A THEORY OF PERFORMANCE MEASURES IN PARAMETER DESIGN

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A THEORY OF PERFORMANCE MEASURES IN PARAMETER DESIGN*

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ABSTRACT

Parameter design is a quality engineering method, popularized by Japanese quality expert G. Taguchi, that aims to reduce sensitivity to hard-to-control variation in products and manufacturing processes. The method finds the settings of design factors that minimize expected loss due to variation. To do the minimization Taguchi uses controversial two-step procedures involving quantities he calls signal-to-noise (SN) ratios. To explain SN ratios Leon, Shoemaker, and Kacker (1987) introduced Performance Measures Independent of Adjustment (PerMIAs) and showed that some of Taguchi's SN ratios are PerMIAs. In this paper we propose a theory to explain the roles of PerMIAs and adjustment factors in the two-step procedures for constrained minimization. We develop conditions for finding PerMIAs and two-step procedures. In addition, the modeling techniques developed for parameter design problems involving quadratic loss are extended to general loss functions. For this purpose, general dispersion, location and off-target measures are introduced. Our results are illustrated with several examples involving quadratic and other loss functions.

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CONTENTS

1. INTRODUCTION AND SUMMARY

1.1 Parameter Design: Reducing Sensitivity to Variation

A step in processing silicon wafers for IC device fabrication is to grow an epitaxial layer on the silicon wafers. For one type of AT&T IC device, the specifications called for a layer thickness between 14 and 15 micrometers. Yet, the variation around the ideal 14.5 micrometers was too large to meet this specification. To reduce this variation, the engineers working with statisticians identified eight crucial process design factors. Then, using a statistically planned experiment it was found that two design factors, nozzle position and susceptor-rotation method, had the most influence on the epitaxial layer's variability. The experiment also showed that one factor, deposition time, had a large effect on average thickness but no effect on variability. Changing the settings of these three factors to settings suggested by the data analysis, the engineers reduced the variability of the epitaxial layer by 60%, as confirmed by a follow-up experiment. The change to new settings did not increase cost. (See Kacker and Shoemaker, 1986).

Problems, such as the above, where the aim is to increase quality by identifying special settings of the design factors were introduced by Taguchi (Taguchi, 1986; Taguchi and Phadke, 1984; Taguchi and Wu, 1980) under the general title of *parameter design.* More specifically following Leon, Shoemaker, and Kacker (1987) (henceforth denoted by LSK), we define parameter design as the operation of choosing settings for the design factors of a product or manufacturing process to reduce sensitivity to *noise.* Noise is hard-to-control variability affecting performance; for example, the followings are considered to be noise: deviations in the raw materials from specifications, changes in the manufacturing or field operating environment such as temperature or humidity, drifts in the settings of the design factors over time, and deviations of design factor settings from nominal settings due to manufacturing variability.

In parameter design, noise is assumed to be uncontrollable. After parameter design, if the loss caused by noise is still excessive, the engineer may proceed to control the noise through relatively expensive countermeasures, such as the use of higher grade raw materials or higher-precision manufacturing equipment.

1.2 Formulation of Parameter Design Problems

Figure 1 from LSK shows a block diagram representation of the type of parameter design problem that includes the wafer fabrication example. Taguchi (Taguchi and Phadke, 1984) calls this type the *static* parameter design problem because the target is fixed. In this block diagram, for given settings of design factors Θ the noise N produces an output Y; that is, the output is determined by some transfer function $f(N;\Theta)$. The noise is assumed random; hence the output is random. A loss is incurred if the output differs from a fixed target t that represents the ideal output. The average loss is given by

$$R(\Theta) = EL(Y,t)$$

where L is a loss function. The goal of parameter design is to choose the settings of the design factors Θ to minimize average loss. In practice, this minimization may be subject to a constraint, such as the unbiasness constraint, $E(Y)=t$. In some situations the maximum loss over the noise conditions may be a more appropriate measure for $R(\theta)$ than the average loss. This change does not affect the results in this paper on $R(\theta)$ since they are independent of how $R(\theta)$ is defined as a function of loss.

As discussed in LSK, other parameter design problems can be represented with block diagrams similar to the one in Figure 1.

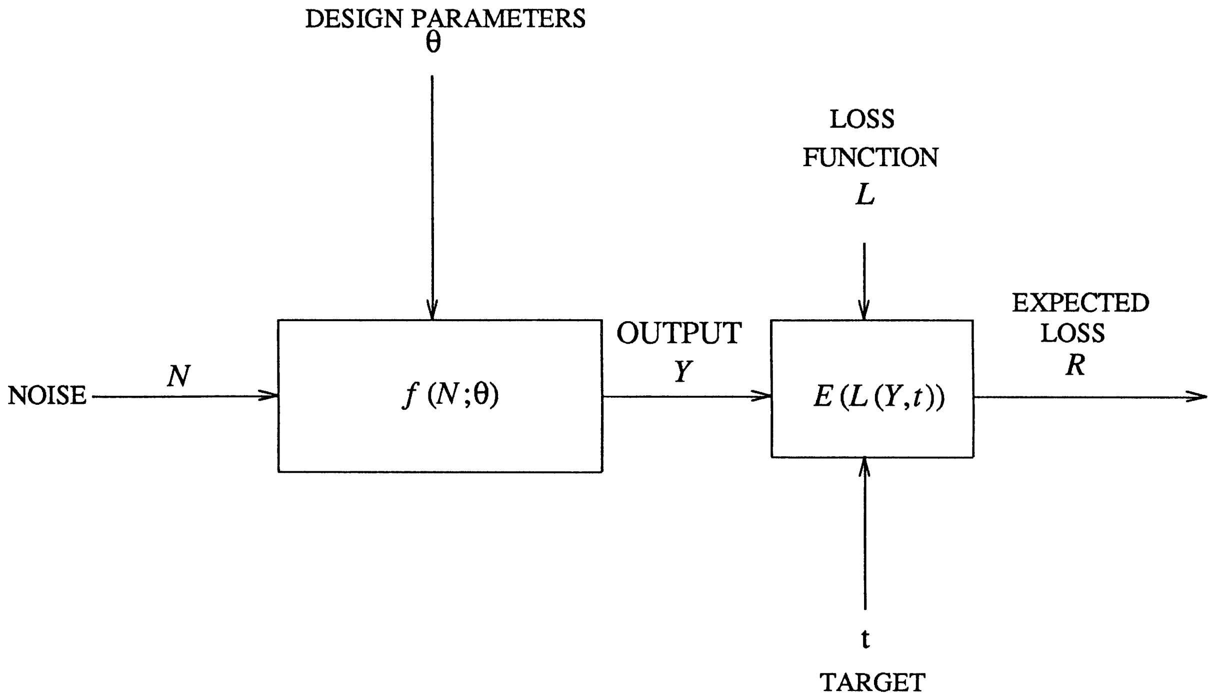

Figure 1. *A Block Diagram Representation of the Static Parameter-Design Problem.* The output Y is determined by the noise N through the transfer function f. The transfer function depends on the design parameters θ. Loss is incurred if the output is not equal to the target t.

1.3 Taguchi's SN Ratios and LSK's PerMIAs

To find the solution to a parameter design problem, Taguchi generally starts by dividing the design factors into two groups, $\Theta=(a, d)$, where a and d are called respectively the adjustment and nonadjustment design factors. Then, to find the optimal settings of the design factors, $\Theta^*=(a^*, d^*)$, he recommends a two-step procedure that can be roughly stated as follows (see Phadke 1982):

Procedure 1 (Taguchi's Generic Two-Step Procedure)

Step 1. Find d^* to maximize a quantity called the *Signal-to-Noise (SN) ratio.*

Step 2. With d fixed at d^*, find a^* by identifying the setting of a that adjusts the output to the target.

For the static parameter design problem with continuous output Y the SN ratio given by Taguchi is

$$SN=10\log[(EY)^2/VarY] \tag{1.1}$$

The use of Taguchi's SN ratios and two-step procedures has been controversial since it has not been clear to many statisticians under what circumstances they should be used. More concretely, a theoretical understanding of the modeling problem that precedes data analysis had been lacking.

Progress in providing this theoretical understanding was made by LSK. They proposed the following two-step procedure to solve the parameter design problem.

Procedure 2 (LSK's Two-Step Procedure)

Step 1. Find d^* to minimize $P(d)=\underset{a}{Min}R(a, d)$.

Step 2. Find a^* to minimize $R(a, d^*)$

LSK called the quantity $P(d)$ in Procedure 2 a *Performance Measure Independent of Adjustment (PerMIA),* since, as discussed above it measures product or process performance independent of adjustment. Then LSK showed that several of Taguchi's SN ratios including the one given in (1.1) coincided with their PerMIAs if different specific transfer functions were chosen.

As an illustration, for the static parameter design problem of Section 1.2 the transfer function LSK identified is: $f(N;a,d)=\mu(a,d)\varepsilon(N,d)$, where $EY=\mu(a,d)$ is a strictly monotone function of each component of a for each d. This model essentially says that there is a factor a so that the SN ratio given in (1.1) does not depend on a.

1.4 Rationale for Using Two-Step Procedures

In parameter design problems an adjustment factor can usually be identified on the basis of engineering knowledge. For example, in many applications there are *scale* factors that are used as adjustment factors. Scale factors can be used to change the "scale" of the product or process. Examples of scale factors are: mold size in tile fabrication, mask dimension in integrated circuit fabrication, and exposure time in window photolithography. In the silicon wafer example of Section 1.1 deposition time is the scale factor. [See Section 2.2 of LSK for more detail on scale factors.]

Methods for identifying adjustment factors based on observed data have also been proposed (Nair and Pregibon, 1986; Box, 1988). But these methods require more data than is usually available in the highly fractionated experiments commonly used in parameter design applications, and perhaps what is more serious they do not incorporate the simple engineering knowledge that enables the investigator to readily identify the adjustment factors. In addition, empirically identified adjustment factors may not hold outside the region of the data. Thus their use to infer optimal product or process behavior around a target outside this region is highly suspect. An illustrative example was given in Wu (1987).

Why use two-step procedures such as Taguchi's or LSK's? We think there are four main advantages in their use in parameter design problems:

1. **Product or process characteristics are used to simplify empirical modeling.** As Phadke (1982) pointed out, it is common to find that there are *adjustment* design factors that have no influence on the SN ratio, which is frequently a measure of variation. As mentioned before, in the epitaxial layer growth example the deposition time is such a factor. Modeling the PerMIA as a function of only the *nonadjustment* design factors simplifies empirical modeling.

2. **Nonadjustment factor settings remain optimal if the target is changed.** The settings of the

nonadjustment design factors d, identified Step 1 remain optimal if design specifications involving the target are changed; for instance, in the epitaxial layer growth example the identified nozzle position and susceptor-rotation method remain optimal if the layer thickness specification is changed. To meet the new specification we simply change the setting of the adjustment design factor, the deposition time.

3. **Behavior around target of a product or process can be inferred from its behavior off target.** In parameter design experiments, settings for the design factors are commonly chosen using design of experiment techniques. Usually, for none of these settings is the product or process output centered around target. Yet, we have to infer from these data what the optimal settings of these design factors would be with product or process output centered around target. Since the PerMIA usually measures product or process performance independent of output (or of adjustment), the difficulties of having "*off-target data*" can be circumvented. For instance, in the epitaxial layer growth example the PerMIA in (1.1) is independent of the scale factor, layer thickness.

4. **Constrained optimization problems are transformed into unconstrained optimization problems.** Using these two-step procedures, the *constrained* optimization problems found in parameter design can be transformed into *unconstrained* optimization problems. For instance, in the continuous output static parameter design problem with unbiasness constraint given above the minimization involves an unbiasness constraint. Yet, the two-step procedure involves no constrained optimization -- the first step of the two-step procedure involves an unconstrained maximization, and the second step is an adjustment. [See Section 3 for details].

These four points are not always true. (See Wu, 1987, for an example.) However, by proper choice of output characteristic and design factors, these four points can often be realized in practice.

1.5 Goals of this Paper

Goal 1. Further Understanding of Two-Step Procedures. LSK's work leaves several questions incompletely answered which we address in this paper. Among them are these three questions:

A. *How does a two-step procedure turn a constrained parameter design problems into an unconstrained one?*

B. *When is the second step of the two-step procedure an adjustment?*

C. *What is a formal definition of PerMIA?*

Goal 2: Development of Modeling Techniques for Non-Quadratic Loss. The quadratic loss function assumed by all previous authors is not adequate for many problems of practical interest; for example, the loss for underfilling or overfilling a container are typically unequal.

1.6 Overview

In Section 2 we give a new definition of PerMIA involving a two-step procedure for solving a constrained minimization problem. The idea is to transform a high dimensional constrained optimization problem into a *high dimensional unconstrained* optimization problem followed by a *low dimensional constrained* optimization problem. We also introduce maximal PerMIAs to describe PerMIAs that identify all solutions to a constrained minimization problem. We then proceed to give geometric and analytic characterizations of PerMIAs and maximal PerMIAs.

In Section 3 we give conditions under which the second step of the two-step procedure of Section 2 is an "adjustment." We introduce *adjustment functions* to describe functions of the design factors which are used to make adjustments. So far the only adjustment function used has been the mean. With the work in this section other adjustment function, such as the median, can be used as we show in Section 7. In Section 4 we prove a number of results that can be used to find PerMIAs for constrained minimization problems. In Section 5 we apply the results of the previous three sections to particular parameter design problems involving quadratic loss. For example, we obtain more general results than those of Box (1988), Nair and Pregibon (1986) and Tsui (1987).

In Section 6 we develop modeling techniques for use in parameter design problems involving non-quadratic loss functions. These techniques exploit special properties found in engineering problems. For many products and manufacturing processes, such as the epitaxial layer growth process of Section 1.1, performance is best measured in terms of a dispersion measure. This follows since it is often easy to center output around the target once dispersion is reduced. To exploit this property when the loss function is non-quadratic, we introduce a general class of dispersion and location measures that is well suited for parameter design applications. Roughly, the dispersion measure measures expected loss around an "ideal" target (location). An associated notion introduced in Section 6 is the off-target measure. In Section 7 we use these general dispersion, location, and off-target measures to extend a useful model considered by Nair and Pregibon (1986), Tsui (1987), and Box (1988). Then, we apply the notions developed in Section 6 to four special cases of practical interest that involve non-quadratic loss. In each case we derive the dispersion, the off-target measure, the adjustment function, and the associated two-step procedure.

The results are given in the context of parameter design but can be used for more general optimization problems which are beyond the scope of this paper.

2. PERMIAS AND CONSTRAINED MINIMIZATION PROBLEMS

Let X be a compact region in R^{n+m} and let A and D be respectively its projection onto R^m and R^n, where R^k is the Euclidean k-space. Here A and D stand for the spaces of adjustment and nonadjustment design factors. The elements of A and D are denoted by a and d.

Let R be a continuous function from X into R^1. Throughout we will refer to the following optimization problem.

Constrained Minimization Problem (CMP): Find $(a^*, d^*) \varepsilon X$ to minimize R (a, d).

The primary example for R in this paper is the expected loss (i.e. risk) in a parameter design problem. As remarked in Section 1.1, other functions of the loss such as the maximum loss can also be used for R. In practice the minimization problem *CMP* may involve constraints on a and d, for example, through requirements on the mean response. To streamline the presentation, we will absorb such constraints into the definition of X but still call it a constrained minimization problem.

In what follows we will show that under some conditions the *CMP* can be solved using a two-step procedure. To introduce the first of these procedures, we use the following notation:

(2.1)
$$X_d = \{a : (a, d) \varepsilon X\} \text{ for } d \varepsilon D;$$
$$R_d(a) = R(a, d) \text{ for } a \varepsilon X_d \text{ and } d \varepsilon D.$$

Throughout let P be a continuous function from D into R^1.

Constrained Two-Step Procedure (C2P):

Step 1. Find $d^* \varepsilon D$ to minimize $P(d)$ over D.

Step 2. Find $a^* \varepsilon X_{d^*}$ to minimize $R_{d^*}(a)$.

Compactness of X ensures the existence of a^* and d^* in CMP and $C2P$.

We now give a more general definition of a *Performance Measure Independent of Adjustment (PerMIA)* than the one originally given in LSK (1987).

Definition 2.1.

a) The function P is a *PerMIA* for the *CMP* if the solutions to the *C2P* involving P are solutions to the CMP.

b) A *PerMIA* P is *maximal* if *every* solution to the *CMP* can be obtained with the *C2P* involving P.

In many problems *PerMIAs* can be chosen that allows the decomposition of the *CMP* into two simpler problems. The first problem deals with d only and involves no constraint. The second problem deals with a only. Although this second problem may involve a constraint, the variable a is of *low* dimension. Examples are given in Section 3.

We restate the notion of *PerMIA* in geometrical terms in Proposition 2.2.

Let $D^*(P)$ be the *minima* set of P and X^* be the *solution set* of the *CMP*, (2.3) $$D^*(P) = \{d^* \varepsilon D: P(d^*) = \min_{d\varepsilon D} P(d)\}$$

(2.4) $$X^* = \{(a^*, d^*) \varepsilon X: R(a^*, d^*) = \min_{(a,d)\varepsilon X} R(a, d)\} .$$

Let D_{X^*} be the projection of X^* onto D, that is

$$D_{X^*} = \{d^* \varepsilon D: (a^*, d^*) \varepsilon X^*\}$$

Proposition 2.2.

a) P is a *PerMIA* for the *CMP* if and only if $D^*(P) \subset D_{X^*}$.

b) P is a maximal *PerMIA* for the *CMP* if and only if $D^*(P) = D_{X^*}$.

The following theorem provides a method for constructing a maximal *PerMIA* for any *CMP*. We will give an application of this construction in Section 4 along with other approaches for identifying *PerMIAs*.

Theorem 2.3. Define the function M to be

(2.5) $$M(d) = \min_{a\varepsilon X_d} R(a, d) \text{ for } d \varepsilon D.$$

Then M is a maximal *PerMIA* for the *CMP*.

Proof. First we show that M is a *PerMIA*. This is quite obvious since for any solution (a^*, d^*) of the $C2P$ involving M,

$$R(a^*,d^*) = \min_{a\varepsilon X_{d^*}} R(a,d^*) = M(d^*) =$$

$$\min_{d\varepsilon D} M(d) = \min_{(a,\ d)\varepsilon X} R(a, d) .$$

To prove that the *PerMIA* M is maximal, we must show that (a^*, d^*) can be obtained using the *C2P* involving M for any (a^*, d^*) solving the CMP.

First we show that the first step of the *C2P* identifies d^*, i.e., (2.6) $$M(d^*) = \min_{d \varepsilon D} M(d) .$$

This follows from

$$M(d) = \min_{a \varepsilon X_d} R(a, d) \geq \min_{(a, d) \varepsilon X} R(a, d) = R(a^*, d^*)$$

$$= \min_{a \varepsilon X_{d^*}} R(a, d^*) = M(d^*).$$

From the definition of R_{d*} the second step of the $C2P$ identifies a^*, thus completing the proof. Q.E.D.

In LSK (1987) the *PerMIA* used throughout is a special case of that given by *Theorem 2.3.* The following corollary of *Proposition 2.2* and *Theorem 2.3* gives another characterization of *PerMIAs.*

Corollary 2.4. Let M be as in Theorem 2.3 and let $D^*(M)$ be defined as in (2.3). Then

(a) P is a *PERMIA* for the *CMP* if and only if $D^*(P) \subset D^*(M)$

(b) P is a *maximal PERMIA* for the *CMP* if and only if $D^*(P) = D^*(M)$.

3. ADJUSTMENT FACTORS AND FUNCTIONS

As we mentioned in the introduction, the second step of the two-step procedures used by Taguchi and subsequent workers, is usually an "adjustment". In this section we develop the concept of adjustability and adjustment function and give conditions under which the second step of the *C2P* is an "adjustment". For convenience we first restate the *CMP* in an equivalent form.

Unlike Section 2 in which we absorb the constraint into the definition of *X*, we now write explicitly the constraint on *a* and *d* as $h(a,d) \varepsilon T$, where h is a continuous function from X into R^k and T is a compact subset of R^k. Then the *CMP* can be restated as follows:

"Find $(a^*, d^*) \varepsilon X$ to minimize $R(a, d)$ subject to the constraint $h(a, d) \varepsilon T$. "

The concept of adjustability and of adjustment functions is developed below.

Definition 3.1. For $t \varepsilon T$ and $d \varepsilon D$, the function h is $(t,\ d)$*–adjustable* if there exists $a \varepsilon X_d$ such that $h(a,\ d) = t$. We refer to the function h as an *adjustment function.*

Define

$$Z = \left\{ (t,\ d) : t \varepsilon T,\ d \varepsilon D \ and \ t = h(a,\ d) \ for\ some\ a \varepsilon X_d \right\},$$

$$Z_d = \left\{ t : (t,\ d) \varepsilon Z \right\} \ for \ \ d \varepsilon D \quad .$$

Note that Z_d is the set of realizable target values for d.

Adjustability Conditions for the Function P:

AC1. There exists a function $H: Z \rightarrow R^1$ such that for $(a,\ d) \varepsilon X$

$$R(a,\ d) = H(h(a,\ d),\ d).$$

AC2. The set of m satisfying

$$H(m,\ d^*) = \min_{t \varepsilon Z_{d*}} H(t,\ d^*)$$

is nonempty and is independent of $d^* \varepsilon D^*(P)$, where $D^*(P)$ is as defined in (2.3).

AC3. The function h is $(m,\ d^*)$*–adjustable* for all $d^* \varepsilon D^*(P)$ and m given in $AC2$.

Under these adjustability conditions for P we define the following two-step procedure.

Two-Step Procedure with Adjustment (2PA)

Step 1. *Unconstrained Step*

Find $d^* \varepsilon D$ to minimize $P(d)$ over D.

Step 2. *Adjustment Step*

Find $a^* \varepsilon X_{d*}$ such that $h(a^*,\ d^*) = m$ for some m given in $AC2$.

In view of Step 2, we call any value of m given in AC3 an *adjustment point.* As shown in the example of Section 5, m can often be identified without solving a complicated optimization problem.

Theorem 3.2. Assume the Adjustability Conditions for the function P. Then

(a) The solutions to the *C2P* and to the *2PA* are identical.

(b) P is a *PerMIA* if and only if the solutions to the *2PA* involving P are solutions to the *CMP*.

(c) A *PerMIA* P is maximal if and only if every solution to the *CMP* can be obtained with the *2PA* involving P.

Proof. Parts (b) and (c) are immediate from Part (a) and the definitions of PerMIA and maximal PerMIA. Hence we only need to prove Part (a) .

Since the first step of the *C2P* and *2PA* are identical, to prove Part (a) we only need to show that a solution of the second step of the *2PA* is a solution of the second step of the *C2P* and vice versa. For $d^* \varepsilon D^*(P)$ and a^* selected in Step 2 of the $2PA$,

$$\begin{aligned} R_{d*}(a^*) &= H(h(a^*, d^*), d^*) = H(m, d^*) \\ &= \min_{h(a,\ d*) \varepsilon T} H(h(a,\ d^*), d^*) \leq H(h(a,\ d^*),\ d^*) \\ &= R_{d*}(a), \end{aligned}$$

which shows that a^* is a solution to the second step of the $C2P$. To show the converse, let $d^* \varepsilon D^*(P)$ and a^* be a solution to the second step of the $C2P$, i.e.,

$$\begin{aligned} R(a^*, d^*) &= H(h(a^*, d^*), d^*) = \min_{h(a,\ d^*) \varepsilon T} H(h(a, d^*), d^*) \\ &= \min_{t \varepsilon Z_{d*}} H(t, d^*). \end{aligned}$$

From the Adjustability Condition AC2, $h(a^*, d^*) = m$ for some m given in AC2, that is, a^* is chosen according to step 2 of the $2PA$.

Q.E.D.

In the important special case of $h(a, d) = a$, the adjustability conditions reduce to

"*The set of* m *satisfying* $R(m, d^*) = \min_{a \varepsilon X_{d^*}} R(a, d^*)$ *exists and is independent of* $d^* \varepsilon D^*(P)$."

In this case the second step of the 2PA is simply to set a equal to some m given by this condition.

4. METHODS FOR FINDING PERMIAS

In this section we give several results that are particularly convenient for finding PerMIAs. First, we state a technical theorem that implies all the other results in this section.

Theorem 4.1. In addition to the Adjustability Conditions for P, assume

1. $\min_{t \varepsilon Z_{d^*}} H(t, d^*) = c$ is a constant independent of $d^* \varepsilon D^*(P)$, where $D^*(P)$ is as defined in (2.3).

2. $\min_{t \varepsilon Z_d} H(t, d) \geq c$ for $d \notin D^*(P)$.

Then

(a) P is a *PerMIA* for the *CMP*;

(b) The solutions to the 2PA involving P are solutions to the CMP.

If, in addition, the inequality in Condition 2 is strict, then

(c) P is a maximal *PerMIA* for the *CMP*;

(d) The solutions to the $2PA$ involving P and to the *CMP* are identical.

Proof. By Theorem 3.2(b), Part (b) follows from Part (a). By Corollary 2.4(a) to show Part (a) it is enough to show that $D^*(P) \varepsilon D^*(M)$, or, equivalently, that $M(d) \geq M(d^*)$ for $d^* \varepsilon D^*(P)$ and $d \varepsilon D$. Noting that the constraint set is $X \cap h^{-1}(T)$,

$$M(d) = \min_{h(a,\, d) \varepsilon T} H(h(a, d), d)$$

$$= \min_{t \varepsilon Z_d} H(t, d) \geq c = \min_{t \varepsilon Z_{d^*}} H(t, d^*)$$

$$= M(d^*).$$

Part (a) follows.

To prove Part (c), note that if the inequality in Condition 2 is strict, then retracing the inequalities above, it can be seen that $d \not\in D^*(P)$ cannot be in $D^*(M)$, or equivalently, $D^*(M) \subset D^*(P)$. It follows that $D^*(M) = D^*(P)$, which by Corollary 2.4(b) implies Part (c).

Part (d) follows from Part (c) by Theorem 3.2 (c). Q.E.D.

Corollary 4.2. (Wu, 1987). Let $R_1 : D \rightarrow R^1$ and $R_2 : X \rightarrow R^1$ be functions such that

$$R(a, d) = R_1(d) + R_2(a, d)$$

for all $(a, d) \; \varepsilon \; X$. Let $D^*_{R_1}$ be the minima set for R_1. Assume that:

1. $\min_{a \,\varepsilon\, X_d} R_2(a, d) = c$ is a constant independent of $d \varepsilon \; D^*_{R_1}$.

2. $\min_{a \varepsilon X_d} R_2(a, d) \geq c$ for $d \not\in D^*_{R_1}$.

Then R_1 is a PerMIA. If, in addition, the inequality in Condition 2 is strict R_1 is a maximal *PerMIA*.

Proof. Since $min \; R(a, d) = R_1(d) + min \; R_2(a, d)$ for $a \varepsilon \; X_d$, the result is immediate from Theorem 4.1. Q.E.D

Wu (1987) pointed out that R_1 is not a PerMIA of the form given in Theorem 2.3 (or in LSK), and gave an application of this type of PerMIA.

Corollary 4.3. Assume that:

1. For each $(a, d) \varepsilon X$, $R(a, d) = Q(h(a, d), P(d))$ for some function $Q: R^{k+1} \rightarrow R^1$.

2. $Z_d \subset Z_{d}^{*}$, i.e., the set of realizable target values for d is contained in the set of realizable target values for $d^* \ \varepsilon D^*(P)$.

3. For each $t \ \varepsilon T$ and $d \ \varepsilon D$, $Q(t, P(d^*)) \leq Q(t, P(d))$.

4. For $d^* \ \varepsilon D^*(P)$, $m \ \varepsilon T$ exists and satisfies

$$Q(m, P(d^*)) = \min_{t \ \varepsilon Z_{d^*}} Q(t, P(d^*)) .$$

5. h is $(m, d^*) - adjustable.$

Then

(a) P is a *PerMIA*

(b) The *2PA* gives a solution to the *CMP*.

Proof: Let $H(t, d) = Q(t, P(d))$ for $t \ \varepsilon T$, $d \ \varepsilon D$. Then clearly the Adjustability Conditions for P and Condition 1 of Theorem 4.1 are satisfied. We now verify Condition 2 of Theorem 4.1. Let $d \varepsilon D^*(P)$. Then

$$\begin{aligned} \min_{t \ \varepsilon Z_d} H(t, d) &= \min_{t \ \varepsilon Z_d} Q(t, P(d)) \\ &\geq \min_{t \ \varepsilon Z_{d^*}} Q(t, P(d^*)) && \text{[by Conditions 2 and 3]} \\ &= \min_{t \ \varepsilon Z_{d^*}} H(t, d^*) = c. && \text{Q.E.D} \end{aligned}$$

Condition 2 of Corollary 4.3 would usually be established by showing that the function Q is increasing in its second argument for each value of the first argument. The constant m would be usually identified by differentiation of the function $f(x) = Q(x, P(d^*))$ with respect to x. This is the case in an example in Section 5.

5. EXAMPLES FOR QUADRATIC LOSS

In this section we illustrate the preceding theory by expanding on results by previous authors.

Nair and Pregibon (1986), Box (1988) and Tsui (1987) consider the parameter design problem under square error loss with the model

$$Var\ Y = \gamma(\mu(a,\ d))P(d)$$

for the variance of the response. Here $\mu(a,\ d)$ is the mean expressed as a function of the design factors, P is a positive function and γ is a positive convex function. Note that

$$E(Y-t)^2 = \gamma(\mu(a,\ d))P(d) + (\mu(a,\ d) - t)^2$$

where t is the target. Define

$$h(a,\ d) = \mu(a,\ d) \text{ for } (a,\ d) \in X.$$

Consider in turn the constraint sets

$$(a)\ \ T = \{t\}$$

and

$$(b)\ \ T = R^1\ .$$

Case (a) corresponds to an unbiasness constraint, and case (b) to no constraint on the mean. In either case it is clear that Corollary 4.3 applies so that P is a *PerMIA*.

In case (a) the adjustment step of the *2PA* is to choose a^* such that

$$\mu(a^*,\ d^*) = t,$$

i.e., to adjust the mean on target.

To find the adjustment step of the *2PA* in case (b), consider the function f given by

$$f(x) = \gamma(x)P(d^*) + (x-t)^2 .$$

From the remark following Corollary 4.3 the adjustment step of the *2PA* is to choose a^* such that $\mu(a^*, d^*) = m$ where m minimizes f.

In the special case of $\gamma(x) = x^2$, which is the model behind Taguchi's SN ratio for the static (stationary target) parameter design problem, the adjustment point is $m = t/(1+P(d^*))$ and the adjustment step of the $2PA$ is to choose a^* such that

$$\mu(a^*, d^*) = \frac{t}{1 + P(d^*)}$$

as LSK showed. Box (1988) coined the term "*aim off factor*" for m. We prefer to call it "shrinkage factor" since the "adjustment" is to something less than the target, not to the target as Taguchi and Phadke (1984) seem to imply.

This *shrinkage phenomenon* holds for very general models. Assume that $\gamma(x)$ is increasing in $|x|$ and strictly convex. Then

$$f'(x) = \gamma'(x)P(d^*) + 2(x-t)$$

has the following properties:

(i) $f'(x)$ is strictly increasing,

(ii) $f'(0) \leq 0 < f'(t)$ for $t \geq 0$,

$f'(0) \geq 0 > f'(t)$ for $t \leq 0$.

These properties imply that the adjustment point m satisfies $|m| \leq |t|$, and consequently that the adjustment step of the *2PA* is to choose a^* so that $\mu(a^*, d^*) = m$ which is less than the target in absolute value.

The conditions on $\gamma(x)$ above are satisfied by the class of power function $\gamma(x) = |x|^\alpha, \alpha \geq 1$. The adustment point m can be readily computed in closed form for several values of α which are given in Table 1. From this table we see that the nature of the shrinkage depends on the value of α. For example, when $\alpha = 1$ the shrinkage is additive and when $\alpha = 2$ it is multiplicative.

α	m
1	$\max\{t - P_* / 2, 0\}, \quad t \ge 0$
	$\min\{t + P_* / 2, 0\}, \quad t < 0$
$\frac{3}{2}$	$t - \frac{9P_*^2}{32}\left[\left(1 + \frac{64t}{9P_*^2}\right)^{1/2} - 1\right], \quad t \ge 0$
	$t + \frac{9P_*^2}{32}\left[\left(1 + \frac{64t}{9P_*^2}\right)^{1/2} - 1\right], \quad t < 0$
2	$t / (1 + P^*)$
3	$t - \frac{1}{3P_*}\left[1 + 3P_*t - (1 + 6P_*t)^{1/2}\right], \quad t \ge 0$
	$t + \frac{1}{3P_*}\left[1 + 3P_*t - (1 + 6P_*t)^{1/2}\right], \quad t < 0$

Table 1. *Adjustment points m for several values of* α *in* $\gamma(x) = x^\alpha$. We write P_* for $P(d^*)$.

LSK identified a model leading to Taguchi's signal-to-noise ratio for a dynamic parameter design problem involving a measuring instrument. (Dynamic means "moving target response".) They provided an engineering justification, which we omit, for the model including a physical interpretation of its adjustment factors. Their model for the response is

$$Y = \alpha(a_1, d) + \beta(a_2, d)(\gamma(d)s + \varepsilon\ (N, d)), \tag{5.1}$$

where s is the target response and (a_1, a_2, d) are the design factors. The objective of parameter design is to find the setting of design factors to minimize $E(Y - s)^2$ over a range of targets s, subject to the unbiasness constraint $EY = s$.

In our framework we can write this constraint as $h(a, d) \varepsilon T$, where

$$h(a, d) = (\alpha\ (a_1, d),\ \beta\ (a_2, d)) \text{ for } (a_1, a_2, d) \varepsilon X$$

and

$$T = \{(0, \gamma(d)^{-1})\ :\ d \varepsilon D\}.$$

Note that

$$E(Y - s)^2 = [\beta\ (a_2, d)]^2\ \mathrm{Var}_N\ \varepsilon\ (N, d) + [\alpha\ (a_1, d) + \beta\ (a_2, d)\gamma(d)s - s]^2$$

can be expressed as H (h (a,d), d). Then by imposing the constraint h(a,d) ε T, we have

$$P(d) = \min_{t \varepsilon Z_d} H(t, d) = \frac{\mathrm{Var}_N \varepsilon\ (N, d)}{[\gamma(d)]^2}.$$

By Theorem 3.2 this P is a PerMIA under condition AC3.

To summarize, the $2PA$ for the problem is

1. Find $d^* \varepsilon D$ to minimize $\dfrac{Var_N \varepsilon\ (N, d)}{[\gamma(d)]^2}$.
2. Find (a_1^*, a_2^*) such that $\alpha\ (a_1^*, d^*) = 0$ and $\beta\ (a_2^*, d^*) = 1/\gamma(d^*)$.

LSK showed that this *PerMIA* is *equivalent* to Taguchi's SN ratio for a dynamic parameter design problem.

6. DISPERSION, LOCATION AND OFF-TARGET MEASURES FOR GENERAL LOSS FUNCTIONS

As discussed in the introduction, for many products and manufacturing processes, performance is conveniently measured in terms of a dispersion measure. This follows since it is often easy to center output around the target once dispersion has been reduced. To exploit this property, when the loss function is non-quadratic, in this section we introduce general dispersion, location and off-target measures. These measures are used in the next section to develop tractable forms of two-step procedures for general loss functions. It is important to develop these methods for non-quadratic loss because quadratic loss is often found to be unrealistic in practical applications. Use of the Taylor series approximation to justify quadratic loss as discussed in Section 1.5 has flaws. Also, for moderate to large deviations from the target, quadratic function provides a poor approximation to the true loss function since it often overpenalizes such deviations. Quadratic loss also ignores the possible asymmetric nature of loss about the target.

The motivation for deriving these dispersion, location and off-target measures comes from the familiar formula for quadratic loss

$$R = E(X - t)^2 = var\, X + (EX - t)^2 \tag{6.1}$$

As shown in Section 5, this formula is exploited to derive two-step procedures for quadratic loss. With the general definition of dispersion, location and off-target measure, a formula similar to (6.1) is available for deriving two-step procedures for general loss function.

Let $L(x, t)$ be the loss accrued when the response is x and the target is t. Let X be the random variable associated with the response x and F be its associated distribution. Then we define the risk R_t by

$$R_t = E_F L(X, t) ,$$

We define the *dispersion measure* for X associated with the loss function L to be the minimum of R_t when the target t is allowed to vary in the space T of physically realizable targets, that is,

$$D = \min_{t \varepsilon T} R_t = R_{t^*} ,$$

where t^* minimizes R_t over T. We call t^* the *location measure* of X associated with the loss function L. Note that t^* is the ideal value of the target when the distribution is F and the loss function is L and that D and t^* do not depend on the target t.

We call the excess risk

$$O_t = R_t - D = R_t - R_{t^*}, \tag{6.2}$$

resulting from t^* not being equal to the intended target t, the *off-target measure* of X from t. Rewriting (6.2) as

$$R_t = D + O_t ,$$

the risk is the sum of the dispersion measure and the off-target measure, a complete analog to (6.1). Note that for the quadratic loss, D is the variance of X, t^* is the mean of X and $O_t = (EX-t)^2$ is the bias square. We consider two other examples below.

First we consider the asymmetric square error loss $L_2(x, t) = w_t(x-t)^2$ with $T = R^1$, where

$$w_t = \begin{cases} b_1 \text{ if } x < t \\ b_2 \text{ if } x > t \end{cases}, \quad b_1, b_2 > 0 . \tag{6.3}$$

This loss function gives different penalties for deviations above and below target, as would be the case in food packaging where an under-fill usually results in a bigger loss to the manufacturer than an over-fill.

To obtain an expression for the dispersion measure D, we differentiate

$$R_t = b_1 \int_{-\infty}^{t} (x-t)^2 dF(x) + b_2 \int_{t}^{\infty} (x-t)^2 dF(x)$$

to get $\frac{\partial R_t}{\partial t} \Big|_{t=t^*} = 0$, which gives

$$b_1 \int_{-\infty}^{t^*} (x-t^*) dF(x) + b_2 \int_{t^*}^{\infty} (x-t^*) dF(x) = 0 \quad ,$$

or, equivalently

(6.4)
$$t^* = \frac{b_1 \int_{-\infty}^{t^*} x dF(x) + b_2 \int_{t^*}^{\infty} x dF(x)}{b_1 F(t^*) + b_2 (1-F(t^*))} \quad .$$

Since $\partial^2 R_t \ / \ \partial t^2 > 0$, t^* is the unique minimizer of R_t. Note that t^* is implicitly defined but can be obtained by iteratively solving (6.4). It is evident from (6.4) that t^* can be interpreted as a location measure of X for the loss L_2. The dispersion measure is $D = R_{t^*}$, with R and t^* given above. The off-target measure $O_t = R_t - R_{t^*}$ can be obtained from the decomposition

(6.5)
$$w_t(x-t)^2 - w_{t^*}(x-t^*)^2 = 2w_{t^*}(x-t^*)(t^*-t) + w_{t^*}(t^*-t)^2 + (w_t - w_{t^*})(x-t)^2 \ .$$

The first term of the right side of (6.5) has expectation zero from (6.4). The factor $w_t - w_{t^*}$ is zero for x outside the interval $(\min\{t, t^*\} \ , \ \max\{t, t^*\})$. Then by taking the expectation of the right side of (6.5), we have

$$O_t = \int_{\min\{t,\ t^*\}}^{\max\{t,\ t^*\}} \{w_t^*[(t^*-t)^2 - (x-t)^2] + w_t(x-t)^2\} \ dF(x)$$

(6.6) $$+ (t^*-t)^2[b_1 F(min\{t,t^*\}) + b_2(1 - F(\max\{t,\ t^*\}))] \quad ,$$

where w_t^* and w_t in the curly bracket are respectively b_1 and b_2 (b_2 and b_1) for $t<t^*(t^*<t)$.

Another important loss function is the absolute error loss

(6.7) $$L_1(x,\ t) = \begin{cases} b_1 \mid x-t \mid & \text{for } x \le t, \\ b_2 \mid x-t \mid & \text{for } x > t. \end{cases}$$

This loss occurs when the penalty is linear in the deviations from target. Note that the penalties for above and below target are allowed to be different. It is proved in the Appendix that the risk $R_t = E_F L_1(X,\ t)$ has the decomposition

(6.8) $$R_t = D + O_t \ ,$$

where

$$D = \int_{-\infty}^{t^*} b_1 \mid x-t^* \mid \ dF(x) + \int_{t^*}^{\infty} b_2 \mid x-t^* \mid \ dF(x)$$

is a dispersion measure of X,

(6.9) $$t^* = F^{-1}(b_2 \ / \ (b_1+b_2)),$$

and $$O_t = (b_1+b_2) \int_{\min\{t,\ t^*\}}^{\max\{t,\ t^*\}} \mid x-t \mid \ dF(x)$$

is an off-target measure which decreases as the location measure t^* gets closer to the intended target t.

Note that t^* is the $100\, b_2 / (b_1 + b_2)$ percentile of X. When $b_1 \neq b_2$ (i.e., the penalties for above and below target are different), t^* is different from the median in the direction with the larger penalty.

7. DEVELOPMENT OF TWO-STEP PROCEDURES FOR GENERAL LOSS FUNCTIONS

In this section we model the general dispersion and off-target measures as functions of the design factors. This model allows us to develop tractable forms of two-step procedures for general loss function. The results of Section 4 are used to derive and justify these two-step procedures.

As shown in Section 6, the risk R_t is the sum of the dispersion measure D and the off-target measure O_t. In practical applications it is often convenient to model separately the dependency of D and O_t on the non-adjustment and adjustment factors d and a. We assume

$$D = P_1(a, d)P(d) \quad , \quad O_t = P_2(a, d) \quad , \tag{7.1}$$

which is analogous to the assumptions adopted by Nair and Pregibon (1986), Box (1988) and Tsui (1987) for the quadratic error loss. Then $R_t = P_1(a, d)P(d) + P_2(a, d)$ is increasing in $P(d)$, so by Corollary 4.3 $P(d)$ is a PerMIA and the two-step procedure holds under appropriate conditions.

In the rest of this section we illustrate this modeling technique with four examples. In particular, for each example we (1) develop formulas for P, P_1, and P_2, (2) identify an adjustment function. and (3) use this adjustment function to derive a two-step procedure, which can be generically stated as follows:

Procedure 3

Step 1. Find d^* to minimize the PerMIA *P(d)*.

Step 2. Find a^* such that $h(a^*, d^*) = t$, where h is an adjustment function.

To give the two-step procedure for each example we only need to identify $P(d)$ and $h(a,d)$.

Example 1: Additive Model and Asymmetric Squared Error Loss

The additive model for the output is

$$x(a,\ d) = \mu(a,\ d) + \varepsilon(d)\ , \tag{7.2}$$

where the error ε has distribution F_d and μ is an arbitrary function of (a,d). (Commonly, the dependency of F_d on d is through its standard deviation $\sigma(d)$, e.g., $F_d(\varepsilon) = F(\varepsilon\ /\sigma\ (d))$ but we do not need this assumption). For the asymmetric square error loss (6.3) the location measure t^* given by (6.4) is equal to $\mu(a,\ d) + \varepsilon^*$, where

$$\varepsilon^* = \frac{b_1 \int_{-\infty}^{\varepsilon^*} \varepsilon\ dF_d(\varepsilon) + b_2 \int_{\varepsilon^*}^{\infty} \varepsilon dF_d(\varepsilon)}{b_1 F_d(\varepsilon^*) + b_2(1-F_d(\varepsilon^*))}. \tag{7.3}$$

Note that ε^* depends on d only. Since $D = R_t*$, the dispersion measure is given by

$$D = P(d) = b_1 \int_{-\infty}^{\varepsilon^*} (\varepsilon - \varepsilon^*)^2 dF_d(\varepsilon) + b_2 \int_{\varepsilon^*}^{\infty} (\varepsilon - \varepsilon^*)^2 dF_d(\varepsilon). \tag{7.4}$$

Note that D is a function of d only. Since $O_t = R_t - R_t*$, by writing $\varepsilon_t = t - \mu(a,\ d)$, the off-target measure is given by

$$O_t = P_2(a,d) = \int_{\min\{\varepsilon_t,\ \varepsilon^*\}}^{\max\{\varepsilon_t,\ \varepsilon^*\}} \left\{ w_{t^*}[(\varepsilon^* - \varepsilon_t)^2 - (\varepsilon - \varepsilon_t)^2] + w_t(\varepsilon - \varepsilon_t)^2 \right\} dF_d(\varepsilon) +$$

$$(\varepsilon^* - \varepsilon_t)^2 [b_1 F_d(\min\{\varepsilon_t,\ \varepsilon^*\}) + b_2(1-F_d(\max\{\varepsilon_t,\ \varepsilon^*\}))].$$

To derive the adjustment function, note that the off-target measure O_t has minimum at zero when $\varepsilon_t = \varepsilon^*$, that is, $\mu(a,\ d) + \varepsilon^*(d) = t$, where $\varepsilon^*(d)$ is given by (7.3). We can choose, for a given d, the adjustment factor a to satisfy the previous equation. Hence, for this problem, we use

(7.5) $$h(a, d) = \mu(a, d) + \varepsilon^*(d)$$

as the adjustment function. Since P and h have been identified, we have the two-step procedures given in Procedure 3. If the function $\mu(a, d)$ allows step 2 of Procedure 3 to be carried out for any d^* in $D^*(P)$, then the conditions of Corollary 4.3 are satisfied. Therefore, Procedure 3 gives solutions to the *CMP*, and *P(d)* is a *PerMIA*.

Example 2. Additive Model and Absolute Error Loss

For the absolute error loss (6.7), the location measure t^* given by (6.4) is equal to $\mu(a, d) + \varepsilon^*(d)$, where

(7.6) $$\varepsilon^*(d) = F_d^{-1}(b_2 \ / \ (b_1+b_2)).$$

The dispersion measure is given by

(7.7) $$D = P(d) = \int_{-\infty}^{\varepsilon^*} b_1 \mid \varepsilon - \varepsilon^* \mid \ dF_d(\varepsilon) + \int_{\varepsilon^*}^{\infty} b_2 \mid \varepsilon - \varepsilon^* \mid \ dF_d(\varepsilon),$$

which is a function of d only. By writing $\varepsilon_t = t - \mu(a, d)$, the off-target measure $O_t = R_t - R_t*$ can be shown to be

$$O_t = P_2(a, d) = (b_1+b_2) \int_{\min\{\varepsilon_t, \varepsilon^*\}}^{\max\{\varepsilon_t, \varepsilon^*\}} \mid \varepsilon - \varepsilon_t \mid \ dF_d(\varepsilon).$$

To derive the adjustment functions, note that $O_t = 0$ if $\varepsilon_t = \varepsilon^*$, that is, $\mu(a, d) + \varepsilon^*(d) = t$, where $\varepsilon^*(d)$ is given by (7.6). Using

(7.8) $$h(a, d) = \mu(a, d) + \varepsilon^*(d),$$

as the adjustment function, we have Procedure 3. Its justification is the same as in Example 1.

We now comment on the adjustment step in Procedure 3 for Examples 1 and 2. Assume that the error $\varepsilon(d)$ in the additive model (7.2) has zero mean in the case of square error loss and zero median in the case of absolute error loss; that is we assume that $\mu(a, d)$ in (7.2) is respectively the mean and median of $x(a, d)$. If $b_1 = b_2$, then $\varepsilon^*(d)$ in (7.3) and (7.6) is zero and the adjustment step is to set the mean or median $\mu(a, d)$ on the target t. If $b_1 \neq b_2$, then $\varepsilon^*(d)$ in (7.3) and (7.6) is in general nonzero and can be either positive or negative depending on whether $b_1 > b_2$ or $b_1 < b_2$. So the adjustment is to set the mean or median $\mu(a, d)$ equal to $t - \varepsilon^*(d)$. We call $\varepsilon^*(d)$ a "*location correction factor*" driven by the loss function. Unlike the shrinkage factor in (5.2), it can adjust $\mu(a, d)$ either above or below the target t.

Next we consider Examples 3 and 4. In these two examples, we assume the multiplicative model

(7.9) $$x(a, d) = \mu(a, d)\, \eta\ (d),\ \eta \sim F_d\ ,$$

where the error η depends on d only, $x \geq 0$. Practical examples of (7.9) can be found in LSK.

Example 3. Multiplicative Model And Asymmetric Square Error Loss

For this problem the location measure t^* given by (6.9) can be shown to be $\mu(a, d)\eta^*$, where

(7.10) $$\eta^* = \frac{b_1 \int_0^{\eta^*} \eta\, dF_d(\eta) + b_2 \int_{\eta^*}^{\infty} \eta\, dF_d(\eta)}{b_1 F_d(\eta^*) + b_2(1 - F_d(\eta^*))}$$

Note that η^* depends on d only. The dispersion measure is given by

$$D = [\mu(a, d)]^2 P(d) \quad ,$$

where

$$\text{(7.11)} \qquad P(d) = b_1 \int_0^{\eta^*} (\eta - \eta^*)^2 dF_d(\eta) + b_2 \int_{\eta^*}^{\infty} (\eta - \eta^*)^2 dF_d(\eta).$$

By writing $\eta_t = t / \mu(a, d)$ and using (6.5), the off-target measure can be shown to be

$$O_t = [\mu(a, d)]^2 Q_t(a, d) \quad ,$$

where

$$Q_t(a, d) = \int_{\min\{\eta_t, \eta^*\}}^{\max\{\eta_t, \eta^*\}} \left\{ w_{t^*}[(\eta^* - \eta_t)^2 - (\eta - \eta_t)^2] + w_t(\eta - \eta_t)^2 \right\} dF(\eta)$$

$$+ (\eta^* - \eta_t)^2 [b_1 F(\min\{\eta_t, \eta^*\}) + b_2(1 - F(\max\{\eta_t, \eta^*\}))].$$

Since $[\mu(a, d)]^2$ appears in both D and O_t, the two-step procedure with adjustment can not be simplified.

If the loss function L_2 in (6.3) is rescaled to $L_2(x, t) / x^2$, then the risk simplifies to

$$R_t = P(d) + Q_t(a, d) \quad .$$

To derive the adjustment function we note that $Q_t(a, d)$, the term involving the adjustment factor a, is zero by chosing a to satisfy $\eta_t = \eta^*$, that is, $\eta^*(d)\ \mu(a, d) = t$, $\eta^*(d)$ given by (7.10). Hence using

$$\text{(7.12)} \qquad h(a, d) = \eta^*(d)\ \mu(a, d),$$

as the adjustment function we have Procedure 3 with *P(d)* given by (7.11)

Example 4. Multiplicative Model And Absolute Error Loss

For this problem the location measure t^* given by (6.9) is $\mu(a,\ d)\eta^*(d)$, where

(7.13) $$\eta^*(d) = F_d^{-1}\ (b_2\ /\ (b_1+b_2))\ .$$

The dispersion measure is given by

$$D = \mu(a,d)P(d)\ ,$$

where

(7.14) $$P(d) = \int_0^{\eta^*} b_1 \mid \eta - \eta^* \mid\ dF_d(\eta) + \int_{\eta^*}^{\infty} b_2 \mid \eta - \eta^* \mid\ dF_d(\eta)\ ,$$

and $\eta^* = \eta^*(d)$ is given by (7.13). By writing $\eta_t = t/\mu(a,\ d)$, the off-target measure is

$$O_t = \mu(a,\ d)\ Q_t(a,\ d)\ ,$$

where

(7.15) $$Q_t(a\ ,d) = (b_1+b_2) \int_{\min\{\eta_t,\eta^*\}}^{\max\{\eta_t,\eta^*\}} \mid \eta - \eta_t \mid dF(\eta),\ \eta^* \textit{ given by } (7.13)\ .$$

As in the previous example, by rescaling the L_1 loss to $L_1(x,\ t)/x$, the risk simplifies to

$$R_t = P(d) + Q_t(a, d) \quad ,$$

where $P(d)$ and $Q_t(a, d)$ are given by (7.14) and (7.15). To derive the adjustment function, note that $Q_t(a, d)$, the term involving the adjustment factor a, is zero by choosing a to satisfy $\eta_t = \eta^*$, that is, when $\eta^*(d)\ \mu(a, d) = t$ where $\eta^*(d)$ is given by (7.15). Using

$$h(a, d) = \eta^*(d)\mu(a, d) \ , \tag{7.16}$$

as the adjustment function, we have Procedure 3 with $P(d)$ given by (7.14).

If the function $\mu(a, d)$ allows step 2 of Procedure 3 to be carried out for any d^* in $D^*(P)$, then from Corollary 4.3, for the two rescaled loss functions in Examples 3 and 4, Procedure 3 gives solutions to the *CMP* and $P(d)$ is a PerMIA for each problem.

We conclude this section with some comments on the adjustment steps to Procedure 3 for Examples 3 and 4. Assume $\mu(a, d)$ in (7.9) is respectively the mean and median of $x(a, d)$. Then $\eta(d)$ has mean one and median one respectively. If $b_1 = b_2$, then $\eta^*(d)$ in (7.10) and (7.13) equals one and the adjustment steps is to set the mean or median $\mu(a, d)$ on the target t. If $b_1 \neq b_2$, then $\eta^*(d)$ can be greater than one or smaller than one, depending on whether $b_1 > b_2$ or $b_1 < b_2$. So the adjustment step is to set the mean or median $\mu(a, d)$ equal to $t/\eta^*(d)$. We call $\eta^*(d)$ a "*scale correction factor*" driven by the loss function. Note that unlike the case with the shrinkage factor in Section 5, it can adjust $\mu(a, d)$ to a value either above or below the target t.

APPENDIX

Proof of (6.8):

For any t', $R_t - R_{t'} = E_F\{L_1(X, t) - L_1(X, t')\}$. Take $t' < t$, $L_1(X, t) - L_1(X, t')$ equals

$$\begin{cases} b_1(t-t') & \text{for } X \le t', \\ -(b_1+b_2)X + (b_1 t + b_2 t') & \text{for } t' \le X \le t, \\ b_2(t'-t) & \text{for } X > t. \end{cases}$$

Therefore

$$R_t - R_{t'} = b_1(t-t')F(t') + b_2(t'-t)(1-F(t))$$
$$+ E\{(b_1 t + b_2 t') - (b_1+b_2)X\}\ I_{(t' \le X \le t)}$$

(A.1) $$= (b_1+b_2)E(t-X)I_{(t' \le X \le t)} + F(t')(b_1+b_2)(t-t') + b_2(t'-t)\ .$$

Similarly for $t < t'$,

$$R_t - R_{t'} = (b_1+b_2)E(X-t)I_{(t \le X \le t')}$$

(A.2) $$+ F(t')(b_1+b_2)(t-t') + b_2(t'-t)\ .$$

By differentiation, it is easy to show that $R_t - R_{t'}$ is maximized by taking $t' = t^*$ with $F(t^*) = b_2\ /\ (b_1+b_2)$. Then

$$O_t = R_t - R_{t^*} = (b_1+b_2)E\ |\ X-t\ |\ I_{(\min\{t,\ t^*\} \le X \le \max\{t,\ t^*\})}\ .$$

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