdot (2v)^1)$.

C. Construction of $L_{nt}(8 \cdot 4^h \cdot 2^{n(t-1)-3h-6} \cdot q_1^{r_1} \cdots q_m^{r_m})$, where $s = \min(n-1, t-1)$, $h = \max(0, s-3)$, n and $t/2$ are Hadamard numbers and an $L_n(2 \cdot q_1^{r_1} \cdots q_m^{r_m})$ exists.

Constructing $L_t(2^{t-1})$ and using the notation as in B, we have

$$L_{nt}(2^{n(t-1)+1} \cdot q_1^{r_1} \cdots q_m^{r_m}) = [L_t(2^{t-1}) * H_n,\ 0_t * L_n(2 \cdot q_1^{r_1} \cdots q_m^{r_m})].$$

Rearrange the rows of $L_n(2 \cdot q_1^{r_1} \cdots q_m^{r_m})$ so that the first 2-level column coincides with b_1. Then the seven columns $[[a_1, a_2, a_{t/2+1}] * [0_n, b_1],\ 0_t * b_1]$ together with the column of zeros form a group under addition modulus 2. Replace these seven columns by an 8-level column. Then there are $s - 3$ (provided $s > 3$) remaining triplets of 2-level columns according to the construction in method B. Replace these remaining triplets to obtain $s - 3$ (if $s > 3$) 4-level columns, thus obtaining $L_{nt}(8 \cdot 4^h \cdot 2^{n(t-1)-3h-6} \cdot q_1^{r_1} \cdots q_m^{r_m})$. Again the constructed array is saturated if $L_n(2 \cdot q_1^{r_1} \cdots q_m^{r_m})$ is saturated.

The following special cases will be used later:
provided the existence of an $L_{4k}(2 \cdot q_1^{r_1} \cdots q_m^{r_m})$, we have

C1. $L_{16k}(8 \cdot 2^{12k-6} \cdot q_1^{r_1} \cdots q_m^{r_m})$ and
C2. $L_{32k}(8 \cdot 4^4 \cdot 2^{28k-18} \cdot q_1^{r_1} \cdots q_m^{r_m})$ for $k \geq 2$.

Remarks. 1. By reversing the assignment in (4), we can replace a 4-level column by three orthogonal 2-level columns, i.e. replacing 4 by 2^3 in an array with 4-level column(s). Similarly 8 in an array can be replaced by $4 \cdot 2^4$ or 2^7. We do not explicitly give arrays that can be obtained in this fashion since they are straightforward. For simplicity we only give arrays with the maximum number of 4-level columns. See also Remarks 2 and 3.

2. For Classes A1, B1, B2, C1 and C2, and general odd $k \geq 3$, the number of 4-level columns in our construction is maximum unless special properties of k can be exploited. Without loss of generality we assume k in $L_{2^m k}$ is odd. This is because $L_{2^n j}$ with even j can be reexpressed as $L_{2^m k}$ with odd k and $m > n$, which contains more 4-level and/or 8-level columns. We do not consider $k = 1$ in these classes because, when the run size is a power of 2, a simple method of constructing $L_{2^k}(2^m \cdot 4^n)$ for *every* possible m and n is available (Wu 1988).

3. For the arrays in Classes A, B and C, the number of 4-level columns is maximized by choosing n and t, for fixed nt, to be as close as possible.

5 Examples

In this section we give a list of asymmetrical orthogonal arrays with fewer than 100 runs. Asymmetrical orthogonal arrays whose run size is a power of 2 or 3 are not given here since they can be constructed by replacement and grouping (Wu 1988). Only arrays with the maximum number of 4-level columns are given, i.e. $L_n(4^s \cdots)$ with maximum s. Arrays of the type $L_n(4^u \cdot 2^{3(s-u)} \cdots)$, $0 \leq u < s$, can be easily obtained as explained in Remark 1 of Section 4. The difference matrices used in the construction are given in the Appendix. Unless otherwise stated, the arrays are apparently new.

1. 18-run arrays.

Using $D_{6,6;3}$, one can obtain an 18-run array from

$$[L_3(3) * D_{6,6;3},\ 0_3 * L_6] .$$

Then $L_{18}(3^7)$, $L_{18}(2 \cdot 3^7)$ and $L_{18}(6 \cdot 3^6)$ can be obtained by using respectively $L_6(3)$, $L_6(2 \cdot 3)$ and $L_6(6)$ for L_6. These arrays have been used extensively by G. Taguchi and others.

2. 24-run arrays.

Using H_{12} one can obtain a 24-run array from

$$[L_2(2) * H_{12},\ 0_2 * L_{12}] .$$

Then from Class A1 we obtain $L_{24}(4 \cdot 2^{20})$ (Dey and Ramakrishna 1977), $L_{24}(6 \cdot 4 \cdot 2^{11})$ (Agrawal and Dey 1982) and $L_{24}(3 \cdot 4 \cdot 2^{13})$ by using respectively $L_{12}(2^{11})$, $L_{12}(6 \cdot 2^2)$ and $L_{12}(3 \cdot 2^4)$ for L_{12}. Here

$$L_{12}(6 \cdot 2^2) = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}^T$$

and

$$L_{12}(3 \cdot 2^4) = \begin{bmatrix} 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

is a new array. Both will be used in Examples 3, 5 and 11. It can be shown that the maximum for x in $L_{12}(3 \cdot 2^x)$ is 4.

3. 36-run arrays.

A 36-run array can be constructed from

$$[L_3(3) * D_{12,12;3}, 0_3 * L_{12}] .$$

By using $L_{12}(3^1)$, $L_{12}(2^{11})$, $L_{12}(12^1)$, $L_{12}(6 \cdot 2^2)$, $L_{12}(3 \cdot 4)$ and $L_{12}(3 \cdot 2^4)$ for L_{12}, we obtain respectively, $L_{36}(3^{13})$ (Seiden 1954), $L_{36}(3^{12} \cdot 2^{11})$ (Taguchi 1987), $L_{36}(3^{12} \cdot 12^1)$ (Taguchi 1987), $L_{36}(3^{12} \cdot 6 \cdot 2^2)$, $L_{36}(3^{13} \cdot 4)$ (Dey 1985, p. 62) and $L_{36}(3^{13} \cdot 2^4)$. Here $L_{12}(6 \cdot 2^2)$ and $L_{12}(3 \cdot 2^4)$ are given in Example 2.

4. 40-run arrays.

A 40-run array can be constructed from

$$[L_2(2) * H_{20}, \ 0_2 * L_{20}] .$$

From Class A1 with $L_{20}(20^1)$, $L_{20}(2^{19})$, $L_{20}(10^1 \cdot 2^2)$ and $L_{20}(5 \cdot 2^8)$ for L_{20}, we obtain respectively, $L_{40}(20^1 \cdot 2^{20})$ (Dey 1985, p. 67), $L_{40}(4 \cdot 2^{36})$ (Dey and Ramakrishna 1977),

$L_{40}(10^1 \cdot 4 \cdot 2^{19})$ (Agrawal and Dey 1982) and $L_{40}(5 \cdot 4 \cdot 2^{25})$. Here

$$L_{20}(10 \cdot 2^2) = \left[L_2(2) * D_{10,2;2},\ 0_2 * L_{10}(10^1)\right]$$

and

$$L_{20}(5 \cdot 2^8) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}^T$$

is a new array. It can be shown that the maximum for x in $L_{20}(5 \cdot 2^x)$ is 8.

5. 48-run arrays.

A 48-run array can be constructed from

$$\left[L_4(2^3) * H_{12},\ 0_4 * L_{12}\right]. \tag{7}$$

From Class B1 with $L_{12}(2^{11})$, $L_{12}(6 \cdot 2^2)$, $L_{12}(3 \cdot 2^4)$, $L_{12}(3 \cdot 4)$ and $L_{12}(12^1)$ for L_{12} in (7), we obtain respectively, $L_{48}(4^3 \cdot 2^{38})$ (Chacko, Dey and Ramakrishna 1979), $L_{48}(6 \cdot 4^3 \cdot 2^{29})$, $L_{48}(4^3 \cdot 3 \cdot 2^{31})$, $L_{48}(4^4 \cdot 3 \cdot 2^{27})$ (Dey 1985, p. 51) and $L_{48}(12^1 \cdot 4^3 \cdot 2^{27})$ (Agrawal and Dey 1982).

From Class C1 with $L_{12}(2^{11})$, $L_{12}(6 \cdot 2^2)$ and $L_{12}(3 \cdot 2^4)$ for L_{12} in (7), we obtain respectively $L_{48}(8 \cdot 2^{40})$, $L_{48}(8 \cdot 6 \cdot 2^{31})$ and $L_{48}(8 \cdot 3 \cdot 2^{33})$.

To obtain more 4-level columns, we use the difference matrix $D_{12,12;4}$ in 4 symbols in the following construction

$$\left[L_4(4) * D_{12,12;4},\ 0_4 * L_{12}\right]. \tag{8}$$

By using $L_{12}(2^{11})$, $L_{12}(6 \cdot 2^2)$, $L_{12}(3 \cdot 2^4)$, $L_{12}(3 \cdot 4)$ and $L_{12}(12^1)$ for L_{12} in (8), we obtain respectively $L_{48}(4^{12} \cdot 2^{11})$, $L_{48}(6 \cdot 4^{12} \cdot 2^2)$, $L_{48}(4^{12} \cdot 3 \cdot 2^4)$, $L_{48}(4^{13} \cdot 3)$ and $L_{48}(12^1 \cdot 4^{12})$.

6. 50-run arrays.

A 50-run array can be constructed from

$$[L_5(5) * D_{10,10;5},\ 0_5 * L_{10}].$$

By using $L_{10}(2 \cdot 5)$ and $L_{10}(10^1)$ for L_{10} we obtain respectively $L_{50}(2 \cdot 5^{11})$ and $L_{50}(10^1 \cdot 5^{10})$ (Masuyama 1957).

7. 54-run arrays.

A 54-run array can be constructed from

$$\left[L_9(3^4) * D_{6,6;3},\ 0_9 * L_6\right].$$

By using $L_6(3 \cdot 2)$ and $L_6(6)$ for L_6 we obtain respectively $L_{54}(3^{25} \cdot 2)$ and $L_{54}(6 \cdot 3^{24})$ (Dey 1985, p. 70). The four columns $L_9(3^4) * 0_6$ in $L_9(3^4) * D_{6,6;3}$ can be replaced by a 9-level column since the nine level combinations of these four columns appear equally often. Therefore we obtain respectively $L_{54}(9 \cdot 3^{21} \cdot 2)$ and $L_{54}(9 \cdot 6 \cdot 3^{20})$.

8. 72-run arrays.

A 72-run array can be constructed from

$$[L_2(2) * H_{36},\ 0_2 * L_{36}].$$

Using $L_{36}(6 \cdot 3^{12} \cdot 2^2)$, $L_{36}(3^{12} \cdot 2^{11})$, $L_{36}(3^{13} \cdot 2^4)$ and $L_{36}(12^1 \cdot 3^{12})$ given in Example 3 for L_{36}, we obtain from Class A1 respectively $L_{72}(6 \cdot 4 \cdot 3^{12} \cdot 2^{35})$, $L_{72}(4 \cdot 3^{12} \cdot 2^{44})$, $L_{72}(4 \cdot 3^{13} \cdot 2^{37})$ and $L_{72}(12^1 \cdot 2^{36} \cdot 3^{12})$. The last two arrays are better than the array $L_{72}(4 \cdot 3^4 \cdot 2^{36})$ in Dey (1985, p. 65). As remarked in Section 4, from $L_{72}(6 \cdot 4 \cdot 3^{12} \cdot 2^{35})$ we can construct an $L_{72}(6 \cdot 3^{12} \cdot 2^{38})$ which is better than the array $L_{72}(6 \cdot 3^{12} \cdot 2^{37})$ given by Gupta and Nigam (1987).

To obtain more 6-level columns, we use the difference matrix $D_{12,6;6}$ in 6 symbols in the following construction

$$[L_6(6) * D_{12,6;6},\ 0_6 * L_{12}].$$

By using $L_{12}(2^{11})$, $L_{12}(12^1)$, $L_{12}(3\cdot 4)$, $L_{12}(3\cdot 2^4)$ and $L_{12}(6\cdot 2^2)$ for L_{12}, we obtain respectively $L_{72}(6^6\cdot 2^{11})$, $L_{72}(12^1\cdot 6^6)$, $L_{72}(6^6\cdot 4\cdot 3)$, $L_{72}(6^6\cdot 3\cdot 2^4)$ and $L_{72}(6^7\cdot 2^2)$.

9. 80-run arrays.

An 80-run array can be constructed from

$$\left[L_4(2^3) * H_{20},\ 0_4 * L_{20}\right].$$

From Classes B1 and C1 with $L_{20}(2^{19})$, $L_{20}(10\cdot 2^2)$, $L_{20}(5\cdot 2^8)$ (given in Example 4) and $L_{20}(20^1)$ for L_{20}, we obtain respectively $L_{80}(4^3\cdot 2^{70})$ (Chacko, Dey and Ramakrishna 1979), $L_{80}(8\cdot 4^3\cdot 2^{63})$, $L_{80}(10\cdot 4^3\cdot 2^{53})$, $L_{80}(8\cdot 10\cdot 2^{55})$, $L_{80}(5\cdot 4^3\cdot 2^{59})$, $L_{80}(8\cdot 5\cdot 2^{61})$ and $L_{80}(20^1\cdot 4^3\cdot 2^{51})$ (Dey 1985, p. 50).

10. 90-run arrays.

$$L_{90}(6\cdot 5\cdot 3^{30}) = [L_3(3) * D_{30,30;3},\ 0_3 * L_{30}(6\cdot 5)].$$

Replacing $L_{30}(6\cdot 5)$ by $L_{30}(5\cdot 3\cdot 2)$, we obtain an $L_{90}(5\cdot 3^{31}\cdot 2)$.

11. 96-run arrays.

A 96-run array can be constructed from

$$\left[L_8(2^7) * H_{12},\ 0_8 * L_{12}\right].$$

From Class B2 with $L_{12}(12^1)$, $L_{12}(2^{11})$, $L_{12}(4\cdot 3)$, $L_{12}(3\cdot 2^4)$ and $L_{12}(6\cdot 2^2)$ for L_{12}, we obtain respectively $L_{96}(12^1\cdot 4^7\cdot 2^{63})$ (Cheng 1987), $L_{96}(4^7\cdot 2^{74})$ (Cheng 1987), $L_{96}(4^8\cdot 3\cdot 2^{63})$, $L_{96}(4^7\cdot 3\cdot 2^{67})$ and $L_{96}(4^7\cdot 6\cdot 2^{65})$. From Class C2 with $L_{12}(2^{11})$, $L_{12}(3\cdot 2^4)$ and $L_{12}(6\cdot 2^2)$ for L_{12}, we obtain respectively $L_{96}(8\cdot 4^4\cdot 2^{76})$, $L_{96}(8\cdot 4^4\cdot 3\cdot 2^{69})$ and $L_{96}(8\cdot 4^4\cdot 6\cdot 2^{67})$. Since the seven 2-level columns $L_8(2^7) * 0_{12}$ in $L_8(2^7) * H_{12}$ together with the column of zeros form a group under addition modulus 2, we can replace them by an 8-level column. Then we can construct an $L_{96}(12^1\cdot 8\cdot 2^{77})$ by using $L_{12}(12^1)$ for L_{12}.

Rearrange rows of an L_{12} having a 2-level column, say b, such that this 2-level column coincides with a nonzero column in H_{12}. Then the fifteen columns $[L_8(2^7) * [0_{12}, b], 0_8 * b]$ together with the column of zeros form a group under addition modulus 2. Replacing these fifteen columns by a 16-level column and using $L_{12}(2^{11})$, $L_{12}(3 \cdot 2^4)$ and $L_{12}(6 \cdot 2^2)$ for L_{12}, we obtain respectively $L_{96}(16^1 \cdot 2^{80})$, $L_{96}(16^1 \cdot 3 \cdot 2^{73})$ and $L_{96}(16^1 \cdot 6 \cdot 2^{71})$.

Another class of 96-run arrays can be constructed by

$$[L_2(2) * H_{48},\ 0_2 * L_{48}], \tag{9}$$

where L_{48} is an array constructed through $D_{12,12;4}$ in Example 5. From $H_{48} = H_4 * H_{12}$ and the construction of such L_{48}, (9) can be rewritten as

$$[L_2(2) * (H_4 * H_{12}),\ 0_2 * (L_4(4) * D_{12,12;4}),\ 0_2 * (0_4 * L_{12})].$$

Let $H_4 = [0_4, b_2, b_3, b_4]$ and $H_{12} = [0_{12}, c_1, c_2, c_3, c_4, \mathbf{B}]$. Now suppose there are s 2-level columns in L_{12}, $1 \leq s \leq 4$, which coincide with s nonzero columns in an H_{12}. Without loss of generality, assume that these s columns are $c_1, \ldots c_s$. Then we have

$$\begin{aligned} L_2(2) * (0_4 * 0_{12}) + L_2(2) * (0_4 * c_1) &\equiv 0_2 * (0_4 * c_1) \mod 2, \\ L_2(2) * (b_i * 0_{12}) + L_2(2) * (b_i * c_i) &\equiv 0_2 * (0_4 * c_i) \mod 2 \text{ for } 2 \leq i \leq s \text{ if, } s > 1. \end{aligned}$$

By replacing each of these s triplets of 2-level columns by a 4-level column, we obtain s 4-level columns. Using $L_{12}(12^1)$, $L_{12}(2^{11})$, $L_{12}(3 \cdot 2^4)$ and $L_{12}(6 \cdot 2^2)$ for L_{12}, we obtain respectively $L_{96}(12^1 \cdot 4^{12} \cdot 2^{48})$, $L_{96}(4^{16} \cdot 2^{47})$, $L_{96}(4^{16} \cdot 3 \cdot 2^{40})$ and $L_{96}(6 \cdot 4^{14} \cdot 2^{44})$. Here we use the fact that the four 2-level columns of the $L_{12}(3 \cdot 2^4)$ in Example 2 are part of an H_{12}. Such an H_{12} is given in the Appendix.

12. 98-run arrays.

$$L_{98}(14 \cdot 7^{14}) = \left[L_7(7) * D_{14,14;7},\ 0_7 * L_{14}(14^1)\right].$$

Replacing $L_{14}(14^1)$ by $L_{14}(7 \cdot 2)$, we obtain an $L_{98}(7^{15} \cdot 2)$ (Masuyama 1957).

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Appendix

Some Difference Matrices

$D_{3,3;3}$

0	0	0
0	1	2
0	2	1

$D_{5,5;5}$

0	0	0	0	0
0	1	2	3	4
0	2	3	4	1
0	3	4	1	2
0	4	1	2	3

$D_{6,6;3}$ (Masuyama 1957)

0	0	0	0	0	0
0	1	2	0	1	2
0	2	1	1	0	2
0	0	2	1	2	1
0	2	0	2	1	1
0	1	1	2	2	0

$D_{10,10;5}$ (Masuyama 1957)

0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	1	2	3	4	0
0	2	4	1	3	0	2	4	1	3
0	3	1	4	2	2	0	3	1	4
0	4	3	2	1	2	1	0	4	3
0	1	0	2	2	3	4	4	3	1
0	2	2	0	1	4	3	1	3	4
0	3	4	3	0	4	1	2	2	1
0	4	1	1	4	3	3	2	0	2
0	0	3	4	3	1	4	1	2	2

H_{12}

0	0	0	1	0	0	1	1	1	1	1	0
0	0	0	0	1	1	0	1	0	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	1	1	0	1	1	0	0	0	1
0	0	1	1	0	1	0	0	1	0	1	1
0	0	1	0	1	1	1	0	1	1	0	0
0	1	0	0	1	0	1	0	1	0	1	1
0	1	0	1	1	1	0	1	1	0	0	0
0	1	0	1	0	1	1	0	0	1	0	1
0	1	1	0	0	0	0	1	1	1	0	1
0	1	1	0	0	1	1	1	0	0	1	0
0	1	1	1	1	0	0	0	0	1	1	0

$D_{12,12;3}$ (Seiden 1954)

0	0	0	1	1	0	0	1	0	2	2	0
0	0	0	0	2	0	2	0	2	0	0	1
0	0	1	0	0	2	1	2	0	0	1	0
0	0	2	2	0	1	0	0	1	1	0	0
0	1	2	2	0	0	1	1	2	0	2	2
0	1	2	1	2	1	2	2	2	2	1	0
0	1	0	0	2	2	0	2	1	1	2	2
0	1	1	2	1	2	2	0	0	2	0	2
0	2	1	2	1	0	0	2	2	1	1	1
0	2	1	0	0	1	2	1	1	2	2	1
0	2	2	1	2	2	1	1	0	1	0	1
0	2	0	1	1	1	1	0	1	0	1	2

$D_{12,12;4}$ (Seberry 1979)

00	00	00	00	00	00	00	00	00	00	00	00
00	00	00	01	01	01	11	11	11	10	10	10
00	00	00	11	11	11	10	10	10	01	01	01
00	11	01	10	01	11	01	10	00	11	00	10
00	11	01	11	10	01	00	01	10	10	11	00
00	11	01	01	11	10	10	00	01	00	10	11
00	01	10	11	00	10	01	00	11	01	11	10
00	01	10	10	11	00	11	01	00	10	01	11
00	01	10	00	10	11	00	11	01	11	10	01
00	10	11	01	10	00	01	11	10	01	00	11
00	10	11	00	01	10	10	01	11	11	01	00
00	10	11	10	00	01	11	10	01	00	11	01

addition table for $D_{12,12;4}$

	00	01	10	11
00	00	01	10	11
01	01	00	11	10
10	10	11	00	01
11	11	10	01	00

$D_{12,6;6}$ adapted from a $D_{12,6;12}$ in Beth, Jungnickel and Lenz (1985, p. 364)

0	0	0	0	0	0
0	1	3	2	4	0
0	2	0	1	5	2
0	3	1	5	4	2
0	4	3	5	2	1
0	5	5	3	1	1
0	0	2	3	2	3
0	1	2	4	0	5
0	2	5	2	3	4
0	3	4	1	1	4
0	4	1	0	3	5
0	5	4	4	5	3

$D_{14,14;7}$ (Masuyama 1957)

0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	1	2	3	4	5	6	0
0	2	4	6	1	3	5	5	0	2	4	6	1	3
0	3	6	2	5	1	4	5	1	4	0	3	6	2
0	4	1	5	2	6	3	1	5	2	6	3	0	4
0	5	3	1	6	4	2	0	5	3	1	6	4	2
0	6	5	4	3	2	1	2	1	0	6	5	4	3
0	1	5	5	1	0	2	6	3	4	2	4	3	6
0	2	0	1	5	5	1	3	6	6	3	4	2	4
0	3	2	4	2	3	0	4	6	5	1	1	5	6
0	4	4	0	6	1	6	2	3	1	3	2	5	5
0	5	6	3	3	6	5	4	4	1	2	0	2	1
0	6	1	6	0	4	4	3	2	5	5	2	3	1
0	0	3	2	4	2	3	6	4	6	5	1	1	5

$D_{30,30;3}$ (de Launey 1986)

0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 2 2 2
0 0 0 0 0 1 1 1 1 1 2 2 2 2 2 0 0 0 0 0 1 1 1 1 1 2 2 2 2 2
0 0 0 0 0 1 1 1 1 1 0 0 0 0 0 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1
0 0 0 0 0 2 2 2 2 2 1 1 1 1 1 2 2 2 2 2 1 1 1 1 1 0 0 0 0 0
0 0 0 0 0 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 0 0 0 0 0 1 1 1 1 1
0 1 1 2 2 0 1 1 2 2 0 1 1 2 2 0 1 1 2 2 0 1 1 2 2 0 1 1 2 2
0 1 1 2 2 1 2 0 1 2 1 2 0 1 2 1 2 0 1 2 1 2 0 1 2 1 2 0 1 2
0 1 1 2 2 1 0 2 2 1 1 0 2 2 1 1 0 2 2 1 1 0 2 2 1 1 0 2 2 1
0 1 1 2 2 2 1 2 1 0 2 1 2 1 0 2 1 2 1 0 2 1 2 1 0 2 1 2 1 0
0 1 1 2 2 2 2 1 0 1 2 2 1 0 1 2 2 1 0 1 2 2 1 0 1 2 2 1 0 1
1 2 0 1 2 0 1 1 2 2 2 2 1 0 1 2 1 2 1 0 1 0 2 2 1 1 2 0 1 2
1 2 0 1 2 1 2 0 1 2 0 1 1 2 2 2 2 1 0 1 2 1 2 1 0 1 0 2 2 1
1 2 0 1 2 1 0 2 2 1 1 2 0 1 2 0 1 1 2 2 2 2 1 0 1 2 1 2 1 0
1 2 0 1 2 2 1 2 1 0 1 0 2 2 1 1 2 0 1 2 0 1 1 2 2 2 2 1 0 1
1 2 0 1 2 2 2 1 0 1 2 1 2 1 0 1 0 2 2 1 1 2 0 1 2 0 1 1 2 2
1 0 2 2 1 0 1 1 2 2 2 1 2 1 0 1 2 0 1 2 2 2 1 0 1 1 0 2 2 1
1 0 2 2 1 1 2 0 1 2 2 2 1 0 1 1 0 2 2 1 0 1 1 2 2 2 1 2 1 0
1 0 2 2 1 1 0 2 2 1 0 1 1 2 2 2 1 2 1 0 1 2 0 1 2 2 2 1 0 1
1 0 2 2 1 2 1 2 1 0 1 2 0 1 2 2 2 1 0 1 1 0 2 2 1 0 1 1 2 2
1 0 2 2 1 2 2 1 0 1 1 0 2 2 1 0 1 1 2 2 2 1 2 1 0 1 2 0 1 2
2 1 2 1 0 0 1 1 2 2 1 0 2 2 1 2 2 1 0 1 1 2 0 1 2 2 1 2 1 0
2 1 2 1 0 1 2 0 1 2 2 1 2 1 0 0 1 1 2 2 1 0 2 2 1 2 2 1 0 1
2 1 2 1 0 1 0 2 2 1 2 2 1 0 1 1 2 0 1 2 2 1 2 1 0 0 1 1 2 2
2 1 2 1 0 2 1 2 1 0 0 1 1 2 2 1 0 2 2 1 2 2 1 0 1 1 2 0 1 2
2 1 2 1 0 2 2 1 0 1 1 2 0 1 2 2 1 2 1 0 0 1 1 2 2 1 0 2 2 1
2 2 1 0 1 0 1 1 2 2 1 2 0 1 2 1 0 2 2 1 2 1 2 1 0 2 2 1 0 1
2 2 1 0 1 1 2 0 1 2 1 0 2 2 1 2 1 2 1 0 2 2 1 0 1 0 1 1 2 2
2 2 1 0 1 1 0 2 2 1 2 1 2 1 0 2 2 1 0 1 0 1 1 2 2 1 2 0 1 2
2 2 1 0 1 2 1 2 1 0 2 2 1 0 1 0 1 1 2 2 1 2 0 1 2 1 0 2 2 1
2 2 1 0 1 2 2 1 0 1 0 1 1 2 2 1 2 0 1 2 1 0 2 2 1 2 1 2 1 0