

**NEARLY ORTHOGONAL ARRAYS WITH  
MIXED LEVELS AND SMALL RUNS**

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# NEARLY ORTHOGONAL ARRAYS WITH MIXED LEVELS AND SMALL RUNS

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## *ABSTRACT*

In running a factorial experiment it may be desirable to use an orthogonal array with different (mixed) numbers of factor levels. Because of the orthogonality requirement, mixed orthogonal arrays may have a large run size. By slightly sacrificing the orthogonality requirement, we can obtain nearly orthogonal arrays with economic run size. Some general methods for constructing such arrays are given. For 12, 18, 20, and 24 runs, a large number of orthogonal and nearly orthogonal arrays with mixed levels are constructed and tabulated.

Key words and phrases: Difference matrix, Experimental design, Graphs, Nearly difference matrix.

# 1 Introduction

Two-level fractional factorial experiments are commonly used in scientific and engineering investigations. They do not require a large run size, with the 8-run and 16-run experiments being the most popular. Planning and analysis of these experiments are relatively easy. As a result they are taught in any beginning course on experimental design. In some situations, however, certain factors have more than two levels. It may be undesirable to reduce the factor levels to two if it would result in severe loss in information. Examples include (i) a categorical factor with three machine types or three suppliers and (ii) a continuous factor with three temperature settings. For (i), if the three machine types all contribute to the current production, they should all be included in the study for the purpose of comparison. For (ii), if the response depends on the temperature in a nonmonotone fashion, choice of two temperature settings would not allow the curvilinear relation to be studied. In these and other scenarios, factorial experiments with mixed numbers of factor levels may be adopted. Orthogonal arrays and nearly orthogonal arrays with mixed levels (to be defined later) provide a rich source of layouts for running such experiments. In this paper we give a systematic construction of such arrays with small runs.

We use two examples to illustrate the previous points and motivate the use of such arrays.

Example 1. To reduce the geometric distortion of critical part characteristics of a rear axle gear, it is suggested that the heat treatment process be studied and improved. Five factors are being considered. Factor  $A$  has 3 levels corresponding to the three sources of the gear. Each of the remaining factors (temperature and time in furnace, quench oil type and temperature) has two levels. Because of cost and time considerations, 12 runs is the limit on run size. Find an array to suit this purpose.

A natural first attempt is to use a fractional factorial design for 12 runs, which is obtained by adding a 3-level column to three copies of the fractional factorial design with 4 runs and 3 factors (see Table 1). As shown in Section 2, it is impossible to add another 2-level column to this design and still retain the orthogonality among the columns. Two columns are *orthogonal* if each of their level combinations appears equally often. An array is orthogonal (of strength two) if any pairs of columns are orthogonal (Rao, 1947).

row	column			
1	0	0	0	0
2	0	0	1	1
3	0	1	0	1
4	0	1	1	0
5	1	0	0	0
6	1	0	1	1
7	1	1	0	1
8	1	1	1	0
9	2	0	0	0
10	2	0	1	1
11	2	1	0	1
12	2	1	1	0

Table 1: The 12-run fractional factorial design  $L_{12}(3 \cdot 2^3)$ .

Our construction method in Section 2 provides a satisfactory answer with a 12-run array with one 3-level column and four 2-level columns (see columns *A* to *E* of Table 2.) It is not a fractional factorial design constructed by a group-theoretic method.

row	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	$F_2$	<i>G</i>
1	0	0	0	0	0	0	0	0
2	0	0	1	0	1	0	1	1
3	0	1	0	1	1	1	0	1
4	0	1	1	1	0	1	1	0
5	1	0	0	1	1	1	1	0
6	1	0	1	1	0	1	0	1
7	1	1	0	0	1	0	1	0
8	1	1	1	0	0	0	0	1
9	2	0	0	1	0	0	1	1
10	2	0	1	0	1	1	0	0
11	2	1	0	0	0	1	1	1
12	2	1	1	1	1	0	0	0

Table 2: Columns *A* to *E* form an  $L_{12}(3 \cdot 2^4)$ . Columns *A* to *F*, and *G* form an  $L'_{12}(3 \cdot 2^6)$  and columns *A* to *E*,  $F_2$ , and *G* form another  $L'_{12}(3 \cdot 2^6)$ .

An alternative is to use the following table,

<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
0	0	0	0	0
0	1	1	1	1
1	0	0	1	1
1	1	1	0	0
2	0	1	0	1
2	1	0	1	0
2	0	1	1	0
2	1	0	0	1

which is obtained from the 8-run fractional factorial design with one 4-level column and four 2-level columns by replacing level 3 of the 4-level column by level 2. It saves four runs over the previous arrays but has some disadvantages. The 3-level column is not balanced, i.e., one level receives half of the runs while the other two levels share the remaining quarters. If the level with the best performance happens to be assigned to one of the latter levels, the experiment may not provide enough information on the best level. Furthermore, the six level combinations of the 3-level column and any 2-level column appear with unbalanced frequencies 2:2:1:1:1:1. The imbalance may lead to some difficulty similar to the above. On the other hand, since the frequencies are proportional,

the main effect for the 3-level factor and the main effect for any 2-level factor are uncorrelated (Addelman, 1962).

Example 2. For the experiment in Example 1, it is suggested that two additional 2-level factors, say, furnace cycle time and operating mode, be included in the study. Can a 12-run array suit this purpose?

It is shown in Section 2 that it is impossible to find a 12-run orthogonal array with one 3-level column and more than four 2-level columns. To retain orthogonality, one must increase the run size to 24 to accommodate these two additional factors. A more economical alternative is to keep the run size at 12 but to sacrifice the orthogonality requirement. Ideally we should find an array that is as closely to orthogonal as possible. The method in Section 3 finds one such array (see columns  $A$  to  $F$  and  $G$  of Table 2 or Table 5) that has the smallest number of pairs of nonorthogonal columns, namely  $(D,F)$  and  $(E,G)$ . For this array the 3-level column  $A$  is orthogonal to all the 2-level columns. Since the investigator is unwilling to reduce the levels of  $A$  to two (which would make the whole exercise trivial),  $A$  is usually an important factor. This choice of nonorthogonality seems appropriate.

If engineering knowledge suggests that among the six 2-level factors, three are likely to be important, the previous array would not be suitable because only two 2-level columns  $B$ , and  $C$  are orthogonal to the other columns. The method in Section 3 finds another array (see columns  $A$  to  $E$ ,  $F_2$ , and  $G$  of Table 2 or the layout in Table 8) that has columns  $B$ ,  $C$ , and  $D$  free from nonorthogonality. This array, however, has three pairs of nonorthogonal columns among the factors  $E$ ,  $F_2$ , and  $G$ .

In case that the nonorthogonality among the 2-level factors is more important than the orthogonality between the 3-level factor and any 2-level factor, a different type of nearly orthogonal array

is constructed for this purpose in Table 10 of Section 3.

Formally an orthogonal array of strength two, denoted by  $L_N(s_1^{k_1} \cdots s_r^{k_r})$ , is an  $N \times k$  matrix,  $k = k_1 + \cdots + k_r$ , having  $k_i$  columns with  $s_i$  levels,  $s_i$  being unequal, such that for any two columns all their level combinations appear equally often. If  $r > 1$ , the arrays are said to be *mixed* or to have mixed levels. Orthogonal arrays of strength  $d > 2$  are not considered here because they require a much larger number of runs and can often be obtained from taking a subset of columns of orthogonal arrays of strength two. A nearly orthogonal array, denoted by  $L'_N(s_1^{k_1} \cdots s_r^{k_r})$ , is one in which the orthogonality requirement is nearly satisfied. We use the generic term “nearly” without a rigorous mathematical definition. In some cases we minimize the number of nonorthogonal pairs of columns of certain types or the canonical correlations between two sets of effects. One may use other measures of nonorthogonality to capture the importance the investigator puts on certain types of nonorthogonality. If it is desirable to measure the overall correlations among the columns, one may use the determinant (D) or the trace (A) criterion. This important issue is beyond the scope of the paper.

In Section 2 we give a class of mixed orthogonal arrays of 12 runs. In Section 3 we construct two types of nearly orthogonal arrays of 12 runs. Analogous constructions of 20-run arrays are given in Section 4. By modifying a general method for constructing mixed orthogonal arrays (Wang and Wu, 1989), we give in Section 5 three methods for constructing nearly orthogonal arrays. One of the methods involves a new notion of nearly difference matrix. As an application, we construct nearly orthogonal arrays of 18 and 24 runs in Section 6.

## 2 Construction of $L_{12}(3 \cdot 2^4)$ and $L_{12}(6 \cdot 2^2)$

To construct  $L_{12}(3 \cdot 2^4)$  we use a method based on an intelligent search. It can also be used for constructing other mixed orthogonal arrays with small runs.

First we use isomorphism of arrays to reduce the amount of search. Two arrays are isomorphic if one can be obtained from the other through permutations of rows and columns and level changes.

Without loss of generality, we can fix the first two columns as

$$A = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \end{bmatrix}, \quad B = \begin{bmatrix} a \\ a \\ a \end{bmatrix},$$

where  $\mathbf{i}$  is the  $4 \times 1$  vector of  $\mathbf{i}$ 's,  $a = [0 \ 0 \ 1 \ 1]^t$ , and let the first component of any 2-level column orthogonal to  $A$  and  $B$  be 0. It can be shown (see Appendix A) that, up to isomorphism,

$$C = \begin{bmatrix} b \\ b \\ b \end{bmatrix}, \quad C_2 = \begin{bmatrix} a \\ a + 1 \\ a + b + 1 \end{bmatrix}$$

are the only two columns orthogonal to  $A$  and  $B$ . Here and for the rest of the paper,

$$a = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad 1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

and  $a + b = [0 \ 1 \ 1 \ 0]^t$  is sum of  $a$  and  $b$  modulus 2.

Now consider the first case, i.e.,  $A$ ,  $B$ , and  $C$  are chosen for the first three columns. Any 2-level column orthogonal to  $A$ ,  $B$ , and  $C$  must be from one of the following sets:

$$S_1 = \{[z, z, z]^t\},$$

$$S_2 = \{[a, a + 1, z]^t \text{ and permutations}\} \text{ or}$$

$$S_3 = \{[b, b + 1, z]^t \text{ and permutations}\},$$

where  $z = a + b$  or  $a + b + 1$ . By fixing the first row at 0, there are 16 columns (see Table 3) that are orthogonal to  $A$ ,  $B$ , and  $C$ . Orthogonality among these columns can be represented by the



graph in Figure 1. Each column is represented by a node labelled from 1 to 16. If two columns are orthogonal, their associated nodes are connected by an edge. Since there is no triangle in the graph, the maximum number of additional 2-level columns is two.

row	column									
	$d_1$	$d_2$	$d_3$	$d_4$	$d_5$	$d_6$	$d_7$	$d_8$	$d_9$	$d_{10}$
1 - 4	$a+b$	$a+b$	$a+b$	$a+b$	$a$	$a$	$a$	$a$	$a+b$	$a+b$
5 - 8	$a+b$	$a+b$	$a+b+1$	$a+b+1$	$a+1$	$a+1$	$a+b+1$	$a+b$	$a$	$a+1$
9 - 12	$a+b$	$a+b+1$	$a+b$	$a+b+1$	$a+b+1$	$a+b$	$a+1$	$a+1$	$a+1$	$a$

row	column					
	$d_{11}$	$d_{12}$	$d_{13}$	$d_{14}$	$d_{15}$	$d_{16}$
1 - 4	$b$	$b$	$b$	$b$	$a+b$	$a+b$
5 - 8	$b+1$	$b+1$	$a+b+1$	$a+b$	$b$	$b+1$
9 - 12	$a+b+1$	$a+b$	$b+1$	$b+1$	$b+1$	$b$

Table 3: Candidate 2-level columns, first row being 0, which are orthogonal to  $A, B, C$ .

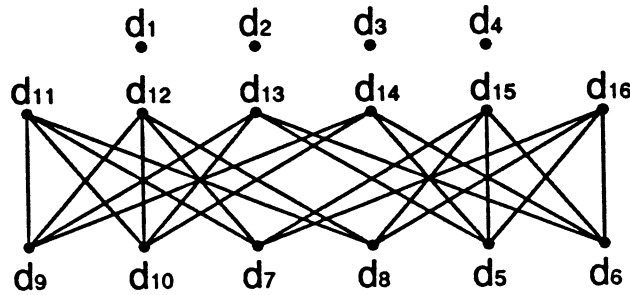


Figure 1: Orthogonality among candidate columns in the first case.

Next we consider the second case, i.e.,  $A, B$ , and  $C_2$  are chosen for the first three columns. By a similar argument, there are 12 columns (see Table 4) orthogonal to  $A, B$ , and  $C_2$ . The orthogonality of these columns is represented by the graph in Figure 2. Again the maximum number of additional 2-level columns is two.

We conclude from both cases that the maximum number of 2-level columns in  $L_{12}(3 \cdot 2^m)$  is 4.

From Figures 1 and 2, there appears to be more than one  $L_{12}(3 \cdot 2^4)$ 's. However, it can be shown

row	column											
	$r_1$	$r_2$	$r_3$	$r_4$	$r_5$	$r_6$	$r_7$	$r_8$	$r_9$	$r_{10}$	$r_{11}$	$r_{12}$
1 - 4	$a+b$	$a+b$	$b$	$b$	$a+b$	$a+b$	$a+b$	$a+b$	$b$	$b$	$b$	$b$
5 - 8	$b$	$b+1$	$a+b+1$	$a+b$	$a+b+1$	$a+b+1$	$a+b$	$a+b$	$b$	$b+1$	$b$	$b$
9 - 12	$b+1$	$b$	$b+1$	$b+1$	$b$	$b+1$	$b+1$	$b$	$b+1$	$b$	$b$	$b+1$

Table 4: Candidate 2-level columns, first row being 0, which are orthogonal to  $A$ ,  $B$ , and  $C_2$ .

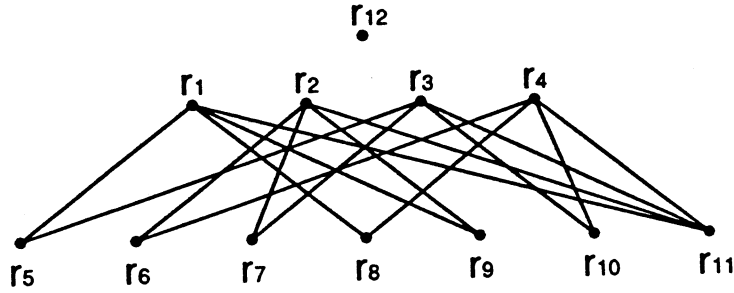


Figure 2: Orthogonality among candidate columns in the second case.

(Wang 1989) that there is a unique  $L_{12}(3 \cdot 2^4)$  up to isomorphism, whose layout is given in Table 2.

For the  $L_{12}(3 \cdot 2^3)$  in Table 1, its third column is one of  $d_1$  to  $d_4$  in Figure 1 for the first case and  $r_{12}$  in Figure 2 for the second case. Since these nodes are isolated from the others, it is not possible to add any 2-level column without sacrificing the orthogonality.

In an  $L_{12}(6 \cdot 2^m)$ , the maximum  $m$  is 2. To see this, let the 6-level column be

$$A = [0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 0 \ 1 \ 2 \ 3 \ 4 \ 5]^t.$$

Then for a 2-level column orthogonal to  $A$ , its last six runs must be the mirror image of the first six runs. That is, it must have the form  $[v, v+1]^t$ , where  $v$  is a  $1 \times 6$  vector. Without loss of generality, denote the first 2-level column by  $B = [v_1, v_1+1]^t$ , where  $v_1 = [0 \ 0 \ 0 \ 0 \ 0 \ 0]$ . Let  $C = [v_2, v_2+1]^t$  and  $C_2 = [v_3, v_3+1]^t$  be two columns orthogonal to  $A$  and  $B$ . From orthogonality,  $v_i + v_1 = 0$  for  $i = 2, 3$ . Therefore each  $v_i$  must have three 0's and three 1's. But then it is impossible that  $v_2$

is orthogonal to  $v_3$ . Therefore, the maximum  $m$  in  $L_{12}(6 \cdot 2^m)$  is 2. By an obvious extension, the maximum  $m$  in  $L_{4k}((2k) \cdot 2^m)$  for odd  $k$  is 2.

### 3 Two Types of Nearly Orthogonal Arrays $L'_{12}(3 \cdot 2^m)$ with $m \leq 9$

Since the maximum number of 2-level columns in  $L_{12}(3 \cdot 2^m)$  is 4, to increase  $m$  beyond 4 we have to sacrifice the orthogonality among some columns. By adding columns to  $L_{12}(3 \cdot 2^4)$ , the resulting array is nearly orthogonal. Two types of arrays are considered, depending on the nature of nonorthogonality.

To construct the first type of arrays, we add to  $L_{12}(3 \cdot 2^4)$  some 2-level columns orthogonal to the 3-level column such that the number of nonorthogonal pairs of 2-level columns is minimized, and subject to this condition, the nonorthogonal pairs have the smallest correlation. Such arrays are referred as type I. The graphs in Figures 1 and 2 provide a convenient tool for choosing such columns. Note that the correlation between the main effects (replace levels 0 and 1 by  $-1$  and  $1$  for the main effects) of any two nonorthogonal columns is  $1/3$  or  $-1/3$ .

In addition to the six degrees of freedom for the main effects in  $L_{12}(3 \cdot 2^4)$ , there are five more degrees of freedom for investigating five additional 2-level factors. For instance, to investigate  $q$  ( $\leq 5$ ) more 2-level factors in addition to those in  $L_{12}(3 \cdot 2^4)$ , we choose  $q$  more columns other than  $d_5$  and  $d_{16}$  from the columns given in Table 3. (Note that  $d_5$  and  $d_{16}$  are the columns  $D$  and  $E$  in  $L_{12}(3 \cdot 2^4)$ .) To minimize the number of nonorthogonal pairs, we need only to choose from the graph in Figure 1 a subgraph that has  $q + 2$  vertices, including  $d_5$  and  $d_{16}$ , and the maximum number of edges. In summary, the layout of nearly orthogonal arrays  $L'_{12}(3 \cdot 2^m)$  of type I for  $m \leq 9$  is given in Table 5.

Minimizing the number of nonorthogonal pairs should not be the sole criterion. Otherwise we could construct  $L'_{12}(3 \cdot 2^{4+x})$ ,  $x \leq 4$ , by repeating columns  $B$ ,  $C$ ,  $D$ , and  $E$  as the last four columns. The resulting arrays have  $x$  nonorthogonal pairs, each of which has correlation 1. For  $x = 1$  and 2, these arrays are inferior to those in Table 5 because the correlations of the nonorthogonal pairs are larger. Although these arrays for  $x = 3$  and 4 have fewer nonorthogonal pairs than those in Table 5, they are much less useful because the nonorthogonal columns are totally confounded.

In an unpublished manuscript “Little Pieces of Mixed Factorials”, Tukey (1959) constructed among many other things a nearly orthogonal array  $L'_{12}(3 \cdot 2^6)$ . For the purpose of comparison, we permute the rows and columns of Tukey’s original array so that its first three columns coincide with  $A$ ,  $B$ , and  $C$ . The resulting array (TK array henceforth) is given in Table 6. Taguchi (1959; 1987, p. 318) gives a nearly orthogonal array  $L'_{12}(3 \cdot 2^6)$  by “partially supplementing” an  $L_8(2^7)$ . This nearly orthogonal array (TG array henceforth), after column permutations, is given in Table 7. Note that the correlation between the main effects of any two nonorthogonal columns in both the TK and TG arrays is also  $\pm 1/3$ .

The  $L'_{12}(3 \cdot 2^6)$  array (WW1 array henceforth) given in Table 5 has several advantages over the TK and TG arrays. First it has only two nonorthogonal pairs whereas each of the TK and TG arrays has three nonorthogonal pairs. Second, its the 2-level columns  $B$  and  $C$  can be used as a *blocking variable* since they are orthogonal to all the other columns. This is impossible for the latter arrays. Furthermore, in the TK array or the TG array, the maximum number of orthogonal 2-level columns is 3, not 4.

We can construct another  $L'_{12}(3 \cdot 2^6)$  of type I by adding the columns  $d_{14}$  and  $d_{15}$  to columns  $A$  to  $E$  in Table 5. We refer to this array as WW2 array and give its layout in Table 8. Although WW2 has 3 nonorthogonal pairs with correlation  $\pm 1/3$ , one more than those of WW1, it has three

row	column										# of non-orthogonal pairs
	A	B	C	D	E	F	G	H	I	J	
				$d_5$	$d_{16}$	$d_6$	$d_{15}$	$d_8$	$d_{14}$	$d_7$	
1 - 4	0	$a$	$b$	$a$	$a+b$	$a$	$a+b$	$a$	$b$	$a$	
5 - 8	1	$a$	$b$	$a+1$	$b+1$	$a+1$	$b$	$a+b$	$a+b$	$a+b+1$	
9 - 12	2	$a$	$b$	$a+b+1$	$b$	$a+b$	$b+1$	$a+1$	$b+1$	$a+1$	
$x$	5			•		•					1
	6				•		•				2
	7			•		•		•			4
	8				•		•		•		7
	9			•		•		•		•	11

Table 5: 12-run  $3 \cdot 2^m$  arrays of type I. The array is orthogonal for  $m \leq 4$ , otherwise it is nearly orthogonal. To get an  $L'_{12}(3 \cdot 2^6)$ , for example, use columns A to G. There are only two nonorthogonal pairs in this array as indicated by paired bullets.

2-level columns free from nonorthogonality. In the latter sense, WW2 is the best. Overall WW2 is better than the TK array and TG arrays since the latter arrays have no 2-level columns free from nonorthogonality.

We next consider the second type of arrays in which all 2-level columns are orthogonal to each other and some 2-level columns are nonorthogonal to the 3-level column. They are referred as type

row	column						
	A	B	C	D	E	F	G
				$d_2$			$d_3$
1 - 4	0	$a$	$b$	$a+b$	$a$	$b$	$a+b$
5 - 8	1	$a$	$b$	$a+b$	$a+1$	$b+1$	$a+b+1$
9 - 12	2	$a$	$b$	$a+b+1$	$a+1$	$b+1$	$a+b$
		•			•		
			•			•	
				•			•

Table 6: Tukey's  $L'_{12}(3 \cdot 2^6)$ . Each nonorthogonal pair is indicated by paired bullets.

row	column						
	A	B	C	D	E	F	G
1 - 4	0	a	b	a+b	a	b	a+b
5 - 8	1	a	b	a+b	a+1	b+1	a+b+1
9 - 12	2	a	b	a+b	a+1	b+1	a+b+1
		•			•		
			•			•	
				•			•

Table 7: Taguchi's  $L'_{12}(3 \cdot 2^6)$ . Each nonorthogonal pair is indicated by paired bullets.

row	column						
	A	B	C	D	E	F <sub>2</sub>	G
1 - 4	0	a	b	a	a+b	b	a+b
5 - 8	1	a	b	a+1	b+1	a+b	b
9 - 12	2	a	b	a+b+1	b	b+1	b+1
					•		•
					•	•	
						•	•

Table 8: WW2 array. Each nonorthogonal pair is indicated by paired bullets.

## II.

Since there are at most four 2-level columns in  $L_{12}(3 \cdot 2^4)$ , any 2-level column orthogonal to the columns  $B, C, D,$  and  $E$  in  $L_{12}(3 \cdot 2^4)$  cannot be orthogonal to  $A$ . Let

$$g_0 = [0 \ 0 \ 0 \ 0]^t, \quad g_1 = [0 \ 1 \ 1 \ 1]^t, \quad g_2 = [0 \ 1 \ 0 \ 0]^t, \quad g_3 = [0 \ 0 \ 1 \ 0]^t, \quad g_4 = [0 \ 0 \ 0 \ 1]^t.$$

Then it can be shown that only 13 columns with the first component at level 0 (see Table 9) are orthogonal to  $B, C, D,$  and  $E$ . The orthogonality among these columns is indicated by the graph in Figure 3. There are two sets of 7 columns orthogonal to  $B, C, D,$  and  $E$ . Each of the two sets can be expanded by adding  $A, B, C, D,$  and  $E$ . Since these two expanded sets are isomorphic (see Appendix B), we consider the first set only.

To obtain a nearly orthogonal array, we can add to the  $L_{12}(3 \cdot 2^4)$  up to five columns chosen

	column						
row	$h_1$	$h_2$	$h_3$	$h_4$	$h_5$	$h_6$	$h_7$
1 - 4	$g_2$	$g_3$	$b$	$a+b$	$g_4$	$g_0$	$g_1$
5 - 8	$a+b$	$g_1$	$g_2+1$	$g_3+1$	$g_4+1$	$a+b+1$	$g_0$
9 - 12	$g_2+1$	$a+1$	$g_1+1$	$g_3$	$a+b$	$g_0+1$	$g_4+1$

	column					
row	$h_8$	$h_9$	$h_{10}$	$h_{11}$	$h_{12}$	$h_{13}$
1 - 4	$g_0$	$g_2$	$b$	$g_4$	$g_3$	$a+b$
5 - 8	$a+b$	$a+b+1$	$g_4+1$	$g_2+1$	$g_3+1$	$g_1$
9 - 12	$g_0+1$	$g_2+1$	$g_3$	$a+1$	$a+b$	$g_1+1$

Table 9: Candidate columns nonorthogonal to  $A$ . Each column is orthogonal to  $B$ ,  $C$ ,  $D$ , and  $E$  but nonorthogonal to  $A$ .

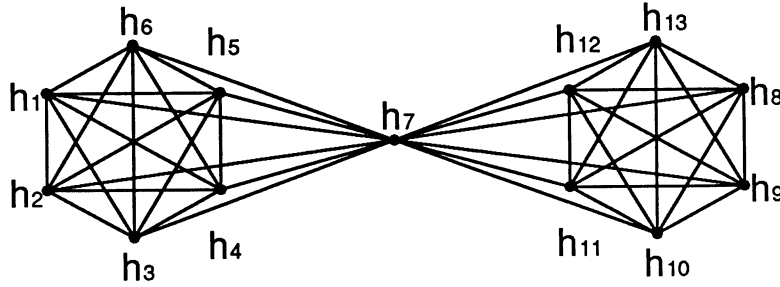


Figure 3: Orthogonality among candidate columns nonorthogonal to  $A$ .

from  $h_1$  to  $h_7$  in Table 9. Let

$$a_l = [-1 \ -1 \ -1 \ -1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1]^t,$$

$$a_q = [1 \ 1 \ 1 \ 1 \ -2 \ -2 \ -2 \ -2 \ 1 \ 1 \ 1 \ 1]^t$$

denote the linear and quadratic effects of  $A$ . Calculating the canonical correlations between  $[a_l, a_q]$  and the vectors formed by  $h_j$ 's reveals that  $[h_1, \dots, h_i]$  gives the *smallest* first canonical correlations among all possible choices of  $i$  columns from  $h_j$ 's for  $i = 1, \dots, 5$ . This holds for all possible level changes of  $A$ . Note that the canonical correlation measures the linear association between two sets of vectors. Among the sets of vectors consisting of the same number of  $h_j$ 's, the one with a smaller canonical correlation with  $[a_l, a_q]$  is considered to be better. We use this criterion to obtain some nearly orthogonal arrays of type II in Table 10.

		column									
		<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	<i>G</i>	<i>H</i>	<i>I</i>	<i>J</i>
row					$d_5$	$d_{16}$	$h_1$	$h_2$	$h_3$	$h_4$	$h_5$
1 - 4		0	$a$	$b$	$a$	$a+b$	$g_2$	$g_3$	$b$	$a+b$	$g_4$
5 - 8		1	$a$	$b$	$a+1$	$b+1$	$a+b$	$g_1$	$g_2+1$	$g_3+1$	$g_4+1$
9 - 12		2	$a$	$b$	$a+b+1$	$b$	$g_2+1$	$a+1$	$g_1+1$	$g_3$	$a+b$

<i>x</i>	5	1st canonical correlation	0.41								
		1st canonical correlation		0.50							
	6	2nd canonical correlation		0.29							
					0.50						
	7				0.50						
	8					0.65					
						0.50					
	9									0.71	
										0.58	

Table 10: 12-run  $3 \cdot 2^m$  arrays of type II. The array is orthogonal for  $m \leq 4$ , otherwise it is nearly orthogonal. To get an  $L'_{12}(3 \cdot 2^6)$ , for example, use columns  $A$  to  $G$ . There are only two nonorthogonal pairs in this array ( $(A, F)$  and  $(A, G)$ ).

#### 4 Construction of $L_{20}(5 \cdot 2^8)$ , $L'_{20}(5 \cdot 2^9)$ and $L'_{20}(5 \cdot 2^{10})$

The technique in the previous sections can be extended to the construction of 20-run arrays.

Using arguments similar to those in Section 2, we can show that there are only three nonisomorphic  $L_{20}(5 \cdot 2^2)$ 's. These arrays are  $[A, B, C_i]$ ,  $i = 1, 2, 3$ , where



$$A = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{bmatrix}, \quad B = \begin{bmatrix} a \\ a \\ a \\ a \\ a \end{bmatrix}, \quad C_1 = \begin{bmatrix} b \\ b \\ b \\ b \\ b \end{bmatrix}, \quad C_2 = \begin{bmatrix} a \\ a+1 \\ a+b+1 \\ a+b+1 \\ a+b+1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} a \\ a \\ a+1 \\ a+1 \\ a+b+1 \end{bmatrix}.$$

Consider the first case where the columns  $A$ ,  $B$ , and  $C_1$  are fixed. Using the method in Section 2 and exhaustive computer search, we can show that at most six mutually orthogonal columns can be added to  $A$ ,  $B$ , and  $C_1$ . Hence, the maximum number of  $m$  in  $L_{20}(5 \cdot 2^m)$  is 8 and can indeed be attained by the array in Table 11. The other two cases of fixing the first three columns also lead to the same conclusion.

run	column								
	$A$	$B$	$C_1$	$D$	$E$	$F$	$G$	$H$	$I$
1-4	0	$a$	$b$	$a+b$	$b$	$b$	$a$	$a+b$	$a+b$
5-8	1	$a$	$b$	$a$	$a+b+1$	$b+1$	$b$	$a+1$	$a+b+1$
9-12	2	$a$	$b$	$a+1$	$a$	$a+b+1$	$b+1$	$a+b+1$	$b+1$
13-16	3	$a$	$b$	$b$	$a+1$	$a$	$a+b+1$	$a+b+1$	$a+b$
17-20	4	$a$	$b$	$b+1$	$b+1$	$a+1$	$a+1$	$a$	$b$

Table 11: An  $L_{20}(5 \cdot 2^8)$ .

To add more 2-level columns to  $L_{20}(5 \cdot 2^8)$ , we have to sacrifice the orthogonality requirement. For simplicity we only consider the case for adding 1 or 2 columns to  $L_{20}(5 \cdot 2^8)$ . (The method for adding more columns is similar.) Four such arrays, denoted by WW3 to WW6, are shown in Table 12. Note that  $q_1$ ,  $q_2$ , and  $q_3$  in Table 12 are orthogonal to  $A$ ,  $B$ ,  $C_1$ ,  $D$ , and  $E$ .

For adding one column, WW3 is better than WW4 in terms of the number of 2-level columns free from nonorthogonality but is worse than WW4 in terms of the largest correlation. Similarly, WW5 is better (worse) than WW6 according to the first (second) criterion. Note that these arrays are of type I.

array name	column(s) added	correlation with			
		$F$	$G$	$H$	$I$
WW3	$q_1$	0.6	0	-0.2	0
WW4	$q_2$	0.2	0	-0.2	0.2
WW5	$q_1$	0.6	0	-0.2	0
	$q_2$	0.2	0	-0.2	0.2
WW6	$q_2$	0.2	0	-0.2	0.2
	$q_3$	0	0.2	0.2	-0.2

Table 12: Comparisons of 20-run nearly orthogonal arrays, where  $q_1 = [b, b+1, a+b, a, a+1]^t$ ,  $q_2 = [b, a+b, b+1, a+b, a+b+1]^t$ , and  $q_3 = [a, a+b, a+b+1, a+1, a+b]^t$ .

## 5 A General Construction Method

Wang and Wu (1989) gave a general method for constructing mixed-level orthogonal arrays.

By modifying their method, we can obtain three classes of nearly orthogonal arrays.

First we review the method. Let  $G$  be an additive group of  $g$  elements denoted by  $\{0, 1, \dots, g-1\}$ . A  $\lambda g \times r$  matrix with elements from  $G$ , denoted by  $D_{\lambda g, r; g}$ , is called a *difference matrix* if among the differences of the corresponding elements of any two columns, each element of  $G$  occurs exactly  $\lambda$  times.

For two matrices  $A = [a_{ij}]$  of order  $n \times r$  and  $B$  of order  $m \times s$  both with entries from  $G$ , define their *Kronecker sum* to be the  $mn \times rs$  matrix

$$A * B = [B^{a_{ij}}]_{1 \leq i \leq n, 1 \leq j \leq r}, \quad (1)$$

where  $B^k = B + kJ$  is obtained from adding  $k$ , over  $G$ , to the elements of  $B$  and  $J$  is the  $m \times s$  matrix of ones.

Let  $L_1 = L_{\mu g}(g^s)$  and  $L_2 = L_{\lambda g}(q_1^{r_1} \cdots q_m^{r_m})$  be two orthogonal arrays,  $D = D_{\lambda g, r; g}$  be a difference matrix, and  $0_{\mu g}$  be the  $\mu g \times 1$  vector of zeros. Then  $L_1 * D$  is a  $\lambda \mu g^2 \times rs$  matrix and  $0_{\mu g} * L_2$  is a  $\lambda \mu g^2 \times (r_1 + \cdots + r_m)$  matrix consisting of  $\mu g$  copies of  $L_2$  as its rows. Wang and Wu (1989) show

that the matrix

$$[L_1 * D, 0_{\mu g} * L_2] \quad (2)$$

is an  $L_{\lambda \mu g^2}(g^{rs} \cdot q_1^{r_1} \cdots q_m^{r_m})$ .

By replacing  $L_1$  or  $L_2$  in (2) by a nearly orthogonal array, or the difference matrix  $D$  by a nearly difference matrix (to be defined below), we obtain the following three classes of nearly orthogonal arrays.

(A) By replacing  $L_2$  by a nearly orthogonal array  $L'_2 = L'_{\lambda g}(q_1^{r_1} \cdots q_m^{r_m})$ , we obtain  $[L_1 * D, 0_{\mu g} * L'_2]$  which is an  $L'_{\lambda \mu g^2}(g^{rs} \cdot q_1^{r_1} \cdots q_m^{r_m})$ . Note that the nonorthogonality of the constructed array is inherited from  $L'_2$  and hence the number of nonorthogonal pairs and the correlation structure of the former array are the same as those of the latter. Several such arrays will be given in detail in Section 6.

(B) A *nearly difference matrix*,  $D'_{n,r;g}$ , is an  $n \times r$  matrix with entries from the group  $G$  such that, among the differences of the entries of any two columns, the elements of  $G$  occur as evenly as possible. In general if the difference matrix in (2) does not exist, we will replace it by a nearly difference matrix. More precisely, if there exist a nearly difference matrix  $D'_{n,r;g}$ , and orthogonal arrays  $L_{\mu g}(g^s)$  and  $L_n(q_1^{r_1} \cdots q_m^{r_m})$ , then the matrix

$$[L_{\mu g}(g^s) * D'_{n,r;g}, 0_{\mu g} * L_n(q_1^{r_1} \cdots q_m^{r_m})] \quad (3)$$

is a nearly orthogonal array  $L'_{n \mu g}(g^{rs} \cdot q_1^{r_1} \cdots q_m^{r_m})$ .

There are two kinds of nearly difference matrices according to the divisibility of  $n$  by  $g$ .

1.  $n$  is not divisible by  $g$ . By definition, no  $D_{n,r;g}$  exists for  $r > 1$ . So we should find

$D'_{n,r;g}$ ,  $r > 1$ , such that each element of  $G$  occurs  $\lfloor n/g \rfloor$  or  $\lfloor n/g \rfloor + 1$  times among the component-wise differences of any two columns. Here  $\lfloor x \rfloor$  is the integral part of  $x$ . For instance, in

$$D'_{3,3;2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

0 and 1 occur once or twice among the differences of the entries of any two columns. The difference matrices in Appendix C are of this kind.

2.  $n$  is divisible by  $g$ . It can be shown (Beth, Jungnickel and Lenz, 1986, p. 365) that no  $D_{n,r;g}$  exists for  $r > 2$  if  $g \equiv 2 \pmod{4}$ . So we should use  $D'_{n,r;g}$  for  $r > 2$  if  $g \equiv 2 \pmod{4}$ . For instance,

$$D'_{6,6;2} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The arrays constructed from (3) with run size  $\leq 24$  are listed in Table 13.

If  $\mu$  in (3) is taken to be 1, the number of nonorthogonal pairs in the array constructed from (3) equals the number of pairs in  $D'$  for which the elements of  $G$  do not occur equally often among the component-wise differences. For instance, in the array

$$L'_6(3 \cdot 2^3) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}$$

nearly orthogonal			
array	$L_g(g)$	$D'_{n,r;q}$	$L_n(q_1^{r_1} \cdots q_m^{r_m})$
$L'_6(3 \cdot 2^3)$	$L_2(2)$	$D'_{3,3;2}$	$L_3(3)$
$L'_{10}(5 \cdot 2^5)$	$L_2(2)$	$D'_{5,5;2}$	$L_5(5)$
$L'_{12}(4 \cdot 3^4)$	$L_3(3)$	$D'_{4,4;3}$	$L_4(4)$
$L'_{12}(3^4 \cdot 2^3)$	$L_3(3)$	$D'_{4,4;3}$	$L_4(2^3)$
$L'_{12}(6 \cdot 2^6)$	$L_2(2)$	$D'_{6,6;2}$	$L_6(6)$
$L'_{12}(3 \cdot 2^9)$	$L_4(2^3)$	$D'_{3,3;2}$	$L_3(3)$
$L'_{15}(5 \cdot 3^5)$	$L_3(3)$	$D'_{5,5;3}$	$L_5(5)$
$L'_{18}(9 \cdot 2^8)$	$L_2(2)$	$D'_{9,8;2}$	$L_9(9)$
$L'_{18}(3^4 \cdot 2^8)$	$L_2(2)$	$D'_{9,8;2}$	$L_9(3^4)$
$L'_{20}(5 \cdot 2^{15})$	$L_4(2^3)$	$D'_{5,5;2}$	$L_5(5)$
$L'_{24}(8 \cdot 3^8)$	$L_3(3)$	$D'_{8,8;3}$	$L_8(8)$
$L'_{24}(3^8 \cdot 2^7)$	$L_3(3)$	$D'_{8,8;3}$	$L_8(2^7)$
$L'_{24}(4 \cdot 3^8 \cdot 2^4)$	$L_3(3)$	$D'_{8,8;3}$	$L_8(4 \cdot 2^4)$
$L'_{24}(3 \cdot 2^{21})$	$L_8(2^7)$	$D'_{3,3;2}$	$L_3(3)$
$L'_{24}(6 \cdot 2^{18})$	$L_4(2^3)$	$D'_{6,6;2}$	$L_6(6)$

Table 13: Nearly orthogonal arrays constructed by equation (3), run size  $\leq 24$ .

(see Table 13), the three pairs of its 2-level columns are nonorthogonal because the corresponding  $D'_{3,3;2}$  has three pairs of columns for which the component-wise differences do not take the elements of  $G$  equally often.

The case of  $\mu > 1$  has a similar structure of nonorthogonality. Let  $L_{\mu g}(g^s) = [c_1, \dots, c_s]$ . It can be shown (Appendix D) that if  $b_1$  and  $b_2$  are any two columns of  $D'$  in (3),  $c_i * b_1$  and  $c_j * b_2$  are orthogonal for  $i \neq j$ . The only nonorthogonal pairs are  $c_i * b_1$  and  $c_i * b_2$ ,  $i = 1, \dots, s$  for those pairs of  $b_1$  and  $b_2$  among whose component-wise differences the elements of  $G$  do not occur equally often. Therefore, the number of nonorthogonal pairs equals  $s$  times the number of pairs of columns in  $D'$  for which the elements of  $G$  do not occur equally frequently among their differences.

- (C) In (2),  $L_1 * D$  is an orthogonal array  $L_{\lambda \mu g^2}(g^{rs})$ . If  $L_1$  is replaced by a nearly orthogonal array  $L'_1$ , what will become of  $L'_1 * D$ ? In some situations we can use this method to construct

nearly orthogonal arrays with good properties. Suppose  $D_{\mu g, r; g}$  is a difference matrix with the column of zeros as its first column, which is always possible because adding a constant to any row of a difference matrix is still a difference matrix. Then the matrix obtained from

$$[0, \dots, (g-2)]^t * D_{\mu g, r; g} \quad (4)$$

is an  $L'_{\mu g(g-1)}(g^{r-1} \cdot (g-1))$ . In (4), the  $(g-1)$ -level column is orthogonal to the  $g$ -level columns while the  $g$ -level columns are not orthogonal among each other. Unlike methods (A) and (B), the right hand portion  $0 * L_2$  in (2) should be added to the array in (4) with caution. It is known that  $r \leq \mu g$  for any difference matrix  $D_{\mu g, r; g}$  (Beth, Jungnickel and Lenz, 1986). If  $r = \mu g$  in  $D_{\mu g, r; g}$ , then the array constructed from (4) is saturated since all the degrees of freedom are used up, i.e.,

$$(\mu g - 1)(g - 1) + (g - 2) = \mu g(g - 1) - 1. \quad (5)$$

Therefore, no additional column can be added. For  $r < \mu g$ , we should choose  $L_2$  such that the total degrees of freedom of its main effects  $\leq (\mu g - r)(g - 1)$ . The  $(g - 1)$ -level column is orthogonal to any added column. A  $g$ -level column constructed by (4) is orthogonal to an added column provided that the corresponding columns in  $D$  and  $L_2$ , respectively, are orthogonal. Arrays with run size  $\leq 36$  obtained by this method are listed in Table 14. The layouts of the difference matrices used in constructing these arrays are given in Appendix C.

The nonorthogonality structure of the constructed array depends on the choice of  $D$  in (4) and the choice of  $L_2$ . Three examples are given to illustrate this point.

1. The array

nearly orthogonal		
array	$D_{\mu g, r; g}$	$L_{\mu g}(q_1^{r_1} \dots q_m^{r_m})$
$L'_{12}(3^5 \cdot 2)$	$D_{6,6;3}$	NA
$L'_{12}(3^4 \cdot 2^2)$	$D_{6,4;3}$	$L_6(3 \cdot 2)$
$L'_{24}(3^{11} \cdot 2)$	$D_{12,12;3}$	NA
$L'_{24}(4 \cdot 3^9 \cdot 2)$	$D_{12,9;3}$	$L_{12}(4 \cdot 3)$
$L'_{24}(3^9 \cdot 2^5)$	$D_{12,9;3}$	$L_{12}(3 \cdot 2^4)$
$L'_{24}(4^7 \cdot 3)$	$D_{8,8;4}$	NA
$L'_{24}(4^6 \cdot 3 \cdot 2^3)$	$D_{8,6;4}$	$L_8(4 \cdot 2^3)$
$L'_{36}(4^{11} \cdot 3)$	$D_{12,12;4}$	NA
$L'_{36}(4^{10} \cdot 3^2)$	$D_{12,10;4}$	$L_{12}(4 \cdot 3)$

Table 14: Nearly orthogonal arrays constructed by (4), run size  $\leq 36$ .

$$L'_{12}(3^5 \cdot 2) = [(0, 1)^t * D_{6,6;3}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 2 & 1 & 1 & 2 & 0 & 1 & 0 & 2 \\ 0 & 2 & 1 & 2 & 0 & 1 & 1 & 0 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 2 & 1 & 2 & 1 & 0 & 2 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 & 1 & 0 & 0 & 2 & 2 & 1 \end{bmatrix}^t$$

is quite good because it is saturated (see (5)) and for any two 3-level columns, six level combinations appear once and three others appear twice. Here all the level combinations appear at least once because, for each component-wise difference in  $D_{6,6;3}$ , there are two distinct pairs associated with it.

## 2. The array

$$L'_6(3^2 \cdot 2) = [(0, 1)^t * D_{3,3;3}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

is poor because among the level combinations of the 3-level columns three do not appear. This is due to the fact that for each component-wise difference in  $D_{3,3;3}$ , there is only one pair associated with it. In general we recommend choosing  $D$  in (4) so that in

the constructed array every level combination appears at least once. All the arrays in Table 14 meet this requirement.

3. By deleting the last two columns of  $D_{6,6;3}$ , we obtain a  $D_{6,4;3}$ . The array

$$L'_{12}(3^4 \cdot 2^2) = [(0,1)^t * D_{6,4;3}, 0_2 * L_6(3 \cdot 2)] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 2 & 1 & 1 & 2 & 0 & 1 & 0 & 2 \\ 0 & 2 & 1 & 2 & 0 & 1 & 1 & 0 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 2 & 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}^t$$

is good since each level combination of any nonorthogonal pair appears at least once. Note that the 2-level column in the added portion (the last column) is also orthogonal to the first 3-level column (the second column) since the 2-level column in  $L_6(3 \cdot 2)$  is orthogonal to the second column in  $D$ . The arrays  $L'_{24}(4 \cdot 3^9 \cdot 2)$ ,  $L'_{24}(3^9 \cdot 2^5)$ ,  $L'_{24}(4^6 \cdot 3 \cdot 2^3)$ , and  $L'_{36}(4^{10} \cdot 3^2)$  in Table 14 are obtained in a similar manner.

## 6 Construction of $L'_{18}(3^7 \cdot 2^m)$ with $m \leq 3$ and $L'_{24}(3 \cdot 2^m)$ with $m \leq 21$

The  $L'_6(3 \cdot 2^3)$  given in Table 13 can also be used to construct the following array

$$L'_{18}(3^7 \cdot 2^3) = [L_3(3) * D_{6,6;3}, 0_3 * L'_6(3 \cdot 2^3)].$$

In the constructed array, only the 2-level columns are not orthogonal among each other. The level combinations between any two 2-level columns appear with the frequencies 3:6:6:3 or 6:3:3:6. Can the imbalance in the frequencies be further improved to be 4:5:5:4 or 5:4:4:5? It is shown in the Appendix E that this is impossible. More precisely, for any  $L'_{18}(3^n \cdot 2^3)$



in which the only nonorthogonal pairs are among the 2-level columns and the frequencies of their level combinations are 4:5:5:4 or 5:4:4:5, the maximum  $n$  is four.

The method for determining the maximum for  $m$  in  $L_{12}(3 \cdot 2^m)$  and  $L_{20}(5 \cdot 2^m)$  can be applied to an analogous problem for  $L_{24}(3 \cdot 2^m)$ . According to Wang and Wu (1989), we can construct

$$L_{24}(3 \cdot 2^{16}) = \begin{bmatrix} H_{12} & L_{12}(3 \cdot 2^4) \\ H_{12} + \mathbf{1} & L_{12}(3 \cdot 2^4) \end{bmatrix},$$

where  $H_{12}$  is the Hadamard matrix of order 12. It is shown in Appendix F that this construction is best in the sense that the maximum  $m$  in any  $L_{24}(3 \cdot 2^m)$  is 16. To go beyond this, we can construct

$$L'_{24}(3 \cdot 2^{16+q}) = \begin{bmatrix} H_{12} & L'_{12}(3 \cdot 2^{4+q}) \\ H_{12} + \mathbf{1} & L'_{12}(3 \cdot 2^{4+q}) \end{bmatrix} \quad (6)$$

by using Method A of Section 5, where  $L'_{12}(3 \cdot 2^{4+q})$ ,  $1 \leq q \leq 5$  are given in Section 3. For the array in (6), nonorthogonal pairs occur on the right hand part which are inherited from  $L'_{12}(3 \cdot 2^{4+q})$ . For example, if the WW1 array is used for  $L'_{12}(3 \cdot 2^{4+q})$ , we obtain from (6) an  $L'_{24}(3 \cdot 2^{18})$  which has only two nonorthogonal pairs of 2-level columns.

From Section 3, after row and column permutations and level changes, we can obtain the following Hadamard matrix

$$H_{12} = [\mathbf{0}, B, C, D, E, h_1, h_2, \dots, h_7]. \quad (7)$$

Using (6) and (7) we obtain an  $L'_{24}(3 \cdot 2^{16+q})$ . Denote the columns of the constructed array by

$$a_1, \dots, a_{12}, a_0, a_{13}, \dots, a_{16}, a_{17}, \dots, a_{16+q}, \quad (8)$$

where  $a_0$  is the 3-level column. Then we have

$$a_1 + a_{i+1} = a_{i+12}, \quad 1 \leq i \leq 4 + q,$$

i.e., for  $1 \leq i \leq 4 + q$ ,  $a_{i+12}$  is the interaction of  $a_1$  and  $a_{i+1}$ . To see this, consider  $a_1$  and  $a_2$  for instance. Their interaction is given by

$$\begin{bmatrix} \mathbf{0} + B \\ \mathbf{1} + (B + \mathbf{1}) \end{bmatrix} = \begin{bmatrix} B \\ B \end{bmatrix},$$

which is identical to  $a_{13}$ . As an application, if we use (7) for the  $H_{12}$  in (6), then by replacing the following three columns

$$\begin{bmatrix} \mathbf{0} & B & B \\ \mathbf{1} & B + \mathbf{1} & B \end{bmatrix}$$

by a 4-level column according to the rule  $(0, 0, 0) \rightarrow 0$ ,  $(0, 1, 1) \rightarrow 1$ ,  $(1, 0, 1) \rightarrow 2$ , and  $(1, 1, 0) \rightarrow 3$ , we obtain an  $L'_{24}(4 \cdot 3 \cdot 2^{13+q})$ . The 4-level column is orthogonal to the other columns.

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## References

- Addelman, S. (1962), "Orthogonal Main-effect Plans for Asymmetrical Factorial Experiments," *Technometrics*, 4, 21–46.

- Beth, T., Jungnickel, D., and Lenz, H. (1985), *Design Theory*, Zürich: Wissenschaftsverlag.
- Bose, R. C., and Bush, K. A. (1952), "Orthogonal Arrays of Strength Two and Three," *The Annals of Mathematical Statistics*, 23, 508–524.
- Masuyama, M. (1957), "On Difference Sets for Constructing Orthogonal Arrays of Index Two and of Strength Two," *Report on Statistical Application and Research, JUSE*, 5, 27–34.
- Rao, C. R. (1947), "Factorial Experiments Derivable from Combinational Arrangements of Arrays," *Journal of the Royal Statistical Society, Supplementary*, 9, 128–139.
- Seiden, E. (1954), "On the Problem of Construction of Orthogonal Arrays," *The Annals of Mathematical Statistics*, 25, 151–156.
- Taguchi, G. (1959), "Linear Graphs for Orthogonal Arrays and Their Application to Experimental Design, with the Aid of Various Techniques," *Report on Statistical Application and Research, JUSE*, 6, No. 4, 1–43.
- Taguchi, G. (1987), *System of Experimental Design*, volumes 1 and 2, White Plains: UNIPUB.
- Tukey, J. (1959), "Little Pieces of Mixed Factorials," unpublished.
- Wang, J. C. (1989), "Orthogonal Arrays and Nearly Orthogonal Arrays with Mixed Levels: Construction and Applications," Ph. D. thesis, University of Wisconsin, Madison, Department of Statistics.

Wang, J. C. and Wu, C. F. J. (1989), "An Approach to the Construction of Asymmetrical Orthogonal Arrays," *Research Report RR-89-01*, Institute for Improvement in Quality and Productivity, University of Waterloo.

## Appendices

### A Nonisomorphic $L_{12}(3 \cdot 2^2)$ 's

There are two nonisomorphic  $L_{12}(3 \cdot 2^2)$ 's. To see this, let  $A$  be the 3-level column and  $B$ , and  $C$  be the 2-level columns. Denote by  $x$  and  $y$  the frequencies of the level combinations  $(0,0,0)$ , and  $(1,0,0)$  for  $(A,B,C)$ , respectively. Then  $3 - (x + y)$  is the frequency of  $(2,0,0)$  since the frequency of the level combination  $(0,0)$  for  $(B,C)$  is 3. So we have

$$0 \leq x \leq 2, 0 \leq y \leq 2, \text{ and } 0 \leq 3 - (x + y) \leq 2.$$

The integer solutions to the inequalities above are  $(1,1)$ ,  $(2,0)$ ,  $(2,1)$ ,  $(1,0)$ ,  $(1,2)$ ,  $(0,1)$ , and  $(0,2)$ . The solution  $(1,1)$  gives one array and all the other solutions give another array. These two arrays are displayed in Figure 4, from which their nonisomorphism is apparent.

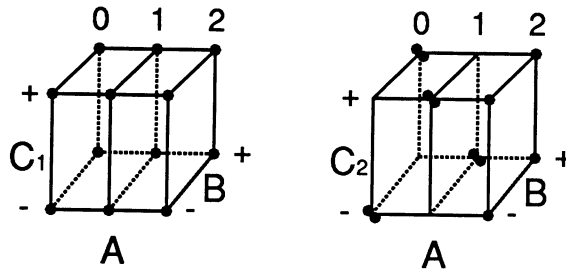


Figure 4: Two nonisomorphic  $L_{12}(3 \cdot 2^2)$ 's.

### B Isomorphism of Sets

We prove the claim in Section 3 that the two sets,

**C1:**  $[A, B, C, D, E, h_1, h_2, h_3, h_4, h_5, h_6, h_7]$

**C2:**  $[A, B, C, D, E, h_7, h_8, h_9, h_{10}, h_{11}, h_{12}, h_{13}]$ ,

are isomorphic.

Denote by  $R_{i,j}$  (and resp.  $C_{ij}$ ) the permutation of rows (and resp. columns)  $i$  and  $j$ . For  $2 \leq i \leq 12$ , let  $\bar{i}$  be the operation of exchanging the levels 0 and 1 of the  $i$ -th column. Let  $A_{i \leftrightarrow j}$  be the operation of exchanging the levels  $i$  and  $j$  of the 3-level column  $A$ . By applying to **C1** the following operations sequentially:  $C_{23}, C_{45}, R_{1,9}, R_{2,10}, R_{3,11}, R_{4,12}, R_{1,4}, R_{5,8}, R_{9,12}, \bar{7}, \bar{6}, \bar{5}, \bar{4}, \bar{3}, \bar{2}, A_{0 \leftrightarrow 2}$ , we obtain **C2**.

### C Some Nearly Difference Matrices and Difference Matrices

$D'_{4,4;3}$	$D'_{5,5;2}$	$D'_{5,5;3}$
0 0 0 0	0 0 0 0 0	0 0 0 0 0
0 1 2 0	0 0 1 1 1	0 0 1 2 2
0 2 1 1	0 1 0 1 1	0 1 2 2 1
0 1 0 2	0 1 1 0 0	0 2 2 1 0
	0 1 1 1 0	0 2 1 0 1

$D'_{8,8;3}$	$D'_{9,8;2}$
0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0
0 1 2 0 1 2 0 1	0 1 0 0 1 1 0 1
0 2 1 0 2 1 0 2	0 0 1 0 1 0 1 1
0 0 0 1 1 1 2 2	0 1 1 0 0 1 1 0
0 1 2 1 2 0 2 0	0 0 0 1 0 1 1 1
0 2 1 1 0 2 2 1	0 1 0 1 1 0 1 0
0 0 0 2 2 2 1 1	0 0 1 1 1 1 0 0
0 1 2 2 0 1 1 2	0 1 1 1 0 0 0 1
	0 0 0 0 0 0 0 0

$D_{6,6;3}$ (Masuyama 1957)
0 0 0 0 0 0
0 1 2 0 1 2
0 2 1 1 0 2
0 0 2 1 2 1
0 2 0 2 1 1
0 1 1 2 2 0

$D_{8,8;4}$ (here $01 + 10 = 11$ )							
00	00	00	00	00	00	00	00
00	01	10	00	01	11	11	10
00	10	00	01	11	11	10	01
00	00	01	11	11	10	01	10
00	01	11	11	10	01	10	00
00	11	11	10	01	10	00	01
00	11	10	01	10	00	01	11
00	10	01	10	00	01	11	11

$D_{12,12;3}$ (Seiden 1954)											
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	2	1	0	2	2	2	1	1	1
0	0	1	2	2	2	1	1	0	1	2	0
0	0	2	1	2	1	0	2	1	2	1	0
0	1	2	1	2	0	1	0	2	1	0	2
0	1	2	0	1	1	2	1	2	0	2	0
0	1	0	2	1	2	0	1	1	2	0	2
0	1	1	1	0	2	2	2	0	0	1	2
0	2	1	1	0	0	0	1	2	2	2	1
0	2	1	2	2	1	2	0	1	0	0	1
0	2	2	0	1	2	1	0	0	2	1	1
0	2	0	0	0	1	1	2	1	1	2	2

$$D_{12,12;4} = \begin{bmatrix} \mathbf{0} & B_1 & B_2 & B_3 \\ B_1^T & K_0 & K_0 & K_0 \\ B_2^T & K_0 & K_1 & K_2 \\ B_3^T & K_0 & K_2 & K_1 \end{bmatrix},$$

where  $\mathbf{0}$  is a  $3 \times 3$  matrix of 00's,  $K_0 = K = [c_1, c_2, c_3]$ ,  $K_1 = [c_2, c_3, c_1]$ ,  $K_2 = [c_3, c_1, c_2]$ ,

$B_i = [c_i, c_i, c_i]$ , for  $i = 1, 2, 3$ , and

$$[c_1, c_2, c_3] = \begin{bmatrix} 01 & 10 & 11 \\ 10 & 11 & 01 \\ 11 & 01 & 10 \end{bmatrix}.$$



## D Orthogonality in (3)

We discuss the orthogonality in (3) according to  $\mu = 1$  and  $\mu > 1$ .

### 1. $\mu = 1$ .

Let  $D'_{n,r;g} = [b_1, \dots, b_r]$ . Then  $L_g(g)*b_i - L_g(g)*b_j = 0_g*(b_i - b_j)$ . It follows then that if the elements of  $G$  do not occur equally often among the differences of  $b_i$  and  $b_j$ , then they also do not occur equally often among the differences of  $L_g(g)*b_i$  and  $L_g(g)*b_j$ , which are not orthogonal. On the other hand  $L_g(g)*b_i$  and  $L_g(g)*b_j$  are orthogonal for any  $b_i$  and  $b_j$  whose differences appear equally often among the elements of  $G$  (Bose and Bush, 1952).

The orthogonality between  $L_g(g)*D'_{n,r;g}$  and  $0_g*L_n(q_1^{r_1} \cdots q_m^{r_m})$  can be proved by showing that  $L_g(g)*b_i$  and  $0_g*a$  are orthogonal, where  $a$  is a  $q_j$ -level column in  $L_n(q_1^{r_1} \cdots q_m^{r_m})$ .

Note that

$$[L_g(g)*b_i, 0_g*a] = \begin{bmatrix} b_i & a \\ b_i + 1 & a \\ \vdots & \vdots \\ b_i + (g-1) & a \end{bmatrix}$$

Let  $f_{k,l;v}$  be the frequency of the level combination  $(k, l)$  for the two columns occurring in  $[b_i + v, a]$  for  $k = 0, \dots, g-1, l = 0, \dots, q_j-1$ , and  $v = 0, \dots, g-1$ . Then the frequency of the level combination  $(k, l)$  of these two columns is

$$\sum_{v=0}^{g-1} f_{k,l;v} = \sum_{v=0}^{g-1} f_{k-v,l;0} = \sum_{w=0}^{g-1} f_{w,l;0} = \text{frequency of level } l \text{ in } a = \text{constant.}$$

Therefore, these two columns are orthogonal.

### 2. $\mu > 1$ .

Let  $L_{\mu g}(g^s) = [c_1, \dots, c_s]$ . Then the orthogonality in  $\mathbf{L} = [c_i*D'_{n,r;g}, 0_{\mu g}*L_n(q_1^{r_1} \cdots q_m^{r_m})]$

is analogous as the previous case. To see this, note that  $\mathbf{L}$  can be written as  $[L_g(g) * D'_{\mu n, r; g}, 0_g * L_{\mu n}]$  by permuting its rows, where  $D'_{\mu n, r; g}$  (and resp.  $L_{\mu n}$ ) is a matrix consisting of  $\mu$  copies of  $D'_{n, r; g}$  (and resp.  $L_n(q_1^{r_1} \cdots q_m^{r_m})$ .) It remains to show the orthogonality in  $\mathbf{A} = [c_i * b_k, c_j * b_l]$  for  $k, l = 1, \dots, r$ , and  $i < j$ . Since  $c_i$  and  $c_j$  are orthogonal,  $\mathbf{A}$  can be rewritten as  $[L_g(g) * b_k^\mu, 0_g * b_l^\mu]$  by permuting its rows, where  $b_k^\mu$  (and resp.  $b_l^\mu$ ) is a column consisting of  $\mu$  copies of  $b_k$  (and resp.  $b_l$ .) Therefore, the columns in  $\mathbf{A}$  are orthogonal.

**E A result on  $L'_{18}(2^m \cdot 3^n)$**

Let  $A$  and  $B$  be the columns with 4:5:5:4 as the frequencies of level combinations, i.e.,

$$A = [000000000111111111]^t,$$

$$B = [0000111111111100000]^t,$$

and  $C$  be a 3-level column orthogonal to  $A$  and  $B$ . Let  $x$  (and resp.  $y$ ) be the numbers of the level combinations  $(0,0,0)$  (and resp.  $(0,0,1)$ ) for  $(A, B, C)$ . Then  $4 - (x + y)$  is the number of the level combination  $(0,0,2)$  for  $(A, B, C)$ . Since each level combination  $(0, i)$  for  $B$  and  $C$ ,  $i = 0, 1, 2$  appears exactly 3 times, we have  $0 \leq x \leq 3$ ,  $0 \leq y \leq 3$ ,  $0 \leq 4 - (x + y) \leq 3$ . There are 12 integer solutions to the inequalities above. By using computer search, the maximum number of mutually orthogonal 3-level columns corresponding to these solutions is four. Therefore, the maximum  $n$  in  $L'_{18}(3^n \cdot 2^2)$  is four. Indeed the array  $L'_{18}(3^4 \cdot 2^8)$  in Table 13 attains the maximum and the frequencies of level combinations in any two 2-level columns are 4:5:5:4 or 5:4:4:5.

**F The maximum  $m$  in  $L'_{24}(3 \cdot 2^m)$  is 16**

We follow the method for  $L_{12}(3 \cdot 2^m)$  in Section 2. Fix the 3-level column as

$$A = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \end{bmatrix},$$

where  $\mathbf{i}$  is the  $8 \times 1$  vector of  $i$ 's. Then any 2-level column orthogonal to  $A$  must have the form

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix},$$

where each  $X_i$  is an  $8 \times 1$  vector from the basic vectors as follows:  $c, c+1, d, d+1, e, e+1, c+d, c+d+1, c+e, c+e+1, d+e, d+e+1, c+d+e, c+d+e+1$ . Here

$$c = [0\ 0\ 0\ 0\ 1\ 1\ 1\ 1]^t, \quad d = [0\ 0\ 1\ 1\ 0\ 0\ 1\ 1]^t, \quad e = [0\ 1\ 0\ 1\ 0\ 1\ 0\ 1]^t.$$

We will fix the first three 2-level columns. Corresponding to level  $i$  of the 3-level column, the 8 runs of the 2-level columns can take one of the following five nonisomorphic choices:  $\{x, y, z\}$ ,  $\{x, y, x+y\}$ ,  $\{x, y, x+y+1\}$ ,  $\{x, x, y\}$ , and  $\{x, x+1, y\}$ , where  $x$  and  $y$  can take any of the basic vectors. Therefore, by a complete enumeration, we can show that there are eight nonisomorphic ways of fixing the first three 2-level columns:  $[B, C, D_i]$ ,  $i = 1, \dots, 8$ , where

$$B = \begin{bmatrix} c \\ c \\ c \end{bmatrix}, \quad C = \begin{bmatrix} d \\ d \\ d \end{bmatrix}, \quad D_1 = \begin{bmatrix} e \\ e \\ e \end{bmatrix}, \quad D_2 = \begin{bmatrix} c \\ c+1 \\ c+d \end{bmatrix}, \quad D_3 = \begin{bmatrix} c \\ c+1 \\ e \end{bmatrix},$$

$$D_4 = \begin{bmatrix} c+d \\ c+d \\ c+d \end{bmatrix}, \quad D_5 = \begin{bmatrix} c+d+1 \\ c+d \\ c+d \end{bmatrix}, \quad D_6 = \begin{bmatrix} e \\ c+d \\ c+d \end{bmatrix}, \quad D_7 = \begin{bmatrix} e \\ c+d+1 \\ c+d \end{bmatrix}, \quad D_8 = \begin{bmatrix} e \\ e \\ c+d \end{bmatrix}.$$

By using essentially the same method as in Section 2, we can show that the maximum number of 2-level columns that can be added in any of the eight cases is not greater than 13. Therefore, the maximum  $m$  in any  $L_{24}(3 \cdot 2^m)$  is 16.