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ABSTRACT

A general method for constructing supersaturated designs is proposed. It is proved that supersaturated designs produced by this method are optimal with respect to the $E(s^2)$ criterion of Booth & Cox (1962). Within this class of designs, further discrimination can be made by minimizing the pairwise correlations and using the generalized D and A criteria proposed by Wu (1993). Efficient designs of 8, 12, 16, 20, and 24 runs are constructed by following this approach.

Key words and phrases: A-efficiency; D-efficiency; Effect sparsity; Plackett-Burman design; Screening designs.

1. INTRODUCTION

Whenever the number of parameters of interest exceeds the number of runs of a design, the design is called supersaturated. For example, when m factors are investigated in a two-level experiment, a supersaturated design has less than $m + 1$ runs. A brief review of early work on supersaturated designs by Satterthwaite (1959), Watson (1961), and Booth & Cox (1962) can be found in Wu (1993).

After thirty dormant years, interest in supersaturated designs has recently been renewed and gaining momentum. Lin (1993) uses half fractions of Hadamard matrices to construct supersaturated designs, taking a column as the branching column and then constructing two half fractions according to the sign of that column. Wu (1993) augments Hadamard matrices by adding interaction columns. Generalized D and A criteria for assessing supersaturated designs are proposed by Wu. Though the methods of Lin and Wu produce some useful designs, they have the limitation that both depend on Hadamard matrices with certain three dimensional properties. (For details, see § 6.)

In this paper, we propose a general method for constructing supersaturated designs also through the use of Hadamard matrices. This method is more flexible in that any Hadamard matrix can be used for construction. We will demonstrate this flexibility by constructing supersaturated designs of 8 and 16 runs for which Lin's and Wu's methods fail. More significantly, the proposed method generates **optimal** supersaturated designs in the sense of minimizing the $E(s^2)$ criterion. The $E(s^2)$ of a supersaturated design having m columns

is defined as

$$E(s^2) = \sum_{1 \le i < j \le m} s_{ij}^2 / \binom{m}{2},\tag{1}$$

where $s_{ij} = \langle c_i, c_j \rangle$ is the inner product of the ith column c_i and the jth column c_j.

In § 2 we present the construction method and establish its optimality. In §§ 3–5 we construct supersaturated designs of 8, 12, 16, 20, and 24 runs. Comparison with Lin's and Wu's methods is made in § 6.

2. CONSTRUCTION METHOD AND ITS $E(s^2)$ OPTIMALITY

We motivate the construction method by a simple result. Let $c_0 = (1, \dots, 1)^t \in R^n$ and $R_0^n = \{x \in R^n \mid \langle x, c_0 \rangle = 0\}$, be the orthogonal complement of c_0. Consider the subset of R_0^n, $C = \{c \in R_0^n \mid$ the components of c are $\pm 1 \}$. Suppose $c_1, \dots, c_{n-1}$ in C are mutually orthogonal. Then for any c in C, we have

$$\sum_{j=1}^{n-1} |\langle c, c_j \rangle|^2 = n^2,\tag{2}$$

which follows from the fact that $c_1/\sqrt{n}, \dots, c_{n-1}/\sqrt{n}$ form an orthonormal basis.

Suppose that we want to start with a set of orthogonal columns $c_1, \dots, c_{n-1}$ in C to construct a supersaturated design by adding more columns $c_n, c_{n+1}, \dots, c_m$ from C. By (2), we have

$$\sum_{1 \le i < j \le m} s_{ij}^2 = (m - n + 1)n^2 + \sum_{n \le i < j \le m} s_{ij}^2.$$

Therefore the $E(s^2)$ value is minimized by requiring $c_n, c_{n+1}, \dots, c_m$ to be mutually orthogonal. We can fulfill this requirement only if $m - (n - 1) \le n - 1$, that is, $m \le 2(n - 1)$. If

3

more columns are to be added, they should also be orthogonal. This motivates the following construction method. Let $H_j = (c_0, c_{j1}, \ldots, c_{j(n-1)})$, $j = 1, \ldots, k$, be k Hadamard matrices of order n, i.e., the n columns in H_j are orthogonal. Then we may obtain a supersaturated design with $m = k(n-1)$ columns,

$$D = (c_{11}, \ldots, c_{1(n-1)}, c_{21}, \ldots, c_{2(n-1)}, \ldots, c_{k1}, \ldots, c_{k(n-1)}), \qquad (3)$$

which has $\mathrm{E}(s^2) = \binom{k}{2}(n-1)n^2 / \binom{m}{2}$. As the Hadamard matrices considered in this paper have c_0 as the first column, we will omit c_0 in presenting them.

A minimal requirement for the k Hadamard matrices is that none of the $k(n-1)$ columns is fully aliased with another. (Two columns c and d are said to be fully aliased if $c = \pm d$.) To ensure that D in (3) is a good supersaturated design, we also want each s_{ij}^2 in (1) to be small. This poses the problem of selecting appropriate Hadamard matrices $H_1, \ldots, H_k$, which will be discussed at the end of the section. We first study some theoretical aspects of supersaturated designs and prove the optimality of our construction method.

To make designs comparable, we standardize D in (3) to be

$$D^* = D/\sqrt{n}. \qquad (4)$$

Each column of D^* is a unit vector in R_0^n. The design D^* has

$$\mathrm{E}(s^2) = (n-1)\binom{k}{2} / \binom{m}{2} = (k-1)/(m-1).$$

Now consider any other competing design $X = (x_1, \ldots, x_m)$, where each x_j is a unit vector

4

in R_0^n. The following theorem gives a lower bound on the $E(s^2)$ value and establishes the $E(s^2)$ optimality of D^*.

Theorem 1. (i). *For any $m \geq n$ and the design $X = (x_1, \ldots, x_m)$, we have*

$$E(s^2) \geq \frac{m - n + 1}{(m-1)(n-1)}. \tag{5}$$

(ii). *For $m = k(n-1)$, D^* are optimal according to the $E(s^2)$ criterion.*

Proof. Let $A = X^t X$. From $\text{tr}(A^2) = \text{tr}(A^t A) = \sum s_{ij}^2$, where the summation is over all i, j, we have

$$E(s^2) = (\text{tr}(A^2) - m)/2 \binom{m}{2}. \tag{6}$$

It can easily be checked that $\text{tr}(A^2) = \lambda_{(1)}^2 + \cdots + \lambda_{(m)}^2$, where $\lambda_{(1)} \geq \cdots \geq \lambda_{(m)}$ are the ordered eigenvalues of A. Since X has rank at most $n - 1$ and A has the same rank as X, we have $\lambda_{(i)} = 0$ for $i \geq n$. Thus $\text{tr}(A^2) = \lambda_{(1)}^2 + \cdots + \lambda_{(n-1)}^2$. We also have

$$\sum_{i=1}^{n-1} \lambda_{(i)} = \sum_{i=1}^{m} \lambda_{(i)} = \text{tr}(A) = m.$$

Under this constraint, $\lambda_{(1)}^2 + \cdots + \lambda_{(n-1)}^2$ is minimized by taking $\lambda_{(1)} = \cdots = \lambda_{(n-1)} = m/(n-1)$. Therefore $\text{tr}(A^2) \geq (n-1)m^2/(n-1)^2 = m^2/(n-1)$. Finally, by (6), we have

$$E(s^2) \geq (m^2/(n-1) - m)/m(m-1) = \frac{m - n + 1}{(m-1)(n-1)},$$

which proves Theorem 1(i). The lower bound in (5) becomes $(k-1)/(m-1)$ for $m = k(n-1)$, which is attained by D^* in (4). This proves Theorem 1(ii).

To construct a supersaturated design with $m = (k-1)(n-1) + j$ columns, $1 \leq j \leq n-2$, we may simply delete the last $(n - 1 - j)$ columns from D^*. Though the resulting design

5

does not achieve the lower bound in (5), it has $E(s^2)$ value very close to the lower bound. It is not known whether this design is still $E(s^2)$-optimal.

We now discuss the choice of appropriate Hadamard matrices for constructing supersaturated designs. For a given Hadamard matrix, we may switch the signs for any column, or permute the columns or rows to obtain many other equivalent Hadamard matrices. It is obvious that only permuting the rows needs to be considered for our purpose. If there are several nonequivalent Hadamard matrices of a given order, we can generate all Hadamard matrices of a given order by permuting the rows of each of the nonequivalent matrices.

To find k Hadamard matrices such that each s_{ij}^2 in (1) is small, we utilize equation (2). It is known that except for $n = 1, 2$, the integer n must be a multiple of 4. Suppose $n \geq 4$, and $n = 4t$. It can easily be shown that $\langle d_1, d_2 \rangle$ is also a multiple of 4, for any $d_1, d_2 \in C$. Let $\langle c, c_j \rangle = 4t_j$. Then (2) reduces to

$$\sum_{j=1}^{n-1} (t_j/t)^2 = 1. \tag{7}$$

The correlation of c and c_j, t_j/t, can only take the values $0, \pm 1/t, \ldots, \pm(t-1)/t, \pm 1$.

Formula (7) is used as a guide in selecting appropriate Hadamard matrices in §§ 3—5. We do not use the conservative criterion $\max s_{ij}^2$, as it will exclude many useful designs.

6

3. SUPERSATURATED DESIGNS OF 8 RUNS

For $n = 8$, there are, in total, $\binom{8}{4} = 70$ columns in C. Once a column is included in a design, its negative must be ruled out. With the provision that a column and its negative are considered identical, there are 35 columns to be considered. These 35 columns together give a supersaturated design of which any two columns are either orthogonal or have correlation $\pm 1/2$, as $t = 2$ in (7). Simple enumeration shows that among the $\binom{35}{2} = 595$ pairs of columns, 315 pairs are orthogonal and the remaining 280 pairs have correlation $\pm 1/2$. Thus for this design $E(s^2) = 280 \times 16/595 = 7.53$.

Using the method proposed in §2, we can construct supersaturated designs with less than 35 columns. Because all Hadamard matrices of order 8 are equivalent, we can take any of them as a starting Hadamard matrix. Permuting the rows of this matrix gives all Hadamard matrices needed for constructing supersaturated designs. The one we choose, denoted by H_1, is given by the first seven columns of the design D in Table 1.

(Table 1)

By permuting the rows of H_1, we can easily find another matrix H_2 such that no column of H_2 is fully aliased with any column of H_1. Therefore we obtain a supersaturated design $D_1 = (H_1, H_2)$ with 14 columns, which has $E(s^2) = 4.92$. We again permute the rows of H_1 to look for an H_3 such that no column of H_3 is fully aliased with any column of D_1. Computer search confirms that no such H_3 exists for any choice of H_2. Because of this fact, we have to modify the previous construction method.

Recall that there are 35 columns in C. Putting the 14 columns in D_1 aside, we denote the set of the remaining 21 columns by C'. Since there do not exist seven columns in C' that are mutually orthogonal, we proceed to find all the subsets of six mutually orthogonal columns. It would be desirable to have three such subsets that are mutually exclusive. Again this is impossible for any choice of H_2. The best we can get is to divide the 21 columns into four groups H_3, H_4, H_5, and H_6 as in Table 1, where H_3 and H_4 have six columns, H_5 has five, and H_6 four, and the columns within each H_i are orthogonal. Write $D = (H_1, H_2, H_3, H_4, H_5, H_6)$. When a design with m columns, $m \leq 20$, is needed, we simply take the first m columns from D. Permuting the columns of H_3 does not make any difference. However, for $m \geq 21$, the order of columns in H_j, $j = 4, 5, 6$, matters. Thus we arrange the six columns of H_4 in the order of increasing correlation with $D_2 = (H_1, H_2, H_3)$. That is, the column a_1 in H_4 giving the smallest $\sum |\langle a_1, d \rangle|^2$, comes first, the column a_2 in H_4 giving the second smallest $\sum |\langle a_2, d \rangle|^2$ comes second, and so on, where the summations are over $d \in D_2$. The orders of columns in H_5 and H_6 are arranged accordingly. ¿From this arrangement, a design with m columns can be obtained by taking the first m columns of D, and has the smallest $E(s^2)$ value.

4. SUPERSATURATED DESIGNS OF 12 RUNS

All Hadamard matrices of order 12 are equivalent. We take $H_1 = (c_1, \ldots, c_{11})$ to be the one given by Paley (1933), which is essentially the same as the Plackett-Burman (1946)

design of 12 runs. For $n = 12$, (7) becomes

$$\sum_{j=1}^{11}(t_j/3)^2 = 1. \tag{8}$$

Note that $t_j/3$ is the correlation of c and c_j. ¿From (8), we see that the relationship of a column c with the matrix $H_1 = (c_1, \ldots, c_{11})$ must be one of the four types:

(i) c is fully aliased with one column of H_1, and orthogonal to the rest,

(ii) c has correlation $\pm 2/3$ with two columns of H_1 and correlation $\pm 1/3$ with one column of H_1, and is orthogonal to the rest,

(iii) c has correlation $\pm 2/3$ with one column of H_1 and correlation $\pm 1/3$ with five columns of H_1, and is orthogonal to the rest,

(iv) c has correlation $\pm 1/3$ with nine columns of H_1, and is orthogonal to the rest.

We now show that type (ii) is impossible. Suppose, without loss of generality, that c has correlation $2/3$ with c_1 and c_2, that is, $\langle c, c_1 \rangle = \langle c, c_2 \rangle = 8$. Let $c = (a_1, a_2, \ldots, a_n)^t$, $c_1 = (a_{11}, a_{21}, \ldots, a_{n1})^t$, and $c_2 = (a_{12}, a_{22}, \ldots, a_{n2})^t$, where $n = 12$. Then we have

$$\langle c, c_1 \rangle = \sum_{i=1}^{12} a_i a_{i1} = 2 \sum_{a_i=+1} a_{i1},$$

which implies that the sum of a_{i1} over i with $a_i = +1$ is 4. Therefore among the six a_{i1}'s with $a_i = +1$, five are $+1$, and the remaining one is -1. A similar result holds among the six a_{i2}'s with $a_i = +1$. Thus at least four of the six pairs in $\{(a_{i1}, a_{i2}) | a_i = +1\}$ are $(+1, +1)$, which contradicts the definition of orthogonality between c_1 and c_2.

Let $H_2 = (d_1, \ldots, d_{11})$ be another Hadamard matrix of order 12. For each d_j, we would like its relationship with H_1 to be of type (iv). The following result shows that no such Hadamard matrix H_2 exists.

Lemma 1. *Let s be the number of columns of H_2 having their relationships with H_1 to be of type (iv). Then $s \leq 10$, and in addtion, for $6 \leq s \leq 10$ at least one column of H_2 is fully aliased with some column of H_1.*

Its proof is given in the Appendix.

We now have two ways to proceed. One way is to allow no pair of columns to have correlation $\pm 2/3$, in which case one column of H_2 must be fully aliased with some column of H_1. Removing that column of H_2, we may obtain a design with 21 columns, which consists of 11 columns of H_1 and 10 columns of H_2. This is essentially equivalent to Wu's construction method (1993) which simply takes $H_2 = (c_1, c_1c_2, c_1c_3, \ldots, c_1c_{11})$, where $(c_1, \ldots, c_{11}) = H_1$, and c_ic_j is the component-wise product of c_i and c_j. If more columns are needed, we should find an H_3 such that for each H_j, $j = 1, 2$, one column of H_3 is fully aliased with some column of H_j and each of the remaining 10 columns of H_3 has its relationship with H_j to be of type (iv). Two columns of H_3 have to be removed in constructing a supersaturated design, and so we can arrive at a design having 30 columns of which 11 columns are from H_1, 10 columns from H_2, and 9 columns from H_3. This is again equivalent to Wu's method which takes $H_3 = (c_2, c_2c_1, c_2c_3, \ldots, c_2c_{11})$. If this procedure is continued, it is seen that we can construct a supersaturated design with $11 + 10 + \cdots + 2 + 1 = 66$ columns. Although our method can lead to a design essentially the same as that of Wu (1993), Wu's method

has the virtue of giving explicit expressions for H_2, H_3, and so on. On the other hand, the present argument serves to provide an alternative justification for Wu's designs.

The other way is to permit pairs of columns to have correlation $\pm 2/3$. In this case, the best is to take $s = 5$ in finding an H_2, that is, five columns of H_2 are of type (iv) and the remaining six columns of type (iii). Simply by permuting the rows of H_1, we can easily find such an H_2. We thus obtain a supersaturated design (H_1, H_2) with 22 columns, and its $E(s^2) = 6.86$ is smaller than that of Wu's design. As a trade-off, (H_1, H_2) has six pairs of columns with correlation $\pm 2/3$.

Lin's (1993) design with 22 columns has the same $E(s^2)$ value, 6.86, but enjoys the advantage of having no pair of columns with correlation $\pm 2/3$. However, among all pairs of columns of (H_1, H_2), only 2.6% (i.e. six pairs) have correlation $\pm 2/3$. The design (H_1, H_2) has the following advantage not shared by Lin's design. Its 22 columns consists of two sets of 11 mutually orthogonal columns. If the 22 factors to be studied in an experiment may be naturally divided into two groups of 11 factors each, such that the small number of significant factors, i.e. those with non-zero effects, must appear in the same group, (H_1, H_2) can actually provide an orthogonal design in this situation. These remarks, in a slightly generalized form, apply to any supersaturated design constructed by the method in § 2.

We have found 10 other Hadamard matrices by permuting the rows of H_1. Altogether we have 12 Hadamard matrices $H_1, H_2, \ldots, H_{12}$ satisfying the condition that for any column $c \in H_i$, its relationship with any H_j, $j \neq i$, is of type (iii) or (iv). Thus we obtain a supersaturated design with 132 columns $D = (H_1, H_2, \ldots, H_{12})$, which has $E(s^2) = 12.09$.

The proportions of pairs of columns having correlation 0, $\pm 1/3$, and $\pm 2/3$ are 43.89%, 49.62%, and 6.49%, respectively.

The matrix H_1 and the corresponding permutations $p_2, \ldots p_{12}$ for generating $H_2, \ldots, H_{12}$ are given in Table 2; the matrix H_i, $2 \leq i \leq 12$, can be obtained by $p_i = (p_{i1}, p_{i2}, \ldots, p_{i12})$, where the jth row of H_i is the p_{ij}th row of H_1.

(Table 2)

We conclude this section by making some recommendations on the use of 12 run supersaturated designs. For $12 \leq m \leq 21$, Wu's designs are recommended. For $m = 22$, the user can employ either Lin's design or ours depending on the situation. For $23 \leq m \leq 66$, both Wu's design and ours are useful, and it appears that each complements the other. For $m \geq 67$, only our designs are available.

5. SUPERSATURATED DESIGNS OF 16, 20 AND 24 RUNS

There are precisely five nonequivalent Hadamard matrices of order 16 as shown by Hall (1961). We have used all of them to construct supersaturated designs, but no advantage over the use of a single matrix shows up. We hence only use the Hadamard matrix H_1 corresponding to a regular fractional factorial design. From (7), it is easy to see that any 16-run supersaturated design with H_1 as its first 15 columns must have some pairs of columns with correlation $\pm 1/2$ or higher. By permuting the rows of H_1, we have found two other Hadamard matrices, H_2 and H_3. The correlation matrix of H_i and H_j, $1 \leq i < j \leq 3$, has an interesting pattern that each of its rows or columns has two entries being $\pm 1/2$, eight

being $\pm 1/4$, and five being 0. The three matrices H_1, H_2 and H_3 together give a design with 45 columns and $E(s^2) = 11.64$. The proportions of pairs of columns with correlations $0, \pm 1/4$, and $\pm 1/2$ are 54.55%, 36.36%, and 9.09%, respectively. The matrix H_1 and the corresponding permutations for H_2 and H_3 are given in Table 3.

(Table 3)

To construct supersaturated designs of 20 runs, we only use one of the three nonequivalent Hadamard matrices of order 20, H_1, the one constructed by Paley (1933). We have obtained two other Hadamard matrices H_2 and H_3 by permuting the rows of H_1. Two columns of H_2 have correlation $\pm 3/5$ with H_1, and nine columns of H_3 have correlation $\pm 3/5$ with (H_1, H_2). The columns of H_2 are so arranged that the last two columns have correlation $\pm 3/5$ with H_1. A similar arrangement is made for the columns of H_3. The $E(s^2)$ value of the design $D = (H_1, H_2, H_3)$ is 14.29.

It is interesting to compare D with Wu's 20-run design. For $m = 36$, Wu's design has the same $E(s^2)$ value as D, but has 17 pairs of columns with correlation $\pm 3/5$. For $m = 38$, Wu's design has a higher $E(s^2)$ value than D, and has 19 pairs of columns with correlation $\pm 3/5$. On the other hand we can obtain a design with 46 columns, in which no pair of columns has correlation $\pm 3/5$, by discarding the last two columns of H_2, and the last nine columns of H_3. The matrix H_1 and the corresponding permutations for H_2 and H_3 are given in Table 4.

(Table 4)

For the construction of 24-run designs, we permute the rows of the Hadamard matrix, H_1, given in Paley (1933), to obtain two other Hadamard matrices H_2 and H_3. The design

13

$D = (H_1, H_2, H_3)$ has $E(s^2)$ value 14.77, and the proportions of pairs of columns with correlations $0, \pm 1/6, \pm 1/3,$ and $\pm 1/2$ are 53.15%, 31.07%, 13.43% and 2.34%, respectively. The matrix H_1 and the corresponding permutations for H_2 and H_3 are given in Table 5.

(Table 5)

6. COMPARISON WITH OTHER METHODS

In §§4 and 5 we have made specific comparisons between our designs and those of Lin (1993) and Wu (1993). Here we make some general remarks on the three methods.

One common feature of the three methods is that they all make use of Hadamard matrices. But they diverge in the flexibility and the quality of designs produced. Both Lin's and Wu's methods rely on some three dimensional properties of an individual Hadamard matrix. To see this, let $H = (c_{ij}) = (c_1, \ldots, c_{n-1})$ be a Hadamard matrix of order n. Suppose we choose c_i as a branching column for Lin's method. The inner product of any two columns of Lin's design can be shown to be

$$\sum_{c_{pi}=+1} c_{pj} c_{pk} = 2^{-1} \sum_{p=1}^{n} c_{pi} c_{pj} c_{pk} = 2^{-1} I(i,j,k),$$

where $I(i,j,k)$ is defined to be $\sum c_{pi} c_{pj} c_{pk}$, the summation being over $p = 1, \ldots, n$. Thus Lin's method produces $n - 2$ distinct columns if and only if there exists an i such that no $I(i,j,k)$, where $j \neq k$ and are different from i, is equal to $\pm n$. A sufficient condition for this requirement is that n is not a multiple of 8. When $I(i,j,k) = \pm n$, the three columns c_i, c_j, c_k are fully aliased in the sense that $c_i c_j c_k = \pm c_0$. Of course, for Lin's method to produce a good supersaturated design, it is also necessary that each $I(i,j,k)$ is small in

14

absolute value. The analysis also holds for Wu's method. The only difference is that Wu's method in addition relies on some four dimensional properties of a Hadamard matrix for constructing supersaturated designs with more than $2n - 3$ columns. Details can be found in the 1992 Waterloo Ph.D. Thesis of B. Tang.

Using Lin's or Wu's methods, we have to evaluate each $I(i, j, k)$, and even each $I(i, j, k, l)$, which is defined similarly, to determine whether a given Hadamard matrix is suitable for constructing a supersaturated design. If not, a nonequivalent Hadamard matrix should be used instead, and computations of $I(i, j, k)$ and $I(i, j, k, l)$ need to be carried out again. In contrast, our method requires no knowledge of the three or four dimensional properties of a Hadamard matrix. Any of the nonequivalent Hadamard matrices can be used for construction. The construction of supersaturated designs of 8 and 16 runs demonstrates the flexibility of our method. For the same cases, Lin's and Wu's methods cannot be applied.

In Table 6, using the $E(s^2)$ criterion we compare the supersaturated designs of 12 and 24 runs constructed by the three methods. For $m \leq 2n - 2$, our designs reach the minimum $E(s^2)$ values of Lin's and Wu's designs, and for $m \geq 2n - 1$, our designs have lower $E(s^2)$ values than Wu's designs. Lin's design is not available for $m \geq 2n - 1$.

(Table 6)

At the screening stage of an experiment, very often there are a large number of factors to be studied. However, in many situations the number of factors with significant effects is small. The idea of supersaturated designs stems from this assumption of effect sparsity. Denote the number of significant factors by f. Then the $E(s^2)$ value of a design provides a

15

measure of efficiency for $f = 2$. Since it does not measure the dependence among three or more columns, for further comparison of designs, we use the D_f and A_f criteria (Wu, 1993),

$$D_f = \sum_i |\frac{1}{n} X_i^t X_i|^{1/f} / \binom{m}{f} , \tag{9}$$

$$A_f = \sum_i \frac{1}{f} tr(\frac{1}{n} X_i^t X_i)^{-1} / \binom{m}{f} , \tag{10}$$

where the summations are over all possible $n \times f$ submatrices X_i of the full design matrix. These are natural extensions of the D and A criteria and provide measures of efficiency for $f \geq 3$. In Figure 1 we plot the values of D_f and A_f, $f = 3, 4$, for the 8-run supersaturated design given in § 3.

(Figure 1)

In Figure 1, we do not give the values of D_4 and A_4 for $m \geq 15$, because for $m \geq 15$, some of the matrices $X_i^t X_i$ in (9) and (10) become singular. Therefore using this design for $m \geq 15$ calls for caution if there are more than three significant factors.

In Figure 2 we plot the D_f and A_f values of the 12-run design given in § 4, with the number of factors m ranging from 12 to 66 for $f = 3$, and from 12 to 30 for $f = 4$. (For $f = 4$ and $m \geq 31$, some matrices $X_i^t X_i$ in (9) and (10) become singular.) Compared with Wu's design, the D_f values of our design are slightly lower, and the A_f values slightly higher. This shows that in terms of the D_f and A_f criteria, Wu's design is superior.

(Figure 2)

Finally we compute the D_f and A_f values of the 20-run design that consists of the 19

16

columns of H_1, columns 1—17 of H_2, and columns 1—6 of H_3, where H_1, H_2, H_3 are given in Table 4. The D_f values of this design, $f = 3, 4, 5$, are higher than those of Wu's 20-run design, and the A_f values are lower. Thus in terms of the D_f and A_f criteria, our design is better. The D_f and A_f values are plotted in Figure 3.

(Figure 3)

APPENDIX: PROOF OF LEMMA 1

Consider the matrix $B = (3\rho_{ij})$, where ρ_{ij} is the correlation of c_i and d_j. It is easy to show that B is an orthogonal matrix. Without loss of generality, suppose the first s columns are of type (iv). Thus each of these s columns has two entries being 0 and nine being ± 1. For any two of the s columns, $a = (a_1, \ldots, a_{11})$ and $b = (b_1, \ldots, b_{11})$, it is necessary that there is exactly one pair out of the 11 pairs, (a_i, b_i), $i = 1, \ldots, 11$, being $(0, 0)$, since a and b are orthogonal. Without loss of generality, assume the first column of B has two 0's in its first two entries. Then for any $2 \le j \le s$, one of the two top entries of the jth column of B must be 0, and the other ± 1. It is tedious but straightforward to show that for $s \ge 4$ the 0's in the top two rows and columns 2 to s of B must appear in the same row, say the first row, for otherwise columns 2 to s cannot be mutually orthogonal. So there are at least s zeros in the first row of B. Note that the relationship of c_1 with H_2 must also be of one of three types (i), (iii), and (iv). Thus $s = 11$ would imply that the first row of B consists entirely of 0's, which is impossible. It is also clear that $s \ge 6$ implies that the relationship of c_1 with H_2 must be of type (i). This finishes the proof.

17

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			H_1								H_2			
-1	-1	-1	1	1	1	-1		-1	-1	-1	1	1	1	-1
-1	-1	1	1	-1	-1	1		-1	-1	1	1	-1	-1	1
-1	1	-1	-1	1	-1	1		-1	1	-1	-1	1	-1	1
-1	1	1	-1	-1	1	-1		1	-1	-1	-1	-1	1	1
1	-1	-1	-1	-1	1	1		1	-1	1	-1	1	-1	-1
1	-1	1	-1	1	-1	-1		1	1	-1	1	-1	-1	-1
1	1	-1	1	-1	-1	-1		-1	1	1	-1	-1	1	-1
1	1	1	1	1	1	1		1	1	1	1	1	1	1

			H_3							H_4		
-1	-1	-1	1	1	1		1	1	-1	-1	-1	-1
-1	-1	1	1	-1	-1		1	-1	1	-1	1	-1
-1	1	-1	-1	1	-1		1	-1	-1	1	-1	1
1	-1	-1	-1	-1	1		-1	-1	1	1	-1	-1
1	1	-1	1	-1	-1		-1	-1	-1	-1	1	1
-1	1	1	-1	-1	1		-1	1	-1	1	1	-1
1	-1	1	-1	1	-1		-1	1	1	-1	-1	1
1	1	1	1	1	1		1	1	1	1	1	1

		H_5						H_6	
1	1	-1	-1	-1		1	1	-1	-1
1	-1	1	-1	-1		-1	-1	1	1
-1	-1	1	1	-1		1	-1	1	-1
1	-1	-1	1	1		1	-1	-1	1
-1	1	-1	1	-1		-1	1	-1	1
-1	-1	-1	-1	1		-1	1	1	-1
-1	1	1	-1	1		-1	-1	-1	-1
1	1	1	1	1		1	1	1	1

Table 1. A supersaturated design of 8 runs, $D = (H_1, H_2, H_3, H_4, H_5, H_6)$.

$$H_1$$

1	1	1	1	1	1	1	1	1	1	1
−1	1	−1	1	1	1	−1	−1	−1	1	−1
−1	−1	1	−1	1	1	1	−1	−1	−1	1
1	−1	−1	1	−1	1	1	1	−1	−1	−1
−1	1	−1	−1	1	−1	1	1	1	−1	−1
−1	−1	1	−1	−1	1	−1	1	1	1	−1
−1	−1	−1	1	−1	−1	1	−1	1	1	1
1	−1	−1	−1	1	−1	−1	1	−1	1	1
1	1	−1	−1	−1	1	−1	−1	1	−1	1
1	1	1	−1	−1	−1	1	−1	−1	1	−1
−1	1	1	1	−1	−1	−1	1	−1	−1	1
1	−1	1	1	1	−1	−1	−1	1	−1	−1

H_2	p_2	1	2	3	4	10	6	5	7	11	8	12	9
H_3	p_3	1	3	9	7	2	4	8	11	6	5	10	12
H_4	p_4	1	2	3	4	5	7	12	11	8	6	9	10
H_5	p_5	1	3	9	7	5	6	2	8	10	11	12	4
H_6	p_6	1	3	9	7	10	8	12	4	6	2	11	5
H_7	p_7	1	2	3	4	7	12	11	10	5	6	8	9
H_8	p_8	1	3	5	8	2	9	7	6	4	11	12	10
H_9	p_9	1	3	5	8	2	4	12	7	11	10	6	9
H_{10}	p_{10}	1	3	5	8	2	6	11	12	7	9	4	10
H_{11}	p_{11}	1	3	5	8	4	6	11	7	12	2	10	9
H_{12}	p_{12}	1	3	5	8	10	7	6	11	12	4	2	9

Table 2. The Hadamard matrix of H_1 used to generate supersaturated designs of 12 runs, and the row permutations corresponding to the matrices, $H_2, H_3, \ldots, H_{12}$.

$$H_1$$

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1
1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1
1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	1	1	1	1
1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1
1	−1	−1	1	1	−1	−1	−1	−1	1	1	−1	−1	1	1
1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1	1	1
1	−1	−1	−1	−1	1	1	−1	−1	1	1	1	1	−1	−1
−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1
−1	1	−1	1	−1	1	−1	−1	1	−1	1	−1	1	−1	1
−1	1	−1	−1	1	−1	1	1	−1	1	−1	−1	1	−1	1
−1	1	−1	−1	1	−1	1	−1	1	−1	1	1	−1	1	−1
−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1
−1	−1	1	1	−1	−1	1	−1	1	1	−1	−1	1	1	−1
−1	−1	1	−1	1	1	−1	1	−1	−1	1	−1	1	1	−1
−1	−1	1	−1	1	1	−1	−1	1	1	−1	1	−1	−1	1

H_2	p_2	1	2	5	8	3	9	11	4	10	7	12	6	13	14	15	16
H_3	p_3	1	2	5	8	11	3	4	9	6	10	7	12	13	14	15	16

Table 3. The Hadamard matrix of H_1 used to generate supersaturated designs of 16 runs, and the row permutations corresponding to the matrices H_2 and H_3.

+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
−	−	+	+	−	−	−	−	+	−	+	−	+	+	+	+	−	−	+
+	−	−	+	+	−	−	−	−	+	−	+	−	+	+	+	+	−	−
−	+	−	−	+	+	−	−	−	−	+	−	+	−	+	+	+	+	−
−	−	+	−	−	+	+	−	−	−	−	+	−	+	−	+	+	+	+
+	−	−	+	−	−	+	+	−	−	−	−	+	−	+	−	+	+	+
+	+	−	−	+	−	−	+	+	−	−	−	−	+	−	+	−	+	+
+	+	+	−	−	+	−	−	+	+	−	−	−	−	+	−	+	−	+
+	+	+	+	−	−	+	−	−	+	+	−	−	−	−	+	−	+	−
−	+	+	+	+	−	−	+	−	−	+	+	−	−	−	−	+	−	+
+	−	+	+	+	+	−	−	+	−	−	+	+	−	−	−	−	+	−
−	+	−	+	+	+	+	−	−	+	−	−	+	+	−	−	−	−	+
+	−	+	−	+	+	+	+	−	−	+	−	−	+	+	−	−	−	−
−	+	−	+	−	+	+	+	+	−	−	+	−	−	+	+	−	−	−
−	−	+	−	+	−	+	+	+	+	−	−	+	−	−	+	+	−	−
−	−	−	+	−	+	−	+	+	+	+	−	−	+	−	−	+	+	−
−	−	−	−	+	−	+	−	+	+	+	+	−	−	+	−	−	+	+
+	−	−	−	−	+	−	+	−	+	+	+	+	−	−	+	−	−	+
+	+	−	−	−	−	+	−	+	−	+	+	+	+	−	−	+	−	−
−	+	+	−	−	−	−	+	−	+	−	+	+	+	+	−	−	+	−

H_2	p_2	12	2	8	3	19	6	10	14	17	4	16	5	7	18	11	9	13	20	15	1
H_3	p_3	9	4	5	17	7	8	10	3	16	14	2	15	6	1	18	12	13	11	19	20

Table 4. The Hadamard matrix of H_1 used to generate supersaturated designs of 20 runs, and the row permutations corresponding to the matrices H_2 and H_3.

H_1

```
+ + + + + + + + + + + + + + + + + + + + + + +
- - - - - + - + - - + + - - + + - + - + + + +
+ - - - - - + - + - - + + - - + + - + - + + +
+ + - - - - - + - + - - + + - - + + - + - + +
+ + + - - - - - + - + - - + + - - + + - + - +
+ + + + - - - - - + - + - - + + - - + + - + -
- + + + + - - - - - + - + - - + + - - + + - +
+ - + + + + - - - - - + - + - - + + - - + + -
- + - + + + + - - - - - + - + - - + + - - + +
+ - + - + + + + - - - - - + - + - - + + - - +
+ + - + - + + + + - - - - - + - + - - + + - -
- + + - + - + + + + - - - - - + - + - - + + -
- - + + - + - + + + + - - - - - + - + - - + +
+ - - + + - + - + + + + - - - - - + - + - - +
+ + - - + + - + - + + + + - - - - - + - + - -
- + + - - + + - + - + + + + - - - - - + - + -
- - + + - - + + - + - + + + + - - - - - + - +
+ - - + + - - + + - + - + + + + - - - - - + -
- + - - + + - - + + - + - + + + + - - - - - +
+ - + - - + + - - + + - + - + + + + - - - - -
- + - + - - + + - - + + - + - + + + + - - - -
- - + - + - - + + - - + + - + - + + + + - - -
- - - + - + - - + + - - + + - + - + + + + - -
- - - - + - + - - + + - - + + - + - + + + + -
```

| H_2 | p_2 | 3 | 12 | 7 | 17 | 9 | 13 | 16 | 15 | 6 | 10 | 23 | 24 | 22 | 11 | 20 | 4 | 5 | 8 | 14 | 18 | 21 | 1 | 19 | 2 |
| H_3 | p_3 | 10 | 4 | 2 | 3 | 21 | 11 | 19 | 15 | 18 | 7 | 23 | 9 | 8 | 20 | 22 | 5 | 14 | 6 | 1 | 13 | 24 | 17 | 12 | 16 |

Table 5. The Hadamard matrix of H_1 used to generate supersaturated designs of 24 runs, and the row permutations corresponding to the matrices H_2 and H_3.

$n = 12$			
m	Lin	Wu	the authors
16	6.80	6.00	6.00
18	6.80	6.59	6.59
21	6.86	6.86	6.86
22	6.86	7.40	6.86
24	—	8.17	7.82
55	—	10.90	10.67
66	—	11.08	11.08
110	—	—	11.89

$n = 24$			
m	Lin	Wu	the authors
30	11.59	9.27	9.27
45	12.80	12.80	12.80
46	12.80	13.29	12.80

Table 6. $E(s^2)$ values of designs due to Lin (1993), Wu (1993), and the authors.

(When no design is available, it is indicated by − in the table.)
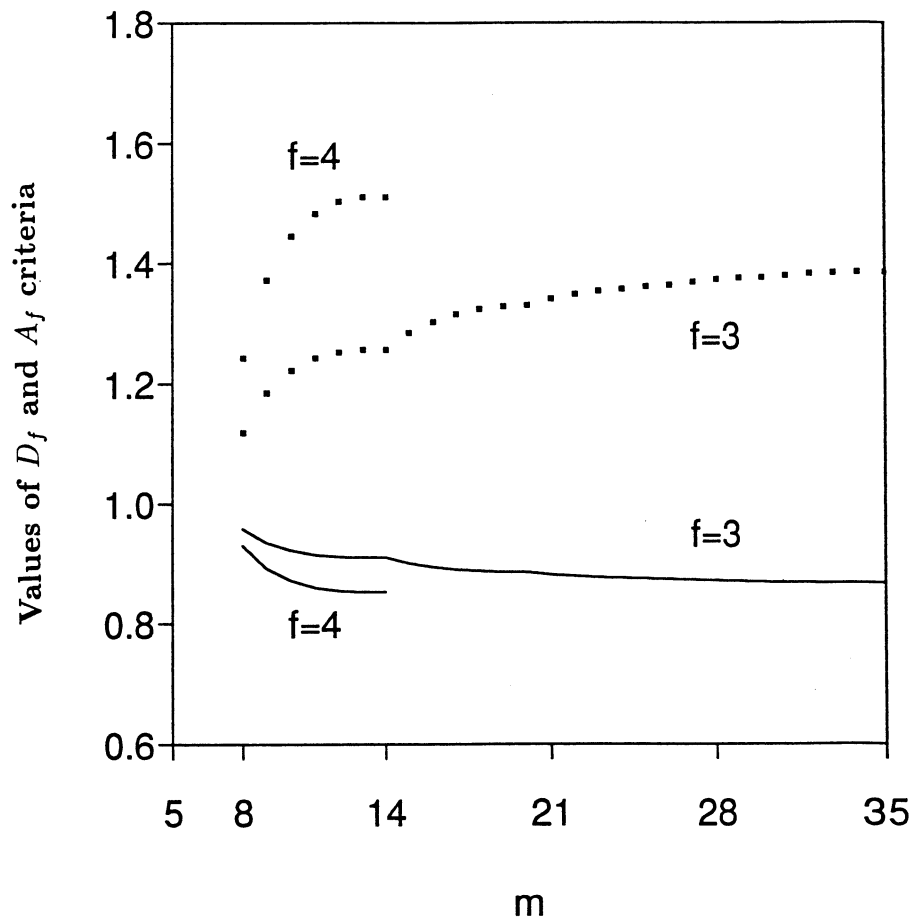
Figure 1: D_f-values (indicated by —) and A_f-values (indicated by $\cdots$) of 8 run-designs with m factors.
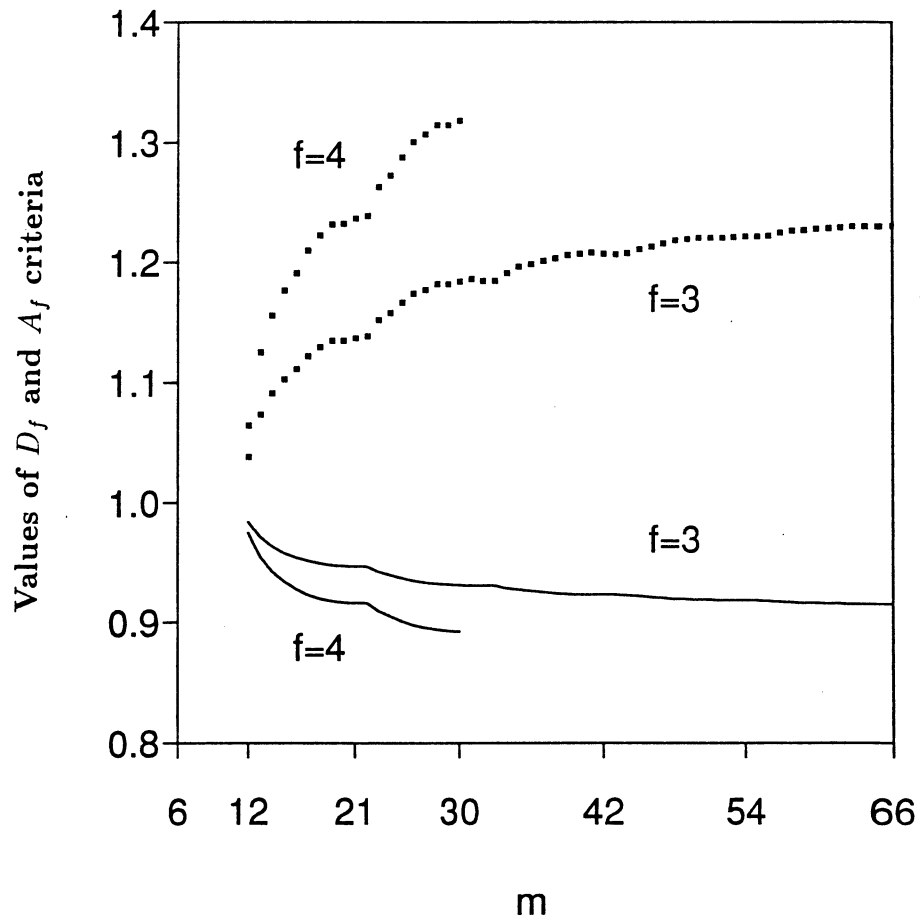
Figure 2: D_f-values (indicated by —) and A_f-values (indicated by $\cdots$) of 12 run-designs with m factors.
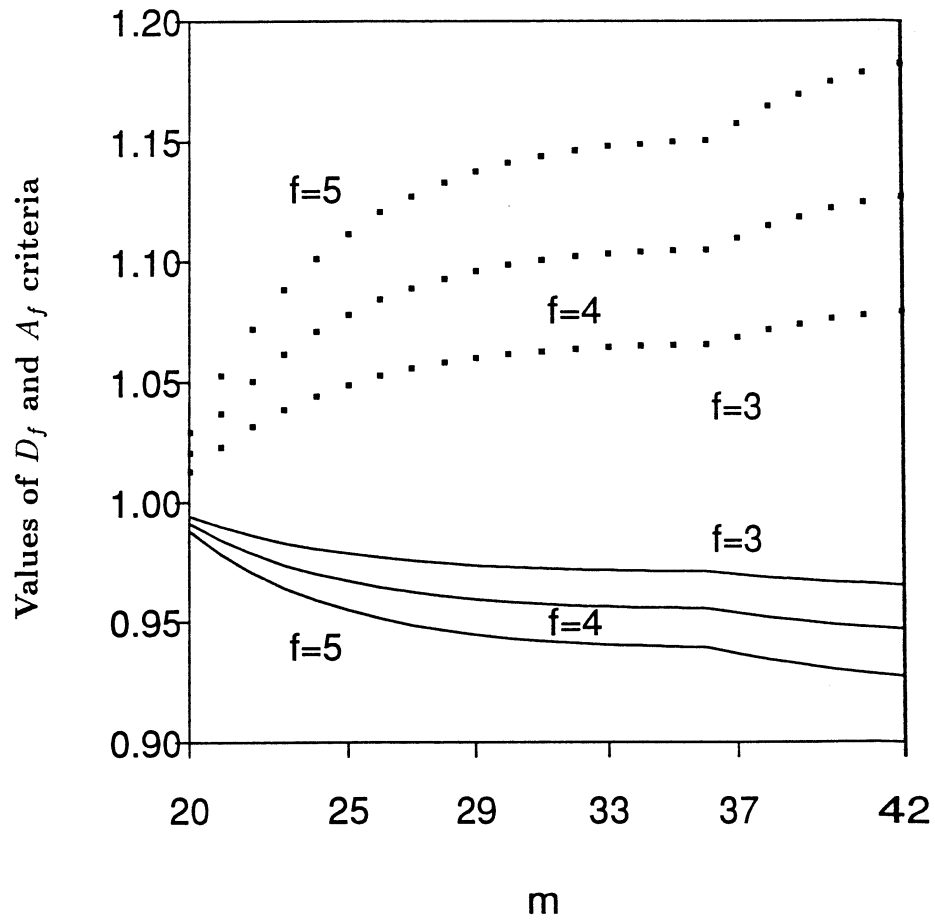
Figure 3: D_f-values (indicated by —) and A_f-values (indicated by $\cdots$) of 20 run-designs with m factors.