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Abstract

There are many practical problems in which finite mixtures of probability distributions arise as models of life lengths. Although the roles of failure rate and mean residual life in modelling life lengths are well established, much work has not been done to characterize mixture distributions in terms of these concepts. In the present paper we establish an identity connecting the failure rate and the mean residual life that characterizes two-component mixtures of exponential, Lomax and beta densities. We also prove a similar result connecting the failure rate and the second moment of residual life. The use of the characteristic property in deriving some quick estimates of the parameters of the mixture model is indicated.

Key Words - Life distributions, mixture, failure rate, mean residual life, characterization.

1 Introduction

Acronyms, Abbreviations and Notations:

MRLF - mean residual life function

IFR, DFR - [increasing, decreasing] failure rate

IMRL, DMRL [increasing, decreasing] MRL

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X	continuous *rv* in the support of the set of non-negative reals
$f(\cdot)$	*pdf* of X
$R(\cdot)$	$Sf\ of\ X$
$h(x)$	failure rate of X $f(x)/R(x)$
$r(x)$	$E(X - x \mid X > x), MRLF$ of X
$m(x)$	$E[(X - x)^2 \mid X > x]$, second moment of residual life
p	a real number in the interval $(0, 1)$
$f_i(\cdot)$	component densities
μ_i	mean corresponding to the density f_i

The role of the failure rate, MRL and second moment of residual life in modelling life time data is well established. Based on the result that these determine the corresponding life distribution uniquely, there have been many attempts in the literature to identify the specific functional forms of the failure rate or MRL that characterize various distributions. Galambos and Kotz [1] discuss this topic extensively.

For many distributions used in life length studies, there is no closed form expression for the failure rate or MRL that permits simple characterizations. However, in such cases, there may exist identities connecting these functions that determine the underlying distribution uniquely. In [2] it is proved that X has gamma distribution if and only if

$$E[X \mid X \geq y] = \mu + yh(y)/\alpha$$

where $\mu = E(X)$. This was extended (see [3]) to cover the Pearson family

$$f'(x) = -(x + d)f(x)/(b_0 + b_1x + b_2x^2)$$

by the identity

$$E[x \mid X > x] = \mu + (a_0 + a_1x + a_2x^2)h(x)$$

where, $a_i = b_i/(1-b_2), i = 0, 1, 2$. Results along the same direction can also be seen in [4] and the references cited therein.

There are many practical problems in which a population of life times allow decomposition into sub populations such as those based on units, in different production periods, with different designs or from different raw materials [5]. Also failure occurs due to various causes and each cause may produce a different density. In all these cases the failure density assumes the form of a finite mixture. We refer to [6], [7] for details and further examples. With mixture distributions as plausible models in life length studies, it is natural to explore the possibilities of characterizing them by means of identities of the type already mentioned. In Section 2 we present two characterizations of the mixtures of exponential, Lomax and beta densities through relationships between (i) failure rate and MRL and (ii) second moment of residual life and failure rate. Some basic properties of these models are also discussed in Section 2. A possible application of the characteristic property to inference on the parameters of the model is pointed out in Section 3.

2 Main results

We give two characterizations of the exponential, Lomax and beta densities.

Theorem 1 *The identity*

$$r(x) = (1+ax)(\mu_1 + \mu_2 + \ a\ \mu_1\mu_2) - \mu_1\mu_2(1+ax)^2 h(x) \tag{2.1}$$

is satisfied for all x for a r.v. X with density

$$f(x) = pf_1(x) + (1-p)f_2(x)$$

if and only if for $i = 1, 2$

$$f_i(x) = \lambda_i \exp[-\lambda_i x], \quad \lambda_i > 0; \quad x > 0, \quad for\ a = 0\ ; \tag{2.2}$$

$$f_i(x) = \alpha_i \beta^{\alpha_i}(x+\beta)^{-(\alpha_i+1)}, \quad \alpha_i, \quad \beta > 0; \quad x > 0, \quad for\ a > 0 \tag{2.3}$$

and $$f_i(x) = \frac{C_i}{R}(1-\frac{x}{R})^{C_i-1}, \quad C_i, \quad R > 0, \ 0 < x < R, \quad for\ a < 0. \tag{2.4}$$

Theorem 2 *The distribution of X will be a mixture of exponentials (Lomax laws; betas) with component densities as in (2.2)((2.3); (2.4)) if and only if for all x*

$$\lambda_1^2\lambda_2^2 m(x) = 2(\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2) - 2(\lambda_1 + \lambda_2)\ h(x) \tag{2.5}$$

$$\Big[(\alpha_2 - \alpha_1)(\alpha_1 - 1)(\alpha_1 - 2)m(x) = 2(x+\beta)^2(\alpha_2 - \alpha_1 u) - 2(x+\beta)^3(1-u)h(x) \tag{2.6}$$

$$where \qquad u \ = \ (\alpha_1 - 1)(\alpha_1 - 2)/(\alpha_2 - 1)(\alpha_2 - 2);$$

$$(C_2 - C_1)(C_1+1)(C_1+2)m(x) \ = \ 2(C_2 - C_1\vartheta)(R-x)^2 - 2(R-x)^3(1-\vartheta)h(x)$$
$$where \qquad \vartheta(C_1+1)(C_1+2)/(C_2+1)(C_2+2)].$$

The proofs of Theorems 1 and 2 are given in the Appendix.

Remarks

1. Although the exponential distributions have constant failure rates (MRL), their mixture has DFR (IMRL). The Lomax densities as well as their mixtures belong to the DFR (IMRL) class. On the other hand even though beta densities are in the IFR class, their mixture can exhibit DFR property. This follows from the fact that the sign of $dh(x)/dx$ for the beta density is the same as that of

$$C_1p^2(1-\frac{x}{R})^{2C_1-2} + C_2(1-p)^2(1-\frac{x}{R})^{2C_2-2} + p(1-p)(1-\frac{x}{R})^{C_1+C_2-2}[C_1+C_2-(C_1-C_2)^2]$$

and a choice of C_1 and C_2 sufficiently apart will lead to $dh(x)/dx < 0$. Thus one has to take care when data belonging to two such beta densities are pooled.

2. Results for single populations can be deduced if we set $\lambda_1 = \lambda_2, \quad \alpha_1 = \alpha_2$ and $C_1 = C_2$.

3. Apart from the increased algebraic calculations, the method of proof remains the same even if we increase the number of components in f.

3 Application

Equation (2.1) in the case of the exponential mixture becomes

$$r(x) = (\mu_1 + \mu_2) - \mu_1\mu_2 \, h(x) \tag{3.1}$$

which implies that a plot of $(h(x), r(x))$ is a straight line. Thus if the values of the MRL and failure rate realised from a random sample of failure times fall along a straight line it indicates that the model is a mixture of exponentials. Moreover, in such cases, the least square estimates of the slope $(\mu_1\mu_2)$ and intercept $(\mu_1 + \mu_2)$ in (3.1) will lead to estimates of the parameters of the component densities. The sample mean is an estimate of $r(0) = E(X)$ and hence using

$$r(0) = p\mu_1 + (1-p)\mu_2$$

and the earlier estimates of μ_1 and μ_2, we can find the mixing parameter p. This procedure presents a simple methodology and can provide some quick estimates of the parameters without many computational difficulties. The properties of these estimates require detailed study and hence will be reported elsewhere.

Appendix

A.1 Proof of Theorem 1

To establish the if part, we note that for the Lomax mixture

$$h(x) = [p\alpha_1\beta^{\alpha_1}(x+\beta)^{-(\alpha_1+1)} + (1-p)\alpha_2\beta^{\alpha_2}(x+\beta)^{-(\alpha_2+1)}]/R_1(x) \quad \text{(A.1)}$$

$$r(x) = [p(\alpha_1-1)^{-1}\beta^{\alpha_1}(x+\beta^{-\alpha_1+1} + (1-p)(\alpha_2-1)^{-1}\beta^{\alpha_2}(x+\beta)^{-\alpha_2+1}]/R_1(x) \quad \text{(A.2)}$$

with

$$R_1(x) = p\beta^{-\alpha_1}(x+\beta)^{-\alpha_1} + (1-p)\beta^{\alpha_2}(x+\beta)^{-\alpha_2}. \quad \text{(A.3)}$$

From (A.1), (A.2) and (A.3),

$$r(x) = \frac{\alpha_1+\alpha_2-1}{(\alpha_1-1)(\alpha_2-1)}(x+\beta) - \frac{(x+\beta)^2}{(\alpha_1-1)(\alpha_2-1)}. \quad \text{(A.4)}$$

Using $\mu_i = \beta(\alpha_i-1)^{-1}, i = 1, 2$ in (A.4) and setting $\beta^{-1} = a > 0$ we obtain (2.1).

In the case of the beta mixture

$$h(x) = [pC_1R^{-1}(1-xR^{-1})^{C_1} + (1-p)(1-xR^{-1})^{C_2}C_2R^{-1}]/R_2(x) \quad \text{(A.5)}$$

$$r(x) = \left[pR(C_1+1)^{-1}(1-xR^{-1})^{C_1+1} + (1-p)R(C_2+1)^{-1}(1-xR^{-1})^{C_2+1}\right] R_2(x) \quad \text{(A.6)}$$

where

$$R_2(x) = p(1-xR^{-1})^{C_1} + (1-p)(1-xR^{-1})^{C_2}.$$

Simplifying using $\mu_i = R(C_i+1)^{-1}$ and $a = -R^{-1} < 0$, we get (2.1). The exponential case is proved in Nassar and Mahmoud [8].

It remains to establish the converse. For this we assume that $a \neq 0$. From the definitions of $h(x)$ and $r(x)$ and (2.1) we write

$$(1+ax)(\mu_1+\mu_2+a\mu_1\mu_2)R(x) - \mu_1\mu_2(1+ax)^2 f(x) = \int_x^\infty R(t)dt \qquad \text{(A.7)}$$

Differentiating (A.7) twice w.r.t x,

$$\mu_1\mu_2(1+ax)^2 f''(x) + (\mu_1+\mu_2+5a\mu_1\mu_2)(1+ax)f'(x)$$

$$+[2a(\mu_1+\mu_2)+4a^2\mu_1\mu_2+1]f(x) = 0. \qquad \text{(A.8)}$$

To solve the differential equation (A.8), we set

$$e^Z = 1+ax \quad \text{and} \quad y = f(x)$$

to yield

$$a^2\mu_1\mu_2\frac{d^2y}{dZ^2} + [(\mu_1+\mu_2)a + 4a^2\mu_1\mu_2]\frac{dy}{dZ} + [1+2a(\mu_1+\mu_2)$$

$$+4a^2\mu_1\mu_2]y = 0, \qquad \text{(A.9)}$$

which is homogeneous with constant coefficients. The auxiliary equation

$$m^2 + \left(\frac{\mu_1+\mu_2}{a\mu_1\mu_2}+4\right) m + \frac{1+2a(\mu_1+\mu_2)}{a^2\mu_1\mu_2} + 4 = 0$$

has roots

$$m_1 = 2+(a\mu_2)^{-1}, \quad m_2 = 2+(a\mu_1)^{-1}.$$

The solution of (A.9) is thus

$$Y = A\ \exp[-(2+a^{-1}\mu_2^{-1})Z]\ + B\ \exp[-(2+a^{-1}\mu_1^{-1})Z] = A(1+ax)^{-(\alpha_2+1)} + B(1+ax)^{-(\alpha_1+1)}\ ;$$

with $\alpha_i = 1+(a\mu_i)^{-1}$. Now choose $A = \alpha_2 q$ and $B = \alpha_1 p$ to read

$$f(x) = \alpha_1 p\beta^{\alpha_1}(x+\beta)^{-(\alpha_1+1)} + \alpha_2 q\beta^{\alpha_2}(x+\beta)^{-(\alpha_2+1)}.$$

Using the conditions, $f(x) \geq 0$ and $\int_0^\infty f(x)dx = 1$, we find $\alpha_1 > 0$ and $q = 1 - p$. Thus we have a mixture of Lomax densities as stated in the Theorem. For $a < 0$, we set $a = -R^{-1}$ and proceed as before. The case for $a = 0$ is treated independently in [8] and our proof is complete.

Proof of Theorem 2

Since the method of proof is the same as that of Theorem 1 we give here only the outline of the proof in one case. Equation (2.5) when differentiated three times after substituting

$$m(x) = 2\int_x^\infty (t - x)R(t)dt$$

takes the form

$$(\lambda_1 + \lambda_2)f''' + (\lambda_1^2\lambda_2^2 + \lambda_1\lambda_2)f'' - \lambda_1^2\lambda_2^2 f = 0. \tag{A.10}$$

The corresponding auxiliary equation is

$$(\lambda_1 + \lambda_2)m^3 + (\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2)m^2 - \lambda_1^2\lambda_2^2 = 0$$

whose solutions are

$$m = -\lambda_1, \ -\lambda_2 \ \text{ and } \ \lambda_1\lambda_2(\lambda_1 + \lambda_2)^{-1}.$$

Accordingly the unique solution of differential equation (A.10) is

$$\begin{aligned} f(x) &= A\ e^{-\lambda_1 x} + B\ e^{-\lambda_2 x} + C\ e^{\lambda_1\lambda_2(\lambda_1+\lambda_2)^{-1}x} \\ &= p\lambda_1\ e^{-\lambda_1 x} + q\ \lambda_2\ e^{-\lambda_2 x} + C\ e^{\lambda_1\lambda_2(\lambda_1+\lambda_2)^{-1}x} \end{aligned} \tag{A.11}$$

For (A.11) to be a density function, C must be zero and hence $q = 1 - p$ and X has mixture exponential distribution. The proofs for (2.6) and (2.7) follow the same pattern with substitutions made as in the solution of (A.8).

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