ASSESSING INFORMATION IN INDUSTRIAL EXPERIMENTS WITH CENSORED AND ORDINAL DATA

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ABSTRACT

In designing experiments with censored and ordinal data, the information provided by such data relative to complete data must be considered. In this paper, information from censored, grouped, ordinal and binary data are evaluated for typical industrial experiments; a unified framework allows these data types arising from various collection schemes to be compared. Two examples are studied which have some useful implications. The information measure, which requires simple calculations and is thus easily programmed, provides one assessment of an experimental design's suitability. Using it to evaluate potential designs is recommended and thereby arms the practitioner with a useful tool for planning experiments with censored and ordinal data.

Key Words: asymptotic relative efficiency, binary data, fractional factorial design, grouped data, maximum likelihood estimation.

Introduction

With the renewed interest in experiments to improve quality and productivity, evaluation of different data collection schemes becomes a necessary part in the planning of such experiments. The types of data collected in these experiments reflect the cost and time constraints which are imposed by today's ever shrinking product cycles and ever increasing reliability of today's products. For example, because an experiment's duration must be limited, some units on test may not fail before the end of the experiment, yielding censored data. Instead of monitoring units continuously, periodic inspection may be done until they fail, yielding grouped data. When continuous measurements are not easily obtained, ordinal data based on subjective assessments such as "none", "slight", "moderate", and "extensive" are often collected; it is far better to experiment immediately and obtain ordinal data which contain information to improve the product than wait for a continuous measurement device to be built. Classifying the product as "good" or "bad" provides an even simpler and faster way to collect data which is known as binary data. All these data types are examples of incomplete data as opposed to complete data whose values are known exactly.

While easier or less expensive to obtain, incomplete data do have costs. Gould and Lawless (1988) and Hamada (1989) studied information loss for parameter estimation in the general regression setting for censored and grouped data relative to complete data. Tse and Wu (1984) studied the advantages of ordinal data over binary data for a specific situation. Also, there is the problem of the non-existence of estimates (Hamada and Tse [1989]) and increased difficulty in analyzing these data (Hamada and Wu [1990, 1991]). In this paper, we focus on the information loss in typical industrial experiments. Here, we are concerned with ensuring that the experiments provide adequate information. Censored, grouped, ordinal and binary data are studied for the first time in a unified framework so that these data types can be compared. Two special cases of grouping, rounding and rounding-censoring, are also considered; the former arises from equally spaced inspection and the latter when equally spaced inspection is combined with censoring.

In this paper, we consider the information loss for the maximum likelihood estimator (MLE) of the vector of factorial effects. For simplicity, we use the exponential regression model, a special case of the popular Weibull regression model used in reliability studies. While the exponential model has been criticized because of its lack of robustness, nevertheless, it has been used successfully to model real data sets (Davis [1952]). Moreover, our focus is a relative comparison between the data types arising from various collection schemes.

We begin by defining asymptotic relative efficiency between two different data collection schemes as a measure of information. A review as well as new results for censored, grouped, ordinal and binary data are given for the general regression situation. Details for ordinal data appear in the Appendix. Assuming a latent model for ordinal and binary data provides a unified framework for studying the different data types. In this way, ordinal data can be viewed as grouped data, except that the group boundaries are unknown and thus have to be estimated.

Next, we consider information loss for some simple full and fractional factorial designs, those typically used in industrial experiments. For censoring and grouping, the structure in such designs results in some simpler expressions that reveal an important property of the information measure. For more complex designs, the formulas remain complicated but are no harder to evaluate so that the different collection schemes can still be compared.

Using these results, we study a few informative examples. First, we consider data from a real experiment. Then, a more complicated experiment is investigated. Finally, we conclude with a discussion of the paper's implications for planning such experiments. The information measure, which requires simple calculations and is thus easily programmed, provides one assessment of an experimental design's suitability. Using it to evaluate potential designs is recommended and thereby arms the practitioner with a useful tool for planning experiments with censored and ordinal data.

Information for the Exponential Regression Model

In this paper, we consider the exponential regression model. Suppose there are m distinct x_i, where x_i is the ith combination of p factorial effects. At each x_i, n replications are taken and are assumed to be exponentially distributed with mean $\theta_i = \exp(x_i\boldsymbol{\beta})$. That is, the data have density $f_{\theta_i}(t) = (1/\theta_i)\exp(-t/\theta_i)$.

Consider two data collection schemes S_1 and S_2. If V_1 and V_2 are their respective asymptotic covariance matrices of the maximum likelihood estimator (MLE) for $\boldsymbol{\beta}$, then we define the asymptotic relative efficiency (ARE) of scheme S_1 to scheme S_2 as

$$\mathrm{ARE}(\hat{\boldsymbol{\beta}};S_1:S_2) = (\,|V_2|/\,|V_1|)^{1/p}\,, \tag{1}$$

where $|\;|$ denotes the determinant of a matrix. Since the determinant of a covariance matrix is related to the volume of the confidence ellipsoid for the true parameters, it provides a measure of the information content for a particular collection scheme. The ARE as defined in (1) allows two collection schemes to be compared and accounts for the number of parameters by taking the pth root. Also, when one parameter is of particular interest, $\mathrm{ARE}(\hat{\beta}_i;S_1:S_2) = V_{2(i,i)}/V_{1(i,i)}$ can be used to study information loss.

When the data are complete or exactly known (E), the asymptotic covariance matrix of $\hat{\boldsymbol{\beta}}$ is

$$V_E = (E\{-\partial^2 l/\partial\beta_i\partial\beta_j\})^{-1} = (X^TX)^{-1}\,, \tag{2}$$

where the design matrix X is $(x_1, \ldots, x_m)^T$ and l is the log likelihood.

Information for Censored and Grouped Data

Let L be the common censoring point for singly Type I censored data. Grouped data are recorded as intervals $\{(a_{i-1}, a_i)\}$, where the interval boundaries partitioning $(0, \infty)$ into k+1 groups are $a_0 = 0, a_1, \ldots, a_{k+1} = \infty$. Rounded and rounded-censored data are special cases of grouped data. Rounded data are recorded as $\{(2i-1)h/2\}$, the midpoints of the intervals $\{[(i-1)h, ih)\}$, where the interval width h determines the degree of rounding. Note that equally spaced inspections also yield rounded data. For rounded-censored data, L is assumed to be a multiple of h with the uncensored data being rounded as above with interval width h.

Hamada (1989) showed that the asymptotic covariance matrix of $\hat{\boldsymbol{\beta}}$ for censored (C) and grouped (G) data is

$$V_C \text{ or } V_G = (E\{-\partial^2 \mathbf{l} / \partial \beta_i \partial \beta_j\})^{-1} = (X^T D X)^{-1} , \tag{3}$$

where $\mathbf{l}$ is the appropriate log likelihood and $D = \text{diag}(d_1, \ldots, d_m)$. d_i can be viewed as the information at x_i since it is the ARE of $\hat{\theta}_i$ (i.e., the MLE of the mean $\theta_i = \exp(x_i \boldsymbol{\beta})$) for censored or grouped data relative to exactly known data. Thus, the information at each x_i is incorporated into the information for $\boldsymbol{\beta}$. For rounded and rounded-censored data, the ARE $\hat{\theta}$ expression for grouped data simplifies. The following are the ARE $\hat{\theta}$ for censored (C), grouped (G), rounded (R) and rounded-censored (RC) data as found in Hamada (1990):

$$\text{ARE}(\hat{\theta}_C) = 1 - \exp(-L / \theta) \tag{4}$$

$$\text{ARE}(\hat{\theta}_G) = \sum_{i=1}^{k+1} p_i^{-1}((\exp(-a_i / \theta)(a_i / \theta) - \exp(-a_{i-1} / \theta)(a_{i-1} / \theta))^2 , \tag{5}$$

where $p_i = (\exp(-a_{i-1} / \theta) - \exp(-a_i / \theta))$ and $\exp(-a_{k+1} / \theta)(a_{k+1} / \theta) = 0$

$$\text{ARE}(\hat{\theta}_R) = g(h / \theta) , \text{ where } g(a) = a^2 \exp(-a) / (1 - \exp(-a))^2 \tag{6}$$

$$\text{ARE}(\hat{\theta}_{RC}) = g(h / \theta)(1 - \exp(-L / \theta)) \tag{7}$$

Note that for the more general Weibull distribution, the rounding and rounding-censoring formulas no longer have this simple form.

Using (1), (2) and (3), the information for grouped or censored data relative to exactly known data is:

$$\text{ARE}(\hat{\boldsymbol{\beta}}) = (|X^T D X| / |X^T X|)^{1/p} . \tag{8}$$

Since the ARE in (4)-(7) are less than one and the covariance matrix for exactly known data can be written with an m×m identity matrix as its middle matrix, it follows that the ARE are less than one. That is, the ARE given in (8) are between zero and one, where values close to one indicate a small loss of information.

Information for Ordinal and Binary Data

In order to compare ordinal and binary data with censored and grouped data, we assume that the former are generated from a latent exponential regression model. That is, for ordinal data, the cutpoints that define the group boundaries for grouped data are no longer known and thus have to estimated. Similarly, for binary data, there is a single cutpoint defining the two groups. Also, for ordinal and binary data, the intercept parameter is confounded with the cutpoints so that the intercept is actually incorporated into the cutpoints.

To obtain the asymptotic covariance matrix of $\hat{\boldsymbol{\beta}}$, we first need to calculate the asymptotic covariance matrix for the MLE of both $\boldsymbol{\beta}$ and the cutpoints $\{a_i\}$. Straightforward algebra yields the form:

$$\begin{bmatrix} X^TDX & X^TW \\ W^TX & Z \end{bmatrix}^{-1} . \tag{9}$$

Note that X^TDX is exactly the same as that from grouped data and W and Z are matrices with dimensions, $m \times k$ and $k \times k$, respectively.

From (9) it follows that the asymptotic covariance matrix of $\hat{\boldsymbol{\beta}}$ is

$$(X^T(D-WZ^{-1}W^T)X)^{-1} . \tag{10}$$

Formulas for W and Z appear in the Appendix. Using (1), (2) and (10), one can obtain the information for ordinal data relative to exactly known data. Observe that the middle matrix is no longer diagonal. This prevents simplification of the information formulas as seen for censored and grouped data in the next section.

For binary data, where a single cutpoint is estimated, (10) simplifies to $(X^TBX)^{-1}$, where

$$B_{ij} = d_i(1-d_i/\sum d_i) \quad \text{for} \quad i=j$$

$$-d_i d_j/\sum d_i \quad \text{for} \quad i \neq j$$

and d_i is the information from two groups.

Information for Factorial Experiments

In this section, we consider information for some simple full and fractional factorial designs, those typically used in industrial experiments. Specifically, some L_4 and L_8 designs are studied, where the subscript denotes the number of runs used in the design. Let 2_c^{a-b} denote both model and design for *a* two level factors and *c* two-factor interactions based on a 2^{-b} fraction of a full factorial design.

The structure in fractional factorial designs simplifies the formulas given in the previous section for censored and grouped data. For exactly known data, since the design matrix X is orthogonal, $|X^T X| = m^p$, where m is the number of design points and p is the number of factorial effects. Here, the model assumes an intercept parameter for exactly known, censored, and grouped data.

Results for the L_4 Design

The L_4 design matrix is given in Table 1. The 2^2 and $2^{3-1}(=2_1^2)$ designs are obtained by using the first two and three columns, respectively.

For 2^2,

$$\text{ARE}(\hat{\beta}) = ((d_1d_2d_3+d_1d_2d_4+d_1d_3d_4+d_2d_3d_4)/4)^{1/3},$$

where the d_i given in the previous section are the individual information at each run. Note that ARE is an average of all 3-tuples of $[d_1d_2d_3d_4]$.

For $2^{3-1}(=2_1^2)$,

$$\text{ARE}(\hat{\boldsymbol{\beta}}) = (d_1 d_2 d_3 d_4)^{1/4},$$

the geometric mean of the information at the four runs of the design. From the properties of the geometric mean, it follows that the ARE is especially sensitive to small information at a particular run, much more than that indicated by taking an arithmetic mean. Also, the form of the ARE for 2^2 suggests the same phenomenon, albeit somewhat less since the information at a particular run enters into three out of the four quantities.

Table 1: Design Matrix for L_4

Run	1	2	12
1	1	1	1
2	1	-1	-1
3	-1	1	-1
4	-1	-1	1

Results for the L_8 Design

For the L_8 design, some of the models with two-factor interactions are identical or isomorphic to main effects models with a larger number of factors. In fact we need to consider only one additional model, 2_1^{3-1}. The design matrices of the following designs are identical: $2_2^3 = 2^{5-2}$, $2_3^3 = 2^{6-3}$, $2_3^{4-1} = 2^{7-4}$, $2_1^{5-2} = (2_2^{4-1}$ or $2^{6-2})$ and $2_2^{5-2} = 2_1^{6-2} = 2^{7-4}$. 2_1^{4-1} and 2_2^{4-1} are isomorphic to 2^{5-1} and 2^{6-2}, respectively. Thus, we need only consider 2^3, 2^{4-1}, 2^{5-2}, 2^{6-3}, 2^{7-4} and 2_1^{3-1}. The L_8 design matrix is given in Table 2 with columns used for different models displayed in Table 3.

Table 2: Design Matrix for L_8

Run	1	2	3	123	12	13	23
1	1	1	1	1	1	1	1
2	1	1	-1	-1	1	-1	-1
3	1	-1	1	-1	-1	-1	1
4	1	-1	-1	1	-1	1	-1
5	-1	1	1	-1	-1	1	-1
6	-1	1	-1	1	-1	-1	1
7	-1	-1	1	1	1	-1	-1
8	-1	-1	-1	-1	1	1	1

Table 3: Columns Used for L_8 Models

Model	1	2	3	123	12	13	23
2^3	x	x	x				
2^{4-1}	x	x	x				x
2^{5-2}	x	x	x	x	x		
2^{6-3}	x	x	x	x	x	x	
2^{7-4}	x	x	x	x	x	x	x
2_1^3	x	x	x	x			

Next we present the information formulas for 2^{5-2}, 2^{6-3} and 2^{7-4} since they simplify. For the remaining designs (models), their formulas are patterned, but not easily described and therefore not included here.

For 2^{5-2},

$$\text{ARE}(\hat{\boldsymbol{\beta}}) = ((1/16)\textstyle\sum \binom{4}{3}\binom{4}{3})^{1/6} ,$$

which averages the product of all 3-tuples of $[d_1d_2d_3d_4]$ and all 3-tuples of $[d_5d_6d_7d_8]$.

For 2^{6-3},

$$\text{ARE}(\hat{\boldsymbol{\beta}}) = ((1/8)\textstyle\sum \binom{8}{7})^{1/7} ,$$

which averages all 7-tuples of $[d_1d_2d_3d_4d_5d_6d_7d_8]$.

Finally, for 2^{7-4},

$$\text{ARE}(\hat{\boldsymbol{\beta}}) = (d_1 d_2 d_3 d_4 d_5 d_6 d_7 d_8)^{1/8} ,$$

the geometric mean of the information at the eight runs of the design. As seen for the L_4 design, the ARE for this design is especially sensitive to small information at a particular design point and to a lesser degree for 2^{5-2} and 2^{6-3}.

For larger designs, such as the L_{16}, the ARE expressions become more complicated. Only a few models such as the fully saturated model yield simple ARE expressions. Nevertheless, a study of information loss for larger designs can still be carried out since the formulas are simple to calculate and consequently easy to program.

A Study of Information for Some Examples

In this section, we look at two examples and compare the information for censored, grouped, ordinal and binary data. First, we look at example using an L_4 design based on a real experiment and then consider a more complicated example using an L_8 design.

Example 1 Involving an L_4 Design

Zelen (1959) presented lifetest data on glass capacitors subjected to various voltage and temperature stresses. Table 4 presents the failure times in hours for four voltage-temperature combinations. Fitting these data with an exponential regression model using the 2_1^2 model (see Table 1) gave the following estimates: $\hat{\boldsymbol{\beta}} = (6.37, .03, -.50, -.06)$, corresponding to intercept, temperature, voltage and temperature×voltage. For this data, voltage is the dominating effect.

Table 4: Example 1 Failure Times for Glass Capacitors

Temp(°C)	Voltage(kV) 200	350
170	439	258
	904	258
	1092	347
	1105	588
180	959	241
	1065	241
	1065	435
	1087	455

Without loss of generality, let β_0=0 since L, h, and $\{a_i\}$ are chosen relative to means $\{\theta_i\}$, i.e., $\boldsymbol{\beta}$ = (0, .03, -.50, -.06). For comparison's sake, we also consider $\boldsymbol{\beta}$ = (0, .10, -.50, -.25). The means at the four design points are given in Table 5 for the two parameter combinations designated by I and II. Note that the means for II are more spread out and that there is a three to five fold difference between θ_{max} and θ_{min} for I and II.

Table 5: Example 1 Means for I and II

	Run 1	2	3	4
I	.59	1.80	.63	1.51
II	.52	2.34	.70	1.16

The amount of censoring L and rounding h used is (.4, .8. 1.2, 1.6). Recall that for censoring, a failure is observed if it occurs before L and that rounding arises from equally spaced inspections of length h. For rounding-censoring, (h, L) is (.2, .8), (.4, .8), (.4, 1.6) and (.8, 1.6), corresponding to 5, 3, 5, and 3 groups, respectively (e.g., (.2, .8) refers to inspections at .2, .4, .6 and .8). Also, two other group schemes are studied, (.1, .2, .4, .8) and (.2, .4, .8, 1.6). Recall that for ordinal data the group cutpoints are unknown and thus have to be estimated. Finally group and binary schemes are compared for a_1 = (.4, .8, 1.2, 1.6); note that a_1 is unknown for the binary scheme. The information for these schemes relative to that for complete data is given in Tables 6 and 7.

Table 6: Example 1 Information for Censored and Rounded Data

Censoring	L			
	.4	.8	1.2	1.6
I	.331	.540	.675	.764
II	.333	.540	.672	.759
Rounding	h			
	.4	.8	1.2	1.6
I	.980	.922	.837	.737
II	.978	.918	.823	.725

Table 7: Example 1 Information for Rounded-Censored, Grouped and Ordinal Data

R-C/O	I		II	
(h, L)	R-C	O	R-C	O
.2 .8	.537	.517	.537	.510
.4 .8	.528	.510	.527	.503
.4 1.6	.748	.728	.741	.716
.8 1.6	.702	.689	.693	.675
G/O	I		II	
$\{a_i\}$	G	O	G	O
.1 .2 .4. 8	.535	.515	.534	.508
.2 .4 .8 1.6	.744	.722	.737	.710
G/B	I		II	
a_1	G	B	G	B
.4	.324	.308	.325	.304
.8	.495	.483	.491	.475
1.2	.558	.555	.549	.542
1.6	.554	.554	.539	.535

In evaluating the information, we use a benchmark ARE of .56 since the ARE for individual parameters is only slightly smaller than the overall criterion and a ratio of variances of .56 corresponds to a ratio of standard deviations of .75, not an appreciable information loss. From Tables 6 and 7, we observe that there is surprisingly little difference between the results for the two parameter combinations I and II. Table 6 reveals that censoring and rounding can be quite heavy (L $\sim .5\theta_{max}$ and h $\sim \theta_{max}$) and still provide adequate information.

Table 7 reveals that even coarse grouping provides adequate information. See, for example, the results for two groups (a_1=1.2). Thus, it is not surprising that given the same censoring point L, the different grouping schemes provide similar information. Finally, it is interesting that ordinal and binary data lose little information (from estimating the group cutpoints) as compared with grouped data.

Example 2 Involving an L_8 Design

We consider a more complicated L_8 design (see Table 2) with a 2^{7-4} (7 factor main effects) model and $\boldsymbol{\beta}$ = (0, 1, .75, .5, .25, .15, .10, .05). Table 8 reveals a 100 fold difference between θ_{max} and θ_{min}. In the previous example, the small three to five fold difference meant that the information at the different design points was more similar. For this example, however, the large 100 fold difference accounts for the different results obtained here.

Table 8: Example 2 Means

Run							
1	2	3	4	5	6	7	8
.06	.30	.61	1.65	1.11	2.46	3.32	6.05

The amount of censoring L is (1.0, 1.5, 2.0, 2.5, 3.0, 4.0) and amount of rounding h is (.1, .25, .5, 1., 1.5, 2). For rounding-censoring, (h, L) is (.25, .5, 1; 2, 3) corresponding to 8, 4, 2, 12, 6, and 3 groups, respectively. Two other group schemes are studied, (.3, .7, 1.4, 2) and (.45, 1.05, 2.1, 3). Also, group and binary schemes are compared for a_1 = (.5, .75, 1, 1.5, 2). The information for these schemes relative to that for complete data is presented in Tables 9 and 10.

Table 9: Example 2 Information for Censored and Rounded Data

Censoring	L					
	1.0	1.5	2.0	2.5	3.0	4.0
	.494	.602	.679	.735	.779	.841
Rounding	h					
	.1	.25	.5	1	1.5	2
	.972	.861	.723	.627	.519	.410

Table 10: Example 2 Information for Rounded-Censored, Grouped and Ordinal Data

R-C/O (h, L)	R-C	O
.25 2	.576	.534
.5 2	.462	.361
1 2	.396	.124
.25 3	.664	.611
.5 3	.543	.413
1 3	.467	.141
G/O $\{a_i\}$	G	O
.3 .7 1.4 2	.538	.496
.45 1.05 2.1 3	.547	.445
G/B a_1	G	B
.5	.201	.171
.75	.240	.133
1.0	.271	.090
1.5	.284	.033
2.0	.258	.010

Table 9 reveals that for this example censoring and rounding can be quite heavy ($L \sim .25\theta_{max}$ and $h \sim .25\theta_{max}$) and still provide adequate information. Table 10 shows that while some grouping schemes provide adequate information, others with coarse grouping do not. Finally, ordinal data loses little information when the information for grouped data is substantial but the loss is significant when the information for grouped data is small.

Tables 9 and 10 can be used to compare two data types directly. For example, the information of ordinal data relative to grouped data is much larger; for (h=.5, L=2), the information is .781 (=.361/462). Although the information of binary relative to two groups is high (.850 =.171/.201) for $a_1 = .5$, one observes that binary data loses substantially more information for larger a_1. Other data types can be compared in a similar manner. For example, the information for grouping in addition to censoring (h=.25, L=2) relative to censoring alone is .839 (=.576/.679).

Discussion

In designing experiments with censored and ordinal data, the amount of censoring, when to do the inspections (i.e., choosing the group cutpoints or the amount of rounding) and the choice to collect ordinal or binary data rather than censored or grouped data needs to be decided. The information from censored, grouped, ordinal and binary data relative to complete data provides one assessment of an experimental design's suitability. In this paper, we have provided a unified framework that allows the various collection schemes to be compared. The information measure is simple to calculate and therefore is easily programmed. Evaluating the information measure for potential designs gives the practitioner a useful tool for planning experiments with censored and ordinal data. In particular, designs which have a significant information loss can be eliminated from consideration.

Simplifications of the information measure for some factorial designs as well as a study of two examples presented in this paper suggest certain things to keep in mind in the planning stage. Recall that the information for a saturated model, where the number of factorial effects (including an intercept) equals the number of design points, is the geometric mean of the individual information measures at each design point. Consequently, this suggests avoiding designs where the individual information at any one design point is very small. Generally, the examples show that rounding and censoring can be quite heavy and still provide adequate information. Moreover, little information is lost for ordinal data relative to grouped data when the means at the different design point means are similar. The examples demonstrate that the information loss does depend on the situation, however. Although insensitive for many situations in the first example, the information loss was substantial in the second. This suggests that several scenarios be evaluated in the planning stage. Since the information measure depends on the true parameters which are not known, several guessed values for the parameters can be tried. If the information loss is substantial or is very sensitive to the guessed parameter values, alternative designs should be considered.

Gould and Lawless (1988) studied information for the log-Burr regression model which includes the exponential (more generally Weibull) regression model as well as the log-logistic regression model. Their study of the log-logistic model (a symmetric distribution as opposed to the asymmetric extreme value distribution for the exponential and Weibull model) found information for grouped and censored data to be somewhat larger. An investigation for the various incomplete data types for this more general model in the industrial context is a topic of future study. Such a study would provide an assessment of the effect of the assumed distribution.

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Appendix: Details on Information for Ordinal Data

Let $\{a_j\}$ be the unknown cutpoints that define k+1 ordered categories. Also, let $r_{i,j} = \theta_i^{-1} a_j$, $e_{i,j} = \exp(-r_{i,j})$ and $\theta_i = \exp(x_i \beta)$ for the ith (out of m) combination of factorial effects. Further, assume that there at least three categories. Details for the matrices W and Z from (10) are as follows.

The m×k matrix W is given by:

$$w_{i,1} = \theta_i^{-1} e_{i,1}(e_{i,1}{}^2 + r_{i,1}\, e_{i,1} - e_{i,1}\, e_{i,2} - r_{i,2}\, e_{i,2})/((1-e_{i,1})(e_{i,1}-e_{i,2}))$$

$$w_{i,j} = \theta_i^{-1} e_{i,j}((r_{i,j}-r_{i,j-1})e_{i,j}\, e_{i,j-1} + (r_{i,j+1}-r_{i,j})e_{i,j}\, e_{i,j+1} - (r_{i,j-1}+r_{i,j+1})e_{i,j-1}\, e_{i,j+1})/((e_{i,j-1}-e_{i,j})(e_{i,j}-e_{i,j+1}))$$

$$w_{i,k} = \theta_i^{-1} e_{i,k}(r_{i,k}\, e_{i,k-1} - r_{i,k-1}\, e_{i,k-1})/(e_{i,k-1}-e_{i,k})$$

The k×k matrix Z is given by:

$$z_{j,j} = \sum_{i=1}^{n} \theta_i^{-2} e_{i,j}{}^2 (e_{i,j-1}-e_{i,j+1})/((e_{i,j-1}-e_{i,j})(e_{i,j}-e_{i,j+1}))$$

$$z_{j,j-1} = z_{j-1,j} = -\sum_{i=1}^{n} \theta_i^{-2} e_{i,j}\, e_{i,j-1}/(e_{i,j-1}-e_{i,j})$$

$$z_{i,j} = 0 \text{ if } |i-j| > 1$$

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