SOME SIMPLE ROBUST METHODS FOR THE ANALYSIS OF RECURRENT EVENTS

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ABSTRACT

Nelson (1988, 1992) has discussed a method of estimating the cumulative mean function for identically distributed process of recurrent events. We show that a similar approach can be used with more general models, including regression. The key idea is to use point estimates based on Poisson models and to develop robust variance estimates that are valid more generally. The methods are illustrated on reliability and warranty data.

Key words: Nonparametric estimation; Point process data; Poisson processes; Regression; Reliability

1. INTRODUCTION

Situations where individuals or systems (henceforth just "systems", for convenience) in some population experience recurrent events are common in areas such as manufacturing, reliability and risk analysis. For example, equipment operating in the field may experience repeated failures and repairs (e.g. Ascher and Feingold 1984), manufactured products may generate warranty claims (e.g. Kalbfleisch, Lawless and Robinson 1991), nuclear power generating systems may have stoppages, and so on. In such cases we may want to study patterns and rates of occurrence of the events in question, compare different systems, assess the effect of explanatory variables, or predict future events.

We consider applications where events are observed for a fairly large number of systems. Suppose that system i in a group of k is observed over the time period $[0, \tau_i]$ and let $N_i(t)$ denote the number of events occurring over $[0, t]$. The cumulative mean function (CMF) for $N_i(t)$ is

$$M_i(t) = E\{N_i(t)\}. \tag{1.1}$$

Depending on whether the time scale is discrete or continuous, $M_i(t)$ is obtained by summing or integrating a mean function $m_i(t)$. When $M_i(t)$ is continuous in t, $m_i(t) = M_i'(t)$ is often called the rate of occurrence or local rate function. Our objectives in the paper are to present simple, robust methods for estimating and studying CMF's.

Poisson processes and renewal processes are often used to model recurrent events, and methods based on them are well established (e.g. Cox and Lewis 1966, Cox and Isham 1980, Crow 1982, Ascher and Feingold 1984, Andersen and Borgan 1985). Poisson or renewal models are unsatisfactory in many applications, however, and other models are also common; Poisson or renewal processes with random effects added (e.g. Lawless 1987, Follman and Goldberg 1988, Aalen and Husebye 1991) have recently received a good deal of attention.

Another approach was taken by Nelson (1988): he observed for cases where the k systems have the same CMF $M(t)$ that the so-called Nelson-Aalen nonparametric estimator $\hat{M}(t)$ given by (2.1) below is rather widely applicable. He subsequently (Nelson and Doganaksoy

1989, Nelson 1992) developed a robust variance estimate for $\hat{M}(t)$ which enables one to obtain confidence limits and to make comparisons.

In this paper we show that a similar approach will handle more general models including regression. The key observation is that parameter estimates based on Poisson models are valid quite generally; Nelson's estimate is of this type. However, Poisson variance estimates and confidence limits are not robust, so we need to develop ones that are. Provided that the end-of-observation times τ_i for individual systems are independent of the recurrent event processes, robust variance estimates are readily obtained, as we show.

The methods presented here focus on mean and cumulative mean functions for processes of recurrent events, and do not involve a full probabilistic specification of the processes. As motivation, we introduce a pair of examples where estimation and comparison of mean functions are important. In both examples time is measured in discrete units (days), but the methods in the paper apply equally when time is treated as a continuous variable. We will return to these examples in section 5.

Example 1. Valve seat replacements

We consider data presented by Nelson and Doganaksoy (1989) and Nelson (1992) which give the times (in days of service) at which valve seats were replaced on 41 diesel engines in a service fleet. For convenience the data are reproduced in Table 1. Each engine had 16 valves but the data do not indicate which ones were replaced. Consequently, Nelson and Doganaksoy considered estimation of the average number of valve replacements $M(t)$ per engine up to a given engine age t, and whether the replacement rate increases over time.

We consider estimation of $M(t)$ in section 5.

Table 1. Valve Seat Replacement Data

Unit	Replacement Times*	Unit	Replacement Times*
1	(761)	22	(593)
2	(759)	23	573, (589)
3	98, (667)	24	165, 408, 604, (606)
4	326, 653, 653, (667)	25	249, (594)
5	(665)	26	344, 497, (613)
6	84, (667)	27	265, 586, (595)
7	87, (663)	28	166, 206, 348, (389)
8	646, (653)	29	(601)
9	92, (653)	30	410, 581, (601)
10	(651)	31	(611)
11	258, 328, 377, 621, (650)	32	(608)
12	61, 539, (648)	33	(587)
13	254, 276, 298, 640, (644)	34	367, (603)
14	76, 538, (642)	35	202, 563, 570, (585)
15	635, (641)	36	(587)
16	349, 404, 561, (649)	37	(578)
17	(631)	38	(578)
18	(596)	39	(586)
19	120, 479, (614)	40	(585)
20	323, 449, (582)	41	(582)
21	139, 139, (589)		

*Numbers in brackets are the ends of the observation periods (i.e. the τ_i's).

Example 2. Automobile Warranty Claims

For products under warranty, age-specific claim frequencies are of considerable interest (e.g. see Kalbfleisch et al. 1991, Robinson and McDonald 1991). Specifically, let t denote the "age" of a product unit, defined as the number of days since the unit was sold, and let $m(t)$ be the average or expected number of warranty claims per unit at age t. Usually $m(t)$ is small and can be thought of as the probability of a claim at age t, but multiple claims on a given day are possible in some situations. The cumulative mean function $M(t) = m(0) + \cdots + m(t)$ is the expected number of claims per unit up to age t. Units are normally sold over time, and an important problem is to estimate $M(t)$ from warranty claims made up to some particular date.

We consider data on warranty claims for a system on a particular car model, discussed by Kalbfleisch, Lawless and Robinson (1991). A total of 36,683 cars were sold over a period of about 60 weeks (see Figure 5 in Kalbfleisch et al., 1991). The claims were reported over an 18 month period, in particular, up to 547 days after the first cars were sold. We consider estimation and analysis of CMF's for these warranty claims in section 5.

An outline of the remainder of the paper is as follows. In section 2 we review estimation of a common CMF for independent recurrent event processes as discussed by Nelson (1988, 1992) and note the essential features that permit extensions to handle regression and other models. Section 3 presents the regression methodology and in section 4 we test the equality of CMF's for two types of processes. In section 5 we reconsider examples 1 and 2 to illustrate the methodology, and in section 6 make a few concluding remarks. Derivations of results are given in an Appendix.

2. NONPARAMETRIC ESTIMATION OF A COMMON CMF

We consider processes $\{N_i(t) : t \geq 0\}$, $i = 1, ..., k$ which are independent and have the same CMF $M(t) = E\{N_i(t)\}$. Our objective is to estimate $M(t)$, having observed the times $t_{i1} \leq \cdots \leq t_{ir_i}$ at which events occur over the interval $[0, \tau_i]$ for each system $i = 1, ..., k$. The treatment here is basically that of Nelson (1988, 1992) but since it sets the notation and motivates the remainder of the paper, we review the approach and note conditions for its validity.

For simplicity we present the results in a discrete time framework where events may occur at times $t = 0, 1, 2, ...$. However, the methods and the formulas also apply fully to continuous time situations, as we note below. Let $n_i(t) \geq 0$ represent the number of events which occur at time t for system i, so that $m(t) = E\{n_i(t)\}$ and $M(t) = \sum_{s=0}^{t} m(s)$. System i is observed over $[0, \tau_i]$, and for notational convenience we define $\delta_i(t) = I(t \leq \tau_i)$ to indicate whether system i is observed at time t. The total number of events and total number of systems observed at time t, respectively, are denoted by $n.(t) = \sum_{i=1}^{k} \delta_i(t) n_i(t)$ and $\delta.(t) = \sum_{i=1}^{k} \delta_i(t)$. We remark that the results and formulas below also hold if system i is observed over an interval $[\tau_{1i}, \tau_{2i}]$ or indeed, any set of times determined independently of the $n_i(t)$'s. For simplicity, however, we will assume throughout that the times at which system i is observed are those in the interval $[0, \tau_i]$.

We assume that the k systems are mutually independent. If the $n_i(t)$'s $(t = 0, 1, 2, \ldots)$ are independent Poisson random variables with means $m(t)$, then the maximum likelihood estimate (mle) of $m(t)$ is $\hat{m}(t) = n.(t)/\delta.(t)$, and the estimate of $M(t)$ for $0 \leq t \leq \tau = \max(\tau_i)$ is

$$\hat{M}(t) = \sum_{s=0}^{t} \frac{n.(s)}{\delta.(s)}. \tag{2.1}$$

This estimator is sometimes referred to as the Nelson-Aalen estimator (e.g. Andersen and Borgan 1985, section 4.1) and it is well known as a nonparametric mle in counting process models. As noted by Nelson (1988), it is also valid more generally. In particular, $\hat{M}(t)$ is an unbiased and consistent estimator of $M(t)$ under conditions that we now discuss.

Since

$$\hat{M}(t) = \sum_{i=1}^{k} \sum_{s=0}^{t} \frac{\delta_i(s) n_i(s)}{\delta.(s)}, \tag{2.2}$$

it is unbiased for $M(t)$ provided $E\{n_i(s) | \delta_i(s) = 1, \delta.(s)\} = m(s)$. If we allow completely arbitrary processes for the $n_i(s)$'s, this rules out certain observational schemes. For example, we could not allow observation of the i'th system to stop upon the occurrence of the r'th event, since the information that $\delta_i(s) = 1$ would then convey the information that $N_i(s - 1) < r$. For the general validity of (2.2), we therefore require that the end-of-observation times τ_i (or for more general observation schemes, the $\delta_i(t)$'s) be independent of the processes $\{N_i(t) : t \geq 0\}$ of events, and we assume this henceforth. This assumption has also been made by Nelson (1988). We remark that if the processes $\{N_i(t)\}$ are truly Poisson, this requirement is not needed (e.g. see Andersen and Borgan 1985).

Nelson and Doganaksoy (1989), Nelson (1992) and Robinson (1990) have discussed variance estimation for $\hat{M}(t)$. A simple treatment is possible: from (2.2) we have directly that

$$\mathrm{Var}\{\hat{M}(t)\} = \sum_{i=1}^{k}\sum_{s=0}^{t}\sum_{u=0}^{t} \frac{\delta_i(s)\delta_i(u)}{\delta.(s)\delta.(u)}\mathrm{cov}\{n_i(s), n_i(u)\}.$$

We assume that $\mathrm{cov}\{n_i(s), n_i(u)\}$ exists for all s, u in $[0, \tau]$. Provided that $\delta.(s) \to \infty$ for all s in $[0, t]$, $\hat{M}(t)$ is then consistent for $M(t)$ and $\mathrm{Var}\{\hat{M}(t)\}$ is under mild conditions consistently estimated by

$$\begin{aligned}\hat{V}(t) &= \sum_{i=1}^{k}\sum_{s=0}^{t}\sum_{u=0}^{t} \frac{\delta_i(s)\delta_i(u)}{\delta.(s)\delta.(u)}\{n_i(s) - \hat{m}(s)\}\{n_i(u) - \hat{m}(u)\} \\ &= \sum_{i=1}^{k}\{\sum_{s=0}^{t} \frac{\delta_i(s)}{\delta.(s)}[n_i(s) - \hat{m}(s)]\}^2. \end{aligned} \tag{2.3}$$

REMARKS

1. The only nonzero terms in (2.1) and (2.3) are for times s at which events occur, i.e. with $n.(s) > 0$. If $t_1 < \cdots < t_r$ are the distinct times at which events occur across all systems combined, then (2.1), (2.3) become

$$\hat{M}(t) = \sum_{j:t_j \le t} \frac{n.(t_j)}{\delta.(t_j)} \tag{2.4}$$

$$\hat{V}(t) = \sum_{i=1}^{k}\{\sum_{j:t_j \le t} \frac{\delta_i(t_j)}{\delta.(t_j)}[n_i(t_j) - \frac{n.(t_j)}{\delta.(t_j)}]\}^2. \tag{2.5}$$

These expressions also define valid nonparametric estimators in the case of continuous time processes. In the continuous case (2.4) and (2.5) may conveniently be written in integral form:

$$\hat{M}(t) = \int_0^t \frac{dN.(s)}{\delta.(s)}, \quad \hat{V}(t) = \sum_{i=1}^{k}\{\int_0^t \frac{\delta_i(s)}{\delta.(s)}[dN_i(s) - \frac{dN.(s)}{\delta.(s)}]\}^2,$$

where $dN_i(s)$ is the number of system i events at time s, and $dN.(s) = \sum_{i=1}^{k} \delta_i(s)dN_i(s)$.

2. Nelson (1992) gives a slightly different estimator for $Var\{\hat{M}(t)\}$; his is unbiased but may give negative values (Robinson 1990).

3. The estimates $\hat{M}(t)$ and $\hat{V}(t)$ are robust because they are simple moment estimates. For example, if all $\tau_i = \tau$ then for $t \leq \tau$, we find $\hat{M}(t) = N.(t)/k$ and

$$\hat{V}(t) = \frac{1}{k^2} \sum_{i=1}^{k} \{N_i(t) - \frac{N.(t)}{k}\}^2,$$

the sample mean and variance for the number of events in $[0, t]$ for the k systems. Operating under Poisson process assumptions, we would on the other hand get $\hat{V}(t) = N.(t)/k^2$. In many situations this tends to underestimate $\text{Var}\{\hat{M}(t)\}$ considerably.

We now develop a similar approach for a flexible family of regression models.

3. REGRESSION MODELS

Let $\boldsymbol{x}_i(t)$ be a vector of covariates associated with system i at time t, and assume that conditional on the covariate values $\boldsymbol{x}_i(t)$ $(t \geq 0)$ we have

$$E\{n_i(t)\} = m_i(t) = m_0(t)P_i(t)g(\boldsymbol{x}_i(t); \boldsymbol{\beta}), \tag{3.1}$$

where $\boldsymbol{\beta}$ is a $p \times 1$ vector of regression parameters, g is positive-valued, $m_0(t) \geq 0$ is a baseline mean function and $P_i(t)$ is a known function. The $P_i(t)$'s are useful when systems are of different known sizes or have different exposures in some sense; an example is given in section 5. Often, of course, the $P_i(t)$'s all equal 1. The specification (3.1) is a natural and flexible way to model covariate effects with recurrent event or count data (e.g. Lawless 1987, McCullagh and Nelder 1989, ch. 6). "Log linear" models with $g(\boldsymbol{x}; \boldsymbol{\beta}) = \exp(\boldsymbol{x}'\boldsymbol{\beta})$ are very useful, but for simplicity and generality we develop results for the general formulation (3.1).

For notational convenience we will write (3.1) as

$$m_i(t) = m_0(t)g_i(t),$$

remembering that $g_i(t) = P_i(t)g(\boldsymbol{x}_i(t); \boldsymbol{\beta})$ is a function of the covariates and the parameter $\boldsymbol{\beta}$.

Our approach, as in section 2, is to obtain point estimates of unknown parameters in (3.1) under the Poisson assumption; in the discrete time case this means that the $n_i(t)$'s are mutually independent Poisson random variables, and in the continuous case that the $N_i(t)$'s are nonhomogeneous Poisson processes. The estimates are once again valid quite generally provided that conditional on the covariate values, the τ_i's are determined independently of the event processes. As before, we want to develop robust variance estimates.

We may specify $m_0(t)$ parametrically as $m_0(t;\boldsymbol{\alpha})$, or treat it nonparametrically. The latter is the natural extension of section 2 and we shall focus on it, but we will note parametric results at the end of the section.

We deal with $m_0(t)$ nonparametrically by treating the unknown values $m_0(t) : t = 0, 1, ..., \tau$ as parameters to be estimated along with $\boldsymbol{\beta}$ of (3.1). The Poisson model gives the following estimating equations for the $m_0(t)$'s and $\boldsymbol{\beta}$ (see the Appendix):

$$\sum_{i=1}^{k} \delta_i(t)\{n_i(t) - m_0(t)g_i(t)\} = 0 \quad t = 0, 1, ..., \tau \tag{3.2}$$

$$\sum_{i=1}^{k}\sum_{s=0}^{\tau} \delta_i(s)\{\frac{n_i(s) - m_0(s)g_i(t)}{g_i(t)}\}\frac{\partial g_i(t)}{\partial \boldsymbol{\beta}} = \mathbf{0}. \tag{3.3}$$

In (3.3) and elsewhere, vectors $\boldsymbol{v}$ are written in column form, T denotes matrix transpose, and if $\boldsymbol{v} = (v_1, ..., v_p)^T$ and $g(\boldsymbol{v})$ is a function of $v_1, ..., v_p$, then we write $\partial g/\partial \boldsymbol{v}$ to denote the vector $(\partial g/\partial v_1, ..., \partial g/\partial v_p)^T$. The left hand sides of the estimating equations (3.2) and (3.3) have expectation zero and so are valid quite generally, provided that (conditional on the covariate values) $E\{n_i(t)|\delta_i(t) = 1\} = m_i(t) = m_0(t)g_i(t)$. To solve (3.2) and (3.3) we note that (3.2) gives

$$m_0(t) = \frac{n.(t)}{R(t;\boldsymbol{\beta})}, \tag{3.4}$$

where we define

$$R(t;\boldsymbol{\beta}) = \sum_{i=1}^{k} \delta_i(t)g_i(t).$$

Two convenient ways of solving (3.2) and (3.3) to obtain the estimates $\hat{m}_0(t)$, $t = 0, 1, ..., \tau$ and $\hat{\boldsymbol{\beta}}$ thus suggest themselves. The first is to alternate between (3.3) and (3.4), treating

$m_0(s)$ as fixed in (3.3) and solving for $\boldsymbol{\beta}$. Since (3.3) are Poisson maximum likelihood (or quasi-likelihood) equations, standard software can be used to solve for $\boldsymbol{\beta}$. The second, computationally faster, approach is to insert (3.4) into (3.3) to obtain the equations

$$\sum_{i=1}^{k}\sum_{s=0}^{T}\delta_i(s)\frac{\partial \log g_i(s)}{\partial \boldsymbol{\beta}}\{n_i(s)-\frac{n.(s)g_i(s)}{R(s;\boldsymbol{\beta})}\}=\mathbf{0}. \tag{3.5}$$

The equations (3.5) can be solved iteratively to give $\hat{\boldsymbol{\beta}}$, and then the $\hat{m}_0(t)$'s may be obtained from (3.4).

The only nonzero terms in (3.5) are for s values at which an event occurs. Let $t_1 < \cdots < t_r$ denote the distinct times at which events occur across all k systems, as in section 2. In addition, let D_j denote the set of systems with events at t_j, including repeats if a system experiences more than one event at t_j. Then (3.5) may be written as

$$\sum_{j=1}^{r}\sum_{l\in D_j}\{\frac{\partial \log g_l(t_j)}{\partial \boldsymbol{\beta}}-\frac{\partial \log R(t_j;\boldsymbol{\beta})}{\partial \boldsymbol{\beta}}\}=\mathbf{0}. \tag{3.6}$$

These equations apply to continuous as well as discrete time cases. For orderly continuous time processes for which only one event may occur at any time point, the equations (3.6) are in fact the Cox partial likelihood equations (Cox 1972). Hence software for the partial likelihood analysis of repeated events may be used to solve (3.6), though the variance estimates for $\hat{\boldsymbol{\beta}}$ given are not valid in the general framework here.

Lawless (1987) and others have noted the connection with the Cox partial likelihood just mentioned and have exploited it for Poisson and mixed Poisson models. However, we want to obtain variance estimates for $\hat{\boldsymbol{\beta}}$ and $\hat{M}_0(t)=\sum_{s=0}^{t}\hat{m}_0(s)$ that are valid more generally. We present an appropriate variance estimate for $\hat{\boldsymbol{\beta}}$ below. Derivations of this and the full covariance matrix for $(\hat{\boldsymbol{\beta}},\hat{M}_0(t))$ for any specified t value are outlined in the Appendix.

Let us define the following vectors and matrices:

$$\boldsymbol{W}_i(\boldsymbol{\beta},s)=\frac{\partial \log g_i(s)}{\partial \boldsymbol{\beta}}-\frac{\partial \log R(s;\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} \tag{3.7}$$

$$\hat{B}_1=\frac{1}{k}\sum_{i=1}^{k}\hat{\boldsymbol{B}}_{1i}\hat{\boldsymbol{B}}_{1i}^T \tag{3.8}$$

where

$$\hat{\boldsymbol{B}}_{1i} = \sum_{s=0}^{\tau} \delta_i(s) \boldsymbol{W}_i(\hat{\boldsymbol{\beta}}, s)[n_i(s) - \hat{g}_i(s)\hat{m}_0(s)],$$

$$\hat{A}_1 = \frac{1}{k} \sum_{i=1}^{k} \sum_{s=0}^{\tau} \delta_i(s) \hat{m}_0(s) \frac{\partial \hat{g}_i(s)}{\partial \hat{\boldsymbol{\beta}}} \boldsymbol{W}_i(\hat{\boldsymbol{\beta}}, s)^T. \tag{3.9}$$

Under mild conditions $\sqrt{k}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ is asymptotically normal with covariance matrix consistently estimated by

$$k \ \widehat{\text{asvar}}(\hat{\boldsymbol{\beta}}) = \hat{A}_1^{-1} \hat{B}_1 (\hat{A}_1^{-1})^T. \tag{3.10}$$

Confidence limits and tests for $\boldsymbol{\beta}$ can be based on $\hat{\boldsymbol{\beta}}$ and (3.10). An example is given in section 5.

Variance estimates and confidence limits for functions of $\boldsymbol{\beta}$ and the $m_0(t)$'s can also be obtained. In the Appendix we indicate how to compute the joint asymptotic covariance matrix for $\hat{M}_0(t)$ and $\hat{\boldsymbol{\beta}}$, for any t value. This allows us to get the asymptotic variance for $\hat{M}_0(t) g(\boldsymbol{x}; \hat{\boldsymbol{\beta}})$, the estimated mean number of events up to time t, for a system with unit exposure and fixed covariates $\boldsymbol{x}$.

The methods just described treat $m_0(t)$ and $M_0(t)$ nonparametrically. If $m_0(t)$ is smooth we may wish instead to model it as $m_0(t; \boldsymbol{\alpha})$, where $\boldsymbol{\alpha}$ is a parameter, usually of low dimension. In this case the Poisson estimating equations for $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ can be written in a simple form: if $\boldsymbol{\theta} = (\boldsymbol{\beta}, \boldsymbol{\alpha})$ represents all of the unknown parameters then $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\alpha}})$ is obtained by solving the equations

$$\boldsymbol{U}(\boldsymbol{\theta}) = \sum_{i=1}^{k} \sum_{s=0}^{\tau} \delta_i(s) \{ \frac{n_i(s) - m_i(s)}{m_i(s)} \} \frac{\partial m_i(s)}{\partial \boldsymbol{\theta}} = \boldsymbol{0} \tag{3.11}$$

in the discrete time case with $m_i(s) = m_0(s; \boldsymbol{\alpha}) g_i(s)$ and, in the continuous time case with $m_i(s)$ a continuous function of s,

$$\boldsymbol{U}(\boldsymbol{\theta}) = \sum_{i=1}^{k} \{ \sum_{j=1}^{r_i} \frac{\partial \log m_i(t_{ij})}{\partial \boldsymbol{\theta}} - \int_0^{\tau} \delta_i(s) \frac{\partial m_i(s)}{\partial \boldsymbol{\theta}} ds \} = \boldsymbol{0}, \tag{3.12}$$

where events for system i occur at times t_{ij} $(j = 1, ..., r_i)$. These estimating equations have expectation zero and are valid under the conditions noted previously. They are readily solved by standard iterative methods such as Newton-Raphson. In many cases existing software can be used; for example, if $m_0(s;\boldsymbol{\alpha}) = \alpha_1 s^{\alpha_2}$, $P_i(s) = 1$ and $g(\boldsymbol{x};\boldsymbol{\beta}) = \exp(\boldsymbol{x}'\boldsymbol{\beta})$ then $m_i(s) = \exp\{\log\alpha_1 + \alpha_2 \log s + \boldsymbol{x}_i'\boldsymbol{\beta}\}$, a log linear model.

Robust variance estimates for $\hat{\boldsymbol{\theta}}$ are readily obtained. Let $\boldsymbol{U}_i(\boldsymbol{\theta})$ denote the i'th term in $\boldsymbol{U}(\boldsymbol{\theta})$ for either (3.11) or (3.12), and define

$$\hat{B} = \frac{1}{k}\sum_{i=1}^{k} \boldsymbol{U}_i(\hat{\boldsymbol{\theta}})\boldsymbol{U}_i(\hat{\boldsymbol{\theta}})^T. \tag{3.13}$$

In addition, define

$$A_i(\boldsymbol{\theta}) = \sum_{s=0}^{\tau} \delta_i(s) m_i(s) \frac{\partial \log m_i(s)}{\partial \boldsymbol{\theta}} \left(\frac{\partial \log m_i(s)}{\partial \boldsymbol{\theta}}\right)^T$$

for the discrete time case and

$$A_i(\boldsymbol{\theta}) = \int_0^{\tau} \delta_i(s) m_i(s) \frac{\partial \log m_i(s)}{\partial \boldsymbol{\theta}} \left(\frac{\partial \log m_i(s)}{\partial \boldsymbol{\theta}}\right)^T ds$$

in the continuous time case, and let

$$\hat{A} = \frac{1}{k}\sum_{i=1}^{k} A_i(\hat{\boldsymbol{\theta}}). \tag{3.14}$$

Then, as discussed in the Appendix, the asymptotic covariance matrix for $\sqrt{k}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is consistently estimated by

$$k\ \widehat{\text{asvar}}(\hat{\boldsymbol{\theta}}) = \hat{A}^{-1}\hat{B}\hat{A}^{-1}. \tag{3.15}$$

4. COMPARISON OF TWO PROCESSES

In comparing processes we often wish to test that their cumulative mean functions are equal. Consider the case of two types of processes $\{N^{(0)}(t)\}$ and $\{N^{(1)}(t)\}$, with observations from k_j processes of type $j = 0, 1$. A simple way to test that their CMF's $M^{(0)}(t)$ and $M^{(1)}(t)$ are equal is to define covariates that model the types of differences in the $M^{(j)}(t)$'s that might be expected. We consider here the simple situation where the $M^{(j)}(t)$'s are expected to be roughly proportional to one another. If we define the fixed scalar covariate x_i to equal 1 if system i is of type 1 and 0 if it is of type 0 and let $g(x_i;\beta) = \exp(\beta x_i)$, then testing equality of $M^{(0)}(t)$ and $M^{(1)}(t)$ is equivalent to testing that $\beta = 0$.

One approach, illustrated in section 5, is to obtain $\hat{\beta}$ as in section 3 and, using the variance estimate given by (3.10), to test that $\beta = 0$. A simpler approach that avoids having to compute $\hat{\beta}$ is to use the pseudo score statistic defined by the left hand side of (3.5) evaluated at $\beta = 0$. This can be shown to equal

$$U = \sum_{s=0}^{\tau} \frac{\delta_{0.}(s)\delta_{1.}(s)}{\delta_{0.}(s)+\delta_{1.}(s)} \{ \frac{n_{1.}(s)}{\delta_{1.}(s)} - \frac{n_{0.}(s)}{\delta_{0.}(s)} \}, \tag{4.1}$$

where $n_{j.}(s)$ is the number of type j events at time s and $\delta_{j.}(s)$ is the number of type j systems observed at time s ($j = 0, 1$). This is the score statistic for testing $\beta = 0$ that arises from a Poisson model, but is valid more generally. To use it we require a robust variance estimate; it follows directly from (4.1) that

$$\widehat{\mathrm{Var}}(U) = \sum_{j=0}^{1} \sum_{i=1}^{k_j} \{ \sum_{s=0}^{\tau} \delta_{ji}(s) \frac{[\delta_{..}(s) - \delta_{j.}(s)]}{\delta_{..}(s)} [n_{ji}(s) - \frac{n_{j.}(s)}{\delta_{j.}(s)}] \}^2 \tag{4.2}$$

is suitable, where, with an obvious notation, $n_{ji}(s)$ is the number of events at time s for the i'th system of type j ($j = 0, 1$), $\delta_{ji}(s)$ indicates whether the system is observed at time s, and dots indicate summation over the appropriate indices.

Note that if $\delta_{ji}(s) = 1$ for $0 \le t \le \tau$ for all (i, j) then (4.1) and (4.2) become the "obvious" statistics

$$U = \frac{k_0 k_1}{k_0 + k_1} \{ \bar{N}_{1.}(\tau) - \bar{N}_{0.}(\tau) \}$$

$$\widehat{\mathrm{Var}}(U) = \sum_{j=0}^{1} \frac{k_{1-j}^2}{(k_0 + k_1)^2} \sum_{i=1}^{k_j} \{ N_{ji}(\tau) - \bar{N}_{j.}(\tau) \}^2.$$

The statistic (4.1) is effective at detecting different CMF's when $M^{(0)}(t)$ and $M^{(1)}(t)$ are proportional, or roughly so. It is analogous to the log rank statistic for testing the equality of survival distributions (e.g. Kalbfleisch and Prentice 1980, ch. 4). Other tests are preferable if other types of departures from equality are expected, for example if the $M^{(j)}(t)$'s may cross at some point. Tests may be readily developed from the results in sections 2 and 3; one approach is to consider general weight functions $w(s)$ and test statistics of the form

$$U = \sum_{s=0}^{\tau} w(s)\{\frac{n_{1\cdot}(s)}{\delta_{1\cdot}(s)} - \frac{n_{0\cdot}(s)}{\delta_{0\cdot}(s)}\}, \tag{4.3}$$

with associated variance estimate

$$\widehat{\text{Var}}(U) = \sum_{j=0}^{1}\sum_{i=1}^{k_j}\{\sum_{s=0}^{\tau}\frac{w(s)\delta_{ji}(s)}{\delta_{j\cdot}(s)}[n_{ji}(s) - \frac{n_{j\cdot}(s)}{\delta_{j\cdot}(s)}]\}^2. \tag{4.4}$$

Tests for the equality of three or more CMF's may be based on generalizations of (4.3) and (4.4). We will not develop this in detail, but consider an example in section 5.

5. EXAMPLES

In this section we discuss further the examples introduced in section 1.

Example 1 Continued

Nelson (1992) obtains point estimates and confidence intervals for $M(t)$, the average number of valve seat replacements per engine up to age t. Since the methods of section 2 are essentially the same as his, we keep discussion to a few points of interest. Nelson's Figure 4.3 shows $\hat{M}(t)$ defined by (2.2) and associated confidence limits based on his unbiased variance estimate. The simpler estimate (2.3) gives virtually the same results. For example, with the data in Table 1 we find that at $t = 400$ days $\hat{M}(t) = .659$, and a standard error (i.e. $\hat{V}(400)^{1/2}$) from (2.3) of .132, in comparison with Nelson's .133. The Poisson variance estimate $\hat{V}(t) = \sum_{s=0}^{t} n_{\cdot}(s)/\delta_{\cdot}(s)^2$ gives an only slightly smaller standard error of .127. The results indicate (see Nelson's Figure 4.3) that the valve replacement rate is fairly constant, possibly with an increase around 600 days.

We should bear in mind the requirement that end-of-observation times τ_i must be independent of the event processes. If this is not the case then $\hat{M}(t)$ may be seriously biased. If the τ_i's vary a good deal, it is sensible to check at least informally on their independence of the event processes. A simple way is to group the systems according to their τ_i's and then to test that the $M(t)$'s in the different groups are equal (see section 4 and example 2 below). For the valve seat data the τ_i's do not vary a great deal and examination of the data shows there is no evidence of non-independence.

Example 2 Continued

This example deals with warranty claims on 36,683 cars which were sold over a period of about 60 weeks. The data include all claims reported to the manufacturer up to 547 days after the first car was sold. Our objective here will be to assess $M(t)$, the expected number of claims per car up to age t (i.e. up to t days after the date of sale).

If car i was sold on day $d_i \geq 0$ and day $T = 547$ is the last day on which data were recorded, then car i was "observed" from age 0 to age $\tau_i = T - d_i$. The dates of sale are known by the manufacturer so for car i the manufacturer in principle observes the number of age t claims $n_i^*(t)$. For a specified population of cars, the age specific expected claims function is then $m(t) = E\{n_i^*(t)\}$.

For the 36,683 cars referred to, the τ_i's ranged from 7 to 547 days. There were 5,701 claims reported by day $T = 547$; the number of claims per car ranged from 0 (for 32,677 cars) to 10 (for 2 cars). There is an additional feature that we will incorporate in our analysis. Car dealers report claims to the manufacturer, who has to validate the claim and enter it into the data base on which our analysis is based; this causes a "reporting" delay equal to the time between the occurrence of the claim and its entry in the data base. For the situation in question most delays are less than 80 days, but almost half exceed 20 days. As a result, claims that are made close to $T = 547$ (i.e. at an age close to a car's τ_i value) may not have been reported yet. To adjust for this and avoid bias we define the expected number of age t claims reported for car i to be

$$m_i(t) = E\{n_i(t)\} = m(t)F(\tau_i - t), \tag{5.1}$$

where $F(r)$ is the probability a claim is reported within r days of its occurrence. Kalbfleisch et al. (1991) discuss reporting delay adjustments at length and give values of $F(r)$ appropriate to the claim reporting system here. In the two parts of our analysis below we use these values.

(i) Plots of $\hat{M}(t)$ provide useful summaries of claims experience and facilitate comparisons. For example, Figure 1 shows separate estimates for cars manufactured in each of 6 two-month production periods over the model year: mid July - mid September (period 1) to mid May - mid July of the following year (period 6). Because of (5.1) the estimated CMF $\hat{M}(t)$ for cars in period j is, instead of (2.1), obtained from (3.2) with $g_i(t) = F(T_i - t)$:

$$\hat{M}^{(j)}(t) = \sum_{s=0}^{t} \frac{n_{\cdot}^{(j)}(s)}{\delta_{\cdot}^{(j)}(s)} \quad j = 1, ..., 6$$

where $n_{\cdot}^{(j)}(s)$ is the number of age s claims reported for cars produced in period j and

$$\delta_{\cdot}^{(j)}(s) = \sum_{i \in P_j} \delta_i(s) F(\tau_i - s),$$

where P_j is the set of cars produced in period j. Similarly, the robust variance estimate for $\hat{M}^{(j)}(t)$ is, instead of (2.3),

$$\hat{V}^{(j)}(t) = \sum_{i \in P_j} \{\sum_{s=0}^{t} \frac{\delta_i(s)}{\delta_{\cdot}^{(j)}(s)} [n_i(s) - \hat{m}_i(s)]\}^2, \tag{5.2}$$

where $\hat{m}_i(s) = F(\tau_i - s)\hat{m}^{(j)}(s) = F(\tau_i - s)n_{\cdot}^{(j)}(s)/\delta_{\cdot}^{(j)}(s)$. Kalbfleisch et al. (1991) discussed estimation of $M(t)$ but obtained variance estimates only under Poisson assumptions; this is undesirable here since the claim processes for cars are quite clearly not identically distributed Poisson processes.

Figure 1 shows strikingly that the expected claims curve for period 3 (November - January) is substantially higher than for the other five periods. Confidence limits based on (5.2) suggest that the difference is too large to attribute to chance alone and it would be interesting to determine its source. For example, at age $t = 364$ days (one year) the estimates and standard errors for $M^{(1)}(t)$ and $M^{(3)}(t)$ are $\hat{M}^{(1)}(364) = .161$ ($s.e. = .0060$) and $\hat{M}^{(3)}(364) = .227$ ($s.e. = .0086$) respectively. Standard errors based on Poisson assumptions are about 25% smaller and make differences appear more significant than they actually are.

(ii) The regression methods of section 3 can be used to study the relationship of claims to explanatory variables. As an illustration we will use a regression model to test the equality of the $M^{(j)}(t)$'s for the 6 production periods. To do so we define a vector $\boldsymbol{x}_i = (x_{i1}, ..., x_{i5})^T$ of dummy covariates, with $x_{ij} = 1$ if car i was produced in period j and 0 otherwise. We consider a model of the form (3.1), with

$$\begin{aligned} m_i(t) &= m_0(t)F(\tau_i - t)\exp(\boldsymbol{x}_i'\boldsymbol{\beta}) \\ &= m_0(t)g_i(t), \end{aligned} \tag{5.3}$$

where $\boldsymbol{\beta} = (\beta_1, ..., \beta_5)^T$ is an unknown vector of regression parameters. We test the equality of the $M^{(j)}(t)$'s by testing $H : \boldsymbol{\beta} = \mathbf{0}$. One way to do this is to estimate $\boldsymbol{\beta}$ and $m_0(t)$ as described in section 3 (see (3.2) and (3.3)). We can then test H by using the statistic $W = k\hat{\boldsymbol{\beta}}^T[\text{Asvar}(\hat{\boldsymbol{\beta}})]^{-1}\hat{\boldsymbol{\beta}}$, where $\text{Asvar}(\hat{\boldsymbol{\beta}})$ is given by (3.10). If H is true W is approximately distributed as $\chi^2_{(5)}$. For the data in question we obtain $W = 85.6$ which gives a p-value much less than .001 and indicates very strong evidence against equality of the CMF's.

An alternative approach is to extend the methods of section 4 to deal with any number of processes. This may be done by considering the regression model (5.3) and defining the statistic $\boldsymbol{U}$ as the left hand side of (3.5) evaluated at $\boldsymbol{\beta} = \mathbf{0}$. The variance of $\boldsymbol{U}$ is estimated either by $\hat{B}_1$ of (3.8) or by (3.8) with the $\hat{\boldsymbol{B}}_{1i}$'s evaluated at $\boldsymbol{\beta} = \mathbf{0}$ instead of $\boldsymbol{\beta} = \hat{\boldsymbol{\beta}}$. If H is true, $k^{-1}\boldsymbol{U}^T\hat{B}_1^{-1}\boldsymbol{U}$ is approximately distributed as $\chi^2_{(5)}$; the first and second choices of variance estimate give observed values of 74.5 and 71.1, respectively, in good agreement with the preceding test.

We note that the model (5.3) is a reasonable basis for comparing the production periods, in view of the patterns in Figure 1. If desired, however, other tests can be devised and, in particular, tests that are not based on a model where the $M^{(j)}(t)$'s are constrained to be proportional to one another.

We conclude this example with a couple of remarks. First, the warranty plan that generated these data had one year and 12,000 mile limits, so some cars were not under warranty for a full year. The definition of $m(t)$ as the average number of claims per car at age t implicitly recognizes this fact, but it should be noted that $m(t)$ is not the average number of age t events that would generate a claim, provided there was no mileage limit. Second, it will be noted from Figure 1 that a few claims at ages over one year were allowed. This likewise does not create any special problems with the analysis.

6. CONCLUDING REMARKS

The methods in this paper are based on Poisson maximum likelihood estimates, along with robust moment-based variance estimates. The Poisson estimates are valid quite generally because they are generalized least squares, or quasi-likelihood, estimates. Similar methods have been used for count data by Breslow (1990), Thall and Vail (1990) and others; Stukel (1993) contains a good review. These methods apply to situations where only the numbers of events in different time intervals is observed, and not the precise event times.

An extension of this paper's methods would be to assume some covariance structure for the processes of events. For example, we might assume (for the discrete time case) that the $n_i(t)$'s are mutually independent with means $m_i(t)$ and variances $\sigma^2 m_i(t)$, where $\sigma^2 > 0$ is an additional parameter. Another approach is taken by Lawless (1987), Kalbfleisch et al. (1991) and others, who use mixed Poisson models that lead to the variance $m_i(t) + \sigma^2 m_i(t)^2$ for $n_i(t)$, with $n_i(s)$ and $n_i(t)$ being correlated for $s \neq t$. Both models have continuous time analogs. If an assumed variance structure is correct, more efficient estimation of parameters in the $m_i(t)$'s is possible, but in most cases would involve substantially more complicated computations. A better practical procedure would be to use the Poisson estimating equations of this paper and to consider the variance specification only when obtaining variance estimates. One advantage of having a variance specification is that interval prediction of future events becomes relatively simple. Variance estimates for parameters would, of course, be less robust than the ones given in this paper, but diagnostic checks on the variance and mean specifications are possible by examining residuals based on observed and expected counts (e.g. see Kalbfleisch et al. 1991).

Various other extensions to the methods in the paper are possible. In particular, smooth estimates of $m(t)$ may be developed, and the regression methods may readily be extended to deal with cumulative cost processes, as done by Nelson (1992) for the case of identically distributed processes. Finally, it would be of interest to compare the efficiency and robustness of these methods with ones based solely on Poisson or renewal models, in different situations.

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APPENDIX

1. Poisson Estimating Equations (3.2), (3.3)

If the $n_i(t)$'s are mutually independent Poisson random variables with means $m_i(t)$ depending on a parameter vector $\boldsymbol{\theta}$, then the log likelihood function for $\boldsymbol{\theta}$ in the discrete time case is

$$l(\boldsymbol{\theta}) = \sum_{i=1}^{k} \sum_{s=0}^{\tau} \delta_i(s)\{n_i(s) \log m_i(s) - m_i(s)\} + \text{constant},$$

giving estimating equations

$$\boldsymbol{U}(\boldsymbol{\theta}) = \frac{\partial l}{\partial \boldsymbol{\theta}} = \sum_{i=1}^{k} \sum_{s=0}^{\tau} \delta_i(s)\{\frac{n_i(s) - m_i(s)}{m_i(s)}\}\frac{\partial m_i(s)}{\partial \boldsymbol{\theta}} = \mathbf{0}. \tag{A.1}$$

When $m_i(t)$ is of the form (3.1) then $\boldsymbol{\theta} = (m(0), ..., m(\tau), \boldsymbol{\beta}^T)^T$ and (A.1) gives (3.2) and (3.3).

2. Robust Asymptotic Variances for $\hat{\boldsymbol{\beta}}$ and $\hat{M}_0(t)$

We consider first the estimate of $\boldsymbol{\beta}$, which may be obtained by solving (3.5). For convenience we will write $\boldsymbol{m} = (m(0), ..., m(\tau))^T$ for the vector of $m(t)$'s. Note that (3.5) can be written as

$$\begin{aligned} \boldsymbol{U}_1(\boldsymbol{\beta}) &= \sum_{i=1}^{k}\sum_{s=0}^{\tau} \boldsymbol{W}_i(\boldsymbol{\beta}, s)\delta_i(s)n_i(s) \\ &= \sum_{i=1}^{k} \boldsymbol{U}_{1i}(\boldsymbol{\beta}), \end{aligned} \tag{A.2}$$

where $\boldsymbol{W}_i(\boldsymbol{\beta}, s)$ is defined in (3.7). Conditional on the $\boldsymbol{x}_i(s)$'s, the $\boldsymbol{U}_{1i}(\boldsymbol{\beta})$'s are independent, so

$$\mathrm{Var}\{k^{-1/2}\boldsymbol{U}_1(\boldsymbol{\beta})\} = \frac{1}{k}\sum_{i=1}^{k}\sum_{s=0}^{\tau}\sum_{u=0}^{\tau} \delta_i(s)\delta_i(u)\boldsymbol{W}_i(\boldsymbol{\beta}, s)\boldsymbol{W}_i(\boldsymbol{\beta}, u)^T \mathrm{cov}[n_i(s), n_i(u)].$$

We assume that $\delta_\cdot(t) \to \infty$ for all t in $[0, \tau]$ as $k \to \infty$, and that $\mathrm{Var}\{k^{-1/2}\boldsymbol{U}_1(\boldsymbol{\beta})\}$ converges to a positive definite matrix $B_1(\boldsymbol{\beta}, \boldsymbol{m})$. We assume in addition that $\hat{\boldsymbol{\beta}}$ and $\hat{M}_0(t)$ are consistent estimates of $\boldsymbol{\beta}$ and $M_0(t)$. A discussion of precise conditions under which consistency holds is beyond the scope of this paper, but one situation where it is easily established is when the τ_i's can take on values only in a finite set, and the $\boldsymbol{x}_i(t)$'s are constant between successive such values. Under these conditions, $\mathrm{Var}\{k^{-1/2}\boldsymbol{U}_1(\boldsymbol{\beta})\}$ is consistently estimated by (3.8).

With mild regularity conditions it follows from standard asymptotic results for estimating equations (e.g. see White 1982 or Breslow 1990) that $\sqrt{k}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ is asymptotically normal with mean $\mathbf{0}$ and covariance matrix

$$V_1 = A_1(\boldsymbol{\beta}, \boldsymbol{m})^{-1} B_1(\boldsymbol{\beta}, \boldsymbol{m})[A_1(\boldsymbol{\beta}, \boldsymbol{m})^{-1}]^T, \tag{A.3}$$

where

$$\begin{aligned} A_1(\boldsymbol{\beta}, \boldsymbol{m}) &= E\{-\frac{1}{k}\frac{\partial \boldsymbol{U}_1(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}^T}\} \\ &= \frac{1}{k}\sum_{i=1}^{k}\sum_{s=0}^{\tau} \delta_i(s)\frac{\partial g_i(s)}{\partial \boldsymbol{\beta}} \boldsymbol{W}_i(\boldsymbol{\beta}, s)^T m_0(s). \end{aligned} \tag{A.4}$$

Under the conditions indicated above, (A.4) is consistently estimated by (3.9), and V_1 is consistently estimated by (3.10).

We remark that although the results here and later are derived in the discrete time framework they also apply, under suitable conditions, for continuous time processes.

The asymptotic covariance matrix for $(\hat{\boldsymbol{\beta}}, \hat{M}_0(t_1), ..., \hat{M}_0(t_k))$ for a fixed set of values $t_1, ..., t_k$ may also be readily obtained. For simplicity and since it suffices for many applications, we will outline the calculations only for $(\hat{\boldsymbol{\beta}}, \hat{M}_0(t))$, for a specified t value.

The estimates $\hat{\boldsymbol{\beta}}$ and $\hat{M}$ are the solution to estimating equations (3.5) along with (see (3.4))

$$k^{-1}U_2(\boldsymbol{\beta}, M_0(t)) = \sum_{s=0}^{t} \frac{n.(s)}{R(s;\boldsymbol{\beta})} - M_0(t) = 0. \tag{A.5}$$

By asymptotic results for estimating equations mentioned above, the asymptotic covariance matrix for $\sqrt{k}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}, \hat{M}_0(t) - M_0(t))$ is

$$V = A(\boldsymbol{\beta}, \boldsymbol{m})^{-1}B(\boldsymbol{\beta}, \boldsymbol{m})[A(\boldsymbol{\beta}, \boldsymbol{m})^{-1}]^T, \tag{A.6}$$

where B is the limiting covariance matrix for $k^{-1/2}(\boldsymbol{U}_1, U_2)$ and $A(\boldsymbol{\beta}, \boldsymbol{m})$ is the limit of

$$\frac{1}{k}E\begin{pmatrix} -\partial \boldsymbol{U}_1/\partial\boldsymbol{\beta} & -\partial \boldsymbol{U}_1/\partial M_0(t) \\ -\partial U_2/\partial\boldsymbol{\beta} & -\partial U_2/\partial M_0(t) \end{pmatrix}.$$

V is estimated consistently by $\hat{V} = \hat{A}^{-1}\hat{B}(\hat{A}^{-1})^T$, where

$$\hat{A} = \begin{pmatrix} \hat{A}_{11} & \mathbf{0} \\ \hat{\boldsymbol{A}}_{21} & 1 \end{pmatrix} \qquad \hat{B} = \begin{pmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & \hat{B}_{22} \end{pmatrix},$$

with $\hat{A}_{11}$ given by (3.9), $\hat{A}_{21}$ the $1 \times p$ matrix $\sum_{s=0}^{t} \hat{m}_0(s)\partial \log R(s;\hat{\boldsymbol{\beta}})/\partial\hat{\boldsymbol{\beta}}^T$, $\hat{B}_{11}$ given by (3.8), $\hat{\boldsymbol{B}}_{12} = \hat{\boldsymbol{B}}_{21}^T = \frac{1}{k}\sum_{i=1}^{k} \hat{\boldsymbol{B}}_{1i}\hat{C}_i$ and $\hat{B}_{22} = \frac{1}{k}\sum_{i=1}^{k} \hat{C}_i^2$, where $\hat{\boldsymbol{B}}_{1i}$ is the same as in (3.8) and

$$\hat{C}_i = k\sum_{s=0}^{t} \frac{\delta_i(s)}{R(s;\hat{\boldsymbol{\beta}})}\{n_i(s) - \hat{g}_i(s)\hat{m}_0(s)\}.$$

The upper left $p \times p$ submatrix of $\hat{V}$ is, of course, the same as (3.10).

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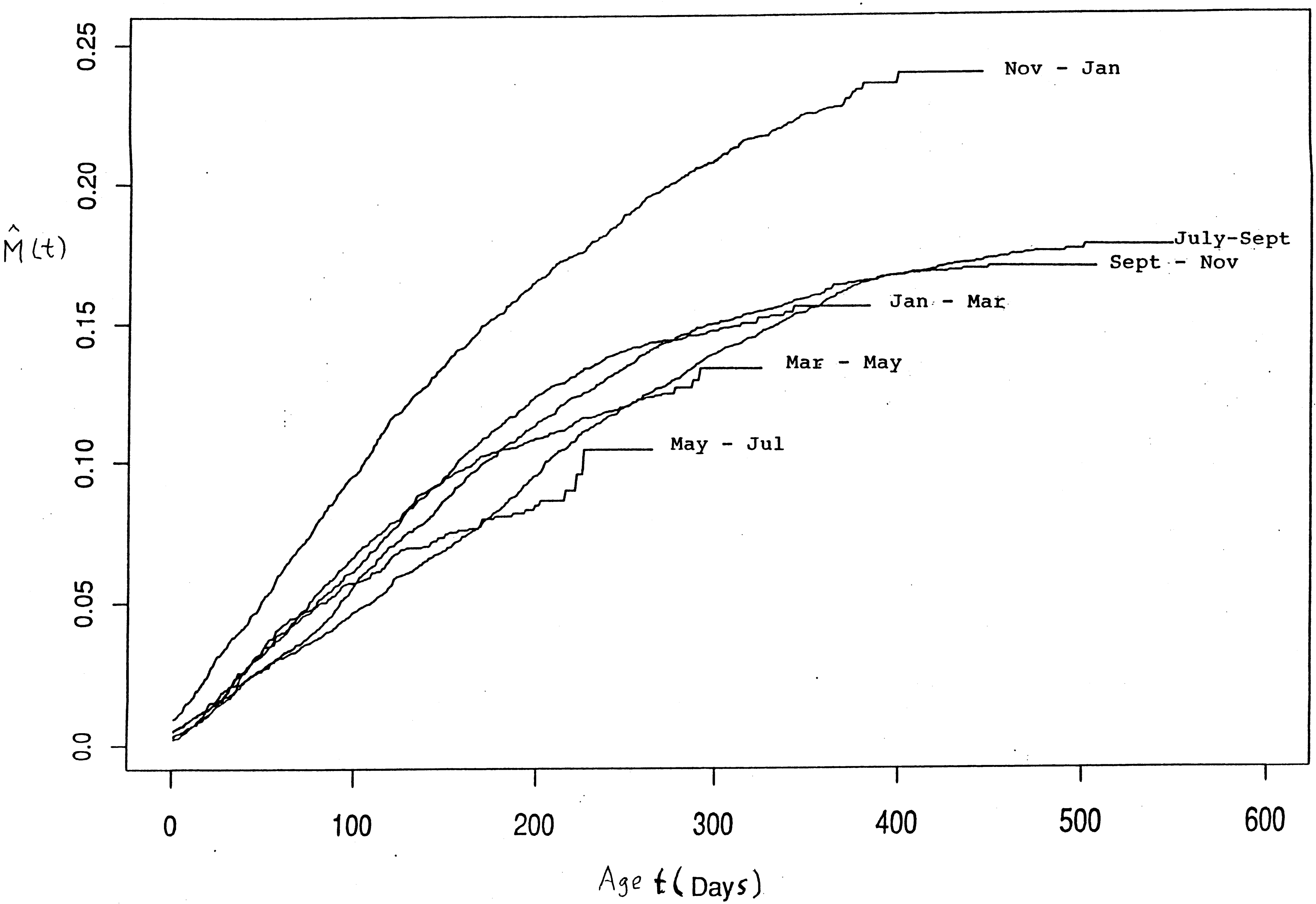
Warranty Claim Frequencies



Data Stratified on Production Date