Analysis of Variation Transmission for Manufacturing Processes in the Presence of Measurement Error

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SUMMARY

Lawless, MacKay and Robinson (1996) have introduced methods for analysing variation in multi-stage manufacturing processes, the idea being to identify stages which contribute most to variation in the final product. Such methods are a valuable prioritization tool in variation reduction studies. Lawless et al. note, however, that when the data are observed with significant measurement error, substantial biases which mislead the investigator can result. The purpose of this article is to present methods that incorporate measurement error. We discuss both maximum likelihood estimation and a simpler "naive" method that is much easier to implement. The naive procedure is shown by simulation studies to provide an effective way of estimating components of variance associated with different stages of a manufacturing process.

KEY WORDS AND PHRASES: moment estimation, autoregressive models, repeated measures, variation reduction

Introduction

Manufacturing processes where items are produced in large quantity consist of various operations or stages. After the final stages the items must meet certain specifications with respect to quality characteristics, and how to control and reduce variation in these characteristics is a crucial issue. A key to reducing variation is to understand how much variation is added at different stages of a process, and to what extent that variation manifests itself downstream.

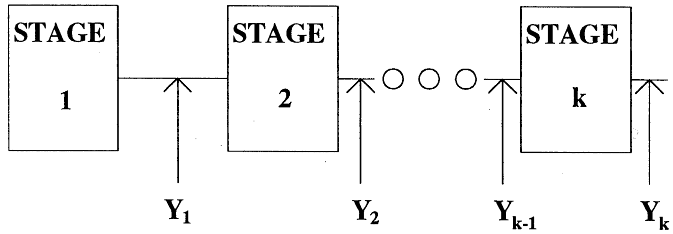
In a recent article, Lawless, MacKay and Robinson (1996) introduced methods for analysing the transmission of variation in multi-stage processes. Their work is based on the idea of measuring individual parts as they progress through stages; using a simple autoregressive model they estimate the amount of variation that is added at each stage, and the amount that is transmitted from upstream. This allows components of variation in the final product's characteristics to be associated with different stages, thus providing guidance for variation reduction activities.

Data on quality characteristics generally involve some measurement error. Lawless et al. (1996) do not include it in their methodology, but point out that ignoring substantial measurement error can lead to wrong conclusions. The purpose of this article is to present methods that incorporate measurement error, thus avoiding such pitfalls. For a model that is an extension of Lawless et al.'s, we consider both maximum likelihood estimation and a simpler "naive" method that is much easier to implement. We also extend previous work by considering confidence intervals for variance components; Lawless et al. (1996) relied on point estimates. In addition, model checking methods are discussed.

Modeling

We start with a brief review of the model in Lawless, MacKay and Robinson (1996).

Consider a k stage process in which a single characteristic Y_i can be measured after each stage i. The final quality characteristic is Y_k. The process can be portrayed as follows:



To model this process, suppose

$$\begin{aligned} Y_1 &\sim N(\mu_1, \sigma_1^2) \\ Y_i | Y_1, \ldots, Y_{i-1} &\sim N(\alpha_i + \beta_i Y_{i-1}, \sigma_{iA}^2) \end{aligned} \tag{1}$$

Note that the distribution of Y_i given the history of the item up to stage $i-1$ depends only on Y_{i-1}. We subsequently refer to this model as the first order autoregressive model (AR(1)). Under this model, the variation in the final quality characteristic is partitioned as

$$Var(Y_k) = \sigma_k^2 = \sigma_{kA}^2 + \beta_k^2 \sigma_{k-1,A}^2 + \ldots + \beta_k^2 \beta_{k-1}^2 \ldots \beta_2^2 \sigma_1^2 \tag{2}$$

The component of variance $\beta_k^2 \beta_{k-1}^2 \ldots \beta_{i+1}^2 \sigma_{i,A}^2$ can be interpreted as the amount of variation in Y_k that is added at stage i and then transmitted through the remaining stages. The term σ_{kA}^2 is the amount of variation added at the final stage. Dividing both sides of (2) by Var(Y_k) gives

$$1 = \frac{\sigma_{kA}^2}{\sigma_k^2} + \frac{\beta_k^2 \sigma_{k-1,A}^2}{\sigma_k^2} + \ldots \frac{\beta_k^2 \beta_{k-1}^2 \ldots \beta_2^2 \sigma_1^2}{\sigma_k^2} \tag{3}$$

This form is useful for identifying stages which contribute a significant proportion of the variation in the final quality characteristic.

An equivalent parameterization of the AR(1) model is

$$\begin{aligned} Y_i &\sim N(\mu_i, \sigma_i^2) \\ Cov(Y_{i-1}, Y_i) &= \rho_{i-1,i} \sigma_{i-1} \sigma_i \qquad i \geq 2 \end{aligned}$$

We note that Cov(Y_{i-s}, Y_i) equals $\rho_{i-1,i} \rho_{i-2,i-1} \ldots \rho_{i-s,i-s+1} \sigma_{i-s} \sigma_i$. In this parameterization, equation (2) can be rewritten as

$$Var(Y_k) = \sigma_k^2(1 - \rho_{k-1,k}^2) + \sigma_k^2 \rho_{k-1,k}^2 (1 - \rho_{k-2,k-1}^2) + \ldots + \sigma_k^2 \rho_{k-1,k}^2 \ldots \rho_{1,2}^2$$

and equation (3) can be rewritten as

$$1 = (1 - \rho_{k-1,k}^2) + (\rho_{k-1,k}^2(1 - \rho_{k-2,k-1}^2)) + \ldots + (\rho_{k-1,k}^2 \cdots \rho_{1,2}^2) \tag{4}$$

To add measurement error to the model, suppose the measured values of the characteristic are $X_1, X_2, \ldots, X_k$ where

$$X_i = Y_i + \epsilon_i, \qquad \epsilon_i \sim N(0, \sigma_{\epsilon_i}^2) \tag{5}$$

We assume that the variances $\sigma_{\epsilon_i}^2$ are known, usually from preliminary studies on the measurement system.

The process $(X_1, X_2, \ldots, X_k)$ is no longer AR(1) if $k > 2$. In fact, the conditional distribution of $X_i|X_1, \ldots, X_{i-1}$ depends on all of $X_1, \ldots, X_{i-1}$.

Given observations $(X_1, X_2, \ldots, X_k)$ on n items, the goal is to estimate the proportions of variance (3). The presence of measurement error substantially complicates this problem.

Effects of Measurement Error if Ignored

We review the effects of ignoring measurement error (Lawless et al., 1996), since this will motivate and set notation for what follows. To demonstrate the effect of ignoring measurement error in the identification of the variance proportions (3), consider first a two stage process in which

$$X_1 \sim N(\mu_1, \sigma_1^2 + \sigma_{\epsilon_1}^2) \qquad X_2 \sim N(\alpha_2 + \beta_2\mu_1, \sigma_{2A}^2 + \beta_2^2\sigma_1^2 + \sigma_{\epsilon_2}^2) \tag{6}$$

with $Cov(X_1, X_2) = \beta_2\sigma_1^2$.

Suppose n items are tracked through the process so that we have data (x_{1j}, x_{2j}; j=1, $\ldots, n$) and we estimate the variance components assuming that the AR(1) model is appropriate, that is, assuming $\sigma_{\epsilon_1} = \sigma_{\epsilon_2} = 0$. Then the maximum likelihood estimates of the parameters in the variance components in model (1) are (Lawless et al., 1996)

$$\hat{\sigma}_1^2 = \frac{S_{x_1x_1}}{n} \qquad \hat{\beta}_2 = \frac{S_{x_1x_2}}{S_{x_1x_1}} \qquad \hat{\sigma}_2^2 = \frac{S_{x_2x_2}}{n}$$

$$\text{where} \qquad S_{x_ix_j} = \sum_{k=1}^{n}(x_{ik} - \bar{x}_i)(x_{jk} - \bar{x}_j).$$

Note that as $n \to \infty$, $\frac{S_{x_i x_j}}{n} \to Cov(X_i, X_j)$, where "$\to$" denotes convergence in probability, so that

$$\hat{\sigma}_2^2 \to \sigma_2^2 + \sigma_{\epsilon_2}^2 \qquad \hat{\beta}_2 \to \beta_2 \frac{\sigma_1^2}{\sigma_1^2 + \sigma_{\epsilon_1}^2} \qquad \hat{\sigma}_1^2 \to \sigma_1^2 + \sigma_{\epsilon_1}^2$$

In the partition (3),

$$1 = \frac{\sigma_{2A}^2}{\sigma_2^2} + \frac{\beta_2^2 \sigma_1^2}{\sigma_2^2}$$

the estimates are such that as $n \to \infty$

$$\frac{\hat{\beta}_2^2 \hat{\sigma}_1^2}{\hat{\sigma}_2^2} \to \frac{\beta_2^2 \sigma_1^2}{\sigma_2^2} \left(\frac{\sigma_1^2}{\sigma_1^2 + \sigma_{\epsilon_1}^2}\right)\left(\frac{\sigma_2^2}{\sigma_2^2 + \sigma_{\epsilon_2}^2}\right)$$

Hence, the variation transmitted from stage 1 is underestimated. Since the estimates of the proportions must also sum to one, this implies that the variation added at stage 2 is overestimated. If the measurement system contributes 20% of the variation in X_1 and X_2, then the asymptotic bias is substantial.

Suppose we expand this to a process with three stages. If we ignore the measurement error, then we would use the estimates

$$\hat{\sigma}_1^2 = \frac{S_{x_1 x_1}}{n} \qquad \hat{\sigma}_{2A}^2 = \frac{S_{x_2 x_2}}{n} - \hat{\beta}_2 \frac{S_{x_1 x_2}}{n} \qquad \hat{\sigma}_{3A}^2 = \frac{S_{x_3 x_3}}{n} - \hat{\beta}_3 \frac{S_{x_2 x_3}}{n}$$

$$\hat{\beta}_2 = \frac{S_{x_1 x_2}}{S_{x_1 x_1}} \qquad \hat{\beta}_3 = \frac{S_{x_2 x_3}}{S_{x_2 x_2}}$$

Then, the proportions of variance contributed according to (3) are

$$1 = \frac{\hat{\sigma}_{3A}^2}{\hat{\sigma}_3^2} + \frac{\hat{\beta}_3^2 \hat{\sigma}_{2A}^2}{\hat{\sigma}_3^2} + \frac{\hat{\beta}_3^2 \hat{\beta}_2^2 \hat{\sigma}_1^2}{\hat{\sigma}_3^2}$$

Using the above estimates,

$$\frac{\hat{\sigma}_{3A}^2}{\hat{\sigma}_3^2} \to \frac{\sigma_{3A}^2}{\sigma_3^2}\left(\frac{\sigma_3^2}{\sigma_3^2 + \sigma_{\epsilon_3}^2}\right) + \frac{\sigma_{\epsilon_3}^2}{\sigma_3^2 + \sigma_{\epsilon_3}^2} + \frac{\beta_3^2 \sigma_2^2 \sigma_{\epsilon_2}^2}{(\sigma_2^2 + \sigma_{\epsilon_2}^2)(\sigma_3^2 + \sigma_{\epsilon_3}^2)}$$

$$\frac{\hat{\beta}_3^2 \hat{\sigma}_{2A}^2}{\hat{\sigma}_3^2} \to \frac{\beta_3^2 \sigma_{2A}^2}{\sigma_3^2}\left(\frac{\sigma_3^2}{\sigma_3^2 + \sigma_{\epsilon_3}^2}\right)\left(\frac{\sigma_2^2}{\sigma_2^2 + \sigma_{\epsilon_2}^2}\right)^2\left(\frac{\sigma_2^2 + \sigma_{\epsilon_2}^2 - \beta_2^2 \sigma_1^2 \left(\frac{\sigma_1^2}{\sigma_1^2 + \sigma_{\epsilon_1}^2}\right)}{\sigma_2^2 - \beta_2^2 \sigma_1^2}\right)$$

$$\frac{\hat{\beta}_3^2 \hat{\beta}_2^2 \hat{\sigma}_1^2}{\hat{\sigma}_3^2} \to \frac{\beta_3^2 \beta_2^2 \sigma_1^2}{\sigma_3^2}\left(\frac{\sigma_3^2}{\sigma_3^2 + \sigma_{\epsilon_3}^2}\right)\left(\frac{\sigma_2^2}{\sigma_2^2 + \sigma_{\epsilon_2}^2}\right)^2\left(\frac{\sigma_1^2}{\sigma_1^2 + \sigma_{\epsilon_1}^2}\right) \tag{7}$$

While it is clear that the proportion of variance transmitted from the first stage is underestimated, the direction of bias for the other two proportions is not obvious. In fact, the bias of the variance added at the third stage is always positive, which can be seen by writing it in the other parameterization.

For a numerical example illustrating these results, see the sixth section.

Two Stages

In the situation described above, it is possible to develop maximum likelihood estimates to take account of the measurement error. Recall that the distribution of (X_1, X_2) was given in equation (6). X_1 and X_2 have a bivariate normal distribution, and there are five functionally independent unknown parameters, μ_1, σ_1, α_2, β_2, σ_{2A} in the model. Equivalently, we may take the parameters to be $\mathrm{E}(X_1)$, $\mathrm{Var}(X_1)$, $\mathrm{E}(X_2)$, $\mathrm{Var}(X_2)$, and $\mathrm{cov}(X_1, X_2)$. The maximum likelihood estimates of these parameters are (Larsen et al., 1986)

$$\hat{E}(X_1) = \bar{x}_1, \qquad \hat{E}(X_2) = \bar{x}_2, \qquad V\hat{a}r(X_1) = \frac{S_{x_1x_1}}{n}, \qquad V\hat{a}r(X_2) = \frac{S_{x_2x_2}}{n}$$

$$C\hat{o}v(X_1X_2) = \frac{S_{x_1x_2}}{n}$$

We then get the following maximum likelihood estimates for the original parameters by the invariance property:

$$\hat{\mu}_1 = \bar{x}_1 \qquad \hat{\sigma}_1^2 = \frac{S_{x_1x_1}}{n} - \sigma_{\epsilon_1}^2 \qquad \hat{\beta}_2 = \frac{S_{x_1x_2}}{S_{x_1x_1} - n\sigma_{\epsilon_1}^2}$$

$$\hat{\alpha}_2 = \bar{x}_2 - \hat{\beta}_2\bar{x}_1 \qquad \hat{\sigma}_{2A}^2 = \frac{S_{x_2x_2}}{n} - \frac{nS_{x_1x_2}^2}{(S_{x_1x_1} - n\sigma_{\epsilon_1}^2)} - \sigma_{\epsilon_2}^2$$

In the other parameterization for this model $(\mu_1, \sigma_1, \mu_2, \sigma_2, \rho_{12})$,

$$\hat{\mu}_1 = \bar{x}_1 \qquad \hat{\mu}_2 = \bar{x}_2 \qquad \hat{\sigma}_1^2 = \frac{S_{x_1x_1}}{n} - \sigma_{\epsilon_1}^2 \qquad \hat{\sigma}_2^2 = \frac{S_{x_2x_2}}{n} - \sigma_{\epsilon_2}^2$$

$$\hat{\rho}_{12} = \frac{S_{x_1x_2}}{\sqrt{(S_{x_1x_1} - n\sigma_{\epsilon_1}^2)(S_{x_2x_2} - n\sigma_{\epsilon_2}^2)}}$$

Recall that we are interested in the estimates of the proportions of the variance of Y_2 (4), which in terms of μ_1, μ_2, σ_1, σ_2 and ρ_{12} is

$$1 = (1 - \hat{\rho}_{12}^2) + \hat{\rho}_{12}^2$$

It is possible to get approximate variance estimates for these proportions, by observing that the cross product matrix has a Wishart distribution (Mardia et al., 1979)

$$S = \begin{bmatrix} S_{x_1x_1} & S_{x_1x_2} \\ S_{x_1x_2} & S_{x_2x_2} \end{bmatrix} \sim W_2(\Sigma, n-1)$$

where

$$\Sigma = \begin{bmatrix} \sigma_1^2 + \sigma_{\epsilon_1}^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 + \sigma_{\epsilon_2}^2 \end{bmatrix}$$

This gives us that (Magnus et al., 1979):

$$E(S) = (n-1)\Sigma$$

and that

$$Var\begin{bmatrix} S_{x_1x_1} \\ S_{x_1x_2} \\ S_{x_2x_2} \end{bmatrix} = \left[\begin{array}{c|c|c} 2(n-1)(\sigma_1^2+\sigma_{\epsilon_1}^2)^2 & 2(n-1)\rho_{12}\sigma_1\sigma_2 * (\sigma_1^2+\sigma_{\epsilon_1}^2) & 2(n-1)\rho_{12}^2\sigma_1^2\sigma_2^2 \\ \hline 2(n-1)\rho_{12}\sigma_1\sigma_2 * (\sigma_1^2+\sigma_{\epsilon_1}^2) & (n-1)\{(\sigma_1^2+\sigma_{\epsilon_1}^2) * (\sigma_2^2+\sigma_{\epsilon_2}^2) + \rho_{12}^2\sigma_1^2\sigma_2^2\} & 2(n-1)\rho_{12}\sigma_1\sigma_2 * (\sigma_2^2+\sigma_{\epsilon_2}^2) \\ \hline 2(n-1)\rho_{12}^2\sigma_1^2\sigma_2^2 & 2(n-1)\rho_{12}\sigma_1\sigma_2 * (\sigma_2^2+\sigma_{\epsilon_2}^2) & 2(n-1)(\sigma_2^2+\sigma_{\epsilon_2}^2)^2 \end{array}\right] \quad (8)$$

Hence we can conclude that

$$Var((1-\hat{\rho}_{12}^2)) \approx F * V * F^T$$

where

$$F = [\frac{(n-1)^2\rho_{12}^2\sigma_1^2\sigma_2^2}{\{(n-1)\sigma_1^2-\sigma_{\epsilon_1}^2\}\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}^2}, \frac{-2(n-1)\rho_{12}\sigma_1\sigma_2}{\{(n-1)\sigma_1^2-\sigma_{\epsilon_1}^2\}\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}}, \frac{(n-1)^2\rho_{12}^2\sigma_1^2\sigma_2^2}{\{(n-1)\sigma_1^2-\sigma_{\epsilon_1}^2\}^2\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}}]$$

and V is the variance-covariance matrix given in (8). Approximate variances for the components of variance can be found analogously, and are given in Agrawal (1997).

Three or More Stages

Maximum likelihood estimation

Maximum likelihood estimates do not have closed form expressions for models with more than two stages. The number of functionally independent parameters in an AR(1) k-stage process observed with measurement error is 3k-1 (two parameters for the initial stage and three more for every additional stage). The number of independent parameters in a general multivariate normal, however, is $k + \frac{k(k+1)}{2}$ (k parameters for the mean, and $\frac{k(k+1)}{2}$ variance-covariance parameters). In the case when k=2, these values are the same and the parameterization (μ_1, σ_1, α_1, β_1, σ_{A1}) is equivalent to (E(X_1), Var(X_1), E(X_2), Var(X_2),Cov(X_1, X_2)). For $k > 2$ the general multivariate normal has more parameters, and a one to one mapping between the two sets of parameters does not exist.

If, in the three stage case, we presume the existence of an underlying AR(1) process (1) for Y_1, Y_2, Y_3, but that what we observe is X_1, X_2, X_3, given by (5), we can parameterize the joint distributions of these variables as follows:

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim MVN\left(\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}, \mathbf{\Sigma_y} = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{12}\rho_{23}\sigma_1\sigma_3 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 \\ \rho_{12}\rho_{23}\sigma_1\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \sigma_3^2 \end{bmatrix}\right)$$

and

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim MVN\left(\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}, \mathbf{\Sigma_x} = \begin{bmatrix} \sigma_1^2 + \sigma_{\epsilon_1}^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{12}\rho_{23}\sigma_1\sigma_3 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 + \sigma_{\epsilon_2}^2 & \rho_{23}\sigma_2\sigma_3 \\ \rho_{12}\rho_{23}\sigma_1\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \sigma_3^2 + \sigma_{\epsilon_3}^2 \end{bmatrix}\right) \quad (9)$$

In this case, the proportions of the variance of Y_3 can be expressed as

$$Var(Y_3) = (1 - \rho_{23}^2) + \rho_{23}^2(1 - \rho_{12}^2) + \rho_{23}^2\rho_{12}^2 \quad (10)$$

where the first term is the proportion of variation added at the third stage, the second term is the proportion of variation added at the second stage and transmitted to the third stage, and the third term is the proportion of variation transmitted from the first stage of the process.

The goal here is to estimate these three proportions based on independent observations (x_{1j}, x_{2j}, x_{3j}), $j = 1, \ldots, n$. This involves estimating the eight unknown parameters in the distribution (9).

The multivariate normal likelihood of (X_1, X_2, X_3) can be expressed as (Johnson et al., 1988)

$$\begin{aligned} L(\mu, \Sigma_x) &= \frac{1}{(2\pi)^{3n/2}|\Sigma_x|^{n/2}} \exp\{-\mathrm{trace}[\Sigma_x^{-1}(\sum_{j=1}^{n}(\mathbf{x_j} - \bar{x})(\mathbf{x_j} - \bar{x})^T)]/2 \\ & -n/2(\bar{x} - \mu)^T \Sigma_x^{-1}(\bar{x} - \mu)\} \end{aligned}$$

where $\mathbf{x_j} = (x_{ij}, x_{2j}, x_{3j})^T$. If we write

$$\sum_{j=1}^{n}(\mathbf{x_j} - \bar{x})(\mathbf{x_j} - \bar{x})^T = \begin{bmatrix} S_{x_1x_1} & S_{x_1x_2} & S_{x_1x_3} \\ S_{x_1x_2} & S_{x_2x_2} & S_{x_2x_3} \\ S_{x_1x_3} & S_{x_2x_3} & S_{x_3x_3} \end{bmatrix} = S_{xx}$$

then the loglikelihood can be written as

$$l(\mu, \Sigma_x) = -(3n/2)\log(2\pi) - (n/2)\log|\Sigma_x| - (1/2)trace\{\Sigma_x^{-1}S_{xx}\} - (n/2)(\bar{x} - \mu)^T\Sigma_x^{-1}(\bar{x} - \mu) \tag{11}$$

It is known (Johnson et al., 1988) that for any Σ, this likelihood is maximized with respect to μ by $\hat{\mu}_i = \bar{x}_i$, (i = 1,2,3). It remains, then, to determine the values of the three variance parameters and the two correlation parameters that will maximize the likelihood. There is not a closed form algebraic expression for any of these parameters, and the estimates must be determined numerically. This is computer intensive and time consuming. If confidence intervals for variance components are also desired, additional computation will be needed. In the next section, we present a simpler method that performs very well.

Naive estimates

Simple estimates for a k-stage process can be obtained by using the two stage maximum likelihood estimates obtained earlier for each pair of consecutive stages. In the parameterization used in Section 5.1, this leads to the following estimates for the three stage case:

$$\tilde{\mu}_i = \bar{x}_i, \qquad \tilde{\sigma}_i^2 = \frac{S_{x_i x_i}}{n} - \sigma^2_{\epsilon_i} \qquad\qquad i = 1,2,3$$

$$\tilde{\rho}_{i,i+1} = \frac{S_{x_i x_{i+1}}}{\sqrt{(S_{x_i x_i} - n\sigma^2_{\epsilon_i})(S_{x_{i+1} x_{i+1}} - n\sigma_{\epsilon^2_{i+1}})}} \qquad\qquad i = 1,2$$

Proving consistency of these estimators is straightforward. Details are given in Agrawal (1997).

In simulations, it was found that the distributions of the estimates of the square roots of the individual variance proportions could be well approximated by normal distributions. This is also true for the estimates of the square roots of the variance components. Hence, it is useful to find confidence intervals in this metric. Calculating the asymptotic variance of these quantities can be done analogously to the method for the results shown in the previous section. See the appendix for the approximate variances of the square root of the proportions of variance at each of the three stages. Similar formulas for the approximate variances of the square root of the components are given in Agrawal (1997). An approximate 98% confidence interval can be computed using the formula

$$\text{estimated proportion} \pm 2.33\sqrt{\hat{Var}(\text{estimated proportion})} \qquad (12)$$

where $\hat{Var}$(estimated proportion) is found using the approximate formula and replacing the true values of the parameters by their estimates.

Parametric bootstrap calculations can also be used to get approximate confidence intervals. Once estimates for the parameters of the model have been found, these values can be used as the "true" values in generating N "bootstrap" samples of size n, the original sample size. Estimates of the variance components can be computed from each of the N samples, and confidence intervals calculated from them. For example, to get a 90% confidence interval, we could take N=99, and select the 5th and 95th values of the ordered estimates as the lower and upper limit for each variance component. For more details on parametric bootstrapping to compute confidence intervals, see Efron and Tibshirani (1986).

Simulation Results

We would like to know how the naive estimators compare to the maximum likelihood estimators. In addition, we want to know how well confidence intervals for variance components perform in terms of giving close to the stated coverage. These questions were addressed in a simulation study in which a three stage process was considered. This section will describe how that study was carried out and its results.

Since the values of the variances at the three stages do not change the properties of the estimators, they were set to always be one. For the same reason, the means at all three stages were set to zero. The variables that were manipulated were ρ_{12}, ρ_{23} and σ_{ϵ_i}. In this simulation, the measurement error was set to be the same at all stages, since this often occurs when the same characteristic is measured at each stage of the process. Three levels for each of ρ_{12} and ρ_{23} were used, $\sqrt{0.2}, \sqrt{0.5}$ and $\sqrt{0.8}$. These values were chosen because they provide a wide range of different values being added and transmitted through the process. Hence, values of the first variance proportion in (10) range from 0.2 to 0.8, while values of the second and third variance proportions range from 0.04 to 0.64. In this case, the values of the proportions are the same as the components. Please see Table 1 for the exact quantities.

Two levels of σ_{ϵ_i} were chosen, 0.1 and 0.3. At the high level of measurement error, the ratio $\sigma_{\epsilon_i}/\sigma_i$ is 30%. This level of measurement error would be unacceptable in some applications in industry; anything higher would call for a different measurement system. Note that even at the low measurement error level, and in the case of three stages, the bias in estimation resulting from ignoring the measurement error can be substantial. Bias here refers to the difference between the mean of a variance proportion estimate in large samples, as given in (7) and the true value of the variance proportion. Table 2 reproduces the variance proportions of Table 1 for each scenario and shows the bias that results if measurement error is ignored.

These combinations of factors were used for an 18 run simulation. At each run, 99 samples (X_1, X_2, X_3) of 99 units were created using the given set of values of ρ_{12}, ρ_{23} and σ_{ϵ_i} as true parameters. For each sample, both the maximum likelihood estimates and the naive estimates were found, and the three variance components were calculated. Then, 99 bootstrap samples were created using each set of estimates as the real parameters. The lowest and the highest values of the estimated variance components from these bootstrap samples were used to specify 98% confidence intervals for the components for each sample. Only 99 bootstrap samples were done here to keep the time limitations of the simulation feasible. In an industrial setting, computing 1000 bootstrap samples would be recommended.

The results of the simulation are given in Tables 3-6. Table 3 shows the average value of the maximum likelihood estimates and the naive estimates for each run, for the first variance proportion. Also included are the standard deviation estimates of the run. Tables 4 and 5 show the same for the second and third variance proportions, respectively. Table 6 gives the coverage frequencies of the bootstrap-based confidence intervals for both the maximum likelihood estimates and the naive estimates for both the first variance component ("Raw") and the first variance proportion ("Proportion"). Recall that this theoretical coverage frequency is 98%. No major discrepancies in coverage frequency are seen. The figures for the second and third variance components and proportions were similar.

These results indicate that the performances of the naive estimates and the maximum likelihood estimates are virtually indistinguishable. In fact, the estimates are very close to each other in most cases. This can be seen in Figures 1-3, which show the naive estimates plotted against the maximum likelihood estimates for each of the variance proportions and for six runs: both levels of measurement error for ρ_{12} and ρ_{23} both at their low levels, ρ_{12} and ρ_{23} both at their medium levels, and ρ_{12} and ρ_{23} both at their high levels. The line on these plots is the Y=X line. These plots represent a sample of such plots for all the runs. Figure 1 illustrates the two runs in which the least difference between naive and maximum likelihood estimates was observed, whereas Figure 3 represents the runs in which this difference was the highest. An interesting feature that can be seen is that regardless of the ρ_{12} or ρ_{23} values, the naive estimates are closer to the maximum likelihood estimates when the measurement error is low, as compared to when it's high. This is expected, since we know that the estimators are the same when there is no measurement error.

Overall, the data suggest that in the three stage case, the naive estimates can be substituted for the maximum likelihood estimates in any situation likely to be encountered in practice. There is little justification for spending time computing the maximum likelihood estimates, when the naive estimates can be found faster and without the use of optimization methods.

Other simulations were done to check the coverage frequencies of the confidence intervals given in equation (12) for various values of the true parameters. For a given set of true parameters, 1000 data sets of sample size 99 were generated. For each data set, the naive estimates of the square root of the variance components and proportions were found. Their approximate variances were calculated using these estimates, and a 98% confidence interval was computed using equation (12). Then, the coverage frequency for that set of real parameters was found by counting how many of the 1000 intervals actually contained the true parameters. See Tables 7-9 for these values. Overall, the coverage frequencies achieved were very close to 98%. This suggests that the approximate variance formulas given in equations (15), (16) and (17) (see the appendix) are useful in finding confidence intervals, which further strengthens the argument for using the naive estimates.

It seems that both the bootstrapping and the approximate variance formulas are satisfactory methods of finding confidence intervals for the sample size considered here (n=99). For small sample sizes, however, one might expect the bootstrap method to be more accurate.

Model Checking

It is important to check whether observed data are consistent with an AR(1) process with known measurement error. As indicated in (9), this model implies that the observed measurements follow a multivariate normal distribution.

As a first step in evaluating the multivariate normal assumption, the normality of the univariate marginal distributions should be checked. This can be done graphically using QQ plots (see Johnson et al, 1988, p.146). If the marginal distributions do not seem normal, then the multivariate normal assumption can be rejected. If they do seem normal, however, the assumption of the linearity of the conditional means should be verified. Given data on $X_1, X_2, \ldots, X_k$, this can be done by plotting X_i against X_j for all i and j, with $i \neq j$, and determining whether a linear model is adequate. Again, if this assumption is contradicted, the multivariate normal assumption should be rejected. Other, more formal tests can be applied to test for multivariate normality (Looney, 1995).

If measurement error is considered negligible we can check the AR(1) assumption by examining the regression of X_i on X_{i-1}, X_{i-2}, etc. More generally, we can test the adequacy of the AR(1) model or the AR(1) with measurement error model via a likelihood ratio test, as follows. Under a general multivariate normal structure, the maximum likelihood estimates are

$$\hat{\mu} = \bar{X} \qquad \hat{\Sigma} = \frac{S_{xx}}{n} \tag{13}$$

[9] and so the maximized loglikelihood takes the form

$$\begin{aligned} l(\hat{\Omega}) &= -\frac{n}{2}log|\frac{S_{xx}}{n}| - \frac{np}{2}log(2\pi) - \frac{1}{2}\text{trace}\{(\frac{S_{xx}}{n})^{-1}S_{xx}\} \\ &= -\frac{n}{2}log|\frac{S_{xx}}{n}| - \frac{np}{2} - \frac{np}{2}log(2\pi) \end{aligned}$$

Under the constraint of being an AR(1) process with measurement error, the maximized loglikelihood takes the form

$$l(\hat{\omega}) = -\frac{n}{2}log|\hat{\Sigma}_x| - \frac{1}{2}\text{trace}\{\hat{\Sigma}_x^{-1}S_{xx}\} - \frac{np}{2}log(2\pi) \tag{14}$$

where Σ_x is of the form given in (9), and an estimate of it has been found by optimizing (11). From the theory of the likelihood ratio test,

$$-2(l(\hat{\omega}) - l(\hat{\Omega})) \approx \chi^2_{\frac{k(k-3)}{2}+1}$$

In simulations for the case k=3, it was found that the distribution of the statistic given above could not be distinguished from χ^2_1, for sample sizes as small as 30. This was true even when $l(\hat{\omega})$ was approximated by evaluating (14) using naive estimates. This means that a simple approximate test can be carried out for the AR(1) model with measurement error without needing to compute the maximum likelihood estimates for the model.

Using the above likelihood expression, the deviance residuals can be examined to see if any observations are particularly influential. See, for example, Williams (1987).

Piston Example

A piston is a part in an automobile located in the engine cylinder, the basic framework of the engine. The piston is essentially a cylinder closed at the top and open at the bottom, where is it connected to a rod. The piston moves in a vertical motion in the engine cylinder, pushing out exhaust on the upstroke and intaking fuel on the downstroke (Crouse, 1970).

A study was done on 96 pistons as they were passing through a production line. Each of the 96 pistons studied had many observations recorded on it, but we will consider only three. These are the diameter of the piston at a height of 4 mm, at each of three stages. The three stages were the final machining stages of the piston production line. It was believed that these stages were not changing the diameters of interest at all. All diameters were measured in millimetres, and it is known that the measurement error standard deviation here is approximately $5 * 10^{-4}$ mm, or 0.5 microns, at each stage. This gives an estimated ratio of $\sigma_\epsilon/\sigma_3 = 22\%$.

Engineering knowledge of the process indicated that the normal AR(1) model with measurement error should adequately describe it. Various model checks were used to determine the adequacy of this model. While the QQ plots at each stage did not reveal any significant departures from normality, the deviance residuals of three pistons proved to be particularly influential. Scatter plots of pairs of measurements also showed these three points as outliers. Hence, these outliers were removed from the subsequent analysis, although some investigation should be done to seek causes for why these particular pistons may have differed from the rest.

The goal of this study is to determine how the variation at the final stage of the process can be attributed to variation transmitted from upstream. When the measurement error is ignored, the proportions of variance contributed according to the AR(1) model are 0.256

at the third stage, 0.244 at the second stage, and 0.500 at the first stage. Using the naive estimates introduced in section five and the known measurement error variance, however, we find instead that the proportion of variance contributed is 0.181 at the third stage, 0.206 at the second stage and 0.613 at the first stage.

Both the bootstrap technique and the approximate variance method described earlier were used to find 98% confidence intervals for these proportions. In the first case, 1000 bootstrap samples were simulated using the naive estimates as the real values, and new estimates for the proportions were computed. The 10th and the 990th ordered values were then found to give the following confidence intervals

$$\begin{aligned} \text{Prop. from 3rd stage}: &\quad (0.088, 0.300) \\ \text{Prop. from 2nd stage}: &\quad (0.105, 0.322) \\ \text{Prop. from 1st stage}: &\quad (0.466, 0.750) \end{aligned}$$

In the case of the approximate variance method, the naive estimates were substituted into equations (15),(16),(17) (see the appendix) and (12) to give the confidence intervals

$$\begin{aligned} \text{Prop. from 3rd stage}: &\quad (0.092, 0.299) \\ \text{Prop. from 2nd stage}: &\quad (0.115, 0.323) \\ \text{Prop. from 1st stage}: &\quad (0.475, 0.769) \end{aligned}$$

The two sets of confidence intervals agree well. The main conclusion is that the first stage contributes most of the variation.

The analysis done using the maximum likelihood estimates yielded the same conclusions as that done with the naive estimates.

Finally, it should be mentioned that the likelihood ratio test given in the previous section was carried out, and was found to yield

$$l(\hat{\omega}) = 1423.91$$

under the assumption of the AR(1) model with measurement error. Under the full model,

$$l(\hat{\Omega}) = 1424.83$$

Using the approximating chi-square distribution on one degree of freedom, this gives a p-value of 0.173, indicating no reason to reject the measurement error model. The likelihood ratio test was also done for the AR(1) model without measurement error, and was found to give a likelihood of 1421.54, which when compared to the full model gives a p-value of 0.010, suggesting that this model does not describe the data adequately.

Future Work

We have seen how variance transmission can be studied in a multi-stage process, and how measurement error in the data can be taken into account in the analysis. In the three stage case, there is no advantage in using the maximum likelihood estimates instead of the naive estimates. This presumably holds for the case of four or more stages as well, and future work could be directed at determining this.

There are several issues left to be looked at in connection with variation transmission analysis. For instance, it is often time consuming and expensive to track items through a process, whereas it is much easier to take measurements on a large sample of items after any given stage. Could such "cross-sectional" data be useful in variation transmission analysis? The naive estimates introduced here can be extended in an obvious way to this situation.

Other important issues are how to deal with missing data, and how to incorporate multivariate data. Fong and Lawless (1996) have proposed the use of the Kalman filter for estimation in multivariate models.

Appendix

The purpose of this appendix is to give the approximate variance estimates of the square root of the proportion of variance contributed at each stage. These approximate variance estimates are computed by finding the parameters of the Wishart distribution of the cross product matrix, and using these in a first order Taylor series expansion of the function. The variance of the square root of the first proportion is

$$Var(\sqrt{1-\tilde{\rho}_{23}^2}) \approx \frac{1}{4f}\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}\begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{12} & v_{22} & v_{23} \\ v_{13} & v_{23} & v_{33} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{15}$$

where

$$\begin{aligned}
f &= 1 - \frac{(n-1)^2\rho_{23}^2\sigma_2^2\sigma_3^2}{\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}\{(n-1)\sigma_3^2-\sigma_{\epsilon_3}^2\}} \\
u_1 &= \frac{(n-1)^2\rho_{23}^2\sigma_2^2\sigma_3^2}{\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}\{(n-1)\sigma_3^2-\sigma_{\epsilon_3}^2\}^2} \\
u_2 &= \frac{-2(n-1)\rho_{23}\sigma_2\sigma_3}{\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}\{(n-1)\sigma_3^2-\sigma_{\epsilon_3}^2\}} \\
u_3 &= \frac{(n-1)^2\rho_{23}^2\sigma_2^2\sigma_3^2}{\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}^2\{(n-1)\sigma_3^2-\sigma_{\epsilon_3}^2\}} \\
v_{11} &= 2(n-1)(\sigma_3^2+\sigma_{\epsilon_3}^2)^2 \\
v_{12} &= 2(n-1)\rho_{23}\sigma_2\sigma_3(\sigma_3^2+\sigma_{\epsilon_3}^2) \\
v_{13} &= 2(n-1)\rho_{23}^2\sigma_2^2\sigma_3^2 \\
v_{22} &= (n-1)\{(\sigma_2^2+\sigma_{\epsilon_2}^2)(\sigma_3^2+\sigma_{\epsilon_3}^2)+\rho_{23}^2\sigma_2^2\sigma_3^2\} \\
v_{23} &= 2(n-1)(\sigma_2^2+\sigma_{\epsilon_2}^2)\rho_{23}\sigma_2\sigma_3 \\
v_{33} &= 2(n-1)(\sigma_2^2+\sigma_{\epsilon_2}^2)^2
\end{aligned}$$

Also,

$$Var(\sqrt{\tilde{\rho}_{23}^2(1-\tilde{\rho}_{12}^2)}) \approx \frac{1}{4f}\begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{12} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{13} & x_{23} & x_{33} & x_{34} & x_{35} \\ x_{14} & x_{24} & x_{34} & x_{44} & x_{45} \\ x_{15} & x_{25} & x_{35} & x_{45} & x_{55} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{bmatrix} \tag{16}$$

where

$$f = \frac{(n-1)^2\rho_{23}^2\sigma_2^2\sigma_3^3}{\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}\{(n-1)\sigma_3^2-\sigma_{\epsilon_3}^2\}} -$$

$$w_1 = \frac{(n-1)^4\rho_{12}^2\rho_{23}^2\sigma_1^2\sigma_2^4\sigma_3^2}{\{(n-1)\sigma_1^2-\sigma_{\epsilon_1}^2\}\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}^2\{(n-1)\sigma_3^2-\sigma_{\epsilon_3}^2\}} \\ \frac{-(n-1)^2\rho_{23}^2\sigma_2^2\sigma_3^2}{\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}\{(n-1)\sigma_3^2-\sigma_{\epsilon_3}^2\}^2} + \\ \frac{(n-1)^4\rho_{12}^2\rho_{23}^2\sigma_1^2\sigma_2^4\sigma_3^2}{\{(n-1)\sigma_1^2-\sigma_{\epsilon_1}^2\}\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}^2\{(n-1)\sigma_3^2-\sigma_{\epsilon_3}^2\}^2}$$

$$w_2 = \frac{2(n-1)\rho_{23}\sigma_2\sigma_3}{\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}\{(n-1)\sigma_3^2-\sigma_{\epsilon_3}^2\}} - \\ \frac{2(n-1)^3\rho_{12}^2\rho_{23}\sigma_1^2\sigma_2^3\sigma_3}{\{(n-1)\sigma_1^2-\sigma_{\epsilon_1}^2\}\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}^2\{(n-1)\sigma_3^2-\sigma_{\epsilon_3}^2\}}$$

$$w_3 = \frac{-(n-1)^2\rho_{23}^2\sigma_2^2\sigma_3^2}{\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}^2\{(n-1)\sigma_3^2-\sigma_{\epsilon_3}^2\}} + \\ \frac{2(n-1)^4\rho_{12}^2\rho_{23}^2\sigma_1^2\sigma_2^4\sigma_3^2}{\{(n-1)\sigma_1^2-\sigma_{\epsilon_1}^2\}\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}^3\{(n-1)\sigma_3^2-\sigma_{\epsilon_3}^2\}}$$

$$w_4 = \frac{-2(n-1)^3\rho_{12}\rho_{23}^2\sigma_1\sigma_2^3\sigma_3^2}{\{(n-1)\sigma_1^2-\sigma_{\epsilon_1}^2\}\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}^2\{(n-1)\sigma_3^2-\sigma_{\epsilon_3}^2\}}$$

$$w_5 = \frac{(n-1)^4\rho_{12}^2\rho_{23}^2\sigma_1^2\sigma_2^4\sigma_3^2}{\{(n-1)\sigma_1^2-\sigma_{\epsilon_1}^2\}^2\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}^2\{(n-1)\sigma_3^2-\sigma_{\epsilon_3}^2\}}$$

$$x_{11} = 2(n-1)(\sigma_3^2+\sigma_{\epsilon_3}^2)^2$$

$$x_{12} = 2(n-1)\rho_{23}\sigma_2\sigma_3(\sigma_3^2+\sigma_{\epsilon_3}^2)$$

$$x_{13} = 2(n-1)\rho_{23}^2\sigma_2^2\sigma_3^2$$

$$x_{14} = 2(n-1)\rho_{12}\rho_{23}^2\sigma_1\sigma_2\sigma_3^2$$

$$x_{15} = 2(n-1)\rho_{12}^2\rho_{23}^2\sigma_1^2\sigma_3^2$$

$$x_{22} = (n-1)\{(\sigma_2^2+\sigma_{\epsilon_2}^2)(\sigma_3^2+\sigma_{\epsilon_3}^2)+\rho_{23}^2\sigma_2^2\sigma_3^2\}$$

$$x_{23} = 2(n-1)\rho_{23}\sigma_2\sigma_3(\sigma_2^2+\sigma_{\epsilon_2}^2)$$

$$x_{24} = (n-1)\{\rho_{12}\rho_{23}\sigma_1\sigma_2^2\sigma_3+\rho_{12}\rho_{23}\sigma_1\sigma_3(\sigma_2^2+\sigma_{\epsilon_2}^2)\}$$

$$x_{25} = 2(n-1)\rho_{12}^2\rho_{23}\sigma_1^2\sigma_2\sigma_3$$

$$x_{33} = 2(n-1)(\sigma_2^2+\sigma_{\epsilon_2}^2)^2$$

$$x_{34} = 2(n-1)\rho_{12}\sigma_1\sigma_2(\sigma_2^2+\sigma_{\epsilon_2}^2)$$

$$x_{35} = 2(n-1)\rho_{12}^2\sigma_1^2\sigma_2^2$$

$$x_{44} = (n-1)\{(\sigma_1^2+\sigma_{\epsilon_1}^2)(\sigma_2^2+\sigma_{\epsilon_2}^2)+\rho_{12}^2\sigma_1^2\sigma_2^2\}$$

$$
\begin{aligned}
x_{45} &= 2(n-1)\rho_{12}\sigma_1\sigma_2(\sigma_1^2+\sigma_{\epsilon_1}^2) \\
x_{55} &= 2(n-1)(\sigma_1^2+\sigma_{\epsilon_1}^2)^2
\end{aligned}
$$

Finally,

$$
Var(\sqrt{\tilde{\rho}_{23}^2\tilde{\rho}_{12}^2}) = \frac{1}{4f}\begin{bmatrix} y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{12} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{13} & x_{23} & x_{33} & x_{34} & x_{35} \\ x_{14} & x_{24} & x_{34} & x_{44} & x_{45} \\ x_{15} & x_{25} & x_{35} & x_{45} & x_{55} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} \tag{17}
$$

where

$$
\begin{aligned}
f &= \frac{(n-1)^4\rho_{12}^2\rho_{23}^2\sigma_1^2\sigma_2^4\sigma_3^2}{\{(n-1)\sigma_1^2-\sigma_{\epsilon_1}^2\}\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}^2\{(n-1)\sigma_3^2-\sigma_{\epsilon_3}^2\}} \\
y_1 &= \frac{-(n-1)^4\rho_{12}^2\rho_{23}^2\sigma_1^2\sigma_2^4\sigma_3^2}{\{(n-1)\sigma_1^2-\sigma_{\epsilon_1}^2\}\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}^2\{(n-1)\sigma_3^2-\sigma_{\epsilon_3}^2\}^2} \\
y_2 &= \frac{2(n-1)^3\rho_{12}^2\rho_{23}\sigma_1^2\sigma_2^3\sigma_3}{\{(n-1)\sigma_1^2-\sigma_{\epsilon_1}^2\}\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}^2\{(n-1)\sigma_3^2-\sigma_{\epsilon_3}^2\}} \\
y_3 &= \frac{-2(n-1)^4\rho_{12}^2\rho_{23}^2\sigma_1^2\sigma_2^4\sigma_3^2}{\{(n-1)\sigma_1^2-\sigma_{\epsilon_1}^2\}\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}^3\{(n-1)\sigma_3^2-\sigma_{\epsilon_3}^2\}} \\
y_4 &= \frac{2(n-1)^3\rho_{12}\rho_{23}^2\sigma_1\sigma_2^3\sigma_3^2}{\{(n-1)\sigma_1^2-\sigma_{\epsilon_1}^2\}\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}^2\{(n-1)\sigma_3^2-\sigma_{\epsilon_3}^2\}} \\
y_5 &= \frac{-(n-1)^4\rho_{12}^2\rho_{23}^2\sigma_1^2\sigma_2^4\sigma_3^2}{\{(n-1)\sigma_1^2-\sigma_{\epsilon_1}^2\}^2\{(n-1)\sigma_2^2-\sigma_{\epsilon_2}^2\}^2\{(n-1)\sigma_3^2-\sigma_{\epsilon_3}^2\}}
\end{aligned}
$$

and where the $x_{ij}'s$ are given as above.

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ρ_{12}	Component	$\rho_{23} = \sqrt{0.2}$	$\rho_{23} = \sqrt{0.5}$	$\rho_{23} = \sqrt{0.8}$
$\sqrt{0.2}$	First	0.80	0.50	0.20
	Second	0.16	0.40	0.64
	Third	0.04	0.10	0.16
$\sqrt{0.5}$	First	0.80	0.50	0.20
	Second	0.10	0.25	0.40
	Third	0.10	0.25	0.40
$\sqrt{0.8}$	First	0.80	0.50	0.20
	Second	0.04	0.10	0.16
	Third	0.16	0.40	0.64

Table 1: Actual values of the three variance components and the proportions in the simulation runs.

ρ_{23}	ρ_{12}	Comp.	Actual	$\sigma_\epsilon = 0.1$ Bias	$\sigma_\epsilon = 0.3$ Bias
		First	0.8	0.00394	0.0317
	$\sqrt{0.2}$	Second	0.16	-0.00238	-0.0200
		Third	0.04	-0.00156	-0.0117
		First	0.8	0.00394	0.0317
$\sqrt{0.2}$	$\sqrt{0.5}$	Second	0.1	-3.88e-05	-0.00251
		Third	0.1	-0.00390	-0.0292
		First	0.8	0.00394	0.0317
	$\sqrt{0.8}$	Second	0.04	0.00230	0.0150
		Third	0.16	-0.00624	-0.0467
		First	0.5	0.00985	0.0792
	$\sqrt{0.2}$	Second	0.4	-0.00595	-0.0500
		Third	0.1	-0.00390	-0.0292
		First	0.5	0.00985	0.0792
$\sqrt{0.5}$	$\sqrt{0.5}$	Second	0.25	-9.71e-05	-0.00627
		Third	0.25	-0.00975	-0.0729
		First	0.5	0.00985	0.0792
	$\sqrt{0.8}$	Second	0.1	0.00576	0.0375
		Third	0.4	-0.0156	-0.117
		First	0.2	0.0158	0.127
	$\sqrt{0.2}$	Second	0.64	-0.00952	-0.0800
		Third	0.16	-0.00624	-0.0467
		First	0.2	0.0158	0.127
$\sqrt{0.8}$	$\sqrt{0.5}$	Second	0.4	-0.000155	-0.0100
		Third	0.4	-0.0156	-0.117
		First	0.2	0.0158	0.127
	$\sqrt{0.8}$	Second	0.16	0.00921	0.0600
		Third	0.64	-0.0250	-0.187

Table 2: Bias of proportions when measurement error is ignored.

ρ_{23}	ρ_{12}	Estimate	σ_ϵ = L	σ_ϵ = H	Real Value
L	L	Mle	0.792 (0.072)	0.790 (0.092)	0.8
		Naive	0.792 (0.072)	0.790 (0.092)	
	M	Mle	0.790 (0.079)	0.789 (0.079)	0.8
		Naive	0.790 (0.079)	0.790 (0.080)	
	H	Mle	0.783 (0.076)	0.792 (0.078)	0.8
		Naive	0.783 (0.076)	0.793 (0.080)	
M	L	Mle	0.503 (0.072)	0.491 (0.087)	0.5
		Naive	0.503 (0.072)	0.490 (0.087)	
	M	Mle	0.499 (0.074)	0.487 (0.079)	0.5
		Naive	0.499 (0.074)	0.486 (0.080)	
	H	Mle	0.499 (0.075)	0.500 (0.094)	0.5
		Naive	0.499 (0.075)	0.501 (0.096)	
H	L	Mle	0.204 (0.039)	0.196 (0.057)	0.2
		Naive	0.204 (0.039)	0.196 (0.056)	
	M	Mle	0.205 (0.042)	0.190 (0.058)	0.2
		Naive	0.205 (0.042)	0.189 (0.058)	
	H	Mle	0.205 (0.043)	0.197 (0.054)	0.2
		Naive	0.205 (0.044)	0.195 (0.054)	

Table 3: Average of 99 values of first proportion estimates for each run. The figures in brackets represent the standard deviation for these values. (Note: σ_ϵ = L and H represents $\sigma_\epsilon = 0.1$ and 0.3 respectively. ρ_{12} and ρ_{23} = L, M and H refer to ρ_{12} and $\rho_{23} = \sqrt{0.2}$, $\sqrt{0.5}$ and $\sqrt{0.8}$.)

ρ_{23}	ρ_{12}	Estimate	σ_ϵ =L	σ_ϵ = H	Real Value
L	L	Mle	0.163 (0.054)	0.167 (0.074)	0.16
		Naive	0.163 (0.054)	0.167 (0.074)	
	M	Mle	0.104 (0.040)	0.101 (0.045)	0.10
		Naive	0.104 (0.040)	0.101 (0.045)	
	H	Mle	0.044 (0.017)	0.041 (0.021)	0.04
		Naive	0.044 (0.017)	0.041 (0.021)	
M	L	Mle	0.396 (0.063)	0.402 (0.074)	0.4
		Naive	0.396 (0.063)	0.402 (0.074)	
	M	Mle	0.253 (0.049)	0.256 (0.052)	0.25
		Naive	0.253 (0.049)	0.256 (0.052)	
	H	Mle	0.100 (0.022)	0.094 (0.027)	0.10
		Naive	0.100 (0.022)	0.094 (0.027)	
H	L	Mle	0.634 (0.064)	0.645 (0.076)	0.64
		Naive	0.634 (0.064)	0.646 (0.076)	
	M	Mle	0.401 (0.052)	0.393 (0.073)	0.40
		Naive	0.401 (0.052)	0.391 (0.073)	
	H	Mle	0.163 (0.034)	0.158 (0.049)	0.16
		Naive	0.164 (0.034)	0.157 (0.050)	

Table 4: Average of 99 values of second proportion estimates for each run. The figures in brackets represent the standard deviation for these values.

ρ_{23}	ρ_{12}	Estimate	σ_ϵ =L	σ_ϵ = H	Real Value
L	L	Mle	0.045 (0.027)	0.043 (0.025)	0.04
		Naive	0.045 (0.027)	0.043 (0.026)	
	M	Mle	0.107 (0.043)	0.110 (0.044)	0.10
		Naive	0.107 (0.043)	0.109 (0.044)	
	H	Mle	0.173 (0.062)	0.167 (0.064)	0.16
		Naive	0.173 (0.062)	0.167 (0.066)	
M	L	Mle	0.100 (0.040)	0.107 (0.045)	0.10
		Naive	0.100 (0.040)	0.108 (0.045)	
	M	Mle	0.247 (0.062)	0.257 (0.066)	0.25
		Naive	0.247 (0.062)	0.258 (0.068)	
	H	Mle	0.401 (0.068)	0.406 (0.084)	0.40
		Naive	0.401 (0.068)	0.405 (0.086)	
H	L	Mle	0.162 (0.060)	0.159 (0.067)	0.16
		Naive	0.162 (0.060)	0.159 (0.066)	
	M	Mle	0.395 (0.067)	0.417 (0.080)	0.40
		Naive	0.395 (0.067)	0.419 (0.080)	
	H	Mle	0.632 (0.054)	0.645 (0.071)	0.64
		Naive	0.631 (0.054)	0.648 (0.074)	

Table 5: Average of 99 values of third proportion estimates for each run. The figures in brackets represent the standard deviation for these values.

ρ_{23}	ρ_{12}	Estimate	Raw	Raw	Proportion	Proportion
			σ_ϵ =L	σ_ϵ = H	σ_ϵ =L	σ_ϵ = H
L	L	Mle	98	91	96	95
		Naive	97	91	98	91
	M	Mle	97	98	95	98
		Naive	98	97	97	98
	H	Mle	91	97	95	98
		Naive	92	94	97	97
M	L	Mle	96	96	96	95
		Naive	94	94	99	98
	M	Mle	97	95	99	96
		Naive	97	95	99	97
	H	Mle	97	91	98	95
		Naive	97	91	95	97
H	L	Mle	96	95	97	95
		Naive	98	96	98	96
	M	Mle	95	93	95	96
		Naive	97	94	97	99
	H	Mle	97	96	95	97
		Naive	98	97	98	97

Table 6: Coverage frequency for first variance term for each run. Note that these figures are not given in percentages - they are the actual number of intervals that cover the real value out of 99 trials. (Nominal confidence coefficient is 98%.)

ρ_{23}	ρ_{12}	Component		Proportion	
		σ_ϵ =L	σ_ϵ = H	σ_ϵ =L	σ_ϵ = H
L	L	96.9	96.8	95.8	96.9
L	M	97.0	97.2	97.2	96.5
L	H	96.9	97.2	97.1	97.4
M	L	96.5	97.7	98.1	97.3
M	M	96.9	97.3	98.1	96.7
M	H	96.8	97.1	98.3	96.9
H	L	96.2	97.3	97.2	97.8
H	M	96.6	97.9	96.6	97.8
H	H	97.2	97.7	98.1	98.4

Table 7: Coverage frequencies of confidence intervals using approximate variance formulas for the square root of the first variance component. Numbers are percentages of 1000 simulations. Theoretical coverage frequency is 98%.

ρ_{23}	ρ_{12}	Component		Proportion	
		σ_ϵ =L	σ_ϵ = H	σ_ϵ =L	σ_ϵ = H
L	L	97.3	98.1	97.3	98.0
L	M	97.6	97.6	98.0	98.1
L	H	97.6	97.8	98.2	98.1
M	L	97.4	97.7	97.4	97.1
M	M	97.8	97.2	97.7	96.8
M	H	96.8	97.0	96.5	98.1
H	L	96.6	97.0	97.1	97.7
H	M	96.6	96.7	96.7	97.0
H	H	96.8	97.7	97.5	98.3

Table 8: Coverage frequencies of confidence intervals using approximate variance formulas for the square root of the second variance component. Numbers are percentages of 1000 simulations. Theoretical coverage frequency is 98%.

ρ_{23}	ρ_{12}	Component		Proportion	
		σ_ϵ =L	σ_ϵ = H	σ_ϵ =L	σ_ϵ = H
L	L	96.3	96.9	96.7	96.9
L	M	97.6	98.1	97.6	97.9
L	H	98.0	98.4	97.5	98.3
M	L	96.8	97.3	97.6	97.9
M	M	96.3	97.0	97.7	96.7
M	H	98.4	97.1	98.3	97.4
H	L	96.8	98.3	96.6	97.6
H	M	97.4	96.7	96.9	97.3
H	H	98.0	97.1	97.5	97.6

Table 9: Coverage frequencies of confidence intervals using approximate variance formulas for the square root of the third variance component. Numbers are percentages of 1000 simulations. Theoretical coverage frequency is 98%.

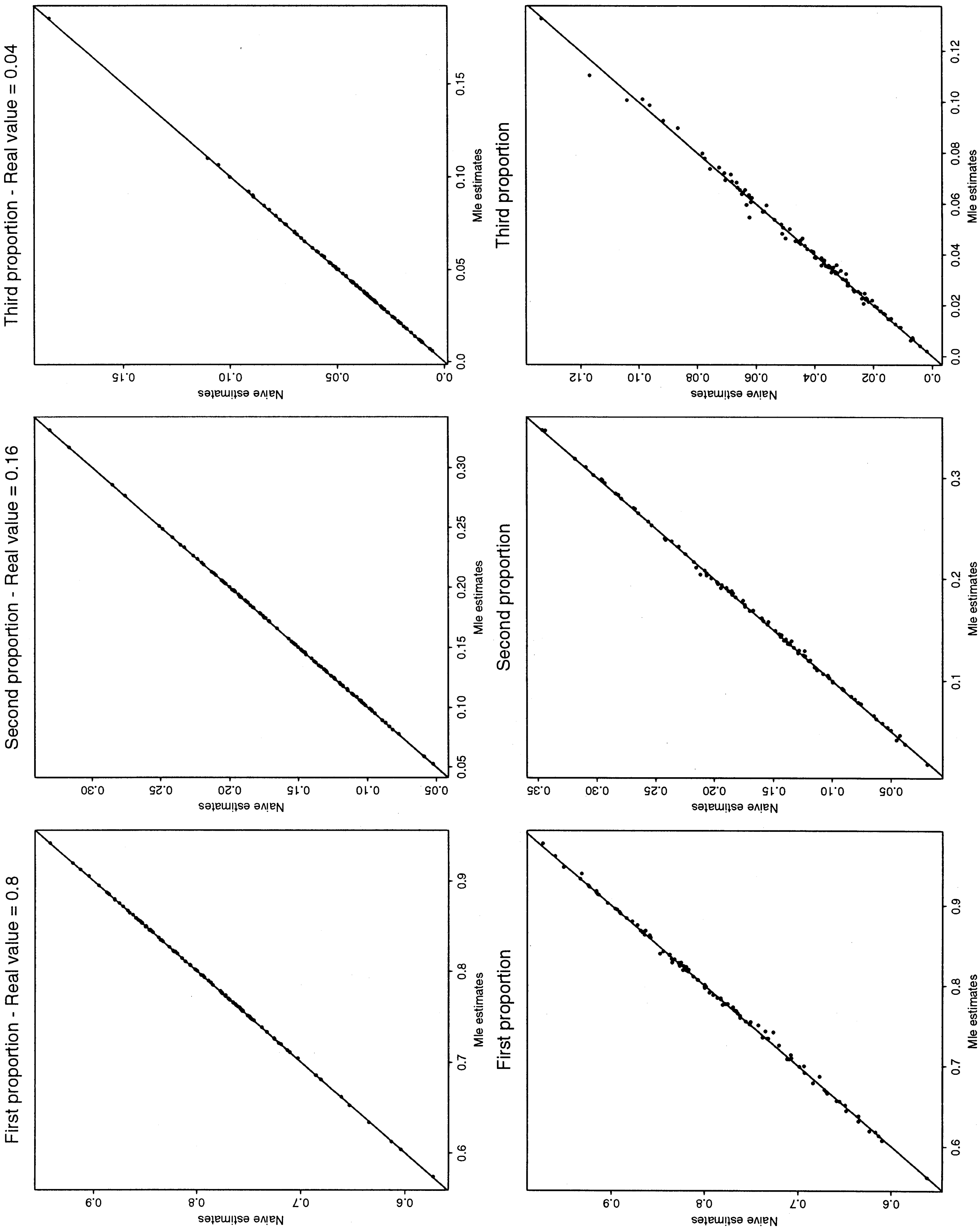

Figure 1: Runs for ρ_{12} and $\rho_{23} = \sqrt{0.2}$. The top row show the run at $\sigma_\epsilon = 0.1$, whereas the bottom row shows $\sigma_\epsilon = 0.3$.

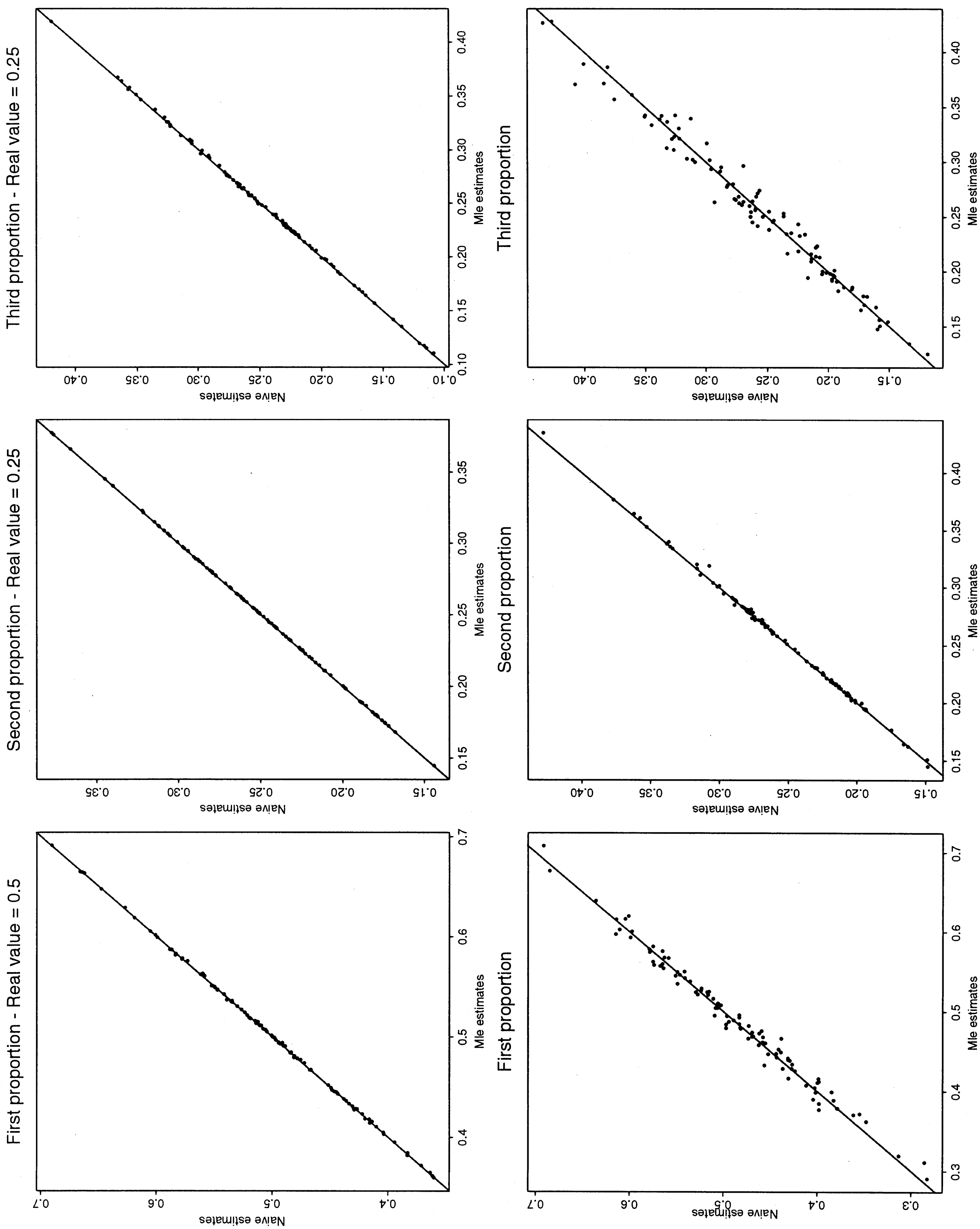

Figure 2: Runs for ρ_{12} and $\rho_{23} = \sqrt{0.5}$.

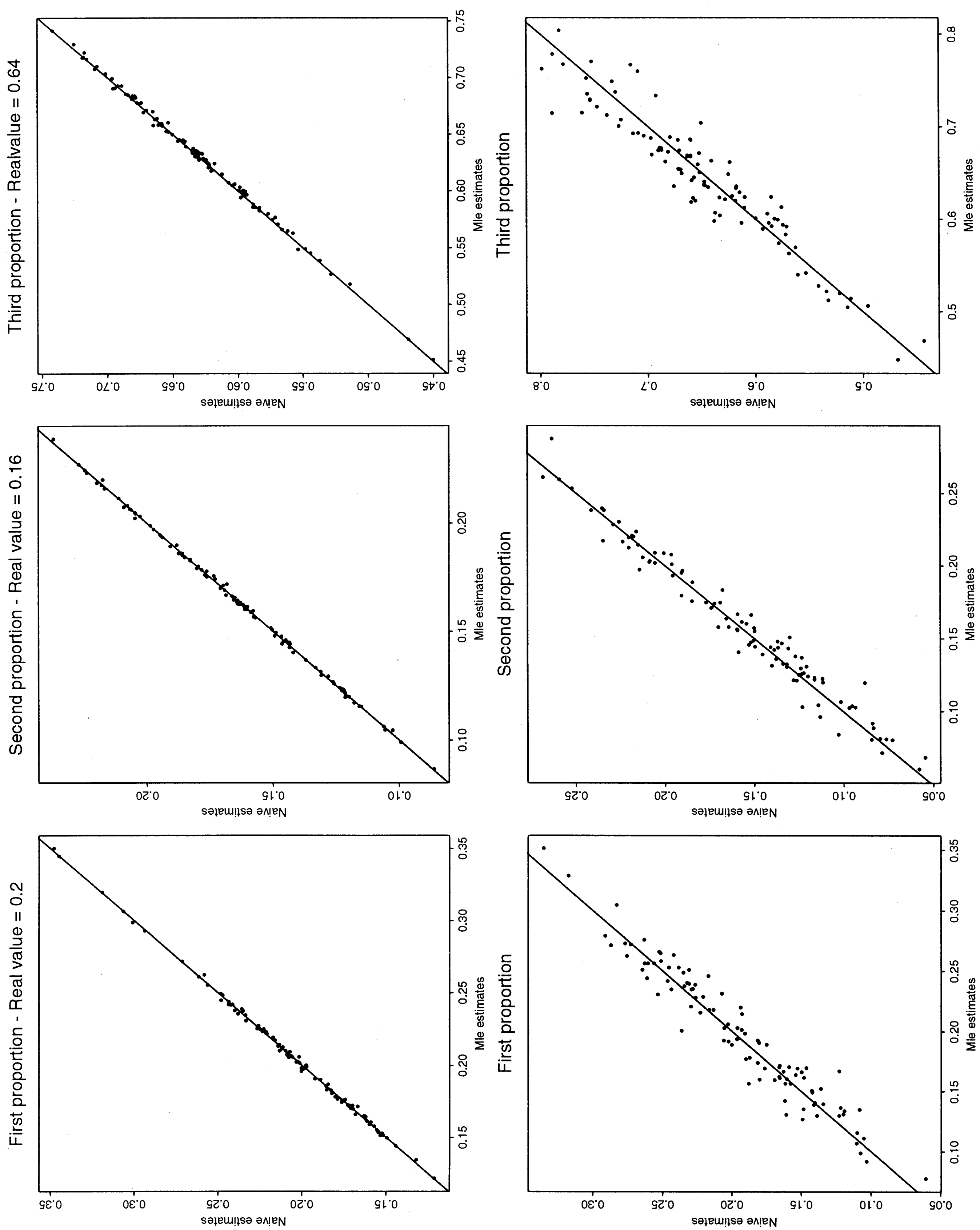

Figure 3: Runs for ρ_{12} and $\rho_{23} = \sqrt{0.8}$.