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Abstract

A first order autoregressive process with one-dimensional inverse Gaussian marginals is introduced. The innovation distributions are obtained under certain special cases. The unknown parameters are estimated using different methods and these estimators are shown to be consistent and asymptotically normal.

Keywords: Conditional least squares, consistent and asymptotically normal, empirical Laplace transform, maximum likelihood, self-decomposable, stable distribution, strong mixing.

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1. INTRODUCTION

The analysis of time series in the classical set up is based on the assumption that an observed series is a realization from a Gaussian sequence. However, there are many situations, where the naturally occurring data showing tendency to follow asymmetric and heavy-tailed distributions which cannot be modelled by Gaussian distributions. The usual techniques of transforming the data for using Gaussian models also fail under certain situations (see Lawrance (1991)). Hence a number of non-Gaussian time series models have been introduced by different researchers during the last two decades. (See for example, Lawrance and Lewis (1985), Adke and Balakrishna (1992) and the references there.)

The literature on non-Gaussian time series mainly deals with finding the innovation distribution for a specified marginal and then discussing the second order properties of the sequences generated by them. Not much attention is given to the problem of statistical inference, which is essential to check the validity of a model in real situations. The problems of estimation for autoregressive sequences with exponential marginals are discussed by

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Smith (1986), Billard and Mohamed (1991), Adke and Balakrishna (1992a). Estimation in Laplacian autoregressive models are considered by Karlsen and Tjϕstheim (1988) and Son and Cho (1996). Another non-Gaussian r.v. which attracted some attention is the inverse Gaussian that have many applications in studying life time or number of event occurrences. Lancaster (1972) showed that the duration of strike in United Kingdom has an inverse Gaussian distribution. Banerjee and Bhattacharyya (1976) studied the purchase incidence model when the inter-purchase time of an individual household is described by an inverse Gaussian distribution. Chhikara and Folks (1977) considered this distribution for a lifetime model and suggest its applications for studying reliability aspects.

In this paper we study the properties of inverse Gaussian autoregressive models and also discuss the related estimation problems. A random variable (r.v) X is said to have an inverse Gaussian (IG) distribution with parameters λ, μ if its probability density function (pdf) is of the form,

$$f(x;\mu,\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\}, \quad x>0,\ \lambda>0,\ \mu>0. \tag{1.1}$$

The properties of (1.1) and its various reparameterisations may be found in Johnson, Kotz and Balakrishnan (1994). If $\lambda = \mu^2$ then (1.1) reduces to

$$f(x;\mu) = \frac{\mu}{(2\pi x^3)^{1/2}} \exp\{-\frac{1}{2x}(x-\mu)^2\}, \quad x>0,\ \mu>0 \tag{1.2}$$

which is the pdf of the first passage time to a point μ in a Brownian motion process with unit drift and unit variance. Another interesting form of pdf can be obtained by letting $\mu \to \infty$ in (1.1). The resulting pdf is given by

$$f(x;\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\{-\frac{\lambda}{2x}\}, \quad \lambda>0,\ x>0. \tag{1.3}$$

This is in fact the pdf of first passage time of drift-free Brownian motion.

In Section 2 we introduce the inverse Gaussian Autoregressive process of order 1, IGAR(1). Section 3 discusses conditional least squares estimation, section 4 considers ML estimation and section 5 contains the estimation based on empirical Laplace Transform. The final section 6 gives a summary and some concluding remarks.

2. AUTOREGRESSIVE MODELS

Let $\{\epsilon_n,\ n \geq 1\}$ be a sequence of independent and identically distributed (iid) non-negative r.v's and X_0 be an inverse Gaussian r.v independent of $\epsilon_j,\ j \geq 1$. Define $\{X_n\}$ by a first order autoregressive (AR(1)) model:

$$X_n = \rho X_{n-1} + \epsilon_n, \quad n = 1, 2, \ldots\ 0 \leq \rho < 1. \tag{2.1}$$

Note that X_n depends on $X_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_n$ and independent of $\epsilon_j,\ j > n$. Suppose that $\{X_n\}$ has inverse Gaussian distribution (1.1) for every $n \geq 0$. Let $\phi_X(s)$ and $\phi_\epsilon(s)$ be the Laplace transforms (LT) of X_n and ϵ_n respectively. Then (2.1) implies that

$$\phi_X(s) = \phi_X(\rho s) \cdot \phi_\epsilon(s), \quad s > 0. \tag{2.2}$$

Pillai and Satheesh (1992) have proved that the inverse Gaussian distribution (1.1) is self-decomposable and hence $\phi_\epsilon(s) = \phi_X(s)/\phi_X(\rho s)$ is the LT of a proper d.f. for every $\rho \in [0, 1)$. The LT of (1.1) is given by

$$\phi_X(s) = \exp\left\{\frac{\lambda}{\mu}\left(1 - \sqrt{1 + \frac{2\mu^2 s}{\lambda}}\right)\right\}. \tag{2.3}$$

Hence

$$\phi_\epsilon(s) = \exp\left[-\frac{\lambda}{\mu}\left\{\sqrt{1 + \frac{2\mu^2 s}{\lambda}} - \sqrt{1 + \frac{2\rho\mu^2 s}{\lambda}}\right\}\right]. \tag{2.4}$$

By inverting $\phi_\epsilon(s)$, we can get the corresponding distribution function of ϵ_n. Thus if $\{\epsilon_n\}$ is an iid sequence with LT (2.4) then $\{X_n,\ n \geq 0\}$ given by (2.1) defines a stationary sequence of inverse Gaussian r.v's with marginal pdf (1.1). For this sequence $E(X_n) = \mu$, $\text{var}(X_n) = \mu^3/\lambda$ for all $n \geq 0$ and the autocorrelation function of order h is

$$\text{Corr}(X_n, X_{n+h}) = \rho^h, \quad h = 1, 2, \ldots\ . \tag{2.5}$$

The LT (2.4) does not seem to have closed form expression for its inverse. Let us now consider a stationary Markov sequence of IG r.v's with pdf (1.3). The LT of (1.3) is given by

$$\psi_X(s) = \exp\{-\sqrt{2\lambda s}\}. \tag{2.6}$$

If we suppose that X_n defined by (2.1) has the pdf (1.3) for every $n \geq 0$ then the LT of ϵ_n becomes

$$\psi_\epsilon(s) = \exp\{-\sqrt{2\lambda s(1-\sqrt{\rho})^2}\}. \tag{2.7}$$

The pdf of ϵ_n corresponding to (2.7) is

$$g(y) = \left(\frac{\lambda(1-\sqrt{\rho})^2}{2\pi y^3}\right)^{1/2} \exp\left\{\frac{-\lambda(1-\sqrt{\rho})^2}{2y}\right\}, \quad y \geq 0. \tag{2.8}$$

Even though, we have a closed form density function for ϵ_n, it is not possible to discuss the second order properties here as $E(X_n^k) = \infty$ if $k > 1/2$.

Remark 1: A r.v X is said to be positive stable if its LT is of the form

$$\psi(s) = \exp\{-(2\lambda s)^\alpha\}, \quad 0 < \alpha < 2, \ s > 0. \tag{2.9}$$

Hence it follows that the IG r.v with LT (2.6) is positive stable with $\alpha = 1/2$. A positive stable r.v has a closed form expression for pdf when $\alpha = 1/2$, which is given by (1.3). So, the above discussion on AR(1) model can be extended for any $\alpha \in (0, 2)$. However, we restrict the details for the case $\alpha = 1/2$ since, the pdf has a closed form expression here.

Remark 2: In AR(1) models for exponential, gamma, Laplace etc. r.v's, the distributions of ϵ_n are entirely different from those of X_n's. But pdf's (2.8) and (1.3) are the same except that λ is replaced by $\lambda(1-\sqrt{\rho})^2$ in (2.8). This is a consequence of one of the characterizations of stable laws (cf. Rao and Shanbhag (1994), p. 154).

Suppose that $\{X_n, \ n \geq 0\}$ is an IGAR(1) sequence with marginal distribution (1.3). Let us consider the distribution of $S_n = X_1 + X_2 + \cdots + X_n$. Using the definition of the model (2.1) we can write

$$S_n = \left(\frac{1-\rho^n}{1-\rho}\right) X_n + \sum_{j=2}^{n} \left(\frac{1-\rho^{n-j+1}}{1-\rho}\right) \epsilon_j.$$

The LT of S_n is computed as,

$$\psi_{S_n}(s) = \exp\left\{-\sqrt{2\lambda s\frac{(1-\rho^n)}{1-\rho}}\right\}\exp\left[-\sqrt{\frac{2\lambda s}{1-\rho}}(1-\sqrt{\rho})\sum_{j=2}^{n}\sqrt{(1-\rho^{n-j+1})}\right].$$

This implies that

$$\psi_{\frac{S_n}{n^2}}(s) \to \exp\left\{-(1-\sqrt{\rho})\sqrt{\frac{2\lambda s}{1-\rho}}\right\} \quad \text{as} \quad n\to\infty.$$

That is, the limit distribution of $\frac{S_n}{n^2}$ as $n \to \infty$ is again an IG with pdf of the type (1.3). For a positive stable AR(1) process it is readily proved that for every $\alpha \in (0,2)$,

$$\psi_{\frac{S_n}{n^{1/\alpha}}}(s) \to \exp\left\{-(1-\rho^\alpha)\left(\frac{2\lambda s}{1-\rho}\right)^\alpha\right\} \quad \text{as} \quad n\to\infty.$$

That is, the positive stable AR(1) process belongs to the domain of attraction of positive stable distribution.

In the rest of the paper we discuss various methods for estimating the parameters.

3. METHOD OF CONDITIONAL LEAST SQUARES

The conditional least square (CLS) estimator of a parametric vector $\theta = (\theta_1, \theta_2, ...\theta_p)$ is obtained by minimising

$$Q_n(\theta) = \sum_{i=1}^{n}[X_{i+1} - E(X_{i+1}|X_i,\ X_{i-1}, ...X_1)]^2 \tag{3.1}$$

with respect to θ. This method and properties of CLS estimators are studied by Klimko and Nelson (1978). We obtain the CLS estimators of $\theta = (\mu, \rho)$ in an IGAR(1) model which generates a sequence with marginal pdf (1.2). That is, pdf (1.1) with $\lambda = \mu^2$. Since AR(1) sequence is Markovian,

$$E(X_{i+1}|X_i, X_{i-1}, ..., X_1) = E(X_{i+1}|X_i) = \rho X_i + (1-\rho)\mu.$$

The CLS estimators of μ and ρ are respectively given by

$$\hat{\mu} = \sum_{i=1}^{n}(X_{i+1} - \hat{\rho}_n X_i)/n(1-\hat{\rho}_n)$$

and

$$\hat{\rho}_n = \frac{\sum_{i=1}^{n-1} X_i X_{i+1} - (\frac{1}{n})\left(\sum_{i=1}^{n} X_i\right)\left(\sum_{i=1}^{n-1} X_{i+1}\right)}{\sum_{i=1}^{n} X_i^2 - n\bar{X}_n^2} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(X_{i+1} - \bar{X})}{\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

(sample lag 1 autocorrelation).

Since $\{X_n,\ n \geq 0\}$ defined by (2.1) is stationary and ergodic, from pointwise ergodic theorem it follows that $\hat{\mu} \overset{\text{a.s.}}{\to} \mu$ and $\hat{\rho}_n \overset{\text{a.s.}}{\to} \rho$ as $n \to \infty$, where $\overset{\text{a.s.}}{\to}$ stands for almost sure convergence. Klimko and Nelson (1978) have proved under certain regularity conditions the asymptotic properties of the CLS estimators. Those conditions are satisfied in the case of AR(1) model generating IG sequence with marginal distributions (1.1) and (1.2). Hence, it follows that

$$\sqrt{n}\begin{bmatrix} \hat{\mu}_n - \mu \\ \hat{\rho}_n - \rho \end{bmatrix} \overset{\mathcal{L}}{\to} N_2(0, \textstyle\sum)\ as\ n \to \infty, \tag{3.2}$$

where $\overset{\mathcal{L}}{\to}$ denotes the convergence in law and $N_2(0, \sum)$ is a bivariate normal vector with mean 0 and dispersion matrix

$$\sum = \begin{pmatrix} \mu(\frac{1+\rho}{1-\rho}) & 0 \\ 0 & 1-\rho^2 \end{pmatrix}.$$

Thus we have CLS estimator of θ, which is consistent and asymptotically normal (CAN).

We cannot apply CLS method for estimating the parameters of AR(1) model generating IG r.v's having marginal pdf (1.3) as it does not possess moments of order greater than 1/2.

4. MAXIMUM LIKELIHOOD ESTIMATION

In this section we obtain the maximum likelihood estimators (mle) of the parameters of AR(1) process having marginal pdf (1.3). The likelihood based inference is not tractable for AR(1) models generating the r.v's with pdf (1.1) and (1.2) since the pdfs of ϵ_n's do not have closed form expressions. Let $\{X_n\}$ be an AR(1) sequence with (1.3) as marginal pdf. The corresponding pdf of ϵ_n is given by (2.8). The transition distribution of X_n given

$X_{n-1} = x_{n-1}$ in this case becomes

$$\begin{aligned} F(x_n|x_{n-1}) &= P[X_n \le x_n | X_{n-1} = x_{n-1}] = P[\epsilon_n \le x_n - \rho x_{n-1}] \\ &= \begin{cases} 0 & \text{if } x_n < \rho x_{n-1} \\ \int_0^{x_n - \rho x_{n-1}} g(u)du & \text{if } x_n \ge \rho x_{n-1} \end{cases} \end{aligned} \tag{4.1}$$

Note that $F(x_n|x_{n-1})$ has density over the region $\{(x_n, x_{n-1}) : \; x_n \ge \rho x_{n-1}\}$ and is given by

$$f(x_n|x_{n-1}) = \begin{cases} \left[\frac{\lambda(1-\sqrt{\rho})^2}{2\pi(x_n - \rho x_{n-1})^3}\right]^{1/2} \exp\{-\lambda(1-\sqrt{\rho})^2/2(x_n - \rho x_{n-1}) & \text{if } x_n \ge \rho x_{n-1} \\ 0 & \text{if } x_n < \rho x_{n-1}. \end{cases} \tag{4.2}$$

The likelihood function of (λ, ρ) based on $(X_0, X_1, ..., X_n)$ can be written as

$$L(\lambda, \rho) = f(X_0) \cdot \prod_{j=1}^{n} f(X_j|X_{j-1}), \quad X_0 > 0, \; X_j > \rho X_{j-1}, \; j = 1, 2, ...n.$$

$$L(\lambda, \rho) = [\frac{\lambda}{2\pi X_0^3}]^{1/2} e^{-\frac{\lambda}{2X_0}} \prod_{j=2}^{n} [\frac{\lambda(1-\sqrt{\rho})^2}{2\pi(X_j - \rho X_{j-1})^3}]^{1/2} \exp[-\frac{\lambda(1-\sqrt{\rho})^2}{2(X_j - \rho X_{j-1})}]$$
$$\text{if } X_j > \rho X_{j-1}, \quad j = 1, 2, ..., n. \tag{4.3}$$

If we fix ρ then mle of $\frac{1}{\lambda} = \beta$ is given by

$$\begin{aligned} \hat{\beta}_n &= \frac{1}{n+1} \left[\frac{1}{X_0} + (1-\sqrt{\rho})^2 \sum_{j=1}^{n} \frac{1}{(X_j - \rho X_{j-1})} \right] \\ &= \frac{1}{n+1} \left[\frac{1}{X_0} + (1-\sqrt{\rho})^2 \sum_{j=1}^{n} \frac{1}{\epsilon_j} \right]. \end{aligned} \tag{4.4}$$

It can be noted here that, if X_0 has the pdf (1.3) then $U = X_0^{-1}$ has a gamma $G(1/2, \lambda/2)$ distribution with pdf

$$h(u) = \begin{cases} \left(\frac{\lambda}{2\pi}\right)^{1/2} u^{-1/2} e^{-\frac{\lambda u}{2}}, & u > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

Further, ϵ_j^{-1} has $G(\frac{1}{2}, \frac{\lambda(1-\sqrt{\rho})^2}{2})$ distribution for each $j \ge 1$ and $X_0, \epsilon_1, \epsilon_2, ..., \epsilon_n$ are mutually independent r.v's. Hence it follows that $\hat{\beta}_n$ is unbiased for β and $\hat{\beta}_n \to \beta$ a.s. as $n \to \infty$.

As far as asymptotic properties of mle are concerned we can ignore the term corresponding to X_0 in (4.4). The summands in the second term of (4.4) are i.i.d. r.v's, $\frac{1}{\epsilon_j}$, $j = 1, 2, ...n$ with

$$E(\frac{1}{\epsilon_j}) = \frac{1}{\lambda(1-\sqrt{\rho})^2} \text{ and } \mathrm{Var}(\frac{1}{\epsilon_j}) = \frac{2}{\lambda^2(1-\sqrt{\rho})^4}. \tag{4.6}$$

Hence an application of central limit theorem (CLT) for iid r.v's leads to the result that

$$\sqrt{n}(\hat{\beta}_n - \beta) = (\frac{n}{n+1})(1-\sqrt{\rho})^2\sqrt{n}\left(\frac{1}{n}\sum_{j=1}^{n}(\frac{1}{\epsilon_j}) - \frac{\beta}{(1-\sqrt{\rho})^2}\right)$$

$$\xrightarrow{\mathcal{L}} N(0, 2\beta^2) \text{ as } n \to \infty.$$

That is $\hat{\beta}_n$ is a CAN estimator of β when ρ is known.

If ρ is unknown, it can be estimated by $\tilde{\rho}_n$ suggested by Feigin and Resnick (1992) where

$$\tilde{\rho}_n = \min_{1 \le j \le n}\left(\frac{X_j}{X_{j-1}}\right). \tag{4.7}$$

This estimator is strongly consistent for ρ. In the following result proved by Feigin and Resnick (1992) the asymptotic distribution of $\tilde{\rho}_n$ in (4.7) is obtained.

Result: Let $\{X_n\}$ be an AR(1) sequence defined by (2.1) and $\tilde{\rho}_n$ be as in (4.7). Suppose that the distribution function G of ϵ_1 satisfies the following regularity conditions:

(i) $\{\epsilon_n\}$ is a sequence of non-negative iid r.v's.

(ii) For some $\epsilon > 0$,

$$\lim_{s\to\infty}\left\{\frac{1-G(sx)}{1-G(s)}\right\} = x^{-\alpha} \text{ for all } x > 0.$$

(iii) $E(\epsilon_n^{-\beta}) < \infty$ for some $\beta > \alpha$.

Then

$$P[b_n(\tilde{\rho}_n - \rho) > x] \to e^{-cx^\alpha}, \text{ as } n \to \infty$$

where $b_n = \left(\frac{1}{1-G}\right)^{\leftarrow}(n)$ (here $H^{\leftarrow}(n) = \inf\{x : H(x) \ge n\}$) and
$c = \int_0^\infty[1 - \prod_{n=0}^{\infty}\{1 - G(\rho^n s)\}]\alpha s^{-\alpha-1}ds.$

It is readily verified that the regularity conditions (i) and (ii) hold with $\alpha = 1/2$. The condition (iii) holds since ϵ_n^{-1} has a gamma distribution and hence all the moments are finite. Thus we have a limit law for $\tilde{\rho}_n$. But we do not have closed form expressions for b_n and c_n.

If ρ is unknown, the mle of β can be obtained by replacing ρ by $\tilde{\rho}_n$ in (4.4). Then the properties of $\hat{\beta}_n$ discussed above will not hold, since it is not possible to extract ϵ_j's from X_j's when ρ is unknown.

In the next section, we propose an estimator for λ, which is free from ρ and hence the above difficulties will not arise.

5. ESTIMATION BASED ON EMPIRICAL LAPLACE TRANSFORM

We estimate λ of the pdf (1.3) when it is the marginal pdf of an AR(1) sequence generated by (2.1) using the empirical LT. The empirical LT of a distribution function based on a sample $(X_1, X_2, .., X_n)$ is defined by

$$\hat{\phi}_n(s) = \frac{1}{n}\sum_{j=1}^{n} e^{-sX_j}, \quad s > 0. \tag{5.1}$$

Note that $E(\phi_n(s)) = \phi(s)$ for $s > 0$. The value of s can be fixed conveniently. We can use method of moments to estimate λ in terms of $\hat{\phi}_n(s)$. The LT of (1.3) is given by

$$\phi(s) = \exp\{-\sqrt{2\lambda s}\}.$$

Equating $\phi(s)$ to $\hat{\phi}_n(s)$ and solving we get an estimator $\hat{\lambda}_n$ of λ as

$$\hat{\lambda}_n = \frac{\{-\log\hat{\phi}_n(s)\}^2}{2s}. \tag{5.2}$$

It will be shown below that $\hat{\lambda}_n$ is a CAN estimator of λ for which we need the following results by Athreya and Pantula (1986).

Result 1: Let $\{X_n,\ n \geq 0\}$ be an AR(1) process (2.1) with marginal pdf (1.3). Assume that

(i) $E[\{\log|\epsilon_1|\}^+] < \infty$ and

(ii) ϵ_1 has a non-trivial absolutely continuous component.

Then for any initial distribution of X_0, the Markov sequence $\{X_n\}$ is Harris recurrent and strong mixing.

The AR(1) process described above satisfies both the conditions. For a Harris recurrent Markov sequence $\{X_n\}$ Athreya and Pantula (1986a) proved that

$$\sup_{A \in \mathcal{F}_0^n,\ B \in \mathcal{F}_{m+n}^\infty} |P(A \cap B) - P(A) \cdot P(B)| = \alpha'(m) \leq 2 \sup_n E[K_{m-1}(X_{n+1})], \tag{5.3}$$

where $\mathcal{F}_0^n$ and $\mathcal{F}_{m+n}^\infty$ are the minimal sigma fields induced by $(X_0, X_1, ...X_n)$ and $(X_{n+m},\ X_{n+m+1}, ...)$ respectively, and

$$K_{m-1}(X_{n+1}) = \parallel P(X_{n+m+1} \in A | X_{n+1}) - \pi(A) \parallel . \tag{5.4}$$

In (5.4), $P(X_{n+m+1} \in A|X_{n+1})$ denotes the m-step transition function of $\{X_n\}$, $\parallel \mu - \nu \parallel$ is the total variation norm of the signed measure $\mu - \nu$ for probability measures μ and ν and $\pi(\cdot)$ is the stationary measure.

In our case, the AR(1) sequence defined by (2.1) is strictly stationary, ergodic, Harris recurrent and strongly mixing. Hence the stationary measure is given by

$$\pi(A) = \int_A f(x)dx,$$

when $f(\cdot)$ is the pdf (1.3). Now, we use the relation (5.4) to determine the mixing coefficients, which is required to study the CAN property $\hat{\lambda}_n$.

Consider

$$\begin{aligned} E[K_{m-1}(X_{n+1})] &= \int_0^\infty K_{m-1}(x) f(x) dx \\ &= \int_0^\infty \parallel P_x(X_{m-1} \in A) - \pi(A) \parallel f(x)dx, \end{aligned} \tag{5.5}$$

where A is an arbitrary event and

$$\begin{aligned} P_x(X_{m-1} \in A) &= P[X_{m-1} \in A | X_0 = x] \\ &= P[\rho^{m-2}\epsilon_1 + \rho^{m-3}\epsilon_2 + \cdots + \epsilon_{m-1} \in A - \rho^{m-1}x], \end{aligned} \tag{5.6}$$

where we have used the recursive relation (2.1) and independence of r.v's X_0, $\epsilon_1, ..., \epsilon_{m-1}$. If we let

$$Z_m = \rho^{m-2}\epsilon_1 + \rho^{m-3}\epsilon_2 + \cdots + \epsilon_{m-1},$$

its pdf is given by

$$f_{Z_m}(z) = [\frac{\lambda\{1-(\sqrt{\rho})^{m-1}\}^2}{2\pi z^3}]^{1/2} \exp[-\frac{\lambda\{1-(\sqrt{\rho})^{m-1}\}^2}{2z}], \quad z \geq 0. \tag{5.7}$$

Now (5.6) implies that

$$\begin{aligned} P_x(X_{m-1} \in A) &= \int_{A-\rho^{m-1}x} f_{Z_m}(z)dz \\ &\leq \int_A f_{Z_m}(z)dz. \end{aligned}$$

Hence for any A, we can write

$$\| P_x(X_{m-1} \in A) - \pi(A) \| \leq \| \int_A [f_{Z_n}(z) - f(z)]dz \| . \tag{5.8}$$

Note that

$$\| \int_A [f_{z_m}(z) - f(z)]dz \| = \| \int_A [f(z) - f_{z_m}(z)]dz \|$$

$$\begin{aligned} \| \int_A \left(\frac{\lambda}{2\pi z^3}\right)^{1/2} & \left[e^{-\frac{\lambda}{2z}} - \{1-(\sqrt{\rho})^{m-1}\}e^{-\frac{\lambda}{2z}\{1-(\sqrt{\rho})^{m-1}\}^2}\right] dz \| \\ &\leq \| \int_A \left(\tfrac{\lambda}{2\pi z^3}\right)^{1/2} e^{-\frac{\lambda}{2z}}[1-1+(\sqrt{\rho})^{m-1}]dz \| \\ &\leq (\sqrt{\rho})^{m-1}. \end{aligned}$$

Thus we have from (5.5)

$$E[K_{m-1}(X_{n+1})] \leq (\sqrt{\rho})^{m-1}.$$

Hence from (5.3), we can take

$$\alpha(m) = 2\rho^{\frac{m-1}{2}} \tag{5.9}$$

as a sequence of mixing parameters for $\{X_n\}$. Clearly $\alpha(m) \to 0$ as $m \to \infty$. So $\{X_n\}$ is a strongly mixing sequence with mixing parameters $\alpha(m)$.

Let $Y_n = e^{-sX_n}$, $s > 0$, $n = 0, 1, 2, \ldots$ then $\{Y_n,\ n \geq 0\}$ is a strictly stationary, strong mixing sequence with mixing parameters $\alpha(m)$. Moreover, $\{Y_n\}$ is a sequence of r.v's which is uniformly bounded by unity and $\sum_{m=1}^{\infty} \alpha(m) < \infty$. Then by Theorem 18.5.4 of Ibragimov and Linnik (1978) we have as $n \to \infty$

$$n^{-1/2} \sum_{j=1}^{n} \{Y_j - E(Y_j)\} \xrightarrow{\mathcal{L}} N(0, \sigma^2),$$

where

$$\begin{aligned} \sigma^2 &= \mathrm{Var}(Y_0) + 2 \sum_{h=1}^{\infty} \mathrm{Cov}(Y_0, Y_h) > 0 \\ &= e^{-\sqrt{4\lambda s}} - e^{-\sqrt{8\lambda s}} + 2e^{-\sqrt{2\lambda s}} \sum_{h=1}^{\infty} \left[e^{-\sqrt{2\lambda s}\{\sqrt{1+\rho^h} - (\sqrt{\rho})^h\}} - e^{-\sqrt{2\lambda s}} \right]. \end{aligned}$$

Note that $0 < \sigma^2 < \infty$. Hence,

$$\sqrt{n}[\hat{\phi}_n(s) - \phi(s)] \xrightarrow{\mathcal{L}} N(0, \sigma^2) \text{ as } n \to \infty.$$

Now applying the well-known results on functions of asymptotic normal variates (cf. Serfling (1980), p. 118), it is proved for the estimator of λ defined by (5.2) that

$$\sqrt{n}(\hat{\lambda}_n - \lambda) \xrightarrow{\mathcal{L}} N\left(0,\ \left\{\frac{\ln \phi(s)}{s\phi(s)}\right\}^2 \sigma^2\right) \quad \text{as } n \to \infty,$$

or, $\sqrt{n}(\hat{\lambda}_n - \lambda) \xrightarrow{\mathcal{L}} N(0, 2\lambda e^{2\sqrt{2\lambda s}} \cdot \frac{\sigma^2}{n})$ as $n \to \infty$. Thus, $\hat{\lambda}_n$ is a CAN estimator of λ.

6. CONCLUDING REMARKS

We introduced a first order autoregressive model which generates a sequence of inverse Gaussian random variables. The existence of innovation distributions in various cases is discussed. Explicit expression for the innovation density is obtained in the case of a one-parameter inverse Gaussian model. The unknown parameters in all models are estimated using appropriate methods of estimation. It is also proved that the estimators obtained are consistent and asymptotically normal. These properties can be utilised for constructing asymptotic tests and confidence intervals for the parameters. The models discussed in this paper can be applied to time series data which show heavy tail behaviour.

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