

**PLANNING ACCELERATED LIFE TESTS FOR
SELECTING THE MOST RELIABLE PRODUCT
UNDER THE WEIBULL-INVERSE POWER MODEL**

S.T. Tseng

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Sheng-Tsaing Tseng

National Taiwan Institute of Technology and
University of Waterloo
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ABSTRACT

At the research and development stage, a decision maker may wish to select the most reliable design from among several competing product designs. This paper proposes a systematic approach to the selection problem for highly reliable products which possess a Weibull-Inverse Power law failure model. First, optimum test plans for both type-I and type-II censoring are derived by minimizing the asymptotic variance of estimated quantiles at the design stress. Next, an intuitively appealing selection rule is proposed. The sample size and censoring time (or number of failures) needed by this selection rule are computed with a predetermined time-saving factor and a minimum probability of correct selection (CS). An example to demonstrate the selection procedure is given. Finally, a cost criterion is used to compare these two censoring plans. Although type-II censoring needs slightly larger sample sizes than type-I censoring, it has a shorter expected life-testing time.

1 Introduction

At the research and development stage, a decision maker usually faces the problem of selecting the most reliable (best) product design from several competing designs. Also, he/she may want to compare the products of several vendors. Referring to the example in section 6, there may be several vendors who provide different competing designs of electrical insulation. The decision maker may wish to select the best vendor. Many selection rules for such problems have been proposed during the the last thirty years. General references may be found in Gibbons et al. (1977), Gupta & Panchapakesan (1979) and Gupta & Huang (1981). Recently, a comprehensive survey of selection procedure in reliability models was given by Gupta & Panchapakesan (1988). Most of those selection rules are based on complete *and/or* censored data. For highly reliable products, a few (or even no) failures can be observed under normal design stress (i.e., normal use condition). Consequently most of these selection rules are not applicable.

Accelerated Life Tests (ALTs) are used to compare and to estimate the life time of highly reliable products within a reasonable testing time. Products are tested at higher stress (such as temperature, voltage, vibration, etc.) and results are extrapolated using an assumed statistical model to estimate the product life at normal design stress. General references may be found in Mann et al. (1974), Lawless (1982), Tobias & Trindade (1986), Peck and Trapp (1987), Viertl (1988), and Nelson (1990).

There is much literature on optimum ALT plans. Chernoff (1962) is a pioneer in this area. Under type-I censoring, Nelson & Kielpinski (1976) and Nelson & Meeker (1978) give an exposition on optimum test plans for lognormal-Arrhenius and Weibull-Inverse Power models respectively. Meeker & Hahn (1977) analyze the linear logistic failure model in a similar way. Escobar & Meeker (1986) and Menzefricke (1988) present optimum test plans under type-II censoring. In addition, Martz & Waterman (1978) and DeGroot & Goel (1979) also discuss the Bayesian analysis of ALT models.

There is no suitable discussion of optimal ALT planning for comparing several highly-reliable, competing product designs. The main reason is that there is no simple estimator even for the single population (design) case. The life of certain products is described with a Weibull life distribution whose characteristic life is an inverse power function of stress (Nelson (1990)). Examples are electrical insulation (voltage as the accelerating variable), ball bearings (load as the accelerating variable) and metal fatigue (mechanical stress as the accelerating variable). This paper presents a systematic approach to this selection problem for highly reliable products possessing a Weibull-Inverse Power model.

When the Weibull shape parameters are known, the optimum test plans for both type-I and type-II censoring are derived by minimizing the asymptotic variance of estimated quantiles at design stress. Based on life data from those plans, we propose an intuitively appealing selection rule to achieve a stated goal. The sample size and censoring time (or number of failures) which are needed by this rule are computed under a predetermined time-saving factor and a minimum probability of correct selection. We use an example to demonstrate this selection procedure.

By using a cost criterion, we compare the relative efficiency of these two censoring plans. It is seen that type-I requires slightly smaller sample sizes than type-II censoring. On the other hand, type-II has a shorter expected life-testing time than type-I censoring.

Although the Weibull shape parameters are usually unknown, it is reasonable to assume that their prior distributions are known over some interval. We use simulation to study the robustness of our rule under a general beta prior distribution. From the results, it is shown that this rule is quite robust when the true shape parameter has a moderate departure from the assumed value.

2 The life-stress model

In this section, we will summarize the statistical life-stress model, censoring mechanism and the optimality criterion.

(A) Weibull-Inverse Power model

The assumptions of this model are:

(1) Product life has a Weibull distribution at any stress. The Weibull reliability function is

$$R(t) = e^{-(t/\theta)^\beta}, t \geq 0,$$

where $\theta > 0$ and $\beta > 0$ are the Weibull scale and shape parameters respectively.

(2) The Weibull shape parameter β is independent of stress (a constant for any stress).

(3) The Weibull characteristic life (scale parameter) θ is an inverse power function of stress S . That is,

$$\theta(S) = e^{\gamma_0} / S^{\gamma_1},$$

where γ_0 and γ_1 are two unknown parameters.

(B) Censoring mechanism

Two well-known censoring plans are often used to shorten the time of life-testing.

(1) Type-I censoring

This involves running each unit for a predetermined time. In this case, the censoring time is fixed while the number of failures is random.

(2) Type-II censoring

This involves simultaneous testing of the units until a predetermined number of them fail. In this case, the common censoring time is random and the number of failures is fixed.

(C) The optimization criterion

Nelson & Kielpinski (1976) describe various criteria for determining optimal ALT plans. In this paper, we use the criterion of minimizing asymptotic variance of estimated quantiles at the design stress to derive the optimal plans.

3 Problem formulation

Suppose Π_1, \dots, Π_k denote k available product designs and S_0 denotes the normal use condition (stress) of those designs. For $1 \leq \ell \leq k$, $R_\ell(t, S_0)$ denotes the reliability function of Π_ℓ under stress S_0 . The design Π_i is said to be the most reliable design at time t^* if

$$R_i(t^*, S_0) = \max_{1 \leq \ell \leq k} R_\ell(t^*, S_0). \quad (1)$$

The goal of the decision maker is to select the most reliable design from among these k available designs.

For highly reliable products, there may be only a few (or even no) failures observed under S_0 . The accelerated life test (ALT) is used to overcome this difficulty. Suppose the tests are conducted at m values of higher stresses $\{S_j\}_{j=1}^m$ and $S_0 < S_1 < \dots < S_m$. It is assumed that the life-stress relation follows a Weibull-Inverse Power model, that is, the lifetime of design Π_i under stress S_j follows a Weibull distribution with an unknown characteristic life (scale parameter) θ_{ij} and a shape parameter β_i , where θ_{ij} with S_j following an inverse power model. This can be expressed as

$$\theta_{ij} = e^{\gamma_{i0}} / S_j^{\gamma_{i1}}, \quad (2)$$

where γ_{i0} and γ_{i1} are unknown parameters of design Π_i .

For each combination of (Π_i, S_j) , there are n_{ij} units which are put on test to perform an ALT. Using a type-I (or type-II) censoring plan for each combination, the experiment terminates when the censoring time η_{ij} (or the number of failures r_{ij}) is reached. Based on these life-testing data to select the most reliable design, some typical decision problems are as follows:

- (1) Which censoring plan is better?
- (2) How many stresses should be used for performing ALT?
- (3) How many observations n_{ij} for each combination of (Π_i, S_j) should be taken?

(4) What is the optimal censoring time η_{ij} (or the optimal number of failures r_{ij}) for each combination of (Π_i, S_j) ?

(5) How to construct a suitable selection rule to achieve the goal of the experimenter?

In the following section, we assume that the Weibull shape parameters are known. In section 8, we will discuss the case of unknown shape parameters.

4 Optimal accelerated life test plan

The maximum likelihood (ML) method is used to estimate the unknown parameters under both type-I and type-II censoring. It is convenient to reparameterize the life-stress model (refer to Nelson & Meeker (1978)). Define the standardized stress v_j as follows:

$$v_j = (\ln S_m - \ln S_j) / (\ln S_m - \ln S_0), \quad 0 \leq j \leq m, \quad (3)$$

It is easily seen that $v_0 = 1$ and $v_m = 0$, while $0 < v_j < 1$, for $1 \leq j \leq (m - 1)$. The relation in equation (2) can be rewritten as:

$$\ln \theta_{ij} = \alpha_{i0} + \alpha_{i1} v_j, \quad (4)$$

where $\alpha_{i0} = (\gamma_{i0} - \gamma_{i1} \ln S_m)$ and $\alpha_{i1} = \gamma_{i1} (\ln S_m - \ln S_0)$.

Let $\{T_{ij\ell}\}_{\ell=1}^{n_{ij}}$ denote a set of observations for the combination of (Π_i, S_j) , and $Z_{ij\ell} = \beta_i (\ln T_{ij\ell} - \alpha_{i0} - \alpha_{i1} v_j)$. It is easily seen that $Z_{ij\ell}$ follows a standard extreme distribution, for i, j, ℓ , where $1 \leq i \leq k$, $1 \leq j \leq m$ and $1 \leq \ell \leq n_{ij}$.

Consider a sample that may be type-I or type-II censored involving observations on the lifetimes of n_{ij} individuals for each combination of (Π_i, S_j) . We shall denote both standardized lifetime and censoring time as $z_{ij\ell} (\ell = 1, \dots, n_{ij})$ and let D_{ij} be the set of individuals for which $z_{ij\ell}$ is an observed lifetime and C_{ij} be the set for which $z_{ij\ell}$ is a standardized censoring time. The likelihood function for the i -th design can be expressed as follows:

$$\prod_{j=1}^m \left\{ \prod_{\ell \in D_{ij}} \beta_i \phi(z_{ij\ell}) \prod_{\ell \in C_{ij}} Q(z_{ij\ell}) \right\}, \quad (5)$$

where $\phi(z)$ and $Q(z)$ denote the probability density function (pdf) and reliability function for the standard extreme distribution respectively.

The maximum likelihood estimators (MLE) for α_{i0} and α_{i1} , $(\hat{\alpha}_{i0}, \hat{\alpha}_{i1})$, can be solved by

$$\sum_{j=1}^m r_{ij} - \sum_{j=1}^m \left\{ \sum_{\ell \in D_{ij}} e^{z_{ij\ell}} + \sum_{\ell \in C_{ij}} e^{z_{ij\ell}} \right\} = 0 \quad (6)$$

and

$$\sum_{j=1}^m r_{ij} v_j - \sum_{j=1}^m v_j \left\{ \sum_{\ell \in D_{ij}} e^{z_{ij\ell}} + \sum_{\ell \in C_{ij}} e^{z_{ij\ell}} \right\} = 0. \quad (7)$$

where r_{ij} = number of individuals in D_{ij} .

From equation (4), we have $\ln \hat{\theta}_{i0} = (\hat{\alpha}_{i0} + \hat{\alpha}_{i1})$, for all $1 \leq i \leq k$, so it is easy to obtain the following lemma.

Lemma 1 $\ln \hat{\theta}_{i0}$ is asymptotically normally distributed with mean $\ln \theta_{i0}$ and variance

$$\frac{1}{\beta_i^2} \left\{ \frac{(\sum_j w_{ij} v_j^2) - 2(\sum_j w_{ij} v_j) + (\sum_j w_{ij})}{(\sum_j w_{ij})(\sum_j w_{ij} v_j^2) - (\sum_j w_{ij} v_j)^2} \right\}, \quad (8)$$

where

$$w_{ij} = \begin{cases} n_{ij} M_{ij} & \text{for type-I censoring} \\ r_{ij} & \text{for type-II censoring} \end{cases} \quad (9)$$

and

$$M_{ij} = 1 - e^{-(\eta_{ij}/\theta_{ij})^{\beta_i}}.$$

Lemma 2 For both type-I and type-II censoring, the necessary condition for minimizing $Var(\ln \hat{\theta}_{i0})$ is $v_1 = v_2 = \dots = v_{m-1}$.

From this lemma, it shows that $m \geq 3$ are non-optimal. This means that only two higher stresses ($m = 2$) are needed to perform accelerated life test.

For simplicity, let L and H denote the low and high stresses. Now, let p_{iL} (p_{iH}) denote the proportion of the sample size allocated to the low (high) stress, and let q_{iL} (q_{iH}) denote the proportion of the number of failures allocated to the low (high) stress. Suppose that n_{i0} and r_{i0} denote the total sample size and the number of failures which are needed by the i -th design (population). Then $n_{ij} = n_{i0}p_{ij}$ and $r_{ij} = r_{i0}q_{ij}$, for $j=L,H$ and eq (8) can be rewritten as

$$Var(\ln \hat{\theta}_{i0}) = \begin{cases} \frac{1}{\beta_i^2 v_L^2 n_{i0}} \left\{ \frac{1}{p_{iL} M_{iL}} + \frac{(1-v_L)^2}{(1-p_{iL}) M_{iH}} \right\} & \text{for type-I censoring} \\ \frac{1}{\beta_i^2 v_L^2 r_{i0}} \left\{ \frac{1}{q_{iL}} + \frac{(1-v_L)^2}{1-q_{iL}} \right\} & \text{for type-II censoring.} \end{cases} \quad (10)$$

It is impossible to find a non-trivial solution (v_L, p_{iL}) (or (v_L, q_{iL})) $\neq (1, 1)$, such that $Var(\ln \hat{\theta}_{i0})$ attains a minimum. Consequently, we shall fix v_L and minimize with respect to p_{iL} (or q_{iL}).

Lemma 3 *For type-I censoring, the optimal proportion of the sample size allocated to the low stress is*

$$p_{iL}^* = \frac{1}{1 + (1 - v_L) \sqrt{(M_{iL}/M_{iH})}}.$$

For type-II censoring, the optimal proportion of the number of failures allocated to the low stress is

$$q_{iL}^* = \frac{1}{2 - v_L}.$$

From Lemma 3, if $M_{iL} = M_{iH}$, then $p_{iL}^* = q_{iL}^*$. Besides, if $r_{i0} = n_{i0} M_{iL}$, then $\ln(\hat{\theta}_{i0}^{\beta_i})$ is asymptotically normally distributed with mean $\ln(\theta_{i0}^{\beta_i})$ and variance $\frac{1}{r_{i0}} \left(\frac{2-v_L}{v_L} \right)^2$.

5 A selection rule

Based on the life data as described above, we propose a selection rule as follows:

δ : select design Π_i if

$$\hat{R}_i(t^*, S_0) = \max_{1 \leq \ell \leq k} \hat{R}_\ell(t^*, S_0), \quad (11)$$

where $\hat{R}_\ell(t^*, S_0) = e^{-(t^*/\hat{\theta}_{\ell 0})^{\beta_\ell}}$, for $1 \leq \ell \leq k$.

This selection rule is completely specified when the sample size and censoring time (or number of failures) are known. In the following, we develop a procedure to determine these values.

For simplicity, we define the i – th preference region as follows:

$$\Omega_i = \{(R_1, \dots, R_k) \mid R_i^\Delta \geq \max_{\ell \neq i} R_\ell\}, \Delta > 1. \quad (12)$$

The selection rule in equation (11) gives a correct selection if Π_i is the most reliable design and we can correctly select it. Let $P_R(CS \mid \delta)$ denote the probability of a correct selection (CS) by using the selection rule δ . It is usually required that the probability of CS exceeds a minimum value P^* (referred to as the P^* -condition), that is,

$$\inf_{R \in \Omega_i} P_R(CS \mid \delta) \geq P^*, \quad (13)$$

where P^* is a value predetermined by the decision maker.

To control the accelerated life-testing time within a specified level, we can compute the sample size in terms of the number of failures (refer to Tseng and Wu (1990)) by

$$E(Y_{ijr_{ij}})/E(Y_{ijn_{ij}}) \leq \zeta_{ij}. \quad (14)$$

where $(Y_{ij1}, \dots, Y_{ijn_{ij}})$ denotes the order statistic of $(T_{ij1}, \dots, T_{ijn_{ij}})$, and ζ_{ij} is a fixed constant.

We now state a lemma to compute $E(Y_{ijr_{ij}})$ as follow:

Lemma 4 For

$$g(a, b, c) = \int_0^1 \frac{(-\ln(1-y))^{1/c} y^{a-1} (1-y)^{b-1}}{\beta(a, b)} dy,$$

$$E(Y_{ijr_{ij}}) = g(r_{ij}, n_{ij} - r_{ij} + 1, \beta_i).$$

Note that Lemma 4 can be easily obtained from David (1981, p.34). We now state two theorems to compute the optimal sampling plan for selecting the most reliable population under both type-I and type-II censoring.

Theorem 1 For type-II censoring, the sample sizes and number of failures $\{(n_{ij}, r_{ij})\}_{j=L}^H$, $1 \leq i \leq k$, can be solved by using the asymptotic approximation

$$\int_0^1 \prod_{j \neq i}^k \left\{ \Phi \left(\sqrt{\frac{r_{j0}}{r_{i0}}} \Phi^{-1}(x) + \sqrt{r_{j0}} (\ln \Delta) \left(\frac{v_L}{2 - v_L} \right) \right) \right\} dx \geq P^*, \quad (15)$$

$$r_{ij} = r_{i0} q_{ij}^*, \quad (16)$$

and

$$\frac{g(r_{ij}, n_{ij} - r_{ij} + 1, \beta_i)}{g(n_{ij}, 1, \beta_i)} \leq \zeta_{ij}, \quad (17)$$

where Φ is the cdf of the standard normal distribution.

Note that equation (15) can be simply derived from Gupta & Panchapakesan (1979, p. 7) and q_{ij}^* in equation (16) comes from Lemma 3. From Theorem 1, it is easily seen that this may lead to an infinite number of solutions for $\{(n_{ij}, r_{ij})\}_{j=L}^H$, $1 \leq i \leq k$. For illustrative purposes, we consider only the case of equal sample size and equal number of failures for each population; that is, $r_{ij} = r_j$ and $n_{ij} = n_j$, for all $1 \leq i \leq k$ and $j = L, H$. We arbitrarily choose $\zeta_{ij} = 0.5$, $P^* = 0.90$, and $v_L = 0.5$. Given various pairs of (R_a, R_b) , where R_a denotes the reliability function of the most reliable population, while R_b denotes the largest reliability function of the other $(k-1)$ less reliable populations, the minimum values of all feasible

solutions $(r_L, n_L; r_H, n_H)$ are computed under $2 \leq k \leq 6$, and $\beta_i = 0.75(0.25)1.50$. The results are given in Table 1. For example, when $k = 4, \beta_i = 1.25$ and $(R_a, R_b) = (.999, .9975)$ (i.e., $\Delta = (\ln 0.9975 / \ln 0.999) = 2.5018$), we obtain from equation (15), $r_{i0} = [64.3 + 1] = 65$, for $1 \leq i \leq k$. From equation (16), we have $r_L = [64.3 * (1/(2 - 0.5)) + 1] = 43$ and $r_H = [64.3 * ((1 - 0.5)/(2 - 0.5)) + 1] = 22$. Finally we obtain $n_L = 52$ and $n_H = 28$ from equation (17).

Similarly, if we let $r'_{iL} = n_{i0}M_{iL}$ and $r'_{iH} = n_{i0}M_{iH}$, for $1 \leq i \leq k$, then we have the following result.

Theorem 2 For type-I censoring, the sample sizes and censoring times $\{(n_{ij}, \eta_{ij})\}_{j=L}^H, 1 \leq i \leq k$, can be solved by using the asymptotic approximation

$$\int_0^1 \prod_{j \neq i}^k \left\{ \Phi \left(\sqrt{\frac{r'_{jL}}{r'_{iL}} \left(\frac{p_{jL}^*}{p_{iL}^*} \right)} \Phi^{-1}(x) + \sqrt{r'_{jL}} (\ln \Delta) (v_L p_{jL}^*) \right) \right\} dx \geq P^*, \quad (18)$$

$$\frac{g(r'_{iL}, n_{i0} - r'_{iL} + 1, \beta_i)}{g(n_{i0}, 1, \beta_i)} \leq \zeta_{i0}, \quad (19)$$

$$n_{ij} = n_{i0} p_{ij}^*, \quad (20)$$

and

$$\eta_{ij} = \xi_j * \theta_{ij}, \quad (21)$$

where

$$\xi_j = \left\{ -\ln \left(1 - \frac{r'_{ij}}{n_{i0}} \right) \right\}^{1/\beta_i}.$$

Again, $\{p_{ij}^*\}$ in equation (18) and (20) come from Lemma 3. We also assume equal sample size and equal censoring time for each population, i.e., $n_{ij} = n_j$ and $\eta_{ij} = \eta_j$, for $j = L, H$ and $1 \leq i \leq k$. Set $M_{iL} = M_{iH}$, then $p_{iL}^* = 1/(2 - v_L)$, for $1 \leq i \leq k$. Under various pairs of (R_a, R_b) , $2 \leq k \leq 6$ and $\beta = 0.75(0.25)1.50$, we compute the minimum feasible solutions of (ξ, n_L, n_H) . The results are given in Table 2. For example, as $k=4, (R_a, R_b) = (0.999, 0.9975)$

and $\beta_i = 1.25$, from equation (18), we have $r'_{iL} = 65$. As $\zeta_{i0} = 0.5$, from equation (19) and (20), we have $n_{i0} = 77$, $n_L = [77 * (2/3) + 1] = 52$, and $n_H = [77 * (1/3) + 1] = 26$. From equation (21), we have $\xi_j = (-\ln(1 - \frac{65}{77}))^{1/1.25} = 1.6421$. So, $\eta_{ij} = 1.6421 * \theta_{ij}$, for all $j = L, H$ and $1 \leq i \leq k$.

Alternatively, we can set $(M_{iL}/M_{iH}) = 0.90$. This gives another feasible solution in $(n_L, n_H, \eta_{iL}, \eta_{iH}) = (51, 24, 1.674 * \theta_{iL}, 2.3547 * \theta_{iH})$, for $1 \leq i \leq k$. Comparing this with the previous case, we find that the case of $M_{iL} = M_{iH}$ requires larger sample sizes but results in a decrease in both censoring times. Therefore, we only consider the case of $M_{iL} = M_{iH}$, for $1 \leq i \leq k$ in Table 2.

6 Example

Suppose that we have four different competing, highly-reliable designs of electrical insulation and that the voltage is the accelerating variable (refer to Nelson & Meeker (1978)). The design stress (use condition), $S_0 = 20KV$. It is also known that 38KV is the highest stress for this problem. The life-stress relation follows a Weibull-Inverse Power model. The decision maker is interested in selecting the most reliable design at a predetermined time $t^*(= 17.0669)$ under the use condition. Some typical decision problems are as follows:

- (1) How many stresses should be chosen for performing an accelerated test?
- (2) Find the suitable accelerated stresses for performing life-testing.
- (3) What is the optimal sampling plan for type-II censoring?
- (4) What is the optimal sampling plan for type-I censoring?
- (5) Construct a suitable selection rule for this problem.
- (6) Which one is a better for the decision maker?

Answer:

(1) From Lemma 2, we know that only two higher stresses are needed for performing an accelerated test.

(2) If we choose $v_L = 0.5$, from equation (3), we have $S_L = 27.57$ KV and $S_H = 38$ KV.

(3) In the beginning, we have no idea about the values of (R_a, R_b) . So we perform a pilot life-testing program. The decision procedure may include the following steps:

- From each population, we randomly select (n_L^o, n_H^o) for performing an ALT under (S_L, S_H) . The experiment terminates when r_L^o and r_H^o are reached. Based on Weibull probability plots and log-log paper plots, suppose that all $\{\beta_i\}_{i=1}^k$ are roughly approximate 0.7764 and

$$(\alpha_{i0}, \alpha_{i1}, R(t^*, S_0)) \simeq \begin{cases} (0.2863, 10.2662, 0.99750) & i=1 \\ (0.3536, 11.3801, 0.99900) & i=2 \\ (0.4052, 10.1217, 0.99745) & i=3 \\ (0.3015, 10.1281, 0.99725) & i=4. \end{cases}$$

So $(R_a, R_b) \simeq (.999, .9975)$. We find that the optimal sample sizes and number of failures for (S_L, S_H) of the type-II censoring plan are $(r_L, n_L; r_H, n_H) = (43, 48; 22, 26)$.

- From each population, we shall take some more observations up to $n_L = 48$ and $n_H = 26$ for performing life-testing under S_L and S_H and the experiment terminates when $r_L = 43$ and $r_H = 22$ are reached.

(4) Under the same conditions in (3), we find that $(\xi, n_L, n_H) = (3.647, 47, 24)$, so we take $n_L = 47$ and $n_H = 24$ from each population for life-testing. The experiment terminates when the censoring times

$$(\eta_{iL}, \eta_{iH}) = \begin{cases} (823.28, 4.8559) & i=1 \\ (1536.88, 5.1937) & i=2 \\ (862.60, 5.4690) & i=3 \\ (780.13, 4.9303) & i=4 \end{cases}$$

are reached.

(5) By using the selection rule in equation (11), for both censoring plans, we have at least 90 % confidence to select the most reliable population correctly if the true configuration is as shown in equation (12).

(6) Comparing decision (3) and (4), we find that type-II censoring is more convenient to perform an ALT than type-I censoring. In the next section, we will use a cost criterion to compare these two plans.

7 Comparison between Type-I and Type-II censoring

MacKay (1977) has suggested some criteria for comparing the two censoring plans, for example, the duration of the experiment, cost of the experiment, etc. In this section we will concentrate on the cost of the experiment.

(A) Product's unit cost

Comparing Table 1 and 2, it shows that a type-I censoring plan needs slightly smaller sample size than that for a type-II censoring plan. If the product's unit cost is very expensive, it seems that type-I will be preferred.

(B) Expected life-testing time

To compare the relative efficiency of type-II censoring with type-I censoring, we define a quantity as follows:

$$\rho = \sum_{i=1}^k \sum_{j=1}^m \{E(Y_{ijr_{ij}}) / \eta_{ij}\}. \quad (22)$$

It is easily shown that ρ can be reduced to:

$$\rho = \sum_{i=1}^k \sum_{j=1}^m \frac{\int_0^1 (-\ln(1-y))^{1/\beta_i} y^{r_{ij}-1} (1-y)^{n_{ij}-r_{ij}} dy}{\beta(r_{ij}, n_{ij} - r_{ij} + 1) (-\ln(1 - \frac{r_{ij}}{n_{ij}}))^{1/\beta_i}}, \quad (23)$$

where

$$\beta(a, b) = \int_0^1 y^{a-1} (1-y)^{b-1} dy.$$

From Table 3, it is seen that the value of ρ is always less than 1. Due to the large separation of R_a from R_b , smaller expected life-testing times are needed by type-II censoring. The value of ρ is very close to 1 when R_a is very close to R_b , and $\{\beta_i\}$ and k are large.

8 Simulation studies

In practical situations, the Weibull shape parameters $\{\beta_i\}_{i=1}^k$ are unknown. It is reasonable to assume that each β_i has a prior distribution. In the following, we will study the “robustness” of this selection rule when each β_i has a known beta prior distribution with (p, q) over the interval $[a_i, b_i]$:

$$f(\beta_i) = \frac{(\beta_i - a_i)^{p-1}(b_i - \beta_i)^{q-1}}{\beta(p, q)(b_i - a_i)^{p+q-1}}, \quad (24)$$

where $a_i \leq \beta_i \leq b_i, \forall 1 \leq i \leq k$.

Let

$$[a_i, b_i] = \beta_i^* \times (1 \pm \epsilon),$$

where $\epsilon > 0$ is a constant.

Three cases are under consideration:

Case I : $\beta_i^* = 0.7764$ and

$$(\theta_{iL}, \theta_{iH}) = \begin{cases} (421.42, 1.4241) & \text{if } \Pi_i \text{ is the most reliable} \\ (233.47, 1.4241) & \text{if } \Pi_i \text{ is less reliable one.} \end{cases} \quad (25)$$

Case II: $\beta_i^* = 1.0$ and

$$(\theta_{iL}, \theta_{iH}) = \begin{cases} (171.70, 1.7284) & \text{if } \Pi_i \text{ is the most reliable} \\ (101.54, 1.5123) & \text{if } \Pi_i \text{ is less reliable one.} \end{cases} \quad (26)$$

Case III: $\beta_i^* = 1.25$ and

$$(\theta_{iL}, \theta_{iH}) = \begin{cases} (82.51, 1.5886) & \text{if } \Pi_i \text{ is the most reliable} \\ (57.68, 1.6136) & \text{if } \Pi_i \text{ is less reliable one.} \end{cases} \quad (27)$$

Under $\epsilon = 0.0$ and $t^* = 17.0669$, we have $(R_a, R_b) = (0.999, 0.9975)$. Thus, as $k=4$, the values $(r_L, n_L; r_H, n_H)$ for these three cases from Table 1 are:

$$\text{Case I : } (r_L, n_L; r_H, n_H) = (43, 48; 22, 26),$$

$$\text{Case II : } (r_L, n_L; r_H, n_H) = (43, 49; 22, 26),$$

and

$$\text{Case III : } (r_L, n_L; r_H, n_H) = (43, 52; 22, 28).$$

When ϵ is nonzero, the performance of this selection rule is investigated by simulation under type-II censoring plan. The basic steps of the simulation are:

- Generate a set of $\{\beta_i\}_{i=1}^4$, where each β_i has a beta prior with (p, q) over $\beta_i^*(1 \pm \epsilon)$.
- From each population, we generate a random sample of size n_L and n_H from Weibull distributions with (θ_{iL}, β_i) and (θ_{iH}, β_i) respectively. The two experiments are terminated when r_L and r_H are reached.
- From equation (6) and (7), we solve $(\hat{\alpha}_{i0}, \hat{\alpha}_{i1})$ by

$$\hat{\alpha}_{i0} = alt_{iH},$$

and

$$\hat{\alpha}_{i1} = (alt_{iL} - alt_{iH})/v_L,$$

where

$$alt_{ij} = \frac{1}{\beta_i^*} \left\{ \ln \left(\sum_{\ell=1}^{r_j} T_{ij\ell}^{\beta_i^*} + (n_j - r_j) T_{ijr_j}^{\beta_i^*} \right) - \ln r_j \right\}. \quad (28)$$

- A correct selection (CS) is made if we can correctly select the most reliable design.

For given (p, q) and each of specified values of ϵ , 500 trials are conducted and the proportion of CS is calculated. The results are given in Table 4. All the computations are done by the MATLAB software. From the results, it is seen that this selection rule is very stable not only for symmetric priors $((p,q)=(1,1)$ and $(5,5))$ but also for skewness to the left $((p,q)=(9,1)$ and $(7,3))$ and for skewness to the right $((p,q)=(1,9)$ and $(3,7))$. All the proportions of CS are very close to the target value $P^* = 0.90$. Even for very large variation (25% from β_i^*), it still has at least 77% of CS (about 85% of the target value). It seems that this selection is insensitive to the $\{\beta_i\}$ values.

9 Conclusion

For highly-reliable products which possess a Weibull-Inverse power model, this paper proposes both type-I and type-II optimal sampling plans for selecting the most reliable product. We use a cost criterion to compare these two plans. Although type-II needs slightly larger sample sizes than that of type-I censoring, it has a shorter expected life-testing time. Finally, when the Weibull shape parameters are unknown but their prior distributions are known, we use simulation to study the “robustness” of our rule. Results show that this rule is quite robust even when the unknown shape parameter has a moderate departure from the assumed value.

The results of this paper can easily be extended to the Weibull-Arrhenius and the lognormal-Arrhenius models.

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11 Appendix : Notation

ALT : accelerated life test.

Π_i : i – th population (product design), for all $1 \leq i \leq k$.

S_j : j – th level of test stress, for all $0 \leq j \leq m$.

$R_i(t, S_j)$: the reliability function of Π_i under stress S_j .

θ_{ij} : Weibull scale parameter for the combination of (Π_i, S_j) .

β_i : Weibull shape parameter for the i – th population.

$(\gamma_{i0}, \gamma_{i1})$: the parameters of inverse power model.

v_j : standardized stress, for all $0 \leq j \leq m$.

$(\alpha_{i0}, \alpha_{i1})$: reparametrization of $(\gamma_{i0}, \gamma_{i1})$.

$T_{ij\ell}$: ℓ – th observed data from (Π_i, S_j) .

$Y_{ij\ell}$: the ℓ – th order statistics of $\{T_{ij\ell}\}$.

$Z_{ij\ell} = \beta_i(\ln T_{ij\ell} - \alpha_{i0} - \alpha_{i1}v_j)$.

$M_{ij} = 1 - \exp(-(\frac{n_{ij}}{\theta_{ij}})^{\beta_i})$.

n_{ij} : sample size for the combination of (Π_i, S_j) .

r_{ij} : number of failures for the combination of (Π_i, S_j) .

D_{ij} : the set of individuals for which $z_{ij\ell}$ is an observed lifetime of (Π_i, S_j) .

C_{ij} : the set for which $z_{ij\ell}$ is a standardized censoring time of (Π_i, S_j) .

η_{ij} : censoring time for the combination of (Π_i, S_j) .

ζ_{ij} : time-saving factor of r_{ij} with respect to n_{ij} for the combination of (Π_i, S_j) .

p_{ij} : proportion of the sample size which allocates to the combination of (Π_i, S_j) .

q_{ij} : proportion of number of failures which allocates to the combination of (Π_i, S_j) .

L : the low stress for ALT.

H : the high stress for ALT.

δ : a natural selection rule.

Ω_i : the i - th preference region.

CS : correct selection.

$P_R(CS | \delta)$: probability of CS of the rule δ under R.

P^* : minimum value of CS probability.

ϕ : pdf for the standard extreme distribution.

Φ : cdf of the standard normal distribution.

$Q(x)$: cdf of the standard extreme distribution.

$$g(a, b, c) = \int_0^1 \frac{(-\ln(1-y))^{1/c} y^{a-1} (1-y)^{b-1}}{\beta(a, b)} dy.$$

$$f(\beta_i) = \frac{(\beta_i - a_i)^{p-1} (b_i - \beta_i)^{q-1}}{\beta(p, q) (b_i - a_i)^{p+q-1}}.$$

$$\beta(a, b) = \int_0^1 y^{a-1} (1-y)^{b-1} dy.$$

Table 1

Sample size, number of failures needed for selecting the most reliable product under type-II

censoring with $P^* = 0.90, \zeta^* = 0.50, v_L = 0.50$.

The values of $(r_L, n_L; r_H, n_H)$ are given in the table.

β	(R_a, R_b)	k=2	k=3	k=4	k=5	k=6
0.75	(.999, .9980)	41 45; 21 24	63 68; 32 35	75 80; 38 42	85 91; 43 47	92 98; 46 50
	(.999, .9975)	24 27; 12 14	36 39; 18 20	43 47; 22 25	49 53; 25 28	53 57; 27 30
	(.999, .9970)	17 19; 9 11	25 28; 13 15	30 33; 15 17	34 37; 17 19	37 40; 19 22
	(.999, .9965)	13 15; 7 9	19 22; 10 12	23 26; 12 14	26 29; 13 15	29 32; 14 16
	(.999, .9960)	11 13; 6 8	16 18; 8 10	19 22; 10 12	22 25; 11 13	23 26; 12 14
1.00	(.999, .9980)	41 47; 21 25	63 71; 32 37	75 84; 38 43	85 94; 43 49	92 102; 46 52
	(.999, .9975)	24 28; 12 15	36 41; 18 22	43 49; 22 26	49 55; 25 29	53 60; 27 31
	(.999, .9970)	17 21; 9 12	25 29; 13 16	30 35; 15 18	34 39; 17 21	37 42; 19 23
	(.999, .9965)	13 16; 7 9	19 23; 10 13	23 27; 12 15	26 30; 13 16	29 34; 14 17
	(.999, .9960)	11 14; 6 8	16 19; 8 10	19 23; 10 13	22 26; 11 14	23 27; 12 15
1.25	(.999, .9980)	41 49; 21 26	63 74; 32 39	75 88; 38 46	85 100; 43 52	92 108; 46 55
	(.999, .9975)	24 30; 12 16	36 44; 18 23	43 52; 22 28	49 58; 25 31	53 63; 27 33
	(.999, .9970)	17 22; 9 12	25 31; 13 17	30 37; 15 20	34 41; 17 22	37 45; 19 24
	(.999, .9965)	13 17; 7 10	19 24; 10 14	23 29; 12 16	26 32; 13 17	29 36; 14 18
	(.999, .9960)	11 15; 6 9	16 21; 8 11	19 24; 10 14	22 28; 11 15	23 29; 12 16
1.50	(.999, .9980)	41 53; 21 28	63 79; 32 42	75 94; 38 49	85 107; 43 55	92 115; 46 59
	(.999, .9975)	24 32; 12 17	36 47; 18 25	43 55; 22 30	49 62; 25 33	53 67; 27 36
	(.999, .9970)	17 24; 9 13	25 33; 13 19	30 39; 15 21	34 44; 17 24	37 48; 19 26
	(.999, .9965)	13 19; 7 11	19 26; 10 15	23 31; 12 17	26 35; 13 19	29 38; 14 20
	(.999, .9960)	11 16; 6 10	16 22; 8 12	19 26; 10 15	22 30; 11 16	23 31; 12 17

Table 2

Censoring time, sample size for (Low) and (High) stress for selecting the most reliable product under type-I censoring with $P^* = 0.90, \zeta^* = 0.50, v_L = 0.50$.

The values of (ξ, n_L, n_H) are given in the table, where $\eta_{ij} = \xi\theta_{ij}, j = L, H$ and $1 \leq i \leq k$.

β	(R_a, R_b)	k=2	k=3	k=4	k=5	k=6
0.75	(.999, .9980)	3.566 45 23	3.972 67 34	4.025 80 40	3.995 90 45	3.934 98 49
	(.999, .9975)	3.464 26 13	3.711 39 20	3.647 47 24	3.848 52 26	3.986 56 28
	(.999, .9970)	2.920 19 10	3.127 27 14	3.403 33 17	3.614 37 19	3.743 40 20
	(.999, .9965)	2.507 15 8	3.154 21 11	3.463 26 13	3.168 29 15	3.328 32 16
	(.999, .9960)	2.856 12 6	2.856 18 9	3.155 21 11	3.315 23 12	3.464 26 13
1.00	(.999, .9980)	2.288 46 23	2.342 70 35	2.343 84 42	2.310 94 47	2.264 103 52
	(.999, .9975)	2.079 28 14	2.165 41 21	2.211 49 25	2.315 54 27	2.280 59 30
	(.999, .9970)	1.981 20 10	2.152 28 14	2.140 34 17	2.114 39 20	2.181 42 21
	(.999, .9965)	1.749 16 8	1.917 22 11	2.079 27 14	2.015 30 15	2.100 33 17
	(.999, .9960)	1.845 13 7	1.946 18 9	1.917 22 11	2.001 24 12	2.079 27 14
1.25	(.999, .9980)	1.666 49 25	1.691 74 37	1.667 85 45	1.653 100 50	1.653 109 55
	(.999, .9975)	1.594 30 15	1.640 43 22	1.642 52 26	1.664 57 29	1.666 62 31
	(.999, .9970)	1.487 21 11	1.564 30 15	1.594 36 18	1.606 41 21	1.651 44 22
	(.999, .9965)	1.433 16 8	1.484 24 12	1.594 28 14	1.579 32 16	1.568 35 18
	(.999, .9960)	1.335 14 7	1.463 20 10	1.483 24 12	1.542 26 13	1.594 28 14
1.50	(.999, .9980)	1.359 52 26	1.364 78 39	1.346 96 48	1.329 108 54	1.306 119 60
	(.999, .9975)	1.313 32 16	1.357 46 23	1.353 55 28	1.355 60 30	1.367 66 33
	(.999, .9970)	1.262 22 11	1.307 32 16	1.344 38 19	1.331 44 22	1.374 47 24
	(.999, .9965)	1.198 18 9	1.275 25 13	1.313 30 15	1.318 34 17	1.324 37 19
	(.999, .9960)	1.190 15 8	1.243 21 11	1.275 25 13	1.272 28 14	1.313 30 15

Table 3

Relative efficiency of type-II censoring to type-I censoring under various values of β , (R_a, R_b) , and $2 \leq k \leq 6$.

β	(R_a, R_b)	k=2	k=3	k=4	k=5	k=6
0.75	(.999, .9980)	0.777	0.823	0.830	0.854	0.899
	(.999, .9975)	0.694	0.810	0.779	0.790	0.793
	(.999, .9970)	0.761	0.796	0.801	0.807	0.760
	(.999, .9965)	0.753	0.672	0.685	0.795	0.793
	(.999, .9960)	0.595	0.795	0.722	0.689	0.685
1.00	(.999, .9980)	0.813	0.864	0.925	0.936	0.964
	(.999, .9975)	0.800	0.837	0.859	0.868	0.884
	(.999, .9970)	0.714	0.796	0.823	0.835	0.853
	(.999, .9965)	0.822	0.781	0.792	0.858	0.819
	(.999, .9960)	0.714	0.852	0.817	0.796	0.792
1.25	(.999, .9980)	0.899	0.929	0.951	0.961	0.976
	(.999, .9975)	0.825	0.863	0.880	0.914	0.923
	(.999, .9970)	0.842	0.863	0.845	0.883	0.877
	(.999, .9965)	0.813	0.840	0.818	0.862	0.876
	(.999, .9960)	0.807	0.853	0.866	0.822	0.818
1.50	(.999, .9980)	0.914	0.951	0.976	0.995	0.979
	(.999, .9975)	0.876	0.882	0.913	0.942	0.929
	(.999, .9970)	0.856	0.873	0.884	0.895	0.895
	(.999, .9965)	0.832	0.850	0.868	0.853	0.881
	(.999, .9960)	0.810	0.886	0.871	0.873	0.850

Table 4

Proportion of correct selections under various values of (p,q) and ϵ

β	(p, q)	ϵ						
		0.1%	0.5%	1.0%	2.5%	5.0%	10%	25%
0.776	(1,1)	0.892	0.898	0.886	0.906	0.888	0.872	0.774
	(5,5)	0.908	0.894	0.888	0.872	0.874	0.894	0.800
	(1,9)	0.888	0.898	0.884	0.872	0.864	0.842	0.772
	(3,7)	0.904	0.868	0.886	0.898	0.880	0.870	0.774
	(7,3)	0.880	0.902	0.902	0.886	0.886	0.850	0.816
	(9,1)	0.910	0.918	0.888	0.900	0.864	0.874	0.784
1.000	(1,1)	0.910	0.906	0.878	0.904	0.870	0.868	0.806
	(5,5)	0.884	0.898	0.898	0.890	0.880	0.852	0.798
	(1,9)	0.888	0.888	0.868	0.920	0.876	0.836	0.812
	(3,7)	0.910	0.884	0.912	0.868	0.884	0.858	0.778
	(7,3)	0.888	0.902	0.902	0.872	0.864	0.880	0.804
	(9,1)	0.894	0.918	0.906	0.870	0.860	0.860	0.770
1.25	(1,1)	0.900	0.886	0.904	0.862	0.890	0.850	0.802
	(5,5)	0.884	0.896	0.884	0.870	0.884	0.872	0.814
	(1,9)	0.892	0.904	0.902	0.896	0.896	0.850	0.772
	(3,7)	0.898	0.906	0.910	0.898	0.886	0.850	0.788
	(7,3)	0.898	0.908	0.884	0.904	0.890	0.856	0.786
	(9,1)	0.894	0.896	0.902	0.890	0.888	0.840	0.800

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