A POINT PROCESS MODEL INCORPORATING RENEWALS AND TIME TRENDS, WITH APPLICATION TO REPAIRABLE SYSTEMS

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ABSTRACT

We discuss models for recurrent event that incorporate both time trends and effects of past events, such as renewal-type behaviour. Inference procedures, including tests for trend, are developed and illustrated on repairable systems failure data. Simulations are used to examine the accuracy of large sample approximations used for tests or interval estimation.

Key words: Estimation; Recurrent events; Poisson process; Renewal process; Simulation; Trend tests.

1. INTRODUCTION

Processes involving recurrent events are common in reliability and many other areas. For simplicity of exposition we will use reliability terminology in this paper and refer to recurrent events (often failures or repairs) in repairable systems, but the models and methods discussed apply to a wide range of problems. The literature on recurrent events in reliability is large. Ascher and Feingold (1984) provide many details and references; for additional examples see Cox and Lewis (1966), Crow (1974, 1982), Lee and Lee (1978), Bain and Engelhardt (1980), Lee (1980) and Crowder, Kimber, Smith and Sweeting (1991). There has been much discussion about modelling and about the appropriateness of Poisson processes and renewal processes as models for repairable systems (e.g. Ascher and Feingold 1984, chapters 2, 8). The purpose of this paper is to present a very useful family of models that incorporates both Poisson and renewal behavior and to illustrate its application to reliability. A major benefit of our work is the ability to assess Poisson and renewal assumptions with a single, comprehensive model.

Consider a repairable system observed over time $t \geq 0$ and suppose that events occur at times $t_1 < t_2 < \ldots$. Let $x_i = t_i - t_{i-1}$ (with $t_0 = 0$ and $i = 1, 2, \ldots$) denote times between events and let $N(s,t)$ denote the number of events in the time interval $(s,t]$. We will also write $N(t)$ for $N(0,t)$. Quite generally, a probability model for such a point process may be specified in terms of its conditional, or "complete" intensity function (CIF) as follows (see e.g. Cox and Isham 1980, p. 9): define $H_t = \{N(s) : 0 \leq s < t\}$ as the "history" of the process up to time t. Then the CIF is

$$\lambda(t; H_t) = \lim_{\Delta t \downarrow 0} \frac{Pr\{N(t, t+\Delta t) = 1 \mid H_t\}}{\Delta t} . \tag{1.1}$$

That is, $\lambda(t; H_t)\Delta t$ is for small Δt the approximate probability of an event in $(t, t + \Delta t]$, given the process history up to t.

Poisson processes are models for which (1.1) is of the form

$$\lambda(t; H_t) = \rho(t), \tag{1.2}$$

in which case $\rho(t)$ is called the intensity or rate function. For Poisson processes it is well known that $N(t)$ has a Poisson distribution with mean $R(t) = \int_0^t \rho(u)du$ and that the numbers of events in non-overlapping time intervals are independent. Renewal processes on the other hand, are models for which (1.1) is of the form

$$\lambda(t; H_t) = h(t - t_{N(t-)}) \,, \tag{1.3}$$

where $t_{N(t-)}$ is the time of the last event prior to t. Thus, (1.3) implies that the times x_i between successive events are independent and identically distributed (i.i.d.) with hazard function $h(x)$, which is the way renewal processes are usually defined.

We can use (1.1) to formulate models incorporating both time trends and renewal-type behavior. Thus, for example, questions concerning whether a system is "bad as old" after a repair (implying a Poisson process) or "good as new" (implying a renewal process) may be addressed. More generally, there is the opportunity to build effects of past events into a model. The purpose of this paper is to study one such class of models and associated statistical methods. We consider processes for which (1.1) is of the form

$$\lambda(t; H_t) = e^{\boldsymbol{\theta}'\mathbf{z}(t)}, \tag{1.4}$$

where $\mathbf{z}(t) = (z_1(t), \ldots, z_p(t))'$ is a vector of functions which may depend on both t and H_t and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)'$ is a vector of unknown parameters. For convenience, all vectors are considered to be column vectors. The model (1.4) is a special case of one considered by Berman and Turner (1992), but our line of investigation is different from theirs.

Many common models are special cases of (1.4). These include Poisson processes with intensity functions (i) $\rho_1(t) = \exp(\alpha + \beta t)$ and (ii) $\rho_2(t) = \alpha t^\beta$, given by (i) $\mathbf{z}(t) = (1, t)'$, $\boldsymbol{\theta} = (\alpha, \beta)'$ and (ii) $\mathbf{z}(t) = (1, \log t)'$, $\boldsymbol{\theta} = (\log \alpha, \beta)'$, respectively. Statistical methods for these models have been considered by many authors (e.g. Cox and Lewis 1966; Bain and Engelhardt 1980; Lee and Lee 1978; Ascher and Feingold 1984; Crowder et al. 1991). Renewal processes are obtained by taking $\mathbf{z}(t)$ as a function of time since the last event, $u(t) = t - t_{N(t-)}$. For example $\mathbf{z}(t) = (1, \log u(t))'$, $\boldsymbol{\theta} = (\log \alpha, \beta)'$ gives a renewal process where the x_i's have a Weibull distribution with hazard function $h(x) = \alpha x^{\beta}$. Models with $\mathbf{z}(t) = (1, g_1(t), g_2(u(t))'$, where g_1 and g_2 are specified functions, incorporate both a time trend and renewal-type behavior.

Assuming that times of events are observed more or less exactly, parametric inference for models of the form (1.4) is shown below to be straightforward, and fast with current computational power. The remainder of the paper develops and investigates inference procedures. It is also easy to incorporate covariates into the analysis, though we do not explore this in the current paper.

Section 2 describes maximum likelihood estimation and hypothesis tests. Section 3 presents examples, including methods of model assessment. Section 4 provides a brief check on large sample methods by simulation. Section 5 considers tests for time trends and gives some simulation results on the adequacy of large sample approximations and on power properties of the tests. Section 6 concludes with some remarks on extensions to this work.

2. MAXIMUM LIKELIHOOD METHODS

We consider the likelihood function for a single process with CIF (1.4) observed over the time interval $(0, T]$. If several independent processes are observed the log likelihood, score equations and information matrices are merely sums of expressions of the form (2.4)-(2.6) below. The likelihood is proportional to the probability density for the observed data, which is of the form $\{n \text{ events, at times } t_1 < ... < t_n \leq T\}$, where $n \geq 0$. Very generally, the likelihood is (e.g. Andersen, Borgan, Gill and Keiding 1993, pp. 57-8; Berman and Turner 1992)

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} \lambda(t_i; H_{t_i}) \cdot \exp\left\{-\int_0^T \lambda(t; H_t)dt\right\} \tag{2.1}$$

where $\lambda(t; H_t)$ is given by (1.4).

It should be remarked that (2.1) is valid under a variety of procedures for choosing T. In particular, T does not have to be pre-specified, but could be chosen as the time of some event (in which case $T = t_n$) or based on past events in the system. It should also be noted that for Poisson or renewal processes it is sometimes possible to derive so-called conditional likelihoods by conditioning on some aspect of the observed data (e.g. see Cox and Lewis 1966, p. 46). This is usually done to remove nuisance parameters. We will not consider such refinements here, except peripherally in section 4. It is easily checked for the special cases of Poisson and renewal processes (see (1.2) and (1.3)) that (2.1) gives the well known results

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} \rho(t_i) \cdot \exp\left\{ - \int_0^T \rho(t)dt \right\} \tag{2.2}$$

and

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f(x_i) \cdot \overline{F}(x_{n+1}), \tag{2.3}$$

respectively. In the renewal process case (2.3), we have $x_1 = t_1, x_i = t_i - t_{i-1} (i = 2, ..., n)$, $x^*_{n+1} = T - t_n$, and $h(x)$, $\overline{F}(x) = \exp\{-\int_0^x h(u)du\}$ and $f(x) = -\overline{F}'(x)$ are, respectively, the hazard, survivor and density functions for the x_i's.

For general models of the form (1.4) the log likelihood function $\ell(\boldsymbol{\theta}) = \log L(\boldsymbol{\theta})$ is, from (2.1),

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{n} \boldsymbol{\theta}'\mathbf{z}(t_i) - \int_0^T e^{\boldsymbol{\theta}'\mathbf{z}(t)} dt \ . \tag{2.4}$$

The maximum likelihood or score equations are, for $r = 1, ..., p$,

$$U_r(\boldsymbol{\theta}) = \frac{\partial \ell(\boldsymbol{\theta})}{\partial \theta_r} = \sum_{i=1}^{n} z_r(t_i) - \int_0^T z_r(t) e^{\boldsymbol{\theta}'\mathbf{z}(t)} dt, \tag{2.5}$$

and the $p \times p$ observed information matrix $I(\boldsymbol{\theta})$ has entries

$$I_{rs}(\boldsymbol{\theta}) = -\frac{\partial^2 l}{\partial \theta_r \partial \theta_s} = \int_0^T z_r(t) z_s(t) e^{\boldsymbol{\theta}'\mathbf{z}(t)} dt \ . \tag{2.6}$$

In special cases we may be able to evaluate the integrals in (2.4)-(2.6) analytically but in general, numerical integration is needed. We discuss their computation below.

If n or, when independent realizations of a process are observed, the total number of events is large and appropriate conditions on the model hold then in order to get interval estimates or tests for $\boldsymbol{\theta}$ we may treat $\hat{\boldsymbol{\theta}}$ as approximately normally distributed with mean $\boldsymbol{\theta}$ and covariance matrix $I(\hat{\boldsymbol{\theta}})^{-1}$. Score or likelihood ratio statistics may also be used in the usual way (e.g. Lawless 1982, Appendix E). Types of conditions needed to prove asymptotic results rigorously are discussed by Andersen et al. (1993, section VI.1.2); a key requirement is that the observed information increase sufficiently fast asymptotically. To be certain that asymptotic approximations are satisfactory in specific finite-sample situations it is best to carry out checks via simulation. We present some simulation results in sections 4 and 5.

The maximum likelihood estimate (m.l.e.) $\hat{\boldsymbol{\theta}}$ may be obtained by solving the equations $U_r(\boldsymbol{\theta}) = \mathbf{0}$ (see (2.5)) using Newton's method. This employs the iteration scheme

$$\boldsymbol{\theta}^{(j+1)} = \boldsymbol{\theta}^{(j)} + I(\boldsymbol{\theta}^{(j)})^{-1} U(\boldsymbol{\theta}^{(j)}) \qquad j = 1, 2, ... \tag{2.7}$$

where $U(\boldsymbol{\theta}) = (U_1(\boldsymbol{\theta}), ..., U_p(\boldsymbol{\theta}))'$ and $\boldsymbol{\theta}^{(1)}$ is an initial guess at $\hat{\boldsymbol{\theta}}$. To calculate $\ell(\boldsymbol{\theta}), u(\boldsymbol{\theta})$ or $I(\boldsymbol{\theta})$ we generally need to use numerical integration. This may be conveniently described as follows: let $0 = a_1 < a_2 < ... < a_m = T$ and associated constants $w_1, ..., w_m$ be defined such that

$$\sum_{j=1}^{m} w_j e^{\boldsymbol{\theta}' \mathbf{z}(a_j)} \doteq \int_0^T e^{\boldsymbol{\theta}' \mathbf{z}(t)} dt \tag{2.8}$$

to a desired degree of accuracy. The w_j's are determined by selecting a particular quadrature rule (e.g. Press, Flannery, Teukolsky and Vetterling, 1986, chapter 4).

There are often discontinuities in the covariates $\mathbf{z}(t)$ at the event times $t_1, ..., t_n$ so it is important when using numerical integration to write

$$\int_0^T e^{\boldsymbol{\theta}' \mathbf{z}(t)} dt = \sum_{i=1}^{n+1} \int_{t_{i-1}}^{t_i} e^{\boldsymbol{\theta}' \mathbf{z}(t)} dt, \tag{2.9}$$

where $t_0 = 0$ and $t_{n+1} = T$, and to evaluate the $n+1$ integrals on the right-hand side of (2.9) separately. More generally, the integral should be split at each discontinuity point for $\mathbf{z}(t)$, but we will assume that jumps occur only at the t_i's. Quadrature formulas tailored to specific types of covariates may be constructed but for general purposes it is simplest to employ general numerical integration software or to program a simple method such as the trapezoidal rule or Simpson's rule (e.g. Press et al. 1986, chapter 4). to approximate the i'th integral in (2.9). Using Simpson's rule, for example, we select a positive integer k_i and define $\Delta_i = (t_i - t_{i-1})/2k_i$. We define a_{ij}'s and w_{ij}'s by

$$a_{ij} = t_{i-1} + (j-1)\Delta_i \qquad j = 1, ..., 2k_i + 1 \tag{2.10}$$

$$w_{i1} = w_{i,2k_i+1} = 1/3; \quad w_{i2} = \ldots = w_{i,2k_i} = 4/3; \quad w_{i3} = \ldots = w_{i,2k_i-1} = 2/3$$

and then

$$\int_{t_{i-1}}^{t_i} e^{\boldsymbol{\theta}'\mathbf{z}(t)} dt \doteq \sum_{j=1}^{2k_i+1} w_{ij} e^{\boldsymbol{\theta}'\mathbf{z}(a_{ij})} . \tag{2.11}$$

Using (2.8), we obtain approximations for $\ell(\boldsymbol{\theta})$, $u_r(\boldsymbol{\theta})$ and $I_{rs}(\boldsymbol{\theta})$ in (2.4)-(2.6) as

$$\ell^A(\boldsymbol{\theta}) = \sum_{i=1}^{n} \boldsymbol{\theta}'\mathbf{z}(t_i) - \sum_{j=1}^{m} w_j e^{\boldsymbol{\theta}'\mathbf{z}(a_j)} \tag{2.12}$$

$$U_r^A(\boldsymbol{\theta}) = \sum_{i=1}^{n} z_r(t_i) - \sum_{j=1}^{m} w_j z_r(a_j) e^{\boldsymbol{\theta}'\mathbf{z}(a_j)} \tag{2.13}$$

$$I_{rs}^A(\boldsymbol{\theta}) = \sum_{j=1}^{m} w_j z_r(a_j) z_s(a_j) e^{\boldsymbol{\theta}'\mathbf{z}(a_j)} . \tag{2.14}$$

Berman and Turner (1992) note that the GLIM software package may be used to maximize (2.4), but a direct approach via (2.7) is often simpler. It should be noted that the main requirement for (2.12)-(2.14) is that the values of $\mathbf{z}(a_1), ..., \mathbf{z}(a_m)$ be available. When $\mathbf{z}(t)$ is a function of the process history there is no difficulty. In cases (not considered in this paper) where $\mathbf{z}(t)$ includes measured time varying covariates it will often be necessary to impute some $\mathbf{z}(a_j)$ values by interpolation.

To make clear the numerical procedures we consider a family of models discussed in sections 3 and 5.

Example. We consider the model (1.4) with

$$\lambda(t; H_t) = e^{\alpha + \beta g_1(t) + \gamma g_2(u(t))}, \tag{2.15}$$

where $\boldsymbol{\theta} = (\alpha, \beta, \gamma)'$ and $g_1(t)$ and $g_2(u(t))$ are specified functions. As in section 1, $u(t)$ is the time since the most recent event, $t - t_{N(t-)}$. Models where $g_1(x)$ is either x or $\log x$ and $g_2(x)$ is either x or $\log x$ are especially useful.

In terms of (2.12)-(2.14) we have $\mathbf{z}(a_j) = [1, g_1(a_j), g_2(u(a_j))]'$. The approximate 3×1 score vector $U^A(\boldsymbol{\theta})$ and 3×3 information matirx $I^A(\boldsymbol{\theta})$ are thus

$$U^A(\boldsymbol{\theta}) = \sum_{i=1}^{n} \begin{bmatrix} 1 \\ g_1(t_i) \\ g_2(u(t_i)) \end{bmatrix} - \sum_{j=1}^{m} h_j(\boldsymbol{\theta})\mathbf{z}(a_j) \tag{2.16}$$

$$I^A(\boldsymbol{\theta}) = \sum_{j=1}^{m} h_j(\boldsymbol{\theta})\mathbf{z}(a_j)\mathbf{z}(a_j)' \ , \tag{2.17}$$

where $h_j(\boldsymbol{\theta}) = w_j \exp[\boldsymbol{\theta}'\mathbf{z}(a_j)]$.

As a numerical illustration we consider the model (2.15) with $g_1(t) = t$ and $g_2(u(t)) = u(t)$, and the data for Plane 6 in section 3, consisting of $n = 30$ events at times $23, ..., 1788$. We used the Newton iteration scheme (2.7) to find $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma})'$. In this case it is possible to evaluate the integrals in (2.4)-(2.6) mathematically and thus numerically. For comparison we also used numerical integration based on Simpson's Rule: (2.10) with $k_i = 6$ for $i = 1, ..., 31$ was sufficient to give $\hat{\boldsymbol{\theta}}$ and $I(\hat{\boldsymbol{\theta}})$ to four significant digits as $\hat{\boldsymbol{\theta}} = (-4.891, 0.0008735, -0.001241)$ and

$$I(\hat{\boldsymbol{\theta}}) = 10^4 \begin{pmatrix} 0.0300 & 3.385 & 0.1788 \\ 3.385 & 4527 & 177.7 \\ 0.1788 & 177.7 & 21.10 \end{pmatrix} .$$

The estimated covariance matrix $I(\hat{\boldsymbol{\theta}})^{-1}$ gives standard errors for $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ of $0.559, 0.000391$ and 0.00322, respectively.

This and another example are discussed further in Section 3. We remark that with various models and data sets, the use of (2.7) combined with numerical integration provided inferences very rapidly. Calculations were programmed in S-Plus (StatSci 1994).

3. EXAMPLES

To illustrate the use of models (1.4) and associated inference procedures we consider a subset of the much discussed data on airplane air conditioning failures given by Proschan

(1963). We will look at the data on just two planes, denoted as Planes 6 and 7 by Cox and Lewis (1966, p. 6). Times between events (air conditioning failures) for each plane are as follows:

Plane 6: 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95 (n=30)

Plane 7: 97, 51, 11, 4, 141, 18, 142, 68, 77, 80, 1, 16, 106, 206, 82, 54, 31, 216, 46, 111, 39, 63, 18, 191, 18, 163, 24 (n=27) .

Times are in operating hours for the equipment.

Figures 1 and 2 display plots of the cumulative number of failures vs. cumulative operating time for each plane. Trend curves, to be described below, are also shown. Plane 6 displays an increasing rate of failure whereas Plane 7 has an approximately constant rate. To investigate the failure process we considered event process models (2.15) with CIF

$$\lambda(t; H_t) = \exp\{\alpha + \beta t + \gamma u(t)\} \tag{3.1}$$

for each plane.

Following the numerical procedures described in the Example of section 2, we obtained maximum likelihood estimates $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ and associated information matrix $I(\hat{\boldsymbol{\theta}})$ for each plane. Table 1 shows, under Model (1), the estimates and standard errors, obtained as the square roots of the diagonal entries in the estimated covariance matrix $I(\hat{\boldsymbol{\theta}})^{-1}$. The maximized log likelihoods $\ell(\hat{\boldsymbol{\theta}})$ are also shown. For both planes it is clear that there is no evidence against the hypothesis $H : \gamma = 0$. Specifically, Wald tests of H may be based on the statistic $W^2 = \hat{\gamma}^2/(I(\hat{\boldsymbol{\theta}})^{-1})_{3,3}$ which, if H is true, is approximately distributed as $\chi^2_{(1)}$. The values for Planes 6 and 7 are .15 and 1.88, respectively. Thus, there is no evidence of renewal-type behavior within the model (3.1).

We also show in Table 1 estimates $\tilde{\alpha}, \tilde{\beta}$ and standard errors when $\gamma = 0$, as Model (2). The maximized log likelihood $\ell(\tilde{\boldsymbol{\theta}}) = \ell(\tilde{\alpha}, \tilde{\beta}, 0)$ is also shown. Another way to test $H : \gamma = 0$ in Model (1) would be to use the likelihood ratio statistic $LR = 2\ell(\hat{\boldsymbol{\theta}}) - 2\ell(\tilde{\boldsymbol{\theta}})$, which, if H is true, is approximately $\chi^2_{(1)}$. We obtain LR=0.15 and 1.80 for Planes 6 and 7, in close agreement with the Wald statistics.

It is also of interest to test for a time trend by considering the hypothesis $H' : \beta = 0$. This can be done either under Model (1) or Model (2). In the former case, for example, the Wald statistic is $\hat{\beta}^2/(I(\hat{\boldsymbol{\theta}})^{-1})_{2,2}$. Observed values are 4.98 for Plane 6 and 0.24 for Plane 7. A comparison of these values with $\chi^2_{(1)}$ quantiles indicates rather strong evidence of a trend for Plane 6 and no evidence against H' for Plane 7.

Table 1. Estimates for Model (3.1) for Air Conditioning Failures

Model		Plane 6	Plane 7
(1)	Estimates $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$	-4.891, .000874, -0.00124	-4.517, -.000162, .00487
	Std. errors	.559, .000391, .00322	.425, .000335, .00355
	$\ell(\hat{\boldsymbol{\theta}})$	-149.40	-143.30
(2)	Estimates $(\hat{\alpha}, \hat{\beta})$	-5.018, .000918	-4.275, -.0000647
	Std. errors	.463, .000377	.379, .000322
	$\ell(\tilde{\boldsymbol{\theta}})$	-149.47	-144.20

Within the model (3.1) we have found that a Poisson process is adequate (i.e. $\gamma = 0$) for each plane. Figures 1 and 2 show the estimated cumulative mean functions, or trend curves,

$$\hat{\Lambda}(t) = \int_0^t e^{\hat{\alpha}+\hat{\beta}u} du = e^{\hat{\alpha}}(e^{\hat{\beta}t} - 1)/\hat{\beta} \tag{3.2}$$

for each plane. They suggest that the parametric form selected in (3.1) is satisfactory, and formal tests do not provide any evidence to the contrary. We have not, however, demonstrated that the full model (3.1) is necessarily satisfactory and consequently some checks are desirable. The generalized residuals

$$\hat{e}_i = \int_{t_{i-1}}^{t_i} \hat{\lambda}(t; H_t)dt \qquad i = 1, \ldots, n \tag{3.3}$$

may be used to do this. If the true intensity is used in (3.3) instead of the estimated one, the $\hat{e}_i$'s are i.i.d. standard exponential random variables (e.g. Cox and Isham 1980). Thus, if the assumed model is satisfactory, the $\hat{e}_i$'s should look roughly like such a sample. Useful checks are to examine index plots (i.e. $\hat{e}_i$ or $\log \hat{e}_i$ vs. i) and exponential probability plots. Figures 3 and 4 illustrate this for Plane 6, using the assumed Model (2) of Table 1. This gives $\hat{e}_i = \hat{\Lambda}(t_i) - \hat{\Lambda}(t_{i-1})$ with $\hat{\Lambda}(t)$ given by (3.2) with $\hat{\alpha} = -5.018$, $\hat{\beta} = .000918$. Figure 3 is a plot of $\hat{e}_i$ vs. i for $i = 1, \ldots, 30$; it shows no unexpected features. Figure 4 is a probability plot of the ordered $\hat{e}_i$'s vs. the expected standard exponential order statistics $\alpha_i = (1/n) + \ldots + (1/(n-i+1))$ with $n = 30$. The plot is reasonably close to a straight line with slope 1, but there is an interesting suggestion of a change in slope, possibly reflecting a mixture of two distributions. In fact, we note in Figure 1 that failures tend to cluster to some extent, which would produce the type of pattern seen in Figure 4. We will not pursue this further, but it could be worthwhile to consider some type of cluster process (e.g. Cox and Isham 1980) for failures.

4. CHECKS ON LARGE SAMPLE APPROXIMATIONS

To obtain confidence limits or tests for parameters we rely on the approximate normality of the maximum likelihood estimates $\hat{\boldsymbol{\theta}}$ in large samples. It is, of course, desirable to check the adequacy of such approximations for the small to medium sample sizes encountered in practice. We present here a limited but useful investigation.

We considered the model (3.1) with parameter values $\alpha = 0.0$, $\beta = 0.03$ and $\gamma = 1.0$; these relative values are plausible ones in application involving 20-100 events. We simulated 2000 series of n events from this process for each of $n = 20,\ 50,\ 100$ and 200, and obtained estimates $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and their standard errors $s(\hat{\alpha})$, $s(\hat{\beta})$, $s(\hat{\gamma})$ for each series, as described in sections 2 and 3. Large sample theory indicates that for large n the variables

$$W_1 = \frac{\hat{\alpha} - \alpha}{s(\hat{\alpha})} \qquad W_2 = \frac{\hat{\beta} - \beta}{s(\hat{\beta})} \qquad W_3 = \frac{\hat{\gamma} - \gamma}{s(\hat{\gamma})}$$

are each approximately standard normal. Table 2 shows the proportion of the 2000 samples for each sample size for which $w_i (i = 1, 2, 3)$ satisfied (i) $-1.96 \le w_i \le 1.96$ (ii) $w_i < -1.96$ (iii) $w_i > 1.96$. The probabilities of these occurrences under the standard normal distribution are $.95, .025, .025$, respectively. We also checked other percentage points, but

Table 2 shows the main pattern in the results.

Table 2. Empirical Probabilities for W_1, W_2, W_3 and Standard Normal Quantiles

		W_1	W_2	W_3
$n = 20$	C	.9385	.9255	.9155
	L	.0130	.0575	.0830
	U	.0485	.0170	.0015
$n = 50$	C	.9430	.9385	.9245
	L	.0160	.0460	.0685
	U	.0410	.0155	.0070
$n = 100$	C	.9475	.9435	.9395
	L	.0180	.0365	.0545
	U	.0345	.0200	.0060
$n = 200$	C	.9500	.9440	.9410
	L	.0165	.0315	.0470
	U	.0335	.0245	.0120

Nominal coverages for $C(-1.96 \leq W_i \leq 1.96)$, $L(W_i < -1.96)$ *and* $U(W_i > 1.96)$ *are .95, .025, .025, respectively.*

Table 2 shows that for smaller sample sizes the two-sided coverages (C) are somewhat less than .95, though quite close for n greater than 50. The one-sided probabilities (L and U) are relatively much further off the nominal .025, even for $n = 200$. The approximations for W_3 are somewhat poorer than those for W_1 and W_2. These results suggest that two-sided tests or confidence intervals based on W_1, W_2 or W_3 should have approximately correct coverage, but that a little more caution is needed for one-sided procedures; Table 2 indicates the degree of over- or under-coverage for each parameter.

5. TESTING FOR A TIME TREND

The presence or absence of time trends in recurrent events is often of interest, and numerous tests for trends have been given. We focus here on monotone trends, in which case events either tend to occur more frequently (increasing trend) or less frequently (decreasing trend) as time passes. Ascher and Feingold (1984, chapter 2) have discussed the difficulty of giving a comprehensive definition of trend, but the most important case of absence of trend is that of a renewal process, the most important sub-case being the homogeneous Poisson process (HPP), for which inter-event times are exponential. Cox and Lewis (1966, chapter 3) and Ascher and Feingold (1984, chapter 5) discuss many tests for monotone trend, usually with the null hypothesis being that the process is either a HPP or a general renewal process.

The models (1.4) provide a way to incorporate or test for time trends with a renewal process. For example, the model (2.15) is a renewal process if $\beta = 0$ but if $\beta \neq 0$ exhibits a time trend. We may test for trend by testing that $\beta = 0$. In this section we consider trend tests based on (2.15), and carry out a small simulation study to compare these tests with two widely used tests (Ascher and Feingold 1984, pp. 73-83). Our objectives are to examine the power and robustness of the tests, and to assess the adequacy of the large sample approximations used for the distributions of the various test statistics.

First we describe trend tests based on (2.15), which amount to tests of $H : \beta = 0$. Letting $\boldsymbol{\theta} = (\alpha, \beta, \gamma)$ as in the example of section 2, we may use any of three asymptotically (large T or n) equivalent test statistics which arise from maximum likelihood theory. The first is the Wald statistic $W = \hat{\beta}/s(\hat{\beta})$, where $s(\hat{\beta}) = (I(\hat{\boldsymbol{\theta}})^{-1})_{2,2}^{1/2}$ is the standard error for the maximum likelihood estimate $\hat{\beta}$. If H is true, W is approximately standard normal, and large values of $|W|$ provide evidence of trend. A second possibility is to use the likelihood ratio statistic $R = 2\ell(\hat{\boldsymbol{\theta}}) - 2\ell(\tilde{\boldsymbol{\theta}})$, where $\tilde{\boldsymbol{\theta}} = (\tilde{\alpha}, 0, \tilde{\gamma})$ maximizes $\ell(\boldsymbol{\theta})$ subject to $H : \beta = 0$. From (2.5), $\tilde{\alpha}$ and $\tilde{\gamma}$ satisfy

$$n - \int_0^T e^{\alpha + \gamma g_2(u(t))} dt = 0 \tag{5.1}$$

$$\sum_{i=1}^{n} g_2(u(t_i)) - \int_0^T g_2(u(t))e^{\alpha+\gamma g_2(u(t))}dt = 0 \ , \tag{5.2}$$

and may be obtained using Newton's method and the approximations (2.13) and (2.14). If H is true, R is approximately $\chi^2_{(1)}$; large values of R provide evidence of trend. A third test statistic based on (2.15) is the standardized partial score statistic (e.g. Lawless 1982, Appendix E). In the interests of brevity we will omit it in our discussion; in simulations it behaved similarly to the likelihood ratio statistic.

The other trend tests which we consider are based on the following statistics:

(i) The Laplace statistic (e.g. Ascher and Feingold 1984, pp. 78-9)

$$LA = \frac{\sum_{i=1}^{n-1} t_i - (n-1)T/2}{\{(n-1)T^2/12\}^{1/2}} \ , \tag{5.3}$$

where $T = t_n$. Under a HPP, i.e. if $\beta = 0$ and $\gamma = 0$ in (2.12), LA is approximately standard normal; large values of $|LA|$ provide evidence of trend; LA arises as a score statistic for testing $\beta = 0$ in a nonhomogeneous Poisson process with $\lambda(t; H_t) = \exp(\alpha + \beta t)$ (e.g. Cox and Lewis 1966, chapter 3). We give the form (5.3) because in our simulations below we generated a fixed number of events (n) for each series. If instead one stops observation at an arbitrary time $T \geq t_n$, a little different form for LA arises, in which $n - 1$ is replaced by n.

(ii) The Lewis-Robinson (1974) statistic, also given for the case where $T = t_n$,

$$LR = LA \left\{ \frac{\bar{x}^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2/(n-1)} \right\}^{1/2} \ . \tag{5.4}$$

This statistic, based on rather heuristic arguments, was constructed to be valid when the null hypothesis is a renewal process, in which case LR is approximately standard normal. Large values of $|LR|$ provide evidence of trend.

The statistics LA and LR should be effective when trends are such that $g_1(t) = t$ in (2.15). We therefore ran simulations for which the true model was of the form (2.15) with $g_1(x) = x$ and $g_2(x) = x$. To make the comparisons with LA and LR "fair", we used the statistics W and R based on (2.15) with $g_1(x) = g_2(x) = x$. We simulated data from two types of models:

Model A: $\lambda(t; H_t) = e^{\alpha+\beta t}$ with $\alpha = 0$ and various β values

Model B: $\lambda(t; H_t) = e^{\alpha+\beta t+\gamma u(t)}$ with $\alpha = 0,\ \gamma = 1$ and various β values

We tested for no trend (i.e. that $\beta = 0$) using $W,\ R,\ LA$ and LR with nominal significance levels of .05 and .10. Two-sided tests were used so that the null hypothesis of no trend was rejected at the .05 level if the absolute value of the test statistics $W,\ LA$ and LR exceeded 1.96, and at the .10 level if they exceeded 1.645. We would expect the statistics W and R to perform well, since they are based on the family of models from which the data were generated. The statistic LA should perform well for Model A, since it arises as a score statistic for $H : \beta = 0$ for that model, but it may perform poorly for Model B, where the null model is not a HPP. The statistic LR, although derived heuristically, should perform well both for Models A and B.

We simulated a fixed number of events n for each process, with $n = 20,\ 50$ and 100. We generated 2000 processes for each model considered, so that empirical probabilities (proportions) should be within .01 of their true values roughly 95% of the time. Table 3 shows the empirical probabilities of rejection for the hypothesis of no trend when $\beta = 0$ in Models A and B. Under Model A all tests have close to the nominal size for each n, except for a mild excess rejection rate for W and R at $n = 20$. For Model B three features are noticeable: (i) The Laplace statistic LA is far off the correct coverage (it is based on the wrong model), (ii) The Lewis-Robinson statistic LR holds close to the nominal level, and (iii) W and R exceed the nominal size a little, particularly at $n = 20$.

Table 3. Empirical Probabilities of Rejection Under Null Hypothesis of No Trend

	n	W	R	LA	LR
Model A	20	.053[1]	.057	.054	.057
		.120[2]	.129	.103	.115
	50	.053	.054	.049	.050
		.107	.108	.102	.105
	100	.053	.055	.055	.051
		.106	.108	.105	.110
Model B	20	.073	.086	.009	.055
		.133	.145	.024	.119
	50	.058	.061	.005	.053
		.114	.115	.021	.108
	100	.061	.062	.008	.056
		.114	.115	.022	.110

[1] *Test with nominal size* .05; [2] *Test with nominal size* .10

We also examined the power of the various statistics. Figure 5 shows the empirical power of W, R , LA and LR for Model A, and Figure 6 compares W, R and LR for Model B, for sample size $n = 50$ and nominal test size .05. For Models A and B the test statistics W and R based on the model (2.15) give somewhat higher power than the Laplace and Lewis-Robinson tests. This is to be expected, since (2.15) is the model from which the data are generated.

The main points arising from our study are as follows. The Laplace test is good at detecting trend departures from a homogeneous Poisson process but may seriously mislead when used to detect trend departures from general renewal processes. The Lewis-Robinson test, on the other hand, is very good generally and its simplicity and robustness make it an important test. Trend tests based on (2.15) are more powerful when that family of models is appropriate, provided $g_1(t)$ and $g_2(u(t))$ are specified more or less correctly. Of course, when the objective is to model the event process and not just test for trend, (2.15) is useful, as demonstrated in previous sections.

6. CONCLUDING REMARKS

Models of the form (2.15) are useful for exploring series of events, particularly when one might expect renewal-type behavior, perhaps with a time trend superimposed. In that regard we have examined the use of (2.15) for trend testing, as well as illustrated its application to two sets of data for which there have been many discussions about renewal vs. Poisson modelling. Processes with CIF (1.4) are considerably more general, and allow time trends, past process history, and external time-varying covariates to be incorporated in a model. We have shown in section 2 how estimation may easily be implemented, and further experience on the application of these models would be valuable.

Inference procedures based on large sample properties of maximum likelihood appear reasonably satisfactory in the situations we have examined. For short series of events (e.g. $n = 20$) and for one-sided procedures in general, an investigation of ways to improve accuracy would be useful. Parametric bootstrap and other resampling methods deserve consideration.

Finally, the model (2.15) is a special type of modulated renewal process (Cox 1972), and it is thus possible to use the semiparametric partial likelihood method described by Cox (1972) and Oakes and Cui (1994) for estimating β and testing for trend. We found in simulations that unless n was very large the large sample approximations associated with this approach was poor and hence did not include in our discussion. Further examination of this approach would be interesting.

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Figure 1

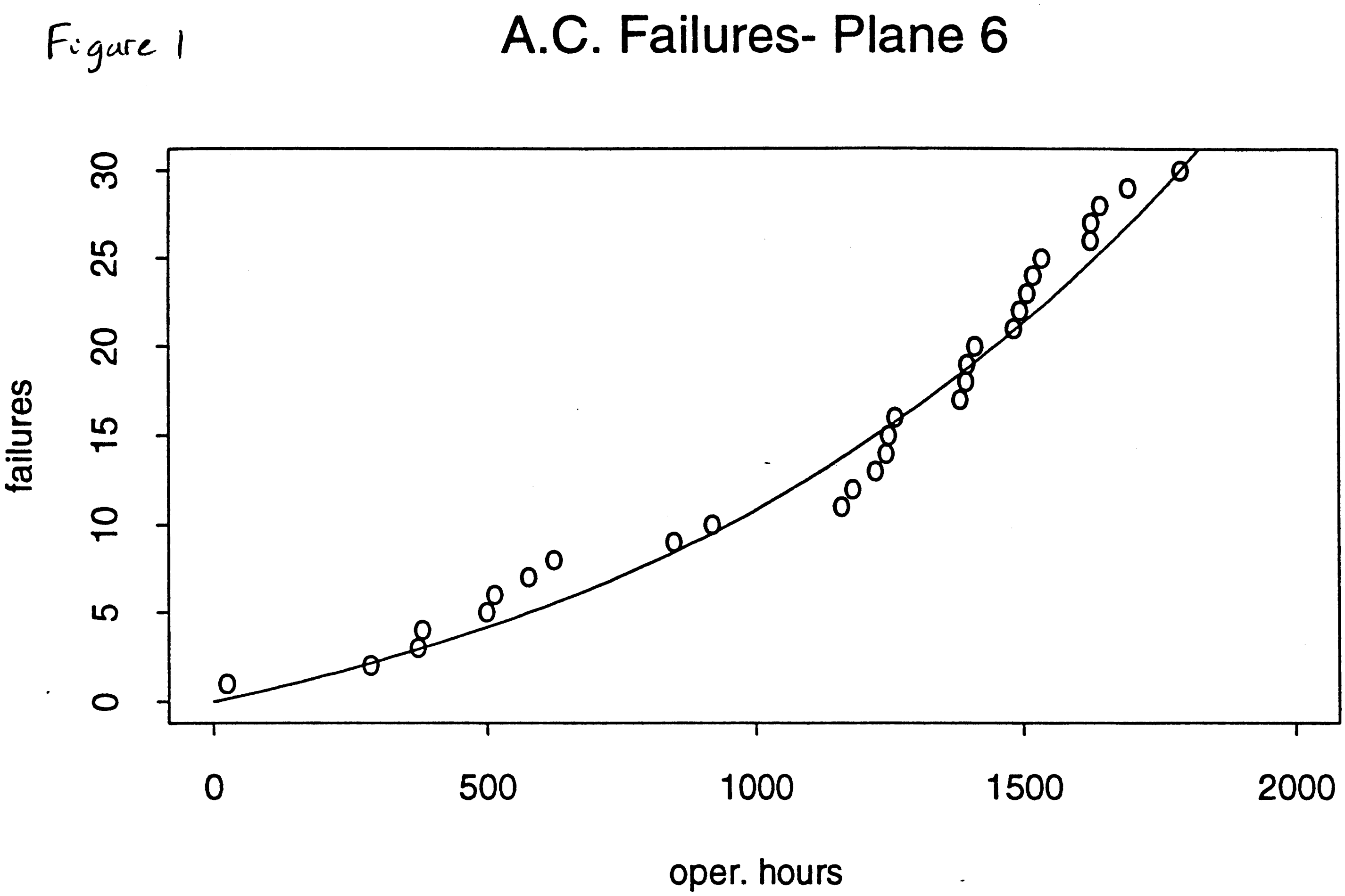


Figure 2

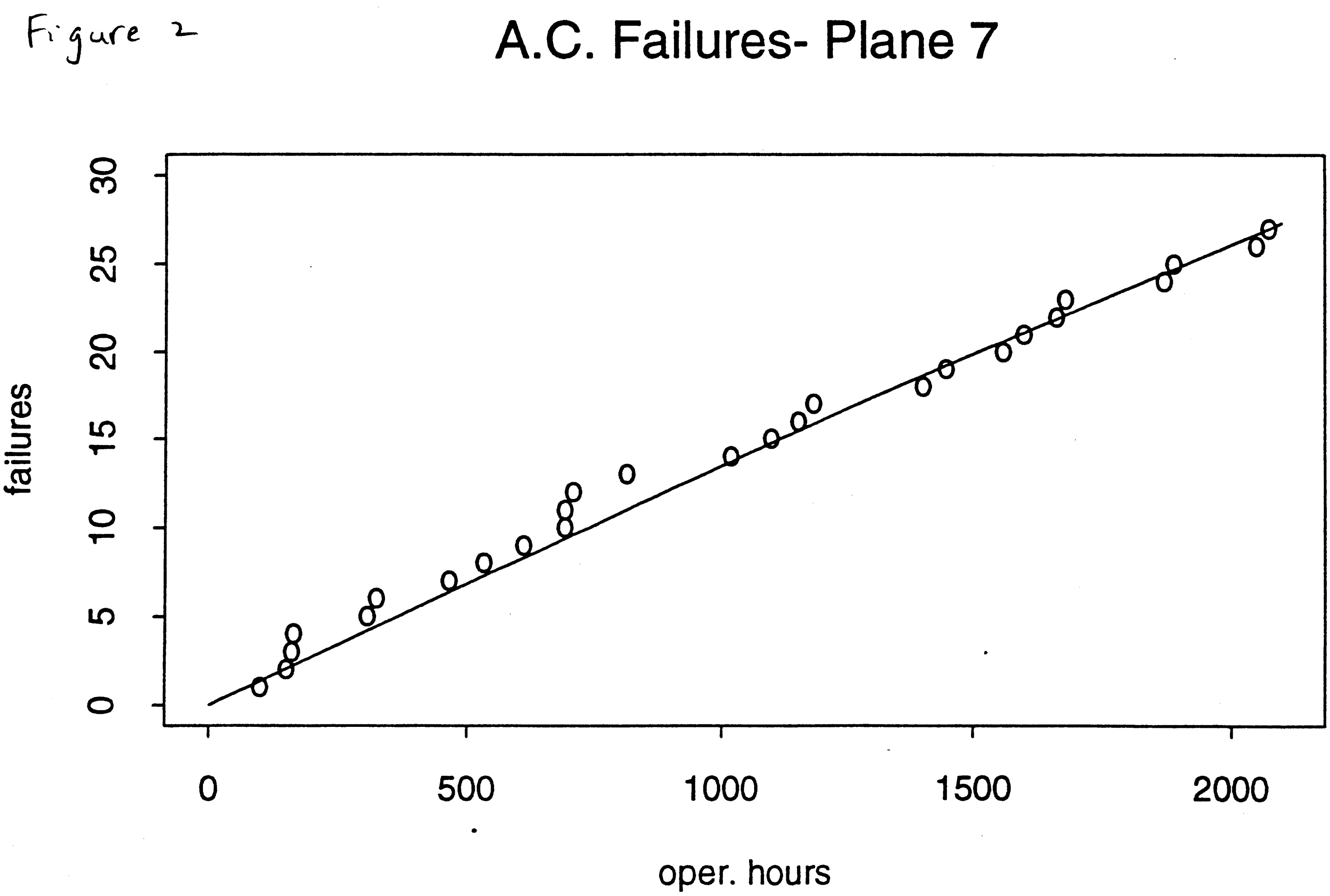


Figure 3

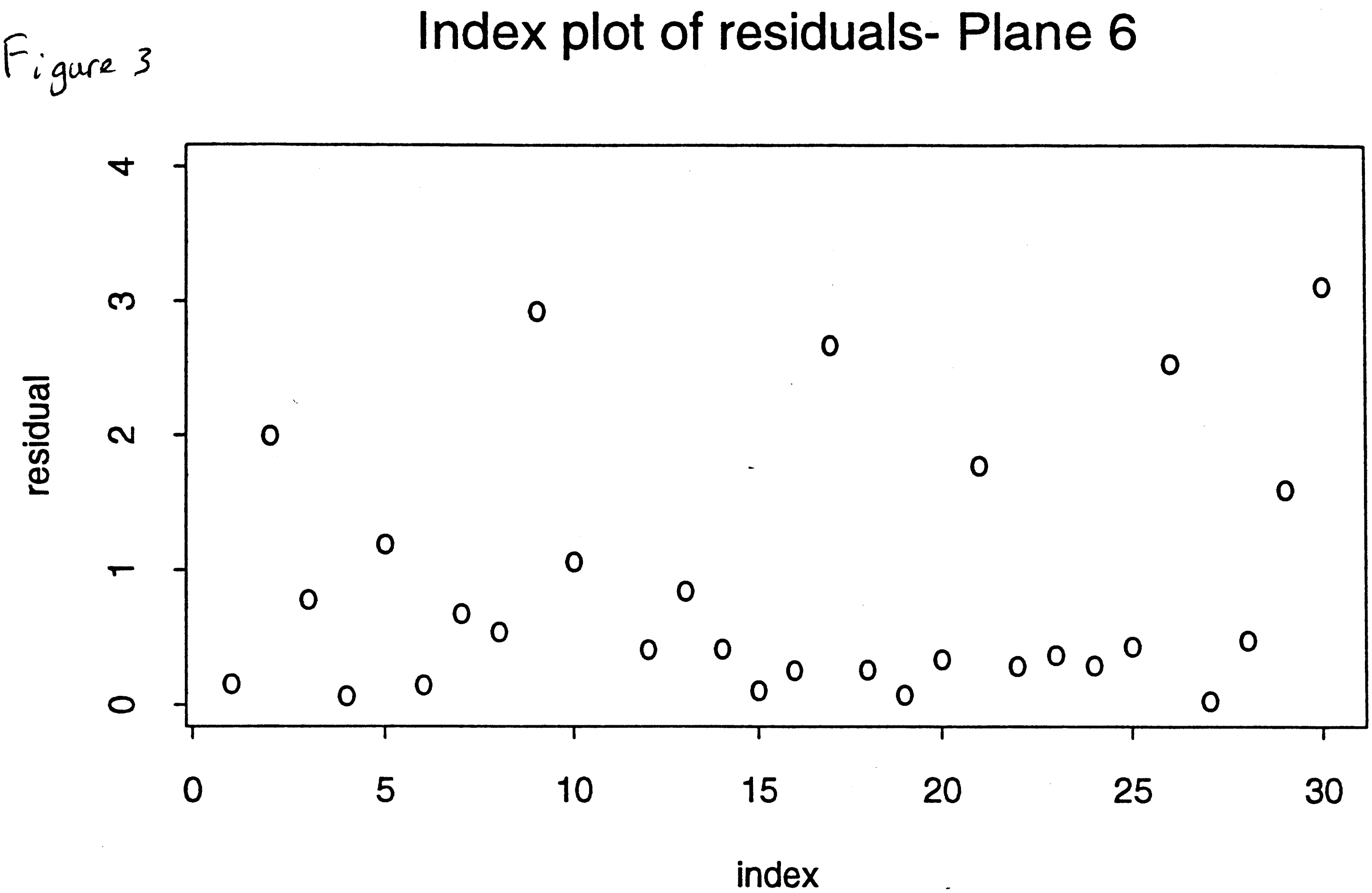
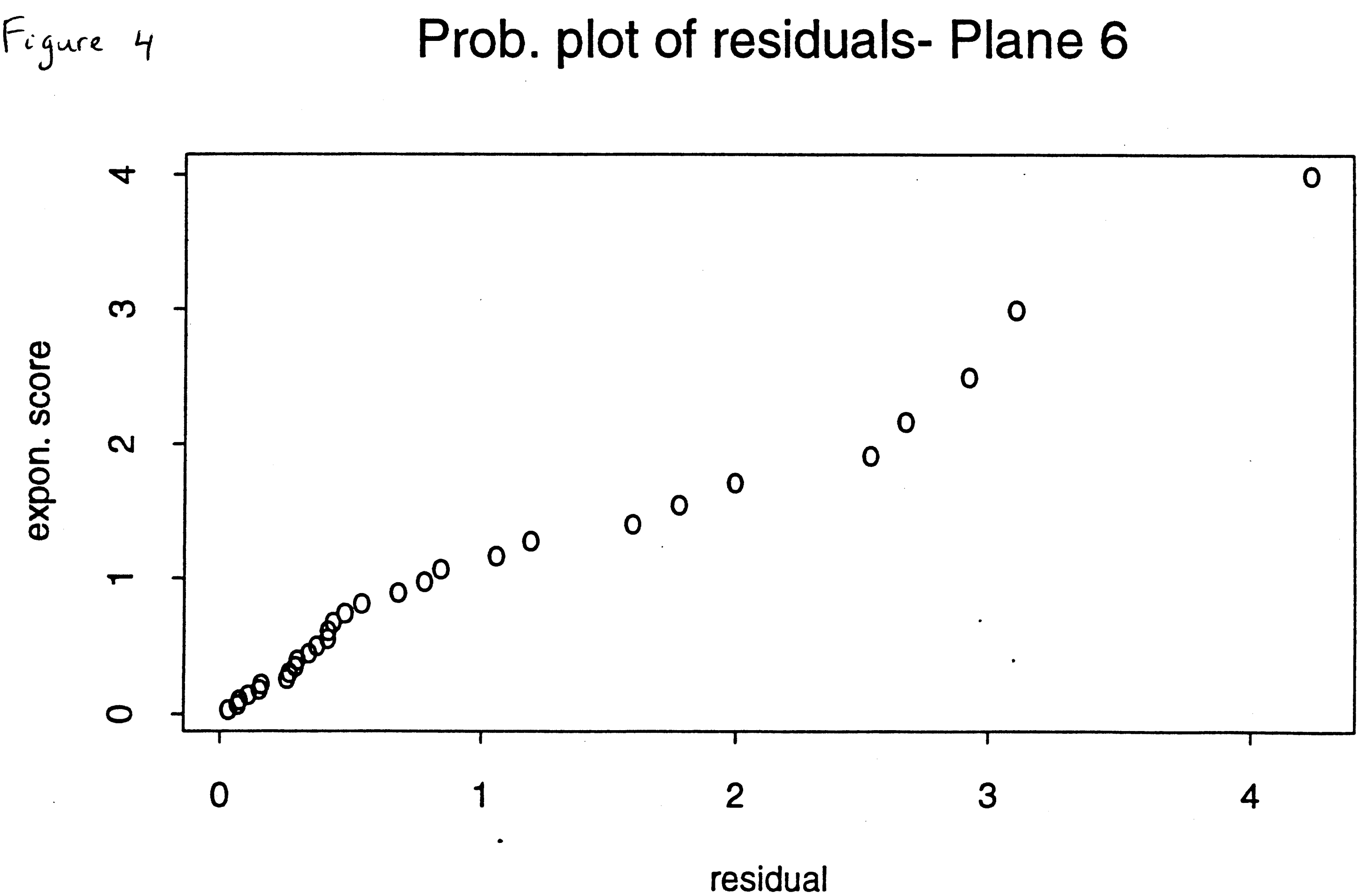
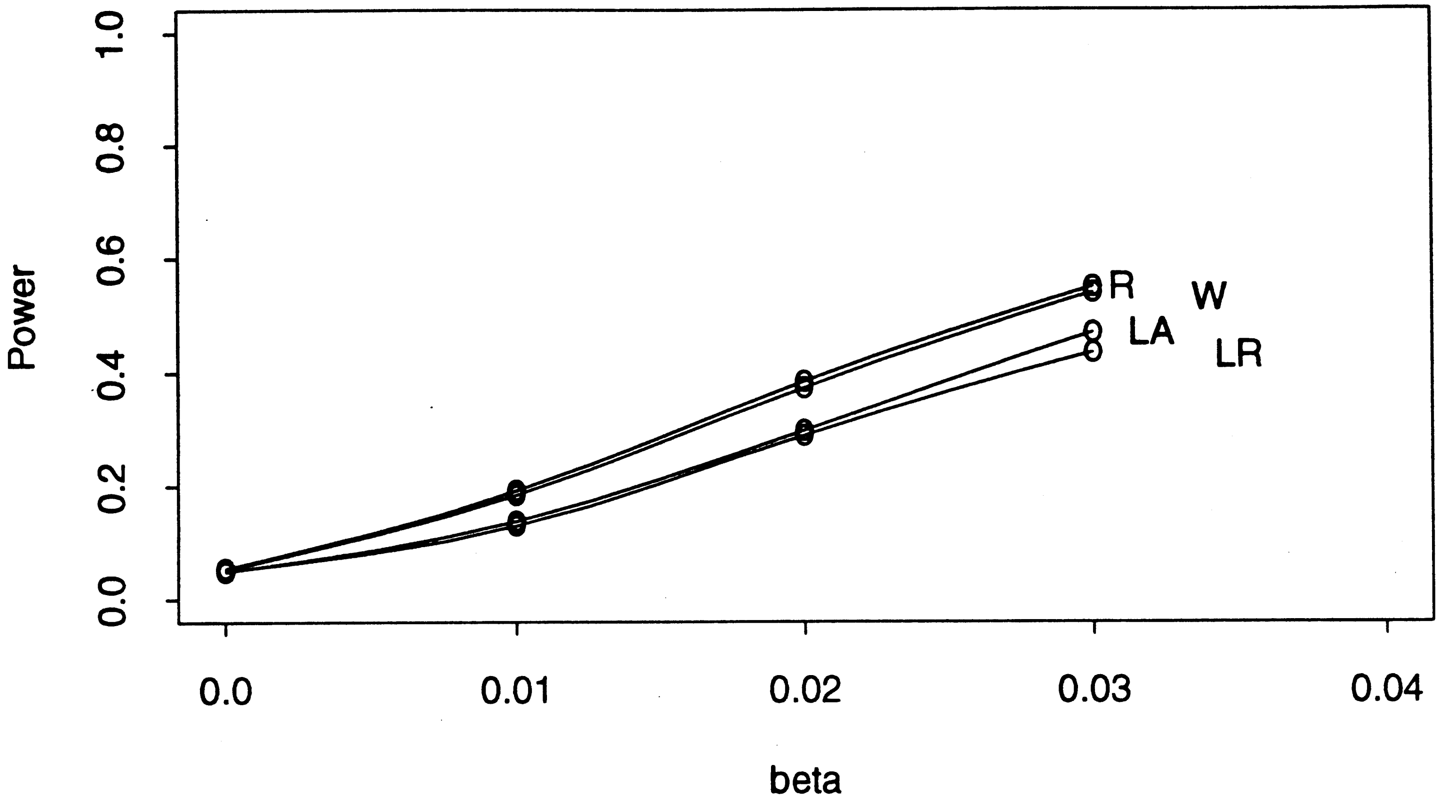


Figure 4



gure 5

Power of 4 tests: Model A, n=50



igure 6

Power of 3 tests: Model B, n=50