Efficient Nested Simulation for Conditional Tail Expectation of Variable Annuities

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Abstract

For valuation of Variable Annuity contracts with a dynamic hedging program using Monte Carlo methods, nested simulation is often required. The process is computationally challenging, sometimes prohibitively so, in many practical applications. We propose a simulation procedure for estimating the Conditional Tail Expectation (CTE) of liabilities of a Variable Annuity dynamic hedging strategy. In a CTE calculation, tail scenarios, i.e., the scenarios that result in extreme losses, are most relevant. Thus, correctly identifying those scenarios would greatly improve the efficiency in a nested simulation. The proposed procedure takes advantage of the special structure of the CTE by first identifying a small set of potential tail scenarios from the first tier of simulation. We then focus the simulation budget on only those scenarios. We conduct extensive numerical experiments on different guarantee types and different stochastic stock return dynamics. The numerical results show that, when given a fixed simulation budget, the proposed procedure can improve the accuracy of CTE estimation by an order of magnitude compared to a standard nested simulation.
1 Introduction

Variable Annuities (VA) are insurance contracts used for wealth management purpose. The premiums of such contracts are invested into sub-accounts or segregated funds whose underlying investments are equities and/or fixed income assets in the capital market. The insurance companies provide various guarantees on the value of the investments, such as minimum value upon death and/or maturity, or minimum level of annual withdraws until deaths. Such guarantees could expose the insurers to significant amount of risk to movement in asset prices.

For insurance companies that issue VAs, it is common practice to dynamically hedge their market risk exposure originated from these contracts. When a dynamic hedging program is in place, the gain and loss from periodic rebalancing of the hedge portfolio contributes to the insurers’ profit and loss (P&L). We refer to such gain and loss as hedging error.

A complete valuation of VA requires a nested simulation consists of inner loop simulation at every rebalancing point of the outer loop simulation. To determine the value of the hedging error associated with the contract, one will need to simulate the value and the composition of the hedge portfolio, e.g. Delta, Gamma, and other applicable Greeks, using risk neutral pricing at every rebalancing point along each real world sample path using Monte Carlo simulation. We denote this process as the inner loop simulation. The periodic gain and loss from rebalancing such hedge portfolio are realized as the real-world sample paths evolve. Other items incorporated in the P&L of VA such as fee income and in-the-money payout of the guarantee are also modeled along the same real-world sample paths. We denote this process as the outer loop simulation. Throughout this paper, we will refer to the outer loop sample paths as “scenarios”.

Nested simulation of VAs is a very time-consuming and complex process, and often leads to results that are difficult to explain. For example, for a typical 60-year monthly projection of a single VA contract using Monte Carlo simulation with 5,000 scenarios and 1,000 inner loop sample paths, it requires

$$5000 \times 60 \times 12 \times 1000 = 3.6 \times 10^9$$

inner simulations of cash flows projected to the maturity of the contract. This number would seem more daunting if we consider a block of business with thousands if not millions of contracts.

A number of other articles have been published to address the computation challenges in nested simulation. The problem was tackled mostly from two directions. One is to reduce the number of individual policies in a portfolio to be processed through simulations without reducing the number of inner loop or outer loop simulation required, e.g. Gan
and Lin (2015). The other direction of the research is to reduce the number of economic scenarios to be processed through nested simulations for the entire portfolio without considering the detail of individual policies. Feng et al. (2016) conducted a comprehensive review of the techniques currently available to improve the computation efficiency of nested simulations from this angle. Gordy and Juneja (2010) and Broadie et al. (2011) presented different methods to strategically allocate computation budget among inner and outer loop simulations. The efficiency in the method by Gordy and Juneja (2010) could be further improved in the application of VA valuation because it considers the same number of simulation for every scenarios. The optimal allocation in Broadie et al. (2011)’s method can be slow and difficult to solve. Other methods address the computation challenge by using some proxies to replace the inner simulation. For example, Broadie et al. (2015) uses the Least-Squares Monte Carlo method and Feng (2014) introduce a PDE method.

This article proposes an Importance-Allocated Nested Simulation (IANS) method that is simple to implement and improves the accuracy by an order of magnitude. It alleviates the computation burden in nested simulation from the second direction mentioned above. The IANS method focus on nested simulation of the Conditional Tail Expectation (CTE) of VA liabilities. It first scans through all scenarios without any inner loop simulation and identifies a small subset of worst scenarios that would produce liabilities in the tail of the distribution. Nested simulation is then carried out only on this subset of scenarios, and the CTE measures are calculated according to the result of the nested simulation.

On a high level, the IANS method is similar to Broadie et al. (2011) in design, but it can be much more efficient in many VA applications. The IANS method allows a large number of scenarios to be considered for nested simulation so that the tail scenarios which produce extreme losses will be captured in the risk measure. Given a fixed computation budget, more inner simulations can be performed on the tail scenarios so that sampling error is reduced. The IANS method results in significant saving in run time due to the reduction in total number of inner simulation, compared to a standard nested simulation.

This article provides some background on Variable Annuities and the valuation of it. It lays out the algorithm for VA valuation using standard nested simulation. It then presents the algorithm of the proposed IANS method. Last but no the least, it illustrates the results from numerical experiments using the IANS method.
2 Variable Annuities: Product, Valuation and Modelling

2.1 Type of Contracts

We will first introduce some notations for describing the VA contracts.

\( F(t) \) Fund value of the sub-account/segregated fund at time \( t \).

\( G(t) \) Guarantee value at time \( t \).

\( T \) Time of maturity of the VA contract.

\( R \) Time of renewal of the VA contract, if applicable.

\( R^- \) The moment at time \( R \) right before the renewal of the contract taking effect, if applicable.

\( R^+ \) The moment at time \( R \) right after the renewal of the contract taking effect, if applicable.

\( R^- \), \( R \), and \( R^+ \) all refer to the same time point \( R \) but we use different notations to distinguish them so that we can clearly identify the sequential financial transactions in the renewal of a VA contract which theoretically occur at the same time.

The major types of benefits provided by Variable Annuities in the market today include:

**Guaranteed Minimum Death Benefit (GMDB)** A GMDB contract pays \( \max(F(t), G(t)) \) upon death of the policyholder at time \( t \). \( G(t) \) is typically 75% of 100% of the original premium, if we ignore any previous partial surrender or subsequent premium paid.

**Guaranteed Minimum Maturity Benefit (GMMB)** A GMMB contract pays \( \max(F(T), G(T)) \) at time \( t = T \).

**Guaranteed Minimum Accumulation Benefit (GMAB)** The GMAB contracts guarantee a minimum fund value at both renewal and maturity of the contract. Upon renewal at time \( t = R \), a new guarantee value \( G(R^+) \) may be set according to the following rules and will be effective from \( t = R \) to \( T \).

- If \( F(R) \leq G(R) \), then the insurer pays \( (G(R) - F(R)) \) into the fund to meet the guarantee value. The new guarantee value \( G(R^+) = G(R) \).
If \( F(R) > G(R) \), then \( G(R^+) = F(R) \).

In essence, the new guarantee value \( G(R^+) = \max(F(R), G(R)) \).

At the maturity of the contract, \( \max(F(T), G(T)) \) is paid out. In practice, minimum term applies (typically 10 years) on renewal and there may be an upper limit to the number of renewals allowed.

**Guaranteed Minimum Income Benefit (GMIB)** A GMIB contract guarantees the minimum annual income rate at which the policyholder can convert the fund value to an annuity benefit. (Hardy, 2003)

**Guaranteed Minimum Withdrawal Benefit (GMWB)** A GMWB contract guarantees the minimum rate up to which the policyholder can take withdrawal from the sub-account or segregated fund without impacting the guarantee value of the contract. The contract typically pays the higher of the fund value and the guarantee value of the contract upon death of the policyholder or maturity of the contract. (Dai et al., 2008)

A typical VA contract invests in mutual funds or sub-accounts, subject to an explicit or implicit guarantee fee in addition to the regular mutual fund management fee. The underlying investment in the mutual funds include equities and fixed income assets. Lapse is allowed and the fund value less any applicable surrender charges are paid upon lapse.

### 2.2 Hedging and Valuation

Because the amount the insurers need to pay to meet the guaranteed obligation in VAs are heavily dependent on the value of the underlying investments, it is now a common risk management practice to hedge the market risk exposures originated from these contracts. Dynamic hedging in particular is widely used among insurers with large portfolio of VAs.

In a dynamic hedging program, a hedge portfolio will be set up for a block of VA contracts using futures or options in accordance with the sensitivity of the VA liabilities to various state variables, i.e. the Greeks. The hedge portfolio needs to be rebalanced periodically due to change in market conditions, business mix (demographics) of the block of contracts. As a result, gain and loss will incur from these rebalancing trades. The rebalancing could occur as frequent as daily in practice, but is commonly modeled as monthly due to modeling constraints such as the computational issues we are trying to address in this article.

Suppose the hedging portfolio consists of a short position on the stock index and a bank account that earns risk-free rate \( r \), and is rebalanced monthly at time \( t = 1, \ldots, T \). We define
$B(t)$ Value invested in the bank account in the hedge portfolio set up at time $t$.

$\Delta(t)$ Units of stock invested in the hedge portfolio set up at time $t$.

$S(t)$ Stock price at time $t$.

$H(t)$ The risk-neutral value of the hedge at time $t$.

$H^+(t - 1)$ The value realized along the real-world paths at time $t$ of the hedge portfolio set up at time $(t - 1)$.

$HE(t)$ The hedging error incurred at time $t$.

Then $HE(t)$ can be expressed as

$$HE(t) = H(t) - H^+(t - 1)$$

where $H(t) = B(t) + S(t)\Delta(t)$ and $H^+(t-1) = B(t-1)e^r + S(t)\Delta(t-1)$.

In VA contracts, the insurer receives management fee and any applicable guarantee fee as income and uses it to pay for fund related expenses, sales commission, overhead expense, and more importantly, any shortfall in fund value to meet the guaranteed obligation. Other typical incomes in an insurance portfolio such as surrender charges, investment income of general account assets, and interest on surplus also apply. These periodic hedging errors, together with any gain and loss from the initial set up and the final unwinding of the hedge are recognized as part of the profit and loss (P&L) of the contract. The present value of net earning loss from these transactions constitutes the liability of the VA to the insurer.

In Canada, the reserve of VAs are generally set between CTE 60 and CTE 80 of the liability values from Monte Carlo simulation. (CIA, 2017) The minimum required capital of VAs are generally the CTE 95 liabilities less any reserves. (OSFI, 2015) In the US, the stochastic component of the reserve of VAs is set at CTE 70 of the Greatest Present Value of Accumulated Deficiencies (GPVAD) whereas the stochastic component of the minimum required capital of VAs is set at CTE 90 of the GPVAD. (AAA, 2011)

### 2.3 Standard Nested Simulation

To model the liability of the aforementioned VA contracts in a dynamic hedging program, a standard nested simulation is often required. The P&L of the contract other than those originated from the hedging program will be projected along the outer loop scenarios over the valuation horizon. The composition of the hedge portfolio need to be calculated
based on inner loop simulations. The inner loop simulation starts at each rebalancing point along the outer loop scenarios and ends at the end of the valuation horizon. Fund value and guarantee payment are projected using risk-neutral measure in each inner loop sample path. Hedging error is then calculated based on the composition of the hedge portfolio and the outer loop economic scenario. It than gets incorporated in the net cashflow of the contract which makes up the liability in each scenario.

To be more specific, Algorithm 1 below describes this nested simulation process. First we define

\( N_{\text{out}} \) Number of scenarios in the nested simulation.

\( N_{\text{in}} \) Number of inner loop sample paths at each projection point along the outer loop sample paths in the nested simulation.

\( S(t) \) Vector of economic state variables at time \( t \).

\( PO(t) \) In-the-money payout of the guarantee benefit at time \( t \), net of the fund value being paid.

\( CF(t) \) Net cashflow of the contract at time \( t \), reflecting the gain/loss from the hedging program

\( L \) Liability of the VA contract, reflecting the gain/loss from the hedging program

In Algorithm 1, the nested structure of the simulation is obvious. The inner loop simulation from line 5 to line 11 is nested within the outer loop simulation from line 1 to line 15.

### 2.4 Using Proxies to Replace Inner Simulation

It has been demonstrated in numerical examples (Gordy and Juneja, 2010) that in a nested simulation of risk measures, the outer loop simulation is more critical than the inner loop simulation to the accuracy of the results. Beyond a certain threshold where the tail events can be captured in the outer loop scenarios with high probability, adding more inner loop simulations provide little improvement to the accuracy of the results.

As a result, a few methods have been proposed to use analytical solution or approximation to replace the inner simulation portion in a standard nested simulation. Using these methods, the run time could be cut down significantly, but the accuracy of these methods depends on the type of VA contracts and asset models being considered.
input : $S(0)$, $F(0)$ and $G(0)$

output: $L_{\text{CTE}(1-\alpha)}$, liability at CTE $(1-\alpha)$

1. for $i \leftarrow 1$ to $N_{\text{out}}$ do
2.   for $t \leftarrow 0$ to $T$ do
3.     Simulate $S_i(t)$ using real-world measures using the input;
4.     Project $F_i(t)$ and $G_i(t)$ using $S_i(t)$ from line 3 and other assumptions such as mortality, lapse, withdrawal, fee deduction, etc.
5.   for $j \leftarrow 1$ to $N_{\text{in}}$ do
6.     for $t' \leftarrow (t+1)$ to $T$ do
7.         Simulate $S_{ij}(t,t')$ using risk-neutral measures;
8.         Project $F_{ij}(t,t')$ and $G_{ij}(t,t')$ using $S_{ij}(t,t')$ from line 7 and other assumptions such as mortality, lapse, withdrawal, fee deduction, etc.
9.     Calculate $PO_{ij}(t,t')$ using $F_{ij}(t,t')$ and $G_{ij}(t,t')$ from line 8
10.   end
11. end
12. Calculate $H_i(t)$ and $\Delta_i(t)$ as
13. \[
    H_i(t) = \frac{1}{N_{\text{in}}} \sum_{j=1}^{N_{\text{in}}} \sum_{t'=t+1}^{T} e^{-r(t'-t)} PO_{ij}(t,t')
\] (2)
14. \[
    \Delta_i(t) = \frac{1}{N_{\text{in}}} \sum_{j=1}^{N_{\text{in}}} \frac{\partial}{\partial S_i(t)} \sum_{t'=t+1}^{T} e^{-r(t'-t)} PO_{ij}(t,t')
\] (3)
15. Project $CF_i(t)$ using $H_i(t)$ and $\Delta_i(t)$ from line 12 and other assumptions.
16. end
17. Calculate $L_i = \sum_{t=0}^{T} e^{rt} \times CF_i(t)$
18. end
19. Sort all $L_i$’s from line 15;
20. Estimate the CTE $(1-\alpha)$ of $L$ as:
21. \[
    L_{CTE(1-\alpha)} = \frac{\sum_{n=(1-\alpha) \times N_{\text{out}} + 1}^{N_{\text{out}}} L(n)}{\alpha \times N_{\text{out}}}
\] (4)
22. where $L(n)$ is the $n$th order statistic of all $L_i$’s.

**Algorithm 1:** Standard Nested Simulation for VAs
For basic asset model such as lognormal returns in stock price, and basic VA contract such as GMDB or GMMB with no riders, closed form solutions can be found to replace the inner loop simulation (Hardy, 2003). However, this type of basic asset model is hardly suitable for modeling long-term stock returns in VA valuations.

In a more general framework, Broadie et al. (2015) use the Least-Squares Monte Carlo method to approximates security prices at each time step using a linear combination of basis function, which replaces the inner simulation. However, the effectiveness of the method depends on the basis function used, and there is no well-defined method or criteria for choosing the basis function, especially for VA valuations. Feng (2014) introduces a method to replace the inner simulation by solving PDEs. This method can be accurate and efficient but solving the PDEs may be challenging for some asset model and liability model.

3 The Importance-Allocated Nested Simulation (IANS) Method

Our proposed IANS method borrows from the idea of using proxies to replace inner simulation. However, we do not actually replace the inner simulation with proxies. Instead, we use the proxies to identify the tail scenarios that are relevant to the risk measure of interest (CTE of liabilities). Then we perform standard nested simulation only on these identified tail scenarios and calculate the CTE of liabilities using output from the standard nested simulation.

The IANS method improves simulation efficiency greatly because the proxy calculation takes negligible computation effort but is able to identify with high accuracy the scenarios that are relevant to the risk measure calculation. Therefore, it allows a wide range of scenarios to be considered without compromising the level of granularity in the inner simulation. As mentioned above, the number of outer loop scenarios used in simulation has significant impact on the accuracy of the risk measure estimation. Given the set up of the IANS method, sufficient computation budget can still be allocated to the inner loop simulation, which reduces the sampling error and further improves the accuracy of the estimation.

Algorithm 2 outlines the general framework of the IANS method. Note that the starting point of Algorithm 2 is after line 4 of Algorithm 1.

The choice of the percentage of scenarios $\beta$ for standard nested simulation mainly depends on the CTE level of the liability we try to estimate. This threshold should be viewed as a design variable in the experiment. Changing the level of such threshold leads to a trade
input : $S_i(t)$, $F_i(t)$ and $G_i(t)$, for $i = 1, \ldots, N_{out}$ and $t = 0, 1, \ldots, T$
output: $L_{CTE(1-\alpha)}$, liability at CTE $(1 - \alpha)$

for $i \leftarrow 1$ to $N_{out}$ do
  for $t \leftarrow 0$ to $T$ do
    Calculate the proxies for the hedge $\hat{H}_i(t)$ and the Delta $\hat{\Delta}_i(t)$;
    Project proxy for cashflows $\hat{CF}_i(t)$ using $\hat{H}_i(t)$ and $\hat{\Delta}_i(t)$ from line 3 and other assumptions;
  end
  Calculate proxy for liabilities $\hat{L}_i = \sum_{t=0}^{T} e^{rt} \times \hat{CF}_i(t)$;
end

Sort all $\hat{L}_i$’s from line 6 and identify a block of $(\beta \times N_{out})$ scenarios whose $\hat{L}_i$ are the worst $(\beta \times N_{out})$ among all $\hat{L}_i$’s. Denote this block of $(\beta \times N_{out})$ scenarios as $S$, i.e.

$$S := \{i | \hat{L}_i \in \{\hat{L}_{((1-\beta) \times N_{out}+1)}, \ldots, \hat{L}_{(N_{out})}\}\}$$

where $\hat{L}_{(n)}$ is the $n$th order statistic of $\hat{L}_i$’s.

for $i \in S$ do
  for $t \leftarrow 0$ to $T$ do
    Simulate $H_i(t)$ and $\Delta_i(t)$ by standard nested simulation using steps in line 5-12 in Algorithm 1;
    Project $CF_i(t)$ using $H_i(t)$ and $\Delta_i(t)$ from line 11 and other assumptions.
  end
  Calculate $L_i = \sum_{t=0}^{T} e^{rt} \times CF_i(t)$;
end

Sort all $L_i$’s from line 14;
Estimate the CTE $(1 - \alpha)$ of $L$ as:

$$L_{CTE(1-\alpha)} = \frac{\sum_{n=(\beta-\alpha) \times N_{out}+1}^{\beta \times N_{out}} L_{(n)}}{\alpha \times N_{out}}$$

where $L_{(n)}$ is the $n$th order statistic of $L_i$’s where $i \in S$.

Algorithm 2: The IANS Method for a CTE $(1 - \alpha)$ Valuation
off between computation effort and the accuracy of results.

4 Numerical Experiments

In this section, we adapt the IANS method to a CTE 80 and CTE 95 valuation on a GMDB contract and a GMAB contract. We illustrate the results from extensive numerical experiments on nested simulation using both the IANS method and standard Monte Carlo simulation.

4.1 Assumptions of the Experiment

4.1.1 Model of Variable Annuities

For both the GMDB and GMAB contracts in consideration, we assume:

- The contracts start at time $t = 0$ and have a term of 20 years (240 months), i.e. time of maturity $T = 240$.
- The starting fund value $F(0) = 1,000$, and there is no subsequent deposits into the fund.
- The guarantee level is at 100%.
- The fund is invested according to a stock index.
- The contract is purchased by the policyholder with an up-front fee that is equivalent to the fair market value of the liability at the time of purchase, i.e. $H(0)$, and no fee will be deducted from the fund.
- There is no lapse.
- The Delta of the contract is dynamically hedged with no basis risk, so that the payoff from the hedge portfolio will exactly cover any in-the-money payout of the contract.
- There is no transaction cost.
- The impact of the hedging is fully reflected in the valuation of the contract.
- For simplicity, best estimate assumptions were used for both inner and outer loop simulations.
More specifically for the GMDB contract, we assume:

- There is no reset, ratchet or other options in the contract.
- The mortality rate is 100% at time \( T \), the end of the 240th month.

For the GMAB contract, we assume:

- There is one renewal at time \( R = 120 \), i.e. the end of the 120th month.
- There is no mortality or other decrements.

The above assumptions may seem limiting, but they can easily be relaxed and the IANS method would still be applicable. For example, we simplified the model by assuming no management fee deduction and no decrements in the contract. Since the management fee deduction and static decrements in a GMDB and GMAB contract are deterministic (Hardy, 2003), their impact or the lack of it does not change the complexity of the Monte Carlo simulation of the stochastic model for stock price and the corresponding contract liability.

Also based on these assumptions, the liabilities of the VA contracts consist only of present value of the hedging error from each month, which we refer to as total hedging error. In practice, fee income, expense, commissions, decrements, and payout due to imperfect hedge are likely to make up a much higher proportion of the liability than hedging error. Nonetheless, the IANS method will still offer some useful insights because the total hedging error is the only quantity we need from the inner loop simulations in a nested simulation.

For the GMDB contract described above, the liability \( L \) is:

\[
L = \sum_{t=1}^{T} e^{-rt} HE(t)
\]

\[
= - \underbrace{(H(0) - \Delta(0)S(0))}_{B(0)} e^{-rT} \underbrace{(H(T) - \Delta(T)S(T))}_{B(T)} + \sum_{t=1}^{T} e^{-rt} S(t)(\Delta(t) - \Delta(t - 1))
\]

The derivation of Equation (6) can be found in Appendix B. Equation (6) shows that the total hedging error can be expressed as the sum of
- Difference between the present value of risk-free bond holding at maturity \((t = T)\) and at inception \((t = 0)\) of the contract.

- Sum of present value of all the changes in market value of the stock future holding as a result of the rebalancing trades.

The expression offers some intuition about the composition of the hedging error. In the bond portion of the hedge, because the interest rate the bond holding accumulates at is the same as the interest rate the (gain)/loss from the bond trades is discounted at, all the interim bond trades between the inception and maturity of the contract can be reduced to one trade, which is to top down/up the bond holding at inception so that it can accumulate to the bond holding required at maturity. Therefore, the (gain)/loss from the bond trades is precisely \((-B(0) + e^{-rT}B(T))\). In the stock portion the hedge, the (gain)/loss from every stock trade contributes to the total hedging error.

Using Equation (6) in simulation saves some computation effort compared to summing over the present value of hedging error from each period explicitly, because the simulation of hedge value between the inception and maturity of the contract is not required in Equation (6).

For the GMAB contract considered in this study, the liability \(L\) is:

\[
L = \sum_{t=1}^{T} e^{-rt} HE(t)
\]

\[
= - \left( H(0) - \Delta(0)S(0) \right) + e^{-rT} \left( H(T) - \Delta(T)S(T) \right)
+ \sum_{t=1}^{T} e^{-rt} S(t) \left( \Delta(t) - \Delta(t - 1) \right) + e^{-rR} \left( H(R^-) - H(R^+) \right)
\]

where

- \(H(R^-)\) represents the hedge portfolio set up at time \(R\) to cover the payout at time \(R\) if there is any, plus the potential payout at time \(T\).

- \(H(R^+)\) represents the hedge portfolio set up at time \(R\), immediately after the payout at time \(R\) if there is any, so that \(H(R^+)\) only covers the potential payout at time \(T\).

- \((H(R^-) - H(R^+))\) is essentially the amount of payout at time \(R\).

- \(\Delta(R) = \Delta(R^+)\), and \(\Delta(R^+)\) is the unit of stock held in hedge \(H(R^+)\).
Compared to Equation (6), Equation (7) has an extra term $e^{-rR}(H(R^-) - H(R^+))$. This extra term is to reverse the hedging error gain that is overstated by assuming $\Delta(R) = \Delta(R^+)$. The derivation with detailed explanation of Equation (7) can be found in Appendix B. Similar to Equation (6), using Equation (7) in simulation also saves some computation effort compared to summing over the present value of hedging error from each period explicitly.

### 4.1.2 Model of Stock Price Dynamic

We used the regime-switching lognormal model for stock price dynamic (Hardy, 2001) in this study. It is a popular model used by practitioners and is well suited for modeling the stock return in the valuation of GMDB and GMAB contracts. The model can capture some important features of stock market returns, including

- Extreme left-tail events
- Volatility clustering
- The association of high volatility and low returns

It is worth noting that the IANS method is not limited to the regime-switching lognormal model for stock price.

Appendix A lists the specific parameters used in our regime-switching model. Given the real-world measure in the regime-switching model, the risk neutral measure we chose to use is the one studied by Bollen (1998), which is also referenced in Hardy (2001). More specifically, the transition probabilities between regimes, $p_{ij}$, for $i, j = 1, 2$, remain the same under the risk neutral measure. The mean of the conditional log-return under the risk neutral measure has changed from the real-world measure as follows:

- In regime-1: from $\mu_1$ to $r - \sigma_1^2/2$
- In regime-2: from $\mu_2$ to $r - \sigma_2^2/2$

Since the market is incomplete in the regime-switching model for stock return, the risk neutral measure is not unique. There are other possible risk neutral measure that can be used for nested simulations but we choose this one because it is easy to implement and straightforward to understand.
4.2 Specification in Standard Nested Simulation

To implement the IANS method, we need to specify the model for a standard nested simulation of the contract because in the second part of the IANS method (line 9 to line 15 in Algorithm 2), the liabilities of the VAs were simulated using standard nested simulation for the chosen scenarios. In this study, we also conducted full standard nested simulation to compare results with the IANS method.

In a standard nested simulation, the value of the hedge in scenario $i$ at time $t$ for the GMDB contract we consider is simulated as:

$$H_i(t) = \frac{1}{N_{in}} \sum_{j=1}^{N_{in}} e^{-r(T-t)} \max(G_{ij}(t, T) - F_{ij}(t, T), 0)$$  \hspace{1cm} (8)

The value of the hedge in scenario $i$ at time $t$ is simulated as pathwise derivative estimates using Monte Carlo method (Glasserman, 2013):

$$\Delta_i(t) = \frac{1}{N_{in}} \sum_{j=1}^{N_{in}} -e^{-r(T-t)} + \mu_{ij}(t, T) 1_{ij}(t, T)$$  \hspace{1cm} (9)

where

$$1_{ij}(t, t') = \begin{cases} 
1 & \text{if } G_{ij}(t, t') > F_{ij}(t, t'), \\
0 & \text{if } G_{ij}(t, t') \leq F_{ij}(t, t'). 
\end{cases}$$

Note that $G(t)$ of the GMDB contract in fact remains constant throughout the projection period under our assumption. The notations allow the guarantee value to vary by time in order to keep the equations general for other type of contracts.

Likewise, for the GMAB contract, the value of the hedge in scenario $i$ at time $t$, where $t < R$, is simulated as:

$$H_i(t) = \frac{1}{N_{in}} \sum_{j=1}^{N_{in}} \left( e^{-r(R-t)} \max(G_{ij}(t, R) - F_{ij}(t, R), 0) + e^{-r(T-t)} \max(G_{ij}(t, T) - F_{ij}(t, T), 0) \right)$$  \hspace{1cm} (10)

The value of the hedge in scenario $i$ at time $t$, where $t < R$, is simulated as pathwise
derivative estimates using Monte Carlo method:

\[
\Delta_i(t) = \frac{1}{N_{in}} \sum_{j=1}^{N_{in}} \left( -e^{-r(R-t)+\mu_{ij}(t,R)}1_{ij}(t, R) \right. \\
+ e^{-r(T-t)+\mu_{ij}(t,T)} \left( 1 - e^{\mu_{ij}(t,T)-\mu_{ij}(t,R)})(1 - 1_{ij}(t, R^)) 1_{ij}(t, T) \right), \quad t < R
\]

For the GMAB contract we consider, if \( t \geq R \), then \( H_i(t) \) and \( \Delta_i(t) \) can be simulated using Equation (8) and (9), respectively.

Equation (8) and (10) show how Equation (2) in Algorithm 1 is adapted to our example. Similarly, Equation (9) and (11) show how Equation (3) in Algorithm 1 is adapted to our example.

With these, \( L_i \)'s can be calculated for each scenario \( i \) by substituting the appropriate \( H_i(t) \), \( \Delta_i(t) \) and \( S_i(t) \) into Equation (6) or (7) depending on the contract that is being modeled. The risk measure at CTE 80 and CTE 95 can then be estimated according to line 18 in Algorithm 1 where \( \alpha = 0.2 \) and \( \alpha = 0.05 \), respectively.

4.3 Specification in the IANS Method

4.3.1 Calculation for the Proxies

In our experiment, we adapt the Black-Scholes pricing formula to the GMDB and GMAB contract and use them as proxies for the value of the hedge and Delta of the hedge referenced in line 3 Algorithm 2. The proxies are calculated as follows.

For a GMDB contract,

\[
\hat{H}_i(t) = G_i(t)e^{-r(T-t)}\Phi(-d_2(t, T, S_i(t), G_i(t))) \\
- S_i(t)\Phi(-d_1((t, T, S_i(t), G_i(t)))
\]

and

\[
\hat{\Delta}_i(t) = -\Phi(-d_1((t, T, S_i(t), G_i(t))))
\]

where \( \Phi(x) \) is the distribution function of a standard Normal distribution, and

\[
d_1(t, t', S_i(t), G_i(t)) = \frac{\log(S_i(t)/G_i(t)) + r(t'-t) + \frac{1}{2}(\hat{\sigma}_i(t, t'))^2(t'-t)}{\hat{\sigma}_i(t, t')\sqrt{t'-t}}
\]

\[
d_2(t, t', S_i(t), G_i(t)) = d_1(t, t', S_i(t), G_i(t)) - \hat{\sigma}_i(t, t')\sqrt{t'-t}
\]
For a GMAB contract, if \( t < R \), then the proxy for the hedge value and Delta at time \( t \) is calculated as:

\[
\tilde{H}_i(t) = \left( G_i(t)e^{-r(R-t)} \Phi(-d_2(t, R, S_i(t), G_i(t))) \right)
\]

\[
- S_i(t) \left( \Phi(-d_1(t, R, S_i(t), G_i(t))) - 1 \right)
\]

\[
\times \left( 1 + e^{-r(T-R)} \Phi(-\tilde{d}_2(R, T, 1, 1)) - \Phi(-\tilde{d}_1(R, T, 1, 1)) \right) - S_i(t)
\]

and

\[
\hat{\Delta}_i(t) = - \left( \Phi(-d_1(t, R, S_i(t), G_i(t))) - 1 \right)
\]

\[
\times \left( 1 + e^{-r(T-R)} \Phi(-\tilde{d}_2(R, T, 1, 1)) - \Phi(-\tilde{d}_1(R, T, 1, 1)) \right) - 1
\]

where \( d_1(t, R, S_i(t), G_i(t)) \) and \( d_2(t, R, S_i(t), G_i(t)) \) are defined in Equation (14) and (15), and

\[
\tilde{d}_1(R, T, 1, 1) = \frac{r(T - R) + \frac{1}{2}(\bar{\sigma}(R, T))^2(T - R)}{\bar{\sigma}(R, T)\sqrt{T - R}}
\]

\[
\tilde{d}_2(R, T, 1, 1) = \tilde{d}_1(R, T, 1, 1) - \bar{\sigma}(R, T)\sqrt{T - R}
\]

For a GMAB contract, if \( t \geq R \), then the proxy for the value of the hedge and Delta at time \( t \) is determined using Equation (12) and (13).

Equation (12) and (13) directly use the Black-Scholes formula for pricing European put options. Equation (16) and (17) are adapted from the solution for value of the hedge and Delta of a GMAB contract under Black-Scholes model, as illustrated in Hardy (2003). The choice of the proxies for value of the hedge and the Delta is not unique. The ones presented above have proved to work sufficiently well in the extensive numerical experiments we conducted. Proxies in different format are likely required for modeling more sophisticated contracts.

### 4.3.2 Volatility Assumption in the Proxy Calculation

The proxy calculations shown above requires two sets of volatility parameters, \( \hat{\sigma}_i(t, t') \) and \( \bar{\sigma}(R, T) \). \( \hat{\sigma}_i(t, t') \) represents an average volatility translated from the variance of log return in stock price over a period of \( (t' - t) \) in a regime-switching model, conditional on the starting regime \( \rho_i(t) \). \( \bar{\sigma}(R, T) \) represents an unconditional average volatility translated from the expected value of variance of log return in stock price over a period of \( (T - R) \) in a regime-switching model. \( \hat{\sigma}_i(t, t') \) and \( \bar{\sigma}(R, T) \) are derived as follows.
We denote

- \( Y(t, t') = \log \frac{S(t')}{S(t)} \), as the random variable of the log return in stock price over the period \((t' - t)\).
- \( \text{Var}(Y(t, t')|\rho(t)) \) as the variance of log return in the regime-switching model over \((t' - t)\) periods, conditional on starting regime being \(\rho(t)\).
- \( R(t, t') \) as the number of sojourns in regime-1 over \((t' - t)\) periods.

The assumptions of the regime-switching model under risk-neutral measure are outlined in Appendix A.

Then we have

\[
(\tilde{\alpha}_i(t, t'))^2 = \frac{\text{Var}(Y(t, t')|\rho(t))}{t' - t}
\]

This calculation translates the multi-period variance in a regime switching model to a single period variance in the Black-Scholes model, which assumes the variances are independent and identically distributed in each period. This translation is only an approximation as our primary focus is not on the exact value of the hedging error, but the ranking of the proxies across all scenarios. The approximation has been proven to work well for our purpose in numerical experiments.

Furthermore,

\[
\text{Var}[Y(t, t')|\rho(t)] = E[\text{Var}(Y(t, t')|(R(t, t')|\rho(t)))] + \text{Var}[E(Y(t, t')|(R(t, t')|\rho(t)))]
\]

\[
= E[(R(t, t')|\rho(t))\sigma_1^2 + (t' - t - (R(t, t')|\rho(t)))\sigma_2^2]
\]

\[
+ \text{Var}[(R(t, t')|\rho(t))(r - \frac{1}{2}\sigma_1^2) + (t' - t - (R(t, t')|\rho(t)))(r - \frac{1}{2}\sigma_2^2)]
\]

\[
= (\sigma_1^2 - \sigma_2^2)E[R(t, t')|\rho(t)] + (t' - t)\sigma_2^2 + \text{Var}\left[\frac{1}{2}(\sigma_2^2 - \sigma_1^2)(R(t, t')|\rho(t))\right]
\]

\[
= (t' - t)\sigma_2^2 + (\sigma_1^2 - \sigma_2^2)E[R(t, t')|\rho(t)] + \frac{1}{4}(\sigma_2^2 - \sigma_1^2)^2\text{Var}[R(t, t')|\rho(t)]
\]

where \(E[R(t, t')|\rho(t)]\) and \(\text{Var}[R(t, t')|\rho(t)]\) are calculated based on the probability mass function of \(R\), which is generated by backward recursion using the algorithm presented in...
Hardy (2001). The expression for the expected value and variance of log return conditional on the number of sojourns in regime-1 are also based on Hardy (2001).

Following the same notation used in the derivation of $\hat{\sigma}_i(t, t')$, 

\[
(\bar{\sigma}(R, T))^2 = E[Var(Y(R, T))] \frac{T - R}{T - R} = \frac{Var(Y(R, T)|\rho(R) = 1) \times \pi_1 + Var(Y(R, T)|\rho(R) = 2) \times \pi_2}{T - R}
\]

where

- $Var(Y(R, T)|\rho(R) = 1)$ and $Var(Y(R, T)|\rho(R) = 2)$ are calculated according to Equation (21).
- $\pi_1 = \frac{p_{21}}{p_{12} + p_{21}}$ and $\pi_2 = \frac{p_{12}}{p_{12} + p_{21}}$ are the unconditional probabilities of the regime-switching Markov chain being in regime-1 and regime-2, respectively.
- $p_{21}$ and $p_{12}$ are transition probability in the regime-switching model. (Hardy, 2001)

$\bar{\sigma}(R, T)$ represents an unconditional average volatility of the regime-switching under the risk-neutral measure over a period of $T - R$. We use $\bar{\sigma}(R, T)$ as a proxy for the volatility over the period of $T - R$ because at any rebalancing point $t$, where $t < R$, on a outer loop sample path, we would not be able to know which regime the stock price under risk neutral measure would be in at time $R$ without simulation. Thus, we calculate the proxy with the unconditional expected value of the volatility for this period.

We try to mimic the volatility that contributes to the stock price projected in a standard Monte Carlo inner loop simulation in the regime-switching model by using $\hat{\sigma}_i(t, t')$ and $\bar{\sigma}(R, T)$ in the proxy calculation. We observe that the absolute size of the discrete hedging error being modeled in the simulation is very sensitive to the volatility of the stock price, particularly when it is close to maturity or renewal. In a regime-switching model, the volatility in regime-2 is much higher than in regime-1, yet the occurrence of regime-2 is much rarer than that of regime-1. The occurrence of regime-2 also tends to cluster. Therefore, $\hat{\sigma}_i(t, t')$ does a much better job than any unconditional volatility assumption in capturing the dynamic of volatility, which is specific to each outer loop scenario.
4.3.3 \( \beta \), the Threshold for Standard Nested Simulation in IANS

In the IANS method described in Algorithm 2, \( \beta \times N_{out} \) number of scenarios are processed through standard nested simulation.

The \( \beta \) we choose for our numerical experiments are:

- For CTE 95 valuation, \( \beta = 10\% \) for both GMDB and GMAB contracts.
- For CTE 80 valuation, \( \beta = 25\% \) for both GMDB and GMAB contracts.

Note that line 1 to line 7 in Algorithm 2 takes less computation effort than a standard nested simulation on a single scenario. Thus, we consider the computation budget taken by line 1 to line 7 in Algorithm 2 negligible. For a CTE 80 valuation, the IANS method will run 25\% of the scenarios through nested simulation, which seemingly reduce the amount of computation effort saved. Nevertheless, even in a hypothetical perfect world where we could exactly pinpoint the scenarios that would generate the worst 20\% liability, we would still have to run nested simulation on these 20\% scenarios to get the actual value of the CTE 80 estimate. Therefore, the additional 5\% scenarios required for nested simulation by the IANS method are only a small increment to the 20\% scenario that could be considered as a sunk cost in computation budget.

4.4 The IANS Method vs. Standard Monte Carlo

The IANS method is more efficient than a full nested simulation. We demonstrate below the numerical results from experiments using the IANS method versus three other standard nested simulations. All four type of experiments have the same computation budget. The results show that the IANS method produces samples of CTE 95 and CTE 80 liability that are much closer to the true value of these two risk measures.

In the absence of analytical solutions, we consider the simulated CTE’s from a comprehensive standard nested simulation with 10,000 scenarios and 10,000 inner loop simulations the true value of the CTE’s. We use it as the benchmark to compare the results from each numerical experiment against. The CTE’s in the benchmark results are shown in Table 1.

For both GMDB and GMAB contract, we repeated each experiment listed in Table 2 100 times.

In Table 2, SMC represents standard Monte Carlo simulation. For the IANS experiments, the numbers in the \( N_{out} \) and \( N_{in} \) column represent the value of these parameters as indicated in Algorithm 2. The numbers in the \( N_{out}^{SMC} \) column represents the number of
<table>
<thead>
<tr>
<th>Contract</th>
<th>CTE95</th>
<th>CTE80</th>
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</thead>
<tbody>
<tr>
<td>GMDB</td>
<td>51.93</td>
<td>30.84</td>
</tr>
<tr>
<td>GMAB</td>
<td>136.07</td>
<td>77.97</td>
</tr>
</tbody>
</table>

Table 1: Benchmark Results

Table 2: Simulations in Numerical Experiments

scenario processed in the standard nested simulation portion of Algorithm 2, i.e. \( \beta \times N_{\text{out}} \).

For the SMC experiments, \( N_{\text{out}} \) and \( N_{\text{in}} \) represents the same parameters as indicated in Algorithm 1 and \( N_{\text{out}}^{\text{SMC}} \) is the same as \( N_{\text{out}} \). The total number of simulations conducted in each experiment is the product of \( N_{\text{out}}^{\text{SMC}} \) and \( N_{\text{in}} \), which is the same among all five experiments listed above.

Results from 100 iterations of each experiment can be found in Table 3 and Table 4. All the experiments and iterations within each experiments are independent of each other. The CTE 80 and CTE 95 liability from the standard Monte Carlo experiments are generated by the same simulations. For both CTE 95 and CTE 80 liability, we calculate the mean of the simulated CTE’s from the 100 iterations of each experiment. We also calculate the mean squared error (MSE) between the CTE’s from the 100 iterations of each experiment and the CTE’s in the benchmark results. Figure 1 and 3 show the distribution of CTE 95 liability from the 100 iterations of each experiment. Figure 2 and 4 show the distribution of CTE 80 liability from the 100 iterations of each experiment. The red line in the graph shows the liabilities from the corresponding CTE level from the benchmark results.

The GMDB results in Table 3 show that the IANS method is superior than standard nested simulation using the same computation budget. Our method produces CTE 95 and CTE 80 liabilities very close to the benchmark result with the least mean squared error from
(a) IANS (2,000/5,000)
(b) SMC (2,000/500)
(c) SMC (1,000/1,000)
(d) SMC (200/5,000)

Figure 1: GMDB - CTE 95 Liability
Figure 2: GMDB - CTE 80 Liability
(a) IANS (2,000/5,000)  
(b) SMC (2,000/500)  
(c) SMC (1,000/1,000)  
(d) SMC (200/5,000)  

Figure 3: GMAB - CTE 95 Liability
Figure 4: GMAB - CTE 80 Liability

(a) IANS (2,000/2,000)
(b) SMC (2,000/500)
(c) SMC (1,000/1,000)
(d) SMC (200/5,000)
<table>
<thead>
<tr>
<th>Experiment</th>
<th>CTE 95</th>
<th>CTE 80</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>MSE</td>
</tr>
<tr>
<td>IANS (2,000/5,000/CTE95)</td>
<td>51.91</td>
<td>3.26</td>
</tr>
<tr>
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<td>n/a</td>
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<tr>
<td>SMC (2,000/500)</td>
<td>54.44</td>
<td>10.47</td>
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<tr>
<td>SMC (1,000/1,000)</td>
<td>53.06</td>
<td>8.13</td>
</tr>
<tr>
<td>SMC (200/5,000)</td>
<td>51.59</td>
<td>33.94</td>
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</table>

Table 3: GMDB - Liability Output from 100 Iterations of Experiment

<table>
<thead>
<tr>
<th>Experiment</th>
<th>CTE 95</th>
<th>CTE 80</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>MSE</td>
</tr>
<tr>
<td>IANS (2,000/5,000/CTE95)</td>
<td>137.35</td>
<td>54.57</td>
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<tr>
<td>IANS (2,000/2,000/CTE80)</td>
<td>n/a</td>
<td>n/a</td>
</tr>
<tr>
<td>SMC (2,000/500)</td>
<td>140.42</td>
<td>68.63</td>
</tr>
<tr>
<td>SMC (1,000/1,000)</td>
<td>137.77</td>
<td>87.36</td>
</tr>
<tr>
<td>SMC (200/5,000)</td>
<td>135.20</td>
<td>343.45</td>
</tr>
</tbody>
</table>

Table 4: GMAB - Liability Output from 100 Iterations of Experiment

the benchmark result compared to the SMC experiments. In the SMC (2,000/500) and the SMC (1,000/1,000) experiments, the CTE estimates are clearly biased even though the mean squared error is moderate. In the SMC (200/5,000) experiments, there is less bias in the CTE estimates but the results are very volatile from experiment to experiment. If we only focus on the mean squared error from each experiment, we can see that the SMC experiments with more outer loop simulations and fewer inner loop simulations, e.g. SMC (2,000/500) and the SMC (1,000/1,000), have much smaller mean squared error than the SMC experiment with fewer outer loop simulations but more inner loop simulations. This echoes the proposition we raised earlier about the outer loop simulation being more important than inner loop simulation in CTE estimations in nested simulation.

The results for GMAB contracts shown in Table 4 commensurate with those for GMDB contracts. In a GMAB contract, which in essence a compound put option, the hedging error could be a lot more volatile because of the structure of the guarantee, which explains why the mean squared errors are much higher than the same experiments on a GMDB
It is interesting to observe that in the GMDB simulation, the mean squared error of the SMC (1,000/1,000) experiment is slightly lower than the SMC (2,000/500) experiment, but it is the other way around in the GMAB simulation. This is due to the fact that the hedging error in GMAB contracts are more volatile, and covers a wider range of outcomes in the distribution. Therefore, it is more important to have sufficient number of outer loop simulations to generate accurate results.

4.5 Benchmark Result

We also make some interesting observations in the benchmark result. Note that the benchmark run for GMDB and GMAB are two independent experiments, i.e. they are based on different scenarios and different inner loop sample paths.

In Figure 5 and Figure 6, we show the full distribution of liability from standard nested simulation against the liability based on the closed-form proxy calculation for all 10,000 scenarios. Both figures show very close relation between the simulated liability and liability based on closed-form calculation. This illustrates how the proxy calculation provides good indication of the ranking of the scenario in terms of simulated liability.

Furthermore, we notice a large number of scenarios overlap between the worst scenarios from the standard nested simulation and the worst scenarios based on the closed-form proxy calculation in terms of liability. The statistics in the benchmark run are shown in Table 5.

<table>
<thead>
<tr>
<th>CTE Level</th>
<th>GMDB</th>
<th>GMAB</th>
</tr>
</thead>
<tbody>
<tr>
<td>CTE 95 (Out of 500 Worst Scenarios)</td>
<td>472</td>
<td>499</td>
</tr>
<tr>
<td>CTE 80 (Out of 2000 Worst Scenarios)</td>
<td>1910</td>
<td>1924</td>
</tr>
</tbody>
</table>

Table 5: Number of Overlap between the Worst Scenarios from Standard Simulation and from Proxy Calculation

We also show in Figure 7 the distribution of the liability from standard nested simulations against the liability calculated from the closed-form proxy calculation of a GMAB contracts with the same term to maturity, but has two renewals of equal periods in between. We use 5,000 scenarios and 5,000 inner loop simulations in this experiment. The results follow the same pattern as the GMAB contracts with only one renewal.
Figure 5: Distribution of Liability of the GMDB Contract from a 10000-Outer-Loop Simulation
Figure 6: Distribution of Liability of the GMAB Contract (1 Renewal) from a 10000-Outer-Loop Simulation
Figure 7: Distribution of Liability of the GMAB Contract (2 Renewal) from a 5000-Outer-Loop Simulation
5 Conclusion

In this article, we illustrate a simulation procedure for estimating the CTE of liabilities of a VA dynamic hedging strategy. The Importance-Allocated Nested Simulation method we propose takes advantage of the special structure of the CTE by first identifying a small set of potential tail scenarios from the first tier of simulation based on a proxy for liabilities calculated from a closed-form solution. We then focus the simulation budget on only those scenarios. We conduct extensive numerical experiments on GMDB and GMAB contracts. The numerical results show significant improvement in efficiency using the IANS method compared to a standard nested simulation.

The IANS method also inspires efficient experiment designs in other financial and actuarial applications where the CTE is estimated by Monte Carlo simulation. For future work, we will consider applying similar experimental design to nested simulation of GMDB and GMAB contracts with dynamic policyholder behavior. We will also explore solutions to improve the efficiency of nested simulation of GMIB and GMWB products.

6 Acknowledgements

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References


A  Parameters in the Model

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<th>Real World</th>
<th>Risk Neutral</th>
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<tr>
<td>Risk-free Rate: $r$</td>
<td>0.002</td>
<td>0.002</td>
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<tr>
<td>Mean - Regime 1 ($\rho = 1$): $\mu_1$</td>
<td>0.0085</td>
<td>0.0013875</td>
</tr>
<tr>
<td>Mean - Regime 2 ($\rho = 2$): $\mu_2$</td>
<td>-0.0200</td>
<td>-0.0012000</td>
</tr>
<tr>
<td>Standard Deviation - Regime 1: $\sigma_1$</td>
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<td>0.035</td>
</tr>
<tr>
<td>Standard Deviation - Regime 2: $\sigma_2$</td>
<td>0.080</td>
<td>0.080</td>
</tr>
<tr>
<td>Transition Probability - from Regime 1: $p_{12}$</td>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td>Transition Probability - from Regime 2: $p_{21}$</td>
<td>0.20</td>
<td>0.20</td>
</tr>
</tbody>
</table>

Table 6: Parameters in the Model for VA Contracts

Table 7: Parameters for Log-Return in the Regime-Switching Model


\section*{B Total Hedging Error}

For the GMDB contract in our study, the liability can be expressed as:

\begin{equation}
L = \sum_{t=1}^{T} e^{-rt} HE(t)
\end{equation}

\begin{align*}
&= \sum_{t=1}^{T} e^{-rt} (H(t) - H^+(t-1)) \\
&= \sum_{t=1}^{T} e^{-rt} \left( [B(t) + \Delta(t)S(t)] - [B(t-1)e^r + \Delta(t-1)S(t)] \right) \\
&= \sum_{t=1}^{T} e^{-rt} \left( [H(t) - \Delta(t)S(t) + \Delta(t)S(t)] \\
&\quad - [(H(t-1) - \Delta(t-1)S(t-1))e^r + \Delta(t-1)S(t)] \right) \\
&= \sum_{t=1}^{T} e^{-rt} \left( H(t) - H(t-1)e^r + \Delta(t-1)[S(t-1)e^r - S(t)] \right) \\
&= -H(0) + \Delta(0)S(0) + e^{-rT}(H(T) - \Delta(T-1)S(T)) \\
&\quad + \sum_{t=1}^{T-1} e^{-rt} S(t)(\Delta(t) - \Delta(t-1)) \\
&= -H(0) + \Delta(0)S(0) + e^{-rT}(H(T) - \Delta(T)S(T)) \\
&\quad + \sum_{t=1}^{T} e^{-rt} S(t)(\Delta(t) - \Delta(t-1))
\end{align*}

Following Equation (23) above, the liability of the GMAB contract in our study can be expressed as:
\[ L = \sum_{t=1}^{T} e^{-rt} \text{HE}(t) \]

\[ = \sum_{t=1}^{R-1} e^{-rt} \text{HE}(t) + e^{-r(R+1)} \text{HE}(R+1) + \sum_{t=R+2}^{T} e^{-rt} \text{HE}(t) \]

\[ = \sum_{t=1}^{R-1} e^{-rt} \text{HE}(t) + e^{-rR} (H(R^-) - H^+(R-1)) + e^{-r(R+1)} (H(R+1) - H^+(R^+)) \]

\[ + \sum_{t=R+2}^{T} e^{-rt} \text{HE}(t) \]

\[ = \sum_{t=1}^{R-1} e^{-rt} \text{HE}(t) + e^{-rR} (H(R^-) - H^+(R-1)) + e^{-r(R+1)} (H(R+1) - H^+(R^+)) \]

\[ + \sum_{t=R+2}^{T} e^{-rt} \text{HE}(t) + e^{-rR} (H(R^-) - H(R^+)) \]

\[ = \sum_{t=1}^{T} e^{-rt} (H(t) - H^+(t-1)) + e^{-rR} (H(R^-) - H(R^+)), \text{ where } H(R) = H(R^+) \]

\[ = - (H(0) - \Delta(0)S(0)) + e^{-rT} (H(T) - \Delta(T)S(T)) \]

\[ + \sum_{t=1}^{T} e^{-rt} S(t) (\Delta(t) - \Delta(t-1)) + e^{-rR} (H(R^-) - H(R^+)) \]

where \( \Delta(R) = \Delta(R^+) \)

Note that \( \text{HE}(R) = (H(R^-) - H^+(R-1)) \) because the hedging error at time \( R \) is the gain/loss from rebalancing the hedge portfolio set up at time \( (R-1) \) to the hedge portfolio required at time \( R^- \), which covers the potential payout at both time \( R \) and time \( T \). In contrast, \( \text{HE}(R + 1) = (H(R + 1) - H^+(R^+)) \) because after the payout (if any) at time \( R \) being made, the hedge portfolio brought forward from time \( R \) is \( H(R^+) \), which is to cover only the potential payout at time \( T \).

The derivations above both assume constant risk-free rate throughout the projection. Nonetheless, it has been verified that the same results hold if risk-free rate varies from
month to month.