# CONSTRUCTION OF RESPONSE SURFACE DESIGNS FOR QUALITATIVE AND QUANTITATIVE FACTORS

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IIQP Research Report
RR-91-03

April, 1991

# CONSTRUCTION OF RESPONSE SURFACE DESIGNS FOR QUALITATIVE AND QUANTITATIVE FACTORS

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*ABSTRACT*

A general approach is proposed for constructing response surface designs of economical size for qualitative and quantitative factors. It starts with a central composite design for the quantitative factors and then partitions them into groups corresponding to different level combinations of the qualitative factors. Good designs are selected to ensure high estimation efficiency for several models that reflect four stated objectives. A fairly complete collection of designs for one qualitative factor with two levels is given. Designs for other situations are constructed and some extensions are considered.

Key Words: Central composite design; Determinant criterion; Factional factorial design; Plackett-Burman design.

# 1 Introduction

An outstanding problem in response surface methodology is the construction of economical designs for both quantitative and qualitative factors and the associated issue of modeling. It may arise in situations in which some of the factors are qualitative by nature. Consider, for example, a machining process for caster rolls in a steel plant. To improve the machining time while keeping a high rating of surface roughness, four factors are identified as potentially important: (i) feed which is the distance the tool advances in one revolution, (ii) speed at which the surface moves past the cutting tool, (iii) lead angle at which the tool meets the work piece, and (iv) insert which is the replaceable part of the cutting tool that does the cutting. The first three factors are quantitative, while the fourth can only take two shapes: round and square. Without the availability of economical designs, the investigator may adopt one of the following two approaches. In the first approach, a second order response surface design such as the central composite design is used for each level of the qualitative factor (i.e., for round and for square). This can take a large number of runs and may therefore be impractical. Alternatively, one may ignore the difference between the two types of factors and use a standard design for both types. As Draper and John (1988) aptly pointed out, standard designs such as the central composite designs may not be suitable because they would require four to five levels which the qualitative factors may not have. Even if this were possible, the quantitative levels among them would be meaningless for the qualitative factors.

The importance of the problem was recognized by Cox (1984), who posed it as one of eleven open problems in design of experiments. Draper and John (1988) appeared to be the first to tackle it seriously. They discussed the relations between designs and models and found designs in some specific situations. In this paper we give a systematic method for constructing designs of economical size and discuss the underlying objectives and models. An obvious requirement for good designs is that when collapsed over the qualitative factors they should possess desirable properties of standard response surface designs for quantitative factors (Box and Draper, 1987; Khuri and Cornell, 1987). These are captured by the objectives C and D of Section 2. Some issues related to the presence of qualitative factors are addressed by the objectives A, B and C. In Section 3 we outline a method of constructing designs to meet these objectives. The main idea is to start with a central composite design for the quantitative factors and then partition them into groups. Each group corresponds to a combination of the qualitative factor levels. From these designs we then select those with high overall efficiencies as measured by the determinant criteria (3.1) and (3.3) for several models. For one qualitative factor with two levels, a fairly complete collection of designs is given in Section 4. A different but related class of designs for one qualitative factor and two quantitative factors is given in Section 5. Strategies for constructing designs with two qualitative and two quantitative factors are briefly discussed in Section 6. Section 7 outlines some extensions.

## 2 Objectives and Supporting Models

The designs will be constructed to meet the following objectives. Denote the quantitative factors by $x_1 \cdots, x_k$ and the qualitative factors by $z_1, \cdots, z_r$.

A. The overall design is efficient for a model that is second order in $x_1, \cdots, x_k$, and has the main effects of $z_1, \cdots, z_r$ and the interactions between $x_i$ and $z_j$.

B. At each combination of the qualitative factors or each level of a qualitative factor $z_j$, it is an efficient first order design in $x_1, \cdots, x_k$.

C. The second order design in A consists of two parts: the first part is a first order design for both $x_1, \cdots, x_k$ and $z_1, \cdots, z_r$, and the second part can be viewed as a sequential addition to the first part to expand it from a first order design to a second order design.

D. When summed over the levels of $z_1, \cdots, z_r$, it is an efficient second order design for $x_1, \cdots, x_k$.

Objective A is self-explanatory. Since the effects of $x_i$ may vary with the levels of $z_j$, it is desirable to have the design at each combination of $z_1, \cdots, z_r$ (or at each level of $z_j$) that allows separate estimation of the first order effects of $x_i$. The second objective accomplishes this. The third objective enables the experiment to be conducted in two stages. The initial experiment allows the estimation of first order effects for both $x_i$ and $z_j$. If warranted, the second experiment can be conducted to ensure the estimation of second order effects in the combined experiment. Objective D ensures that, when there is no significant difference among the $z_j$'s, the combined design is a second order design with good overall properties. The issue of estimation efficiency will be addressed in the next section.

These objectives can be stated more precisely with the aid of regression models. For simplicity we only consider $r$=1, the case of one qualitative factor. A second order model for $x_i$ and $z$ is given by

$$E(y) = \sum_{z=1}^{m} W_z \left(\beta_{oz} + \sum_{i=1}^{k} \beta_{iz} x_i\right) + \sum_{i,j=1}^{k} \beta_{ij} x_i x_j, \tag{2.1}$$

where $m$ is the number of levels of the qualitative factor, $W_z$ is 1 when $y$ is taken at level $z$ and 0 otherwise, $\beta_{oz}$ is the constant term and $\beta_{iz}$ is the slope of $x_i$, both depending on the choice of $z$. If the run size is small, we may only be able to entertain the following submodels of (2.1),

$$E(y) = \sum_{z=1}^{m} W_z \left(\beta_{oz} + \sum_{i=1}^{k} \beta_{iz} x_i\right) + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \text{ some of } \beta_{ij} x_i x_j (i < j), \tag{2.2$a$}$$

$$E(y) = \sum_{z=1}^{m} W_z \beta_{oz} + \sum_{i=1}^{k} \left(\beta_i x_i + \beta_{ii} x_i^2\right) + \text{ some of } W_z \beta_{iz} x_i \text{ and } \beta_{ij} x_i x_j (i < j). \tag{2.2$b$}$$

Model (2.2a) excludes some interaction terms $\beta_{ij}x_ix_j$, $i < j$, in model (2.1). Model (2.2b) further excludes some $\beta_{iz}x_i$ terms . Objective $A$ stipulates that the overall design allows one of these models to be fitted with high efficiency.

Objective B requires that the coefficients in the model

$$E(y) = \beta + \sum_{i=1}^{k} \beta_i x_i + \text{ some of } \beta_{ij} x_i x_j (i < j), \tag{2.3}$$

be estimated with high efficiency from the design at each level of the qualitative factor. The first order submodels of (2.2a) and (2.2b) are respectively

$$E(y) = \sum_{z=1}^{m} W_z \left( \beta_{oz} + \sum_{i=1}^{k} \beta_{iz} x_i \right) + \text{ some of } \beta_{ij} x_i x_j (i < j), \tag{2.4a}$$

$$E(y) = \sum_{z=1}^{m} W_z \beta_{oz} + \sum_{i=1}^{k} \beta_i x_i + \text{ some of } W_z \beta_{iz} x_i \text{ and } \beta_{ij} x_i x_j (i < j). \tag{2.4b}$$

Objective C requires that the coefficients in (2.4a) or (2.4b) be estimated with high efficiency from the first order design.

# 3 Selection Criteria and Construction Method

For $r = 1$ and $m = 2$ the objectives stated in Section 2 can be met by a class of designs to be constructed in this section. Such designs are generically represented in Table 1. For the quantitative factors $x_1, \cdots, x_k$ the first $t(= 2^{k-p})$ runs are chosen according to a $2^{k-p}$ fractional factorial design with high resolution and $x_i = \pm 1$. The $(t+1)$st and $(t+2)$nd runs are the center points. The last $2k$ points are the "star points" with $\alpha = t^{\frac{1}{4}}$ to make the overall design for $x_1, \cdots, x_k$ a rotatable central composite design (Box and Draper, 1987, p. 488). This design for $x_i$ satisfies objective D in Section 2.

**Table 1. Second Order Designs for Quantitative Factors and One Qualitative Factor**

| run | $x_1$ | $x_2$ | $\cdots$ | $x_k$ | $z$ | |
|---|---|---|---|---|---|---|
| 1 | | | | | | |
| 2 | $\pm 1$ | according | to | $a$ | see | |
| $\vdots$ | $2^{k-p}$ | design | | | note 1 | |
| $t(= 2^{k-p})$ | | | | | | |
| $t+1$ | 0 | 0 | $\cdots$ | 0 | 1 | -1 |
| | | | | | or | |
| $t+2$ | 0 | 0 | $\cdots$ | 0 | -1 | 1 |
| $t+3$ | $\alpha$ | 0 | $\cdots$ | 0 | | |
| $t+4$ | -$\alpha$ | 0 | $\cdots$ | 0 | see | |
| $t+5$ | 0 | $\alpha$ | $\cdots$ | 0 | | |
| $t+6$ | 0 | -$\alpha$ | $\cdots$ | 0 | | |
| $\vdots$ | | | | | note 2 | |
| $t+2k+1$ | 0 | $\cdots$ | | $\alpha$ | | |
| $t+2k+2$ | 0 | $\cdots$ | | -$\alpha$ | | |

Notes

1. $z = 1$ or -1 by (i) equating column $z$ to an interaction column among the $x_i's$, or (ii) search over different combinations of $\pm 1's$ according to some criteria.

2. See 1 (ii).

For the first $t$ runs, $z = 1$ or -1 can be chosen according to an interaction column among the $x_i's$ or by searching over different combinations of $\pm 1$'s. The $z$ values for the $(t+1)$st and the $(t+2)$nd runs are (1, -1) or (-1, 1). The $z$ value for the last $2k$ runs are chosen by searching over different combinations of $\pm 1$'s. The search in both cases is directed by objectives A, B and C of Section 2.

We use the determinant criterion (D-criterion) for efficiency comparison,

$$|X^T X|^{1/n}, \tag{3.1}$$

where the model is described by $Ey = X\beta$, $y$ is the vector of observations, $\beta$ is the vector of parameters and $n$ is the dimension of $\beta$, i.e., the number of parameters in the model.

According to A, the overall design with $t + 2k + 2$ runs should allow the paramenters in models (2.1) or (2.2) to be estimated with high efficiency in terms of (3.1). When the fraction $2^{-p}$ in the $2^{k-p}$ design is small, we can usually find a design to accommodate the estimation of parameters in the complete model (2.1). Otherwise, we look for designs that can accommodate the estimation of the largest number of parameters in models (2.2). The same principle is used for models (2.3) and (2.4).

According to B, the runs at each level of $z$ should allow the parameters in model (2.3) to be estimated with high $D$-efficiency.

According to C, the first $t+1$ runs, including one center point, should allow the parameters in models (2.4a) or (2.4b) to be estimated with high $D$-efficiency. If the first $t + 1$ runs and the remaining $2k + 1$ runs are conducted at different times, the D-criterion for the overall design should accommodate the possible difference between the mean levels of the two sets of runs, that is a "block" effect. The model is

$$E(y) = [X, u] \begin{bmatrix} \beta \\ b \end{bmatrix} = X\beta + ub, \tag{3.2}$$

where $y, X, \beta$ are the same as in (3.1), $u = (u_i)$ is the $(t+2k+2) \times 1$ vector with $u_i = 1$ for $i \leq t+1$ and $-1$ for $i > t+1$, and $b$ is the block effect. Then the determinant criterion for $\beta$ is

$$|X^T X - (X^T u)(u^T u)^{-1}(u^T X)|^{1/n}, \tag{3.3}$$

where $u^T u = t+2k+2$. In the optimal design literature, (3.3) is called the $D_s$-criterion. We will use $\tilde{D}_1$ and respectively $\tilde{D}_{-1}$ to denote the value of (3.3) when the $(t+1)$th run (center point) is at $z = 1$ and respectively $z = -1$.

Note that the parameters in the linear model are estimable if and only if the determinant value (3.1) or (3.3) for the design is positive.

**Table 2. Coefficient Matrix of the Design for $x_1$,$x_2$ and $z$**

| $x_1$ | $x_2$ | $x_1x_2$ | $x_1^2$ | $x_2^2$ | $z$ | $x_1z$ | $x_2z$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | | | |
| 1 | -1 | -1 | 1 | 1 | | | |
| -1 | 1 | -1 | 1 | 1 | | | |
| -1 | -1 | 1 | 1 | 1 | | | |
| 0 | 0 | 0 | 0 | 0 | 1 | | |
| 0 | 0 | 0 | 0 | 0 | or | | |
| $\sqrt{2}$ | 0 | 0 | 2 | 0 | | | |
| $-\sqrt{2}$ | 0 | 0 | 2 | 0 | -1 | | |
| 0 | $\sqrt{2}$ | 0 | 0 | 2 | | | |
| 0 | $-\sqrt{2}$ | 0 | 0 | 2 | | | |

We now illustrate the method by constructing designs for $x_1$, $x_2$ and $z$. There are ten runs whose coefficient matrix is given in Table 2. Since the coefficients for the $x_i$'s are determined by the requirement that they form a rotatable central composite design, it remains to choose the levels of $z$ for the ten runs. Therefore the choice of designs amounts to the choice of the vector $\mathbf{z} = (z_1, \cdots, z_{10})$, where $z_i$ is the $z$ level of the $ith$ run. According to the requirements stated above, first, the designs should allow the estimation of the parameters in (2.1), which can be rewritten as

$$E(y) = \beta_o + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \gamma z + \gamma_1 x_1 z + \gamma_2 x_2 z$$

because $z$ takes the values 1 or -1. We denote this model by

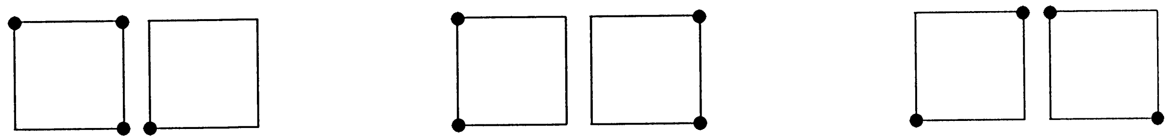
$$(1, x_1, x_2, x_1 x_2, x_1^2, x_2^2, z, x_1 z, x_2 z),$$

where 1 is the constant term, and the value of its determinant criterion (3.1) by $D$. Second, the design at each level of $z$ should allow the estimation of the parameters in the model $(1, x_1, x_2, x_1 x_2)$. The corresponding value of the determinant criterion is denoted by $d_{-1}$ for $z = -1$ and $d_1$ for $z = 1$. There are only ten designs with positive values of $D$, $d_{-1}$ and $d_1$ (see Section 4). Third, the first order design, consisting of the first five runs, should be able to entertain a multitude of models represented by $(1, x_1, x_2, z || x_1 x_2, x_1 z, x_2 z)$, where the first four variables to the left of the double bar must be included in the model and one of $x_1 x_2, x_1 z$ or $x_2 z$ can be added as the fifth variable. We use $D_1(i)$ (and respectively $D_{-1}(i)$) to denote the value of the $D$-criterion for the first order design whose center point takes $z = 1$ (and respectively $z = -1$) and the model consisting of the first four variables and the $ith$ variable to the right of the double bar. We denote the value of the $D_s$-criterion (3.3) by $\tilde{D}_1$ or $\tilde{D}_{-1}$, depending on the $z$ value of the center point in the first order design.

Since good designs must satisfy multi-objective criteria, we list these designs in Table 3.1 of Section 4 according to their values of $D, d_{-1}, d_1, \tilde{D}_1, \tilde{D}_{-1}, D_1(i)$, and $D_{-1}(i)$. A high $D$ value is desired because of the primary importance of fitting the overall model to the data. Subject to this, we may choose designs with approximately equal $d_1$ and $d_{-1}$ values if the information at each level of $z$ is deemed equally important. Or, we may choose those with a large value of $d_1$ or $d_{-1}$ if information at one level of $z$ is more important for investigation (e.g., a more commonly used machine, or a major supplier). There are ten designs with positive values of $D, d_1$, and $d_{-1}$. If sequential design is contemplated, then the first order design should be efficient for some selected models as reflected by high values of $D_1(i)$ or $D_{-1}(i)$ for some $i$, and the overall design should be efficient for fitting the model (3.2) with a block effect, which is reflected by high values of $\tilde{D}_1$ or $\tilde{D}_{-1}$. Recommendations on the choice of designs according to these criteria are given in the next section.

(Figure 1 here)

In Figure 1 we represent these ten designs graphically. The geometry of these designs provides some insight into their values of $d_1, d_{-1}, D_1(i)$ and $D_{-1}(i)$ given in Table 3.1. For example, design 8 is the only one with $d_1 = d_{-1}$. Its designs at $z = 1$ and at $z = -1$ are identical after a $90^o$ rotation. In general designs with equal numbers of points at $z = \pm 1$ have similar $d_1$ and $d_{-1}$ values. The patterns of $D_1(i)$ and $D_{-1}(i), i = 1, 2, 3$ are determined by the distribution of the corner points. From Table 3.1, there are three groups of designs according to the values of $D_1(i)$ and $D_{-1}(i)$. The best is designs 1 to 6, the second best is design 7, and the worst is designs 8 to 10. Their distributions of the corner points over $z = -1$ and 1 are, respectively, of the following types:



There is, however, no good geometric explanation for the $D$ values in Table 3.1. The high values of $\tilde{D}$ in Table 3.1 can be explained partially by the orthogonality between the block effect and some second order effects. This orthogonality is apparent from the geometry in some situations. Take, for example, designs 6 and 8, which have the best and second best $\tilde{D}$ values. From Figure 2, the estimation of $x_1^2$ and $x_2^2$ is based solely on the observations at $z = -1$ of the supplementary design. It is therefore orthogonal to the block effect since the latter measures the overall difference between the first-order design and the supplementary design. A similar explanation holds for design 8.

(Figure 2 here)

# 4 Designs For One Qualitative Factor

In this section we apply the previous construction method to obtain useful designs for $x_1, \cdots, x_k$ with $k \leq 7$ and one qualitative factor $z$. Since $z$ has two levels $\pm 1$, models (2.1) and (2.2a) can be represented respectively by the vectors

$$(1, x_i, x_i^2, x_ix_j, z, x_iz), i = 1, \cdots, k, 1 \leq i < j \leq k, \tag{4.1}$$

and

$$(1, x_i, x_i^2, x_ix_j, z, x_iz), i = 1, \cdots, k, \tag{4.2}$$

with $i < j$ chosen from *selected* pairs. Similarly, models (2.3) and (2.4a) can be represented respectively by the vectors

$$(1, x_i, x_ix_j), \tag{4.3}$$

and

$$(1, x_i, x_ix_j, z, x_iz), \tag{4.4}$$

where $i = 1, \cdots, k$ and $i < j$ take all possible pairs or are chosen from selected pairs as specified in each case. A similar representation was used and explained in Section 3 for the case of $k = 2$.

In each case we give the information on the designs and models in five parts.

(a) A $2^{k-p}$ design for $x_1, \cdots, x_k$ with its defining relations, or a $2^k$ full factorial design. (This is for the upper left part of Table 1.) $N = 2^{k-p} + 2k + 1 =$ total run size, and $N_1 = 2^{k-p} + 1 =$ size of first order design, $\alpha = (2^{k-p})^{\frac{1}{4}}$ in Table 1 that makes the collapsed design (over $z$) rotatable.

(b) If the first $2^{k-p}$ components of the $z$ column in Table 1 are defined by an interaction among the $x_i$'s, say $x_1x_2x_3$, we will write $z = x_1x_2x_3$. Otherwise, this part is omitted.

(c) Overall model (from (4.1) or (4.2)) for the whole design.

(d) Model for $z = 1$ and for $z = -1$ (from (4.3)).

(e) Model for the first order design (from (4.4)) with optional terms to the right of the double bar.

The information on each design is completed by the values of $z$. If the first $t(= 2^{k-p})$ components of $z$ are defined by an interaction among the $x_i$'s (see (b) above), we only give values of $\mathbf{z} = (z_{t+3}, \cdots, z_{t+2k+2})$ for the star points. Otherwise, we give $\mathbf{z} = (z_1, \cdots, z_{t+2k+2})$. For each design, the values of $D, d_1$ and $d_{-1}$ are given. For $k = 2$ and 3, the values of $\tilde{D}_1, \tilde{D}_{-1}, D_1(i)$ and $D_{-1}(i)$ are also given.

<u>General Comments on Table 3.</u>

(i) In each case only designs with positive values of $D, d_1$, $d_{-1}, D_1$ and $D_{-1}$ are given, where $D_1$ and $D_{-1}$ are the $D$ value (3.1) for the first order design with the center point at $z = 1$ and -1 respectively. Recall that positive values of these criteria guarantee that the parameters in the models for the criteria are estimable. These designs are generally good since there is no marked difference among their values of $D, d_1, d_{-1}, D_1$ and $D_{-1}$. Choice of designs depends on the relative importance of these criteria and those in (ii).

(ii) Values of the $D_s$-criterion (3.3) are only given for $k = 2$ and 3. For $k \geq 4$, the rankings of designs according to $\tilde{D}$ are essentially the same as $D$. They do not give new information for design selection and therefore are omitted. Note also that, for $k \geq 4, D_1$ and $D_{-1}$ are constant for the competing designs because the $z$ values for the first order design, being defined by an interaction among the $x_i$'s, are the same for the competing designs. Recall that objective $C$ in Section 2 is manifested by the criteria $D_s, D_1$ and $D_{-1}$. In view of the remarks above, it is not necessary to use them to further discriminate the designs for $k \geq 4$.

The eight cases in Table 3 are referred as Table 3.$i$, $i = 1, \cdots, 8$. Recommendations on the choice of designs are only given for $k = 2$ and 3. Recommendations for the remaining cases can be similarly made.

**Table 3. Constructed Designs for $x_1, \cdots, x_k$ with $k \leq 7$ and one $z$**

1. $k = 2$

   (a) $2^2$ design for $x_1$ and $x_2$, $N = 10$, $N_1 = 5, \alpha = 1.414$.

   (c) Overall model: (4.1).

   (d) Model for $z = -1$ and for $z = 1$: $(1, x_1, x_2, x_1x_2)$.

   (e) First order model: $(1, x_1, x_2, z || x_1x_2, x_1z, x_2z)$.

**Table 3.1**

| design no. | $z = (z_1, \cdots, z_{10})$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | -1 | -1 | -1 | 1 | 1 | -1 | 1 | -1 | 1 | -1 |
| 2 | -1 | -1 | -1 | 1 | 1 | -1 | 1 | -1 | 1 | 1 |
| 3 | -1 | -1 | -1 | 1 | 1 | -1 | -1 | 1 | 1 | -1 |
| 4 | -1 | -1 | -1 | 1 | 1 | -1 | -1 | 1 | 1 | 1 |
| 5 | -1 | -1 | -1 | 1 | 1 | -1 | -1 | 1 | -1 | 1 |
| 6 | -1 | 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 |
| 7 | -1 | -1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | 1 |
| 8 | -1 | 1 | 1 | -1 | 1 | -1 | -1 | -1 | 1 | 1 |
| 9 | -1 | 1 | 1 | -1 | 1 | -1 | -1 | -1 | -1 | 1 |
| 10 | -1 | 1 | 1 | -1 | 1 | -1 | -1 | 1 | -1 | 1 |

| design no. | $D$ | $d_1$ | $d_{-1}$ | $d_1(x_1^2)$ | $d_1(x_2^2)$ | $d_{-1}(x_1^2)$ | $d_{-1}(x_2^2)$ | $d_{-1}(x_1^2, x_2^2)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 5.66 | 1.41 | 4.29 | 0.00 | 0.00 | 3.74 | 3.74 | 3.38 |
| 2 | 5.64 | 2.21 | 3.35 | 0.00 | 2.30 | 2.85 | 2.00 | 0.00 |
| 3 | 4.18 | 1.41 | 3.92 | 0.00 | 0.00 | 2.73 | 3.35 | 2.52 |
| 4 | 3.95 | 2.21 | 2.43 | 0.00 | 2.30 | 1.41 | 2.00 | 0.00 |
| 5 | 3.32 | 1.41 | 2.87 | 0.00 | 0.00 | 2.51 | 2.51 | 1.88 |
| 6 | 5.58 | 2.00 | 2.99 | 0.00 | 0.00 | 3.29 | 3.29 | 3.17 |
| 7 | 4.18 | 1.68 | 3.19 | 0.00 | 0.00 | 3.10 | 2.49 | 2.52 |
| 8 | 5.04 | 2.63 | 2.63 | 0.00 | 2.64 | 2.64 | 0.00 | 0.00 |
| 9 | 4.88 | 1.68 | 3.42 | 0.00 | 0.00 | 3.39 | 2.86 | 2.83 |
| 10 | 4.88 | 2.21 | 2.74 | 2.00 | 2.00 | 2.00 | 2.00 | 0.00 |

| design no. | $\tilde{D}_1$ | $\tilde{D}_{-1}$ | $D_1(1)$ | $D_1(2)$ | $D_1(3)$ | $D_{-1}(1)$ | $D_{-1}(2)$ | $D_{-1}(3)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 4.58 | 4.58 | 3.57 | 3.03 | 3.03 | 2.30 | 0.00 | 0.00 |
| 2 | 3.90 | 4.75 | 3.57 | 3.03 | 3.03 | 2.30 | 0.00 | 0.00 |
| 3 | 3.10 | 3.10 | 3.57 | 3.03 | 3.03 | 2.30 | 0.00 | 0.00 |
| 4 | 3.90 | 3.21 | 3.57 | 3.03 | 3.03 | 2.30 | 0.00 | 0.00 |
| 5 | 2.09 | 2.09 | 3.57 | 3.03 | 3.03 | 2.30 | 0.00 | 0.00 |
| 6 | 4.21 | 5.38 | 2.30 | 0.00 | 0.00 | 3.57 | 3.03 | 3.03 |
| 7 | 3.10 | 3.10 | 3.03 | 0.00 | 3.03 | 3.03 | 0.00 | 3.03 |
| 8 | 4.92 | 4.92 | 3.03 | 0.00 | 0.00 | 3.03 | 0.00 | 0.00 |
| 9 | 3.61 | 4.61 | 3.03 | 0.00 | 0.00 | 3.03 | 0.00 | 0.00 |
| 10 | 3.61 | 4.61 | 3.03 | 0.00 | 0.00 | 3.03 | 0.00 | 0.00 |

First we compare the designs in terms of $D, d_1$, and $d_{-1}$. Obviously designs 1, 2 and 6 are the best, with design 2 being slightly better than 6. Design 8 has a high $D$ value and is the only one with equal values of $d_1$ and $d_{-1}$. Between designs 1 and 2, design 1 is preferred if the efficiency at one level of $z$ is much more important than at the other. In general the choice depends on the relative importance of the three criteria. If a sequential design is contemplated, the designs need to be compared in terms of $\tilde{D}_1, \tilde{D}_{-1}, D_1(i)$ and $D_{-1}(i)$. The first six designs have the best values of $D_1(i)$ or $D_{-1}(i)$, that is, their first order design can entertain any one of $x_1x_2, x_1z$, or $x_2z$ in addition to 1, $x_1, x_2$ and $z$. Design 6 emerges as the best choice since it has the highest $\tilde{D}$ value and the best values of $D_{-1}(i)$. Designs 2, 1, and 8 are the next best three.

2. $k = 3$

(a) $2^3$ design for $x_1, x_2$ and $x_3$, $N = 16$, $N_1 = 9, \alpha = 1.682$.

(c) Overall model: (4.1).

(d) Model for $z = 1$ and for $z =$ -1: (4.3) with all possible pairs of $i$ and $j$.

(e) First order model: $(1, x_1, x_2, x_3, z || \; x_1x_2, x_1x_3, x_2x_3, x_1z, x_2z, x_3z)$.

Thirty choices of $z$ are found to give positive values of $D$, $d_1, d_{-1}$, from which the best three are given below.

**Table 3.2**

| design no. | $\boldsymbol{z} = (z_1, \cdots, z_{16})$ | $D$ | $d_1$ | $d_{-1}$ | $\tilde{D}_1$ | $\tilde{D}_{-1}$ |
|---|---|---|---|---|---|---|
| 1 | 1 1 -1 1 1 -1 -1 -1 1 -1 -1 1 -1 1 -1 -1 | 9.46 | 2.44 | 3.83 | 8.03 | 8.03 |
| 2 | 1 1 -1 1 1 -1 -1 -1 1 -1 -1 -1 -1 1 1 1 | 9.25 | 3.15 | 3.15 | 8.37 | 7.95 |
| 3 | 1 1 1 -1 -1 -1 -1 -1 1 -1 -1 1 -1 1 1 1 | 8.67 | 3.00 | 4.27 | 7.36 | 7.36 |

Let the six variables to the right of the double bar in (e) be denoted by 1, 2, 3, 4, 5, and 6, and $D_1(\boldsymbol{i})$ be the $D$ value for the first order design including the center point at $z = 1$ and the model consisting of 1,$x_1, x_2, x_3, z$ and four additional variables in the set $\boldsymbol{i}$, for example, $x_1x_2, x_1x_3, x_2x_3$ and $x_1z$ for $\boldsymbol{i} =$ (1,2,3,4).

For design 1, $D_1(\boldsymbol{i}) = D_{-1}(\boldsymbol{i}) = 5.44$ for $\boldsymbol{i} =$ (1,2,3,4), (1,2,3,5), (2,3,4,5); $D_1(\boldsymbol{i}) = D_{-1}(\boldsymbol{i})$ = 4.67 for $\boldsymbol{i} =$ (1,2,4,5), (1,2,4,6), (1,2,5,6), (1,3,4,5), (1,3,4,6), (1,3,5,6), (1,4,5,6), (2,4,5,6), (3,4,5,6). For design 2, $D_1(\boldsymbol{i}) = D_{-1}(\boldsymbol{i}) = 5.44$ for the last nine choices of $\boldsymbol{i}$ in design 1 and $D_1(\boldsymbol{i}) = D_{-1}(\boldsymbol{i}) = 4.67$ for the first three choices of $\boldsymbol{i}$ in design 1. For design 3, $D_1(\boldsymbol{i}) = D_{-1}(\boldsymbol{i}) = 5.44$ for $\boldsymbol{i} =$ (1,2,3,5), (1,2,3,6), (1,3,5,6), (2,3,5,6) and $D_1(\boldsymbol{i}) = D_{-1}(\boldsymbol{i}) = 4.67$ for $\boldsymbol{i} =$ (1,2,3,4), (1,3,4,6), (2,3,4,5), (3,4,5,6).

Designs 1 and 2 have the largest $D$ values. Design 3 has a smaller $D$ value, but its value 4.27 of $\max(d_1, d_{-1})$ is much larger than that of design 2. On the other hand design 3 has smaller $\tilde{D}$ values and can entertain a much smaller number of first order models.

For the same case Draper and John (1988) gave three designs in their Figure 8. Using our definitions of $D, d_1$ and $d_{-1}$, their design 8a has $D = 0, d_1 = d_{-1} = 3.56$; design 8b has $D = 7.90, d_1 = d_{-1} = 0$; design 8c has $D = 11.76$, $d_1 = 0$ and $d_{-1} = 8.13$. Their designs 8a and 8b are inferior to the three designs given above. Design 8c has larger values of $D$

and $d_{-1}$ at the expense of having $d_1 = 0$. This can be seen from the distribution of points between $z = 1$ and -1. All the star points and one center point are assigned to $z = 1$. For the design at $z = 1$, the columns of $x_1x_2, x_1x_3$ and $x_2x_3$ are zero. Therefore the coefficients of these three variables in (4.3) are not estimable.

3. $k = 4$

   (a) $2^4$ design for $x_1, x_2, x_3$ and $x_4$, $N = 26, N_1 = 17, \alpha = 2$.

   (b) Define $z = x_1x_2x_3$ for the first 16 runs.

   (c) Overall model = (4.1).

   (d) Model for $z = 1$ and for $z =$ -1: $(1, x_1, x_2, x_3, x_4, x_1x_4, x_2x_4, x_3x_4)$.

   (e) First order model: (4.4) with $(i,j) =$ (1,4), (2,4), (3,4).

The following six designs are found to give positive values of $D$, $d_1$ and $d_{-1}$ and the same value of $D_1 = D_{-1} = 16.15$.

**Table 3.3**

| design no. | $z = (z_{19}, \cdots, z_{26})$ | | | | | | | | $D$ | $d_1$ | $d_{-1}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 16.64 | 8.13 | 12.43 |
| 2 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 16.48 | 9.08 | 11.22 |
| 3 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 | 16.42 | 10.11 | 10.11 |
| 4 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 16.42 | 8.62 | 11.87 |
| 5 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 15.38 | 9.62 | 10.70 |
| 6 | -1 | -1 | -1 | -1 | -1 | 1 | -1 | 1 | 15.21 | 9.14 | 11.32 |

4. $k = 5$

   (a) $2^5$ design for $x_1, \cdots, x_5$, $N = 44$, $N_1 = 33, \alpha = 2.378$.

(b) $z = x_1x_2x_3x_4x_5$ for the first 32 runs.

(c) Overall model: (4.1).

(d) Model for $z$ = 1 and for $z$ = -1: (4.3) with all possible pairs of $i$ and $j$.

(e) First order model: (4.4) with all possible pairs of $i$ and $j$.

The following twelve designs are found to give positive values of $D$, $d_1$ and $d_{-1}$ and the same value of $D_1 = D_{-1} = 32.09$.

**Table 3.4**

| design no. | $z = (z_{35}, \cdots, z_{44})$ | | | | | | | | | | $D$ | $d_1$ | $d_{-1}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 38.39 | 16.06 | 19.54 |
| 2 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 38.23 | 16.72 | 18.81 |
| 3 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 | 38.17 | 17.40 | 18.09 |
| 4 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 38.09 | 16.41 | 19.20 |
| 5 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 37.97 | 17.08 | 18.47 |
| 6 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 | 1 | 37.93 | 17.77 | 17.77 |
| 7 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | -1 | 1 | 37.81 | 16.77 | 18.86 |
| 8 | -1 | -1 | -1 | -1 | -1 | 1 | -1 | 1 | 1 | 1 | 37.74 | 17.45 | 18.15 |
| 9 | -1 | -1 | -1 | -1 | -1 | 1 | -1 | 1 | -1 | 1 | 37.55 | 17.13 | 18.53 |
| 10 | -1 | -1 | -1 | 1 | -1 | 1 | -1 | 1 | 1 | 1 | 37.53 | 17.82 | 17.82 |
| 11 | -1 | -1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | 37.31 | 17.50 | 18.20 |
| 12 | -1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | 37.09 | 17.87 | 17.87 |

5. $k = 5$

(a) $2^{5-1}$ design with $x_5 = x_2x_3x_4$ for $x_1, \cdots, x_5$, $N = 28$, $N_1 = 17, \alpha = 2$.

(b) $z = x_1x_2x_3$ for the first 16 runs.

(c) Overall model: (4.2) with $(i, j)$ = (1,2), (1,3), (1,4), (1,5), (2,3), (2,4) and (2,5).

(d) Model for $z$ = 1 and for $z$ = -1: $(1, x_i, x_2x_4, x_2x_5)$, $i = 1, \cdots, 5$.

(e) First order model: (4.4) with $(i, j)$ = (2,4), (2,5).

The following six designs are found to give positive values of $D$, $d_1$ and $d_{-1}$ and the same value of $D_1 = D_{-1} = 16.14$.

**Table 3.5**

| design no. | $z = (z_{19}, \cdots, z_{28})$ | | | | | | | | | | $D$ | $d_1$ | $d_{-1}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 15.82 | 8.12 | 13.75 |
| 2 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 15.67 | 9.08 | 12.43 |
| 3 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 | 15.63 | 10.11 | 11.22 |
| 4 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 | 1 | 14.61 | 10.70 | 10.70 |
| 5 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 14.61 | 9.62 | 11.87 |
| 6 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 14.61 | 8.62 | 13.14 |

6. $k = 6$

(a) $2^{6-1}$ design with $x_6 = x_1x_2x_3x_4x_5$ for $x_1, \cdots, x_6$, $N = 46$, $N_1 = 33, \alpha = 2.378$.

(b) $z = x_1x_2x_3$ for the first 32 runs.

(c) Overall model: (4.1).

(d) Model for $z = 1$ and for $z =$ -1: (4.3) with $(i, j)$, $i = 1, 2, 3$, and $j = 4, 5, 6$.

(e) First order model: (4.4) with $(i, j)$, $i = 1, 2, 3$, and $j = 4, 5, 6$.

The following two designs are found to give positive values of $D$, $d_1$ and $d_{-1}$ and the same value of $D_1 = D_{-1} = 32.08$.

**Table 3.6**

| design no. | $z = (z_{35}, \cdots, z_{46})$ | | | | | | | | | | | | $D$ | $d_1$ | $d_{-1}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 | 1 | 1 | 29.04 | 18.09 | 18.09 |
| 2 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 | 1 | 27.63 | 17.77 | 18.47 |

7. $k = 7$

(a) $2^{7-1}$ design with $x_7 = x_1x_2x_3x_4x_5$ for $x_1, \cdots, x_7$, $N = 80$, $N_1 = 65, \alpha = 2.828$.

(b) $z = x_1x_2x_3$ for the first 64 runs.

(c) Overall model: (4.1).

(d) Model for $z = 1$ and for $z =$ -1: (4.3) with $(i,j) =$ (4,6), (5,6), (6,7) and $i =$ 1, 2, 3, and $j =$ 4, 5, 6 7.

(e) First order model: (4.4) with the same set of $(i,j)$ as in (d).

The following ten designs are found to give positive values of $D$, $d_1$ and $d_{-1}$ and the same value of $D_1 = D_{-1} = 64$.

**Table 3.7**

| design no. | $z = (z_{67}, \cdots, z_{80})$ | | | | | | | | | | | | | | $D$ | $d_1$ | $d_{-1}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 57.45 | 32.04 | 36.81 |
| 2 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 57.36 | 32.70 | 36.10 |
| 3 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 | 57.30 | 33.36 | 35.40 |
| 4 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 | 1 | 1 | 57.27 | 34.03 | 34.03 |
| 5 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 57.15 | 33.05 | 35.77 |
| 6 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 54.65 | 34.39 | 34.39 |
| 7 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 | 1 | 54.65 | 33.71 | 35.08 |
| 8 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 54.65 | 32.39 | 36.48 |
| 9 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | -1 | 1 | 1 | 1 | 54.46 | 33.40 | 35.45 |
| 10 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | -1 | 1 | 54.46 | 32.75 | 36.15 |

8. $k = 7$

(a) $2^{7-2}$ design with $x_6 = x_1x_2x_3x_4, x_7 = x_1x_2x_3x_5$ and $x_6x_7 = x_4x_5$, for $x_1, \cdots, x_7$, $N = 48$, $N_1 = 33, \alpha = 2.378$ .

(b) $z = x_1x_2x_3x_4x_5$ for the first 32 runs.

(c) Overall model: (4.2) with $(i, j)$ = (1,2), (1,3), (2,3) and $i$ = 1, 2, 3 and $j$ = 4, 5, 6, 7.

(d) Model for $z = 1$ and for $z = -1$: (4.3) with $(i, j)$ = (1,2), (1,3), (2,3) and $i$ = 1, 2, 3, and $j$ = 6, 7.

(e) First order model: $(1, x_i, x_j x_k, z, x_1 z, x_2 z, x_3 z)$, where $i = 1, \cdots, 7, j = 1$ and $k = 2, \cdots, 7$, $j = 2$ and $k = 3, \cdots, 7, j = 3$ and $k = 4, \cdots, 7$.

The following six designs are found to give positive values of $D$, $d_1$ and $d_{-1}$ and the same value of $D_1 = D_{-1} = 32.07$.

**Table 3.8**

| design no. | $\boldsymbol{z} = (z_{35}, \cdots, z_{48})$ | | | | | | | | | | | | | | $D$ | $d_1$ | $d_{-1}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | -1 | 1 | 1 | 1 | 1 | 1 | 33.44 | 16.83 | 15.90 |
| 2 | -1 | -1 | -1 | -1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | 1 | 1 | 33.30 | 13.29 | 16.43 |
| 3 | -1 | -1 | -1 | -1 | -1 | 1 | -1 | 1 | -1 | 1 | 1 | 1 | 1 | 1 | 33.30 | 17.17 | 15.63 |
| 4 | -1 | -1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | 33.17 | 16.53 | 16.99 |
| 5 | -1 | -1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | 1 | 1 | 33.17 | 17.01 | 16.15 |
| 6 | -1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | 33.04 | 16.86 | 16.69 |

# 5 Nine-Point Designs With One Center Point

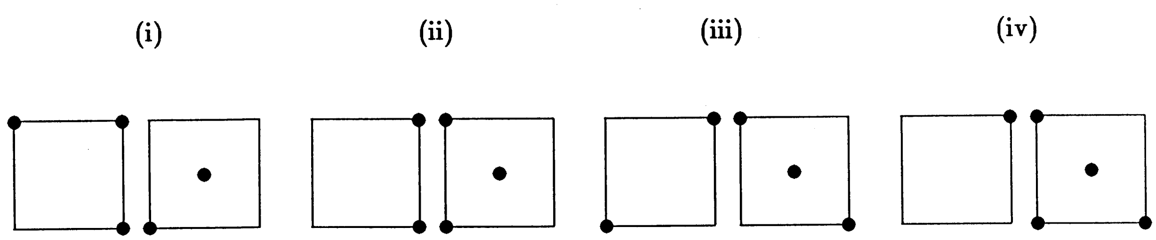
Draper and John (1988) gave a design for $r = 1$ and $k = 2$ in their Figure 5, which consists of one center point, four corner points and four star points. By following the same search procedure for ten-point designs with two center points, we are able to find all possible nine-point designs with desirable properties in terms of $D, D_1(i), d_1$, and $d_{-1}$, whose definitions and models are exactly the same as in Table 3.1. Without loss of generality, we can assume that the center point is assigned to $z = 1$. There are eleven such designs given in Table 4 and Figure 3 which have positive values of $D, d_1$ and $d_{-1}$. The $x_i$ values of these designs are the same as before (see Table 2). The values of $z_1$ to $z_4$ are the $z$ values for the corner points, $z_5$ for the center point, and $z_6$ to $z_9$ for the star points. From their values of $D, d_1, d_{-1}, D_1(1), D_1(2)$ and $D_1(3)$, design 1 is the best overall choice. If $D_1(i)$ are less important, design 7 is comparable to design 1. Both designs are quite unbalanced in the distribution of the star points over $z = \pm 1$. Other designs, particularly 2, 6, 9, 10, may be considered depending on the relative importance of $D, d_1, d_{-1}$ and $D_1(i)$. Note that the design given in Draper and John is equivalent to design 4.

(Figure 3 here)

**Table 4. Eleven Designs with One Center Point, $r = 1$, $k = 2$**

| design no. | $z_1$ | $z_2$ | $z_3$ | $z_4$ | $z_5$ | $z_6$ | $z_7$ | $z_8$ | $z_9$ | $D$ | $d_1$ | $d_{-1}$ | $D_1(1)$ | $D_1(2)$ | $D_1(3)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | -1 | -1 | -1 | 1 | 1 | 1 | -1 | 1 | 1 | 4.87 | 2.21 | 3.11 | 3.57 | 3.03 | 3.03 |
| 2 | -1 | -1 | -1 | 1 | 1 | -1 | 1 | 1 | -1 | 3.43 | 1.41 | 3.64 | 3.57 | 3.03 | 3.03 |
| 3 | -1 | -1 | -1 | 1 | 1 | -1 | 1 | 1 | 1 | 3.29 | 2.21 | 1.29 | 3.57 | 3.03 | 3.03 |
| 4 | -1 | -1 | -1 | 1 | 1 | 1 | -1 | 1 | -1 | 3.29 | 2.21 | 1.29 | 3.57 | 3.03 | 3.03 |
| 5 | -1 | -1 | -1 | 1 | 1 | -1 | 1 | -1 | 1 | 2.32 | 1.41 | 1.69 | 3.57 | 3.03 | 3.03 |
| 6 | -1 | -1 | 1 | 1 | 1 | -1 | -1 | -1 | 1 | 3.43 | 1.68 | 2.99 | 3.03 | 0.00 | 3.03 |
| 7 | -1 | 1 | 1 | 1 | 1 | -1 | -1 | 1 | -1 | 4.87 | 3.35 | 2.00 | 2.30 | 0.00 | 0.00 |
| 8 | -1 | 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | 4.67 | 2.00 | 2.83 | 2.30 | 0.00 | 0.00 |
| 9 | -1 | 1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | 4.00 | 2.21 | 2.38 | 3.03 | 0.00 | 0.00 |
| 10 | -1 | 1 | 1 | -1 | 1 | -1 | -1 | -1 | 1 | 4.00 | 1.68 | 3.13 | 3.03 | 0.00 | 0.00 |
| 11 | -1 | 1 | 1 | 1 | 1 | -1 | -1 | -1 | 1 | 3.29 | 2.43 | 2.00 | 2.30 | 0.00 | 0.00 |

Based on the values of $D_1(i)$, we can group the 11 designs in decreasing order as follows: (i) designs 1 to 5, (ii) design 6, (iii) designs 9 and 10, (iv) designs 7, 8 and 11. From Figure 3 their distributions of the four corner points and the center point are, respectively, of the following types:




The worst group (iv) has the least balanced distribution of points.

Finally we note that the 11 designs in Figure 3 can all be obtained from designs 1 - 7 and 9 - 10 in Figure 1 by deleting a center point. For example, design 1 in Figure 3 is the same as design 2 in Figure 1 with the left center point removed; design 7 in Figure 3 can be obtained from deleting the right center point of design 2 in Figure 1, and then exchanging $x_1$ and $x_2$, as well as $z = -1$ and 1. However, good designs with two center points are not necessarily good when a center point is deleted. For example, the "best" design 1 in Figure 1 corresponds to design 4 in Figure 3 after the left center point is removed. The latter design is not among those we recommended.

# 6 Designs For $r = 2$ and $k = 2$

When there are two or more qualitative factors, construction of designs becomes more complicated because objective B in Section 2 can take several forms. We only consider the simplest case of $r = 2$ and $k = 2$. The following are some reasonable criteria for selecting designs. Since objective C in Section 2 is essentially the same as in $r = 1$, we will not discuss it here for brevity.

(i) The model $(1, x_i, x_i^2, x_1x_2, z, x_iz)$, where $i = 1, 2$ and $z = z_1, z_2, z_1z_2$, can be fitted with high $D$-efficiency from the overall design.

(ii) For each level of $z_1$, the model $(1, x_i, x_i^2, x_1x_2, z_2, x_iz_2), i = 1, 2$, can be fitted with high $D$-efficiency. For each level of $z_2$, the model $(1, x_i, x_i^2, x_1x_2, z_1, x_iz_1), i = 1, 2$, can be fitted with high $D$-efficiency.

(iii) For each of the four combinations of $z_1 = \pm 1$ and $z_2 = \pm 1$, the model $(1, x_1, x_2, x_1x_2)$ can be fitted with high $D$-efficiency.

For each level of $z_1$ and of $z_2$, the criteria (ii) and (iii) are exactly the same as for $r = 1$ and $k = 2$ (see (4.1) and (4.3)). Therefore we can use any of the eleven designs in Section 5 for, say, $z_2 = 1$. Once a design is chosen for $z_2 = 1$, the design at $z_2 = -1$ is obtained by changing the sign of $z_1$ of the design at $z_2 = 1$. This is done so that the design at each of $z_2 = \pm 1$ and of $z_1 = \pm 1$ consists of four corner points, four star points and one center point. An example is given in Figure 4. Altogether there are eleven such designs based on those in Figure 3.

(Figure 4 here)

For each design the $D$ values for the criteria in (ii) and (iii) are the same as those in Table 4. For the overall design, we consider the $D$-efficiency for three models. The first is the model given in (i) with 15 terms. The second and third models are obtained from adding $(x_1^2 z_1, x_2^2 z_1, x_1 x_2 z_1)$ and respectively $(x_1^2 z_2, x_2^2 z_2, x_1 x_2 z_2)$ to the first model. The $D$ values (3.1) for these three models are denoted by $D(1), D(2)$ and $D(3)$ in Table 5 for the eleven designs.

**Table 5.**
**$D$ values of eleven designs with $r = 2$ and $k = 2$.**
**The design number corresponds to the design number in Table 4.**

| design no. | $D(1)$ | $D(2)$ | $D(3)$ |
|---|---|---|---|
| 1 | 11.71 | 9.73 | 9.73 |
| 2 | 8.82 | 6.86 | 6.86 |
| 3 | 8.66 | 6.58 | 6.58 |
| 4 | 12.09 | 10.15 | 10.15 |
| 5 | 6.55 | 4.64 | 4.64 |
| 6 | 8.82 | 6.86 | 6.86 |
| 7 | 11.71 | 9.73 | 9.73 |
| 8 | 11.26 | 9.33 | 9.33 |
| 9 | 9.85 | 8.00 | 8.00 |
| 10 | 9.64 | 8.00 | 8.00 |
| 11 | 8.66 | 6.58 | 6.58 |

In terms of the $D(i)$ in Table 5, the best designs are 4, 1, 7 and then 8 and 9. When criteria (ii) and (iii) are also considered, we refer to the values of $D, d_1$ and $d_{-1}$ in Table 4 for the purpose of comparison. On balance, we would recommend designs 1, 7, 8, 4 and 9 in descending order.

Normally two center points will be enough for the overall design. If four center points with one at each of $z_1 = \pm 1$ and $z_2 = \pm 1$ are required, we can use any of the 10 designs in Figure 1 for $z_2 = 1$ and the rest of the construction is the same.

If the run size is much smaller than 18, we must be content with less ambitious objectives such as the following.

(i) The model $(1, z_1, z_2, z_1z_2, x_1, x_2, x_1x_2, x_1^2, x_2^2)$ can be fitted with high $D$-efficiency from the overall design.

(ii) If the effect of $z_1$ (and resp. of $z_2$) is not significant, the model $(1, z_2, x_1, x_2, x_1x_2, x_1^2, x_2^2, x_1z_2, x_2z_2)$ (and resp. the model $(1, z_1, x_1, x_2, x_1x_2, x_1^2, x_2^2, x_1z_1, x_2z_1)$) can be fitted with high $D$-efficiency from the overall design.

(iii) For each level of $z_1$ (and respectively of $z_2$), the model $(1, x_1, x_2, x_1x_2)$ can be fitted with high $D$-efficiency.

When the design is collapsed over the levels of $z_1$ (and respectively $z_2$), we call the resulting design a marginal design in $z_2$ (and respectively $z_1$), which has one qualitative factor and two quantitative factors. It is clear that, for the marginal design in $z_1$ or in $z_2$, the models in (ii) and (iii) just stated are identical to those in (c) and (d) for the case of $r = 1$ and $k = 2$ (see Table 3.1). If the overall design consists of four corner points, four star points and two center points, then the two marginal designs must be chosen from the 10 designs given in Table 3.1 or Figure 1. Once the marginal designs are specified, the allocation of the corner and star points to $z_1$ and $z_2$ is uniquely determined. There are two choices of the two center points: $(z_1, z_2) = \{(-1, -1), (1, 1)\}$ or $\{(-1, 1), (1, -1)\}$. One such example is given in Figure 5.

(Figure 5 here)

From the previous remarks, the $D$-efficiency in (ii) is given by the $D$ value in Table 3.1. Similarly, the $D$-efficiency in (iii) of each marginal design is given by the values of $d_1$ and $d_{-1}$ in Table 3.1. So we need only to evaluate the $D$ value of the model in (i). The design in Figure 5 has the largest $D$ value 6.16 for the model in (i) among all possible combinations of the marginal designs. In general we can compare designs by these three sets of $D$-efficiency values. To save space, the details are omitted.

# 7 Some Extensions

(i) **Run size reduction through smaller plans for the cube points**

Sometimes we can achieve dramatic reduction of run size by this means without greatly sacrificing the capacity for effect estimation. Take, for example, the case of $r = 1$ and $k = 4$. The designs in Table 3.3 have 26 points. If we choose instead a $2^{4-1}$ design with $x_4 = x_1x_2$ and define $z = x_2x_3$ for the first 8 runs, the total run size is 18 (= 8 + 8 + 2). By further choosing $(z_{11}, \cdots, z_{18}) = (-1, 1, -1, 1, -1, 1, -1, 1)$ for the eight star points, the resulting design can entertain the 18 terms $(1, x_i, x_i^2, x_1x_2, x_1x_4, x_2x_3, x_2x_4, z, zx_i)$, $i$=1,2,3,4, with $D = 5.66$. Obviously it is not as efficient per observation as the designs in Table 3.3. It can entertain $(1, x_1, x_2, x_3, x_4, x_1x_2, x_1x_4, x_2x_3, x_2x_4)$ for $z = 1$ and for $z = -1$, with $d_1 = d_{-1} = 2.52$. If the first order model is $(1, x_1, x_2, x_3, x_4, x_1x_3, z, x_1z)$, the design has $D_1 = D_{-1} = 8.23$. In practice the choice between smaller and larger designs depends on the cost of runs, time limitation, importance of estimation efficiency, etc. If the effects of interest are large, they can be detected by less efficient (but smaller) designs.

To reduce the number of corner points, we can employ the Plackett-Burman designs for the corner points. For example, the design in Table 3.6 has 46 runs including 32 cube points and can entertain an overall model with 35 terms. By taking a Plackett-Burman design with 28, 24 or 20 runs for the corner points, the total run size is reduced by 4, 8, or 12. The estimation efficiency for such designs will be investigated later. Note that the Plackett-Burman designs have been employed to reduce the size of central composite designs (see Draper and Lin, 1990, and its references).

(ii) **Additional variables to be entertained**

For most designs discussed so far, the models we have considered, i.e., (4.1) - (4.3), are not saturated. There is often the possibility of entertaining one or more variables. Here we only discuss three cases. Finding such variables for general designs is computationally straightforward. An illuminating example is the designs for $r = 1$ and $k = 2$. From Figure 1, it is clear that for some designs $x_1^2$ or $x_2^2$ can be entertained at $z = 1$ or $z = -1$. Let $d_{-1}(*)$ and $d_1(*)$ be respectively the $D$ value (3.1) for the design at $z = -1$ and at $z = 1$ and the model consisting of $1, x_1, x_2, x_1x_2$ and the additional variables indicated by $*$. For $z = -1, *$ can be $x_1^2, x_2^2$ or $\{x_1^2, x_2^2\}$. For $z = 1, *$ cannot take both $x_1^2$ and $x_2^2$. The values are given in the second part of Table 3.1. The best two are designs 1 and 6 at $z = -1$. Design 6 having all the four star points at $z = -1$ clearly has a high value of $d_{-1}(x_1^2, x_2^2)$ for entertaining both $x_1^2$ and $x_2^2$ because the quadratic curvature of $x_1$ and of $x_2$ can be efficiently estimated by the two star points, one corner point and the center point.

Another example concerns the 16-run designs for $r = 1$ and $k = 3$. Since the overall model (4.1) has 14 terms, there are two remaining degrees of freedom for fitting additional terms. Consider the possibility of fitting one or two of the six cubic terms, $x_1x_2z, x_1x_3z, x_2x_3z, x_1^2z, x_2^2z$ and $x_3^2z$, which we denote by 1, 2, 3, 4, 5 and 6. Draper and John (1988) stated that their design 8b allows the pairs (2,4), (2,5) and (2,6) to be fitted. In fact, (4,5) and (5,6) can also be fitted, but not others. In contrast, design 2 in Table 3.2 allows any of the $\binom{6}{2} = 15$ pairs of cubic terms to be fitted. Design 1 allows 13 out of 15 pairs to be fitted with the exception of (1,6) and (2,3). Design 3 does not fare as well but still allows (1,3), (1,4), (1,5), (1,6), (3,4), (3,5), (3,6), (4,5) and (5,6) to be fitted. Overall, our designs 1 and 2 are better.

Finally we consider the six designs in Table 3.3. Since the run size at $z = 1$ ranges from 9 to 13 and the model for $z = 1$ has eight terms, one would expect more variables to be entertained in some of the designs. It turns out that each of designs 1, 2, 4, 5, 6 can entertain three additional terms $x_1x_2, x_1x_3$ and $x_2x_3$ for $z = -1$. Designs 5 and 6 can also entertain $x_1x_2$ for $z = 1$.

## Acknowledgements

We would like to thank Xiaodong Sun for his meticulous assistance in computing. The research was supported by the Natural Sciences and Engineering Research Council of Canada, General Motors of Canada Limited, and the Manufacturing Research Corporation of Ontario.

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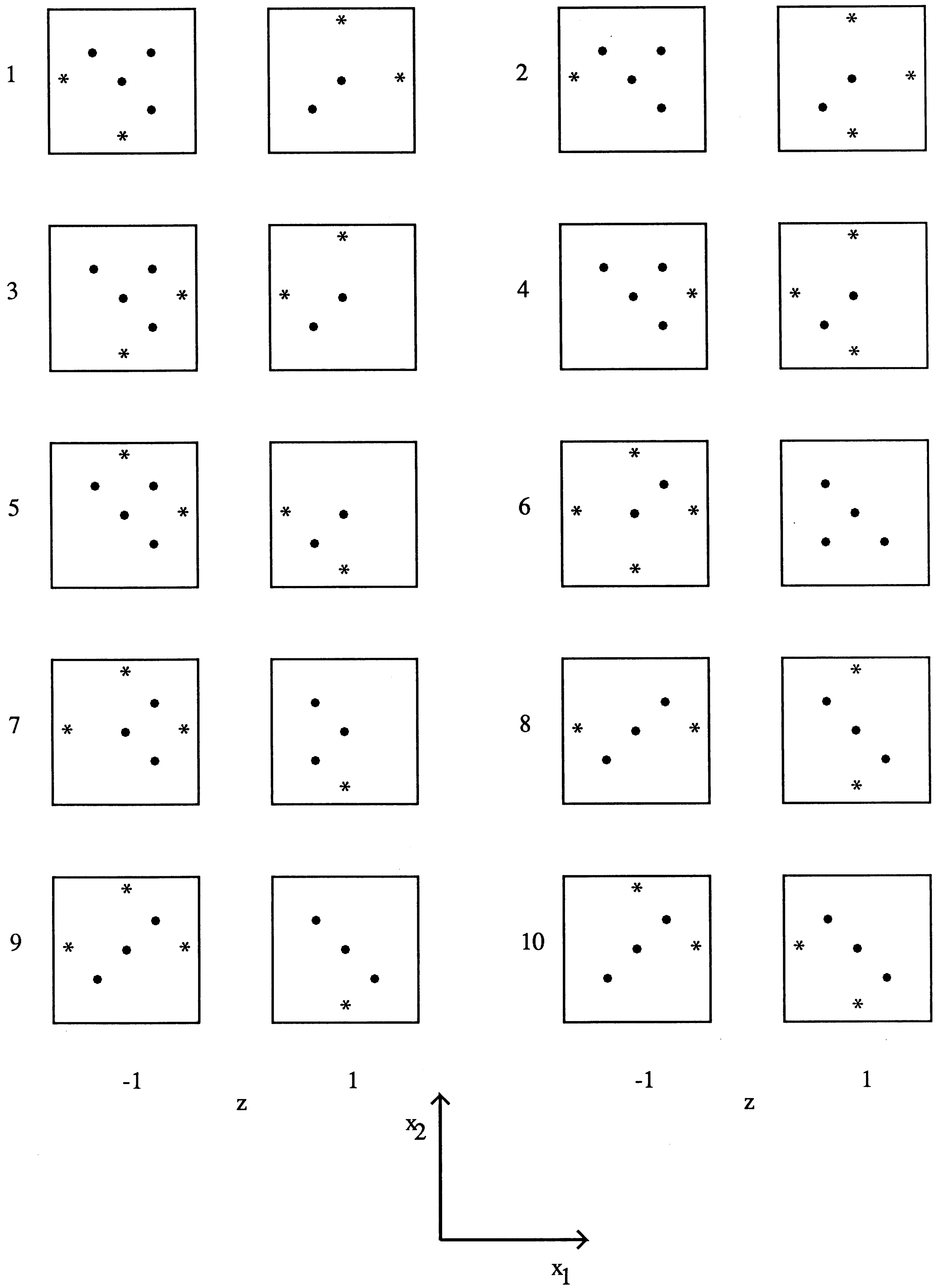


Figure 1. Ten designs for $x_1$, $x_2$ and z with two center points.
The four star points are denoted by $*$.

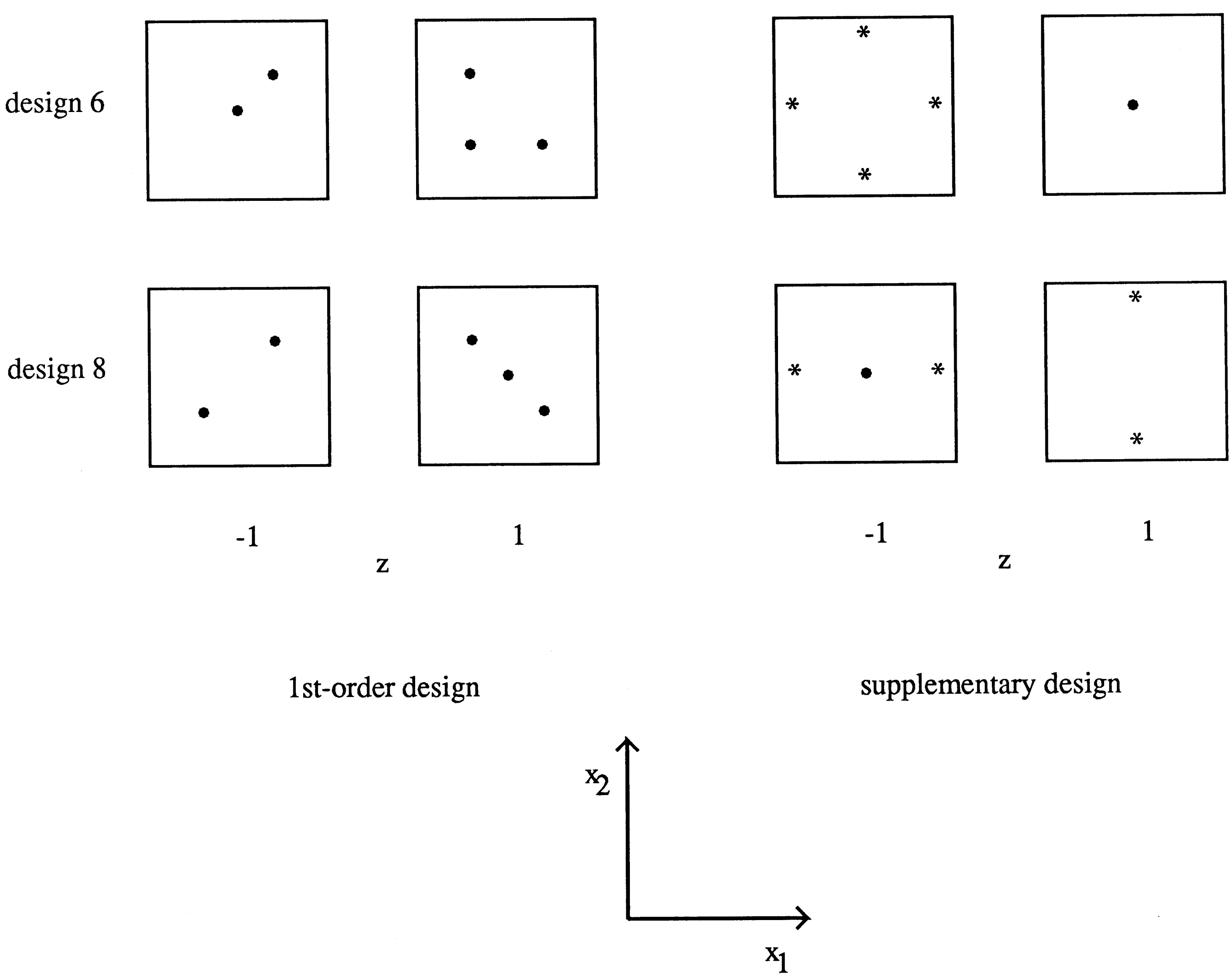


Figure 2. Decomposition of designs 6 and 8 (in Figure 1).

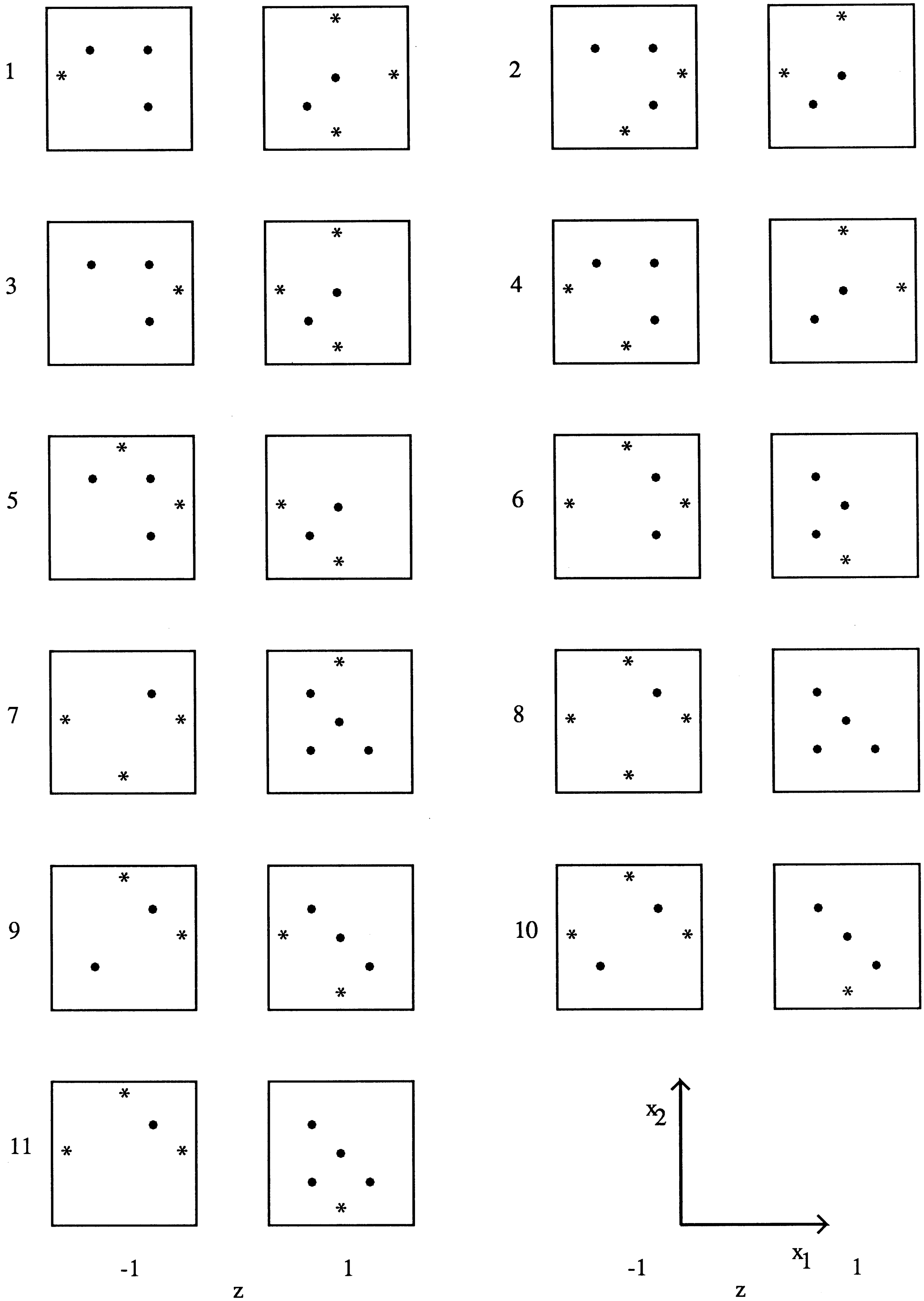


Figure 3. Eleven designs for $x_1$, $x_2$ and z with one center point at z=1.
The four star points are denoted by $*$.

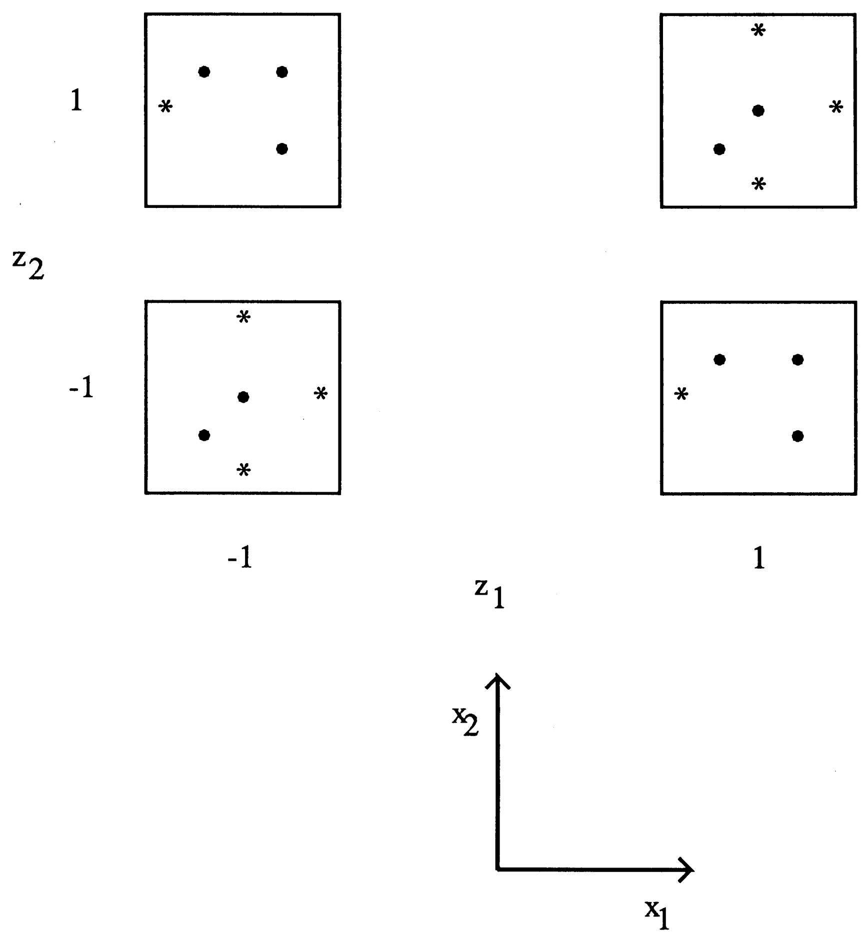


Figure 4. A design for $x_1$, $x_2$, $z_1$, and $z_2$. Its top part ($z_2$=1) is the same as design 1 in Figure 3. Its bottom part ($z_2$= -1) is obtained from changing the sign of $z_1$ of the top part.

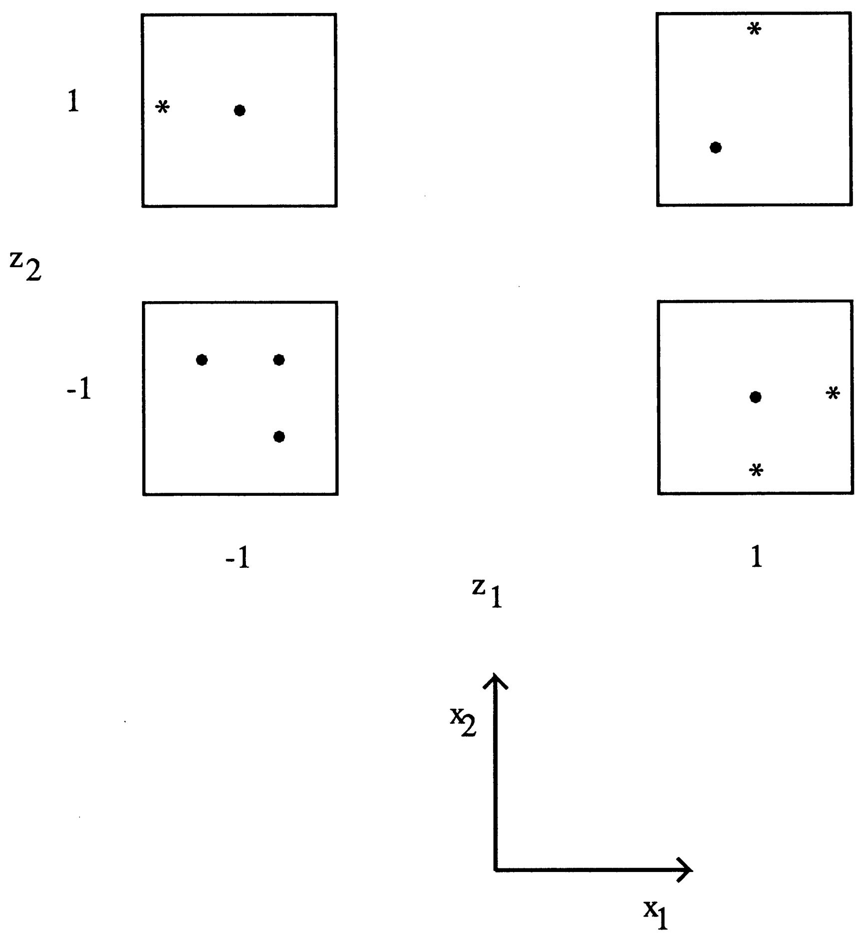


Figure 5. When collapsed over $z_1 = \pm 1$ ( and resp. $z_2 = \pm 1$) , it becomes design 3 (and resp. design 2) in Figure 1.