Risk Management of Policyholder Behavior in Equity-Linked Life Insurance

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Abstract

The financial guarantees embedded in variable annuity (VA) contracts expose insurers to a wide range of risks, lapse risk being one of them. When policyholders’ lapse behavior differs from the assumptions used to hedge VA contracts, the effectiveness of dynamic hedging strategies can be significantly impaired. By studying how the fee structure and surrender charges affect surrender incentives, we obtain new theoretical results on the optimal surrender region and use them to design a marketable contract that is never optimal to lapse. Using numerical examples, we show that this contract is simpler to hedge, and that the hedge is robust to different surrender behaviors.

Keywords: Variable annuities, pricing, GMMB, dynamic hedging, surrender behavior.
1 Introduction

Variable annuities (VAs) and other types of equity-linked insurance products have grown in popularity over the last 20 years. They protect the policyholder against market downturns while offering participation in equity performance. The financial guarantees embedded in these products incorporate risks that can be complex to identify, price and manage (see Boyle and Hardy (2003), Palmer (2006), Bauer, Kling, and Russ (2008)). In fact, the recent financial crisis revealed that insurers must improve their risk management strategies because VAs expose insurers to large systematic losses, which can threaten their solvency.

VAs can be seen as hybrid products combining insurance and investment components. The policyholder pays an initial premium,\(^1\) which is invested in one or more mutual funds. However, unlike mutual funds, VAs offer various financial guarantees at the death of the policyholder or at maturity of the contract. Some VAs also guarantee minimum periodic withdrawals or income amounts for a fixed or varying period of time (see Hardy (2003) for more details). The financial guarantees embedded in VAs can have payoffs similar to standard options available on stock exchanges. However, options embedded in VAs differ from these standard options for at least three reasons. First, they have a longer horizon as contract maturities generally exceed five years. Second, they are financed by a fee, which is typically paid as a fixed proportion of the account value, as opposed to being paid upfront. Third, the insurer can charge penalties if the policyholder surrenders the contract before maturity. These features complicate the risk management of financial options embedded in VAs, because there is uncertainty in both the payoff to the policyholder and the income of the insurer. Consequently, the issuer is exposed to various risks. To mitigate market risk, Coleman, Kim, Li, and Patron (2007) propose to add jump and stochastic volatility risks in the model used to price and hedge the VA guarantees, while Coleman, Li, and Patron (2006) include interest rate risk. Kling, Ruez, and Ruß (2011b) incorporate stochastic volatility and study its impact on the effectiveness of VA hedging strategies. Ngai and Sherris (2011) analyze the significance of mortality and longevity risks on pricing and hedging.

In this paper, we focus on policyholder behavior risk, specifically on the risk that policyholders terminate their contract prior to maturity. We will refer to this risk as a “lapse”\(^2\)

\(^{1}\)In some cases, additional premium amounts can be deposited in the VA account at regular intervals.
risk or “surrender” risk.\textsuperscript{2} Surrenders may impair the insurer’s business for several reasons. First, the insurance company may not be able to fully recover the initial expenses and upfront investments for acquiring new business (Pinquet, Guillén, and Ayuso (2011)), as well as setting up a hedge for the guarantees in the contract. Second, surrenders may cause liquidity issues (and loss of future profits) (Kuo, Tsai, and Chen (2003)) as there is potential for large cash demands in very short timeframes. Finally, surrenders may give rise to an adverse selection problem because policyholders with insurability issues tend not to lapse their policies. To discourage surrenders, VAs typically include surrender charges (Milevsky and Salisbury (2001)), which are relatively high in the first few years of the contract because they provide a way for the insurer to recover acquisition expenses. Although surrender charges act as a disincentive for policyholders to lapse, there are many situations in which surrender can be advantageous for the policyholder, even after accounting for surrender penalties.

Various methods have been proposed to model policyholder lapse behavior. They range from simple, deterministic lapse rates to sophisticated models such as De Giovanni (2010)’s rational expectation and Li and Szimayer (2010)’s limited rationality. Knoller, Kraut, and Schoenmaekers (2013) show that the moneyness of the embedded option plays a role in surrender behavior. Kuo, Tsai, and Chen (2003) and Tsai (2012) study the relationship between lapse rates of life insurance policies and the level of the interest rate. A recent empirical study by Eling and Kiesenbauer (2014) based on the German life market shows that product design and policyholder characteristics have a statistically significant impact on lapse rates, but finds that unit-linked contracts are not surrendered more often than traditional life insurance policies, which do not incorporate any equity-linked insurance components. This suggests that the decision to surrender a VA contract is generally not driven by a rational financial decision. Pinquet, Guillén, and Ayuso (2011) claim that suboptimal lapses are caused by insufficient knowledge of insurance products. Eling and Kochanski (2013) provide an extensive recent review of the existing literature on lapsation.

\textsuperscript{2}Some authors distinguish between “lapse” and “surrender”. A lapse may refer to the failure by the policyholder to accept the insurer’s offer to renew an expiring policy (by ceasing premium payments and without receiving any payout from the insurer), or any voluntary cessation without a surrender payment to the policyholder (Dickson, Hardy, and Waters (2013)). The “surrender” (or cancellation) refers to the specific action by a policyholder during the policy term to terminate the contract and recover the surrender value. See for instance Eling and Kochanski (2013) for details. Throughout this paper, as in Eling and Kochanski (2013), we will not make this distinction given that we do not consider the renewing option but only the surrender option.
and highlight the existing challenges faced by insurers in terms of modeling lapse behavior and mitigating this risk. Overall, identifying the appropriate lapse model is a significant challenge because of lack of data and the difficulty to identify the real motivation behind a policyholder’s decision to surrender the contract by observing data on lapses.

Given that modeling lapses is complex and far from being well-understood, an alternative is to hedge the VA contract as if all policyholders were rational and surrendered their policies optimally. The optimal surrender decision is an optimal stopping problem that can be technically challenging. Assuming optimal surrenders prices a VA contract conservatively, as it represents the worst policyholder behavior scenario from the insurer’s perspective (Bernard, MacKay, and Muehlbeyer (2014)). This is the approach adopted by Grosen and Jørgensen (2002), Bacinello (2003), Bernard and Lemieux (2008) and Bacinello, Biffis, and Millossovich (2009), among others, but in these cases all fees are included in the initial premium. However, VA guarantees are typically financed via fees paid out as a fixed percentage of the account value throughout the life of the contract. When pricing and hedging VA contracts, it is important to account for this particular way of paying fees as it raises the value of the surrender option by encouraging surrenders in certain cases (Bauer, Kling, and Russ (2008), Milevsky and Salisbury (2001) and Bernard, Hardy, and MacKay (2014)). Pricing and hedging VAs assuming optimal policyholder behavior can result in a product that may be too expensive to be marketable, and that is very complicated to manage and hedge. To simplify risk management, the insurer may decide to implement a hedging strategy that ignores lapses. We will show that this simplification significantly impairs hedging effectiveness. A similar conclusion is reached by Kling, Ruez, and Ruß (2011a) who conclude that the effectiveness of hedging strategies can be highly compromised when the lapse experience deviates from the VA issuer’s assumptions. We thus propose to adjust the design of the VA to eliminate this issue.

Our main contribution is to develop a VA design for fees and surrender charges, which allows the contract be correctly priced and hedged without directly accounting for surrenders, while still being marketable (simple design, low fees and low surrender charges). By eliminating the need to model surrender behavior for pricing and hedging purposes, the risk management of the VA contract is simplified, and the risk of having an inappropriate lapse

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3 This is analogous to dealing with an American option, while pricing the maturity guarantee while ignoring optimal lapses is similar to pricing a European option.
model is mitigated. In the proposed design, lapse assumptions will impact the profitability analysis of the product, but will have little influence on the hedging strategy.

We start from a standard VA for which the fee is paid as a constant percentage of the fund throughout the term of the contract. We then find an explicit closed-form expression for a model-free minimal surrender charge that eliminates the surrender incentive for all account values. However, these surrender penalties are very high and generally lead to a product that is not marketable. For this reason, we consider the state-dependent fee structure introduced by Bernard, Hardy, and MacKay (2014), where the fee is paid only when the account value is below a certain threshold. We analyze the optimal surrender behavior under such a fee structure in the presence of surrender charges. We show how to solve for the minimal surrender charge function, which eliminates the surrender incentive during the whole length of the contract. We explore different product designs that are able to eliminate this incentive while keeping the contract marketable and attractive to the policyholder. By combining a state-dependent fee with surrender charges, we find that it is possible to design a contract that can be effectively hedged and managed, while remaining attractive to policyholders, with relatively low fees. In particular, when the surrender incentive is eliminated, the hedging strategy is simpler to implement since it only requires replication of the maturity benefit, not the surrender option.

In Section 2, we introduce the market model, the VA contract, and the partial differential equation (PDE) approach used for pricing. Section 3 presents an analysis of the optimal surrender incentive when the fee is paid as a constant percentage of the fund throughout the term of the contract, and shows how this incentive can be eliminated. In Section 4, we perform a similar analysis in the state-dependent fee case, and also present an example of a contract design that eliminates the surrender incentive. In Section 5, we analyze the effectiveness of dynamic hedging for this contract under different assumptions for surrender behavior, both optimal and sub-optimal and show that product design can help insurers mitigate hedging risk. Section 6 concludes.
2 Pricing the VA contract

2.1 Market and Notation

We consider a VA contract with maturity $T$ and underlying account value at time $t$ denoted by $F_t$, $t \in [0, T]$. Suppose that the initial premium $F_0$ is fully invested in an index whose value process $\{S_t\}_{0 \leq t \leq T}$ has real-world ($P$-measure) dynamics

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW^P_t,$$

where $W^P_t$ is a $P$-Brownian motion.\(^4\) Suppose also that the usual assumptions of the Black-Scholes model are satisfied. Therefore, the market is complete and there exists a unique risk-neutral measure $Q$ under which the index $S_t$ follows a geometric Brownian motion with drift equal to the risk-free rate $r$, so that

$$\frac{dS_t}{S_t} = r dt + \sigma dW^Q_t.$$

We assume that the financial guarantee embedded in the VA contract is financed by a fee paid continuously as a constant percentage $c$ of the account value. This characteristic differentiates the VA from standard options available on stock exchanges, since these are generally financed by a charge paid up-front. This constant fee structure, which is typical for VAs, is problematic because it gives rise to a mismatch between the liability of the insurance company (the financial guarantee) and its income (the future fees that will be collected before maturity or surrender). For instance, when the value of the underlying fund increases, the liability of the insurer decreases while the expected value of the future fee income moves in the opposite direction, i.e., increases. This mismatch creates an important surrender incentive since the policyholder is paying a high price for a guarantee that has little value (for more details, see Milevsky and Salisbury (2001) and Bernard, MacKay, and Muehlbeyer (2014)).

\(^4\)We work on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, P)$ where $(\Omega, \mathcal{F})$ is a measurable space, $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ is the natural filtration generated by the Brownian motion (with $\mathcal{F}_t = \sigma(\{W_s\}_{0 \leq s \leq t})$) and $P$ is the real-world measure. We assume that the probability space is complete ($\mathcal{F}_0$ contains the $P$-null sets) and right-continuous.
To address this problem, we modify the VA design and allow for a state-dependent fee, as first proposed by Bernard, Hardy, and MacKay (2014). Under a state-dependent fee structure, the insurer only charges the fee when the account value is below a given level $\beta$, called the fee barrier threshold. Since the evolution of the account value depends on this threshold $\beta$, we add the superscript ($\beta$) to the symbol $F_t$. The $\mathbb{P}$-dynamics of the account value are given by

$$\frac{dF_t^{(\beta)}}{F_t^{(\beta)}} = (\mu - c)\mathbb{1}_{\{F_t^{(\beta)} < \beta\}} dt + \sigma dW_t^\mathbb{P}, \quad (1)$$

where $\mathbb{1}_{\{A\}}$ is the indicator function of the set $A$. Without loss of generality, we assume that $F_0^{(\beta)} = S_0$. When $\beta = \infty$, the fee is paid throughout the term of the contract regardless of the account value, and equation (1) simplifies to

$$\frac{dF_t^{(\infty)}}{F_t^{(\infty)}} = (\mu - c) dt + \sigma dW_t^\mathbb{P}.$$ 

The case $\beta = \infty$ corresponds to the typical fee structure considered for VA contracts in the literature, and will be referred to as the “constant fee case” in this paper.

### 2.2 Maturity benefit and surrender charges

We focus on a $T$-year VA contract with a guaranteed minimum accumulation benefit (GMAB) given by

$$\max(G, F_T^{(\beta)}).$$

The symbol $G$ denotes a pre-determined guaranteed amount equal to

$$G = e^{gT}F_0^{(\beta)},$$

where $0 \leq g < r$ is the guaranteed roll-up rate. If the policyholder surrenders the contract at any time $0 < t < T$, she receives $(1 - \kappa_t)F_t^{(\beta)}$: the account value diminished by the surrender charge $\kappa_t F_t^{(\beta)}$, where $0 \leq \kappa_t < 1$. Typically, $\kappa_t$ is a decreasing function of time to discourage policyholders from lapsing in the first years of the contract. Early surrenders affect insurers more significantly since VA contracts have front-loaded expenses that are recouped from fees during the first few years of the contract. Since the contract cannot be
surrendered at maturity, we set $\kappa_T = 0$.

We consider two decreasing surrender charge functions, in addition to the case $\kappa_t = 0$ for all $t$. First, we use the function $\kappa_t = 1 - e^{-\kappa(T-t)}$, studied by Bernard, MacKay, and Muehlbeyer (2014). Second, we consider a ‘vanishing’ surrender charge function, $\kappa_t = \kappa(1 - t/T)^3$. This function mimics surrender penalties found on the market, which are typically high in the first years of the contract, and drop rapidly to add liquidity to the VA investment. Both surrender charge functions are illustrated in Figure 1.

![Figure 1: Examples of the surrender charge function $\kappa_t$.](image)

### 2.3 Valuation of the VA contract

We let $V(t, F_t^{(\beta)})$ denote the value of the contract at time $t$, $0 \leq t \leq T$. Since the VA contract can be surrendered at any time before maturity, its pricing becomes an optimal stopping problem. To define this problem, we must introduce further notation. Denote by $\mathcal{T}_t$ the set of all stopping times $\tau$ greater than or equal to $t$ and bounded by $T$. Then, define the continuation value at time $t$ of the VA contract with surrender as

$$ V^*(t, F_t^{(\beta)}) = \sup_{\tau \in \mathcal{T}_t} E_Q[ e^{-r(\tau-t)}\psi(\tau, F_\tau^{(\beta)}) | \mathcal{F}_t], $$
where,
\[ \psi(t, F_t^{(\beta)}) = \begin{cases} (1 - \kappa_t)F_t^{(\beta)}, & \text{if } t \in (0, T), \\ \max(G, F_T^{(\beta)}), & \text{if } t = T, \end{cases} \]
is the payoff of the contract at surrender or at maturity.

Let \( S_t \) be the optimal surrender region at time \( t \in [0, T] \) and define it by
\[ S_t = \{ F_t^{(\beta)} < \infty : \psi(t, F_t^{(\beta)}) \geq V^*(t, F_t^{(\beta)}) \}. \]

That is, the optimal surrender region is defined as the fund values for which the surrender benefit is worth at least as much as the continuation value of the VA contract, because a rational policyholder would surrender her contract if the surrender value is greater than the value if she continues to maintain it, for at least a small period of time. The complement of \( S_t \), denoted by \( C_t \), will be referred to as the continuation region. When the VA fee is paid regardless of the account value (\( \beta = \infty \)), the surrender region at time \( t \), if it exists, is of the threshold type, that is \( S_t = \{ F_t^{(\infty)} \geq B_t \} \), with or without surrender penalties (see Bernard, MacKay, and Muehlbeyer (2014)). The symbol \( B_t \) represents the fund threshold, which induces a rational policyholder to surrender her VA contract at time \( t \). This threshold is usually referred to as optimal surrender boundary. Our analysis in Section 4 shows that the surrender region in the case of a state-dependent fee is not necessarily of the threshold type.

Finally, we can define the price of a VA contract with GMAB and surrender option as
\[ V(t, F_t^{(\beta)}) = \begin{cases} V^*(t, F_t^{(\beta)}), & \text{if } F_t^{(\beta)} \in C_t, \\ \psi(t, F_t^{(\beta)}), & \text{if } F_t^{(\beta)} \in S_t. \end{cases} \]

Throughout the paper, unless otherwise indicated, we assume that VA contracts are fairly priced. The fair fee is defined as the fee rate \( c^* \) satisfying
\[ F_0^{(\beta)} = V(0, F_0^{(\beta)}; c^*), \] (2)
where \( V(0, F_0^{(\beta)}; c^*) \) is the price of the contract evaluated at the fee rate \( c^* \).
2.4 PDE representation of the VA contract price

We derive the price and the optimal exercise region using numerical PDE techniques. In the Black-Scholes framework, under the usual no-arbitrage assumptions, \( V(t, F_t^{(β)}) \) must satisfy the following PDE in the continuation region \( C_t \),

\[
\frac{∂V}{∂t} + \frac{1}{2} \frac{∂^2V}{∂F_t^{(β)^2}} F_t^{(β)^2} \sigma^2 + \frac{∂V}{∂F_t^{(β)}} F_t^{(β)} \left( r - c 1_{ \{ F_t^{(β)} < β \} } \right) - rV = 0,
\]

for \( 0 ≤ t ≤ T \) (and \( F_t^{(β)} ∈ C_t \)). In the optimal surrender region \( S_t \),

\[
V(t, F_t^{(β)}) = ψ(t, F_t^{(β)}),
\]

for \( 0 ≤ t ≤ T \) (and \( F_t^{(β)} ∈ S_t \)). The derivation of (3) is similar to the derivation of the Black-Scholes PDE, and can be found in Appendix 4.A of MacKay (2014). We solve the PDE in (3) with the following boundary conditions:

\[
V(T, F_T^{(β)}) = \max(G, F_T^{(β)}),
\]

\[
\lim_{F_t^{(β)} → 0} V(t, F_t^{(β)}) = V(t, 0) = Ge^{-r(T-t)}.
\]

The first boundary condition reflects the payoff of the VA at maturity. The second condition comes from the fact that when the account value approaches 0, only the maturity guarantee is valuable. To solve the PDE in (3), we must also specify an upper boundary. However, the behavior of the contract price for high account values depends on the fee structure, and it is generally not possible to specify this boundary exactly for a finite value of \( F_t^{(β)} \).

2.4.1 Upper boundary in the constant fee case

In the constant fee case, when the optimal strategy is to lapse whenever the account value is above a certain threshold, we can specify an exact upper boundary because the price of the contract corresponds to the surrender benefit for sufficiently high fund values. However, in Section 3.2, we show that there exists a minimal surrender charge function such that the optimal strategy is to maintain the contract until maturity in all cases. When this happens, the asymptotic behavior of the contract price is the same as if only the maturity
benefit was considered, so that

\[
\lim_{F_t^{(\infty)} \to \infty} \frac{V(t, F_t^{(\infty)})}{F_t^{(\infty)}} = e^{-c(T-t)}. \tag{4}
\]

Intuitively, the result in (4) is due to the fact that the maturity benefit is worth close to nothing at very high fund values \(F_t^{(\infty)}\), which implies that we can write:

\[
E_Q \left[ e^{-r(T-t)} \max(G, F_T^{(\infty)}) | \mathcal{F}_t \right] \approx E_Q \left[ e^{-r(T-t)} F_T^{(\infty)} | \mathcal{F}_t \right] = F_t^{(\infty)} e^{-c(T-t)}. 
\]

In this case, the European contract price has a closed-form expression similar to the Black-Scholes formula.

### 2.4.2 Upper boundary in the state-dependent fee case

With a state-dependent fee, the following asymptotic behavior holds regardless of the form assumed for the surrender charge function:

\[
\lim_{F_t^{(\beta)} \to \infty} \frac{V(t, F_t^{(\beta)})}{F_t^{(\beta)}} = 1. \tag{5}
\]

The proof of (5) can be found in Appendix A. This result allows us to use \(F_t^{(\beta)}\) as an upper boundary for \(V(t, F_t^{(\beta)})\) when solving the PDE in (3) numerically. Intuitively, this limiting behavior stems from the fact that when the account value is very high, the maturity benefit is worth close to nothing, and the policyholder does not expect to pay any more fees. Thus, the value of the contract can be estimated by

\[
E_Q \left[ e^{-r(T-t)} F_T^{(\beta)} | \mathcal{F}_t \right] \approx E_Q \left[ e^{-r(T-t)} F_t^{(\beta)} \frac{S_T}{S_t} | \mathcal{F}_t \right] = F_t^{(\beta)}, \quad \text{when } F_t^{(\beta)} \gg \beta.
\]

Under a state-dependent fee, Bernard, Hardy, and MacKay (2014) derived integral representations for the prices of guaranteed minimum accumulation (maturity) and death benefits (GMAB and GMDB) but they are only valid when surrenders are not allowed. Throughout the paper, we will use the PDE methodology presented in this section to price VA contracts because it allows us to consider all possible surrender assumptions, while
the integral representations only apply when surrenders are ignored. See Appendix B for additional details on the implementation of the PDE approach.

3 VA contract in the constant fee case

In this section, we analyze the impact of surrender charges on the shape of the optimal surrender region in the constant fee case. Throughout this section, \( \beta = \infty \), so we will omit the superscript and write \( F_t \) instead of \( F_t^{(\beta)} \). Understanding the interplay between the fee structure and the surrender incentive in this case is the first step towards designing a contract that eliminates the surrender incentive while offering reasonable fee rates and surrender charges. We present conditions under which the surrender incentive is completely eliminated, and show that the surrender charges that are needed to remove this incentive generally lead to infeasible contract terms.

3.1 Numerical illustrations

We consider a 10-year VA contract guaranteeing an amount of \( G = F_0 = 100 \) at maturity (in other words, the guaranteed roll-up rate of the GMAB is \( g = 0 \)). The market parameters were fitted to a data set of weekly percentage log-returns on the S&P500 from October 28, 1987 to October 31, 2012, from which we obtained \( \mu = 0.07 \) and \( \sigma = 0.165 \). We further assume \( r = 0.03 \).

**Fair fee**

Table 1 presents the fair fees calculated for the constant fee case under different assumptions for the surrender charge function and policyholder behavior (optimal behavior or no surrenders). We consider the two surrender charge functions introduced in Section 2.2, in addition to the case \( \kappa_t = 0 \).

<table>
<thead>
<tr>
<th>No Surrender</th>
<th>Optimal Surrender</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \kappa_t = 0 )</td>
<td>( \kappa_t = 1 - e^{-0.005(T-t)} )</td>
</tr>
<tr>
<td>0.0106</td>
<td>0.0350</td>
</tr>
</tbody>
</table>

Table 1: Fair fee based on \( T = 10 \), \( r = 0.03 \), and \( \sigma = 0.165 \).
We observe that the fair fee under the optimal surrender assumption is significantly higher than in the no surrender case when there are no surrender charges (0.0350 versus 0.0106). When the insurer does not charge a penalty for early surrender, the fee income represents its only revenue. This income compensates the insurer for both the guarantee offered and early surrender risk (the risk of not being able to collect future fees on the account value). To fully mitigate lapse risk, it must charge a fee for which the value of the guarantee offered assuming optimal policyholder behavior equals the amount invested. Under the assumptions stated above, this fair fee corresponds to 0.0350, which is very high. One way to decrease it while still fully mitigating lapse risk is to introduce surrender penalties in the product design. These penalties represent an additional revenue for the insurer, enabling it to reduce the constant fee charge, and also work as a disincentive to lapse. Table 1 shows that the surrender charge schedules that we consider reduce the fair fee by more than 50%. However, they are not able to eliminate the surrender incentive completely, which is why the fair fees are still higher than for the no surrender case.

**Optimal surrender region when there are no surrender charges**

In the constant fee case, when there are no surrender charges, there exists an optimal lapsation boundary above which a rational investor should surrender the VA contract. Figure 2 illustrates four such boundaries, each of which is associated with a given fee rate $c$ (displayed on the curve). Since a lower fee $c$ lessens the incentive to surrender, this boundary shifts upwards when $c$ decreases.

Notice that when the fair fee is set assuming that the policyholder surrenders the contract optimally ($c = 0.0350$), the optimal surrender boundary reaches $F_0 = 100$ at $t = 0$. In fact, when the contract is fairly priced, $F_0 = V(0, F_0)$. This is also the definition of the optimal surrender boundary at time 0. Thus, without surrender charges, it is optimal for the policyholder to surrender the contract as soon as the account value increases. This was previously explained by Milevsky and Salisbury (2001).

**Impact of surrender charges on the optimal surrender region**

Figure 3 plots the optimal surrender boundaries for the two surrender charge schedules introduced in Section 2.2. As stated earlier, the addition of surrender charges lowers the fair fee rate. By decreasing the fair fee, the fund threshold at which the guarantee becomes less valuable than the expected future fees increases. In addition, surrender penalties reduce
Figure 2: Optimal surrender boundary when there are no surrender charges. Each of the four curves is associated with a fee rate $c$. Note that the fair value of $c$ is 0.0350 (bottom curve) when assuming optimal policyholder behavior and 0.0106 (top curve) when assuming the policyholder holds her contract until maturity.

Figure 3: Optimal lapsation boundary for different surrender charge functions. The fair fee $c = 0.0139$ is associated with $\kappa_t = 1 - e^{-0.005(T-t)}$, while the fair fee $c = 0.0170$ results from $\kappa_t = 0.05(1 - t/T)^3$. 

14
the amount received on surrender, further increasing this threshold. The combination of these two effects therefore results in an upward shift of the surrender boundary.

### 3.2 Minimal surrender charge to eliminate surrender incentives

In Section 3.1, we showed that the introduction of surrender penalties reduces the surrender incentive, but does not necessarily eliminate it. We now determine the minimal surrender charge schedule such that the optimal behavior for a VA policyholder is to hold the contract until maturity. The motivation for such a contract design is to simplify the hedging strategy associated with optimal policyholder behavior. In this context, the optimal hedge is to hedge the maturity benefit only (i.e., a European-type hedge), and any surrenders necessarily result in a profit for the insurer as they are sub-optimal. To design such a policy, we look for a surrender charge function \( \kappa_t \) satisfying,

\[
V^*(t, F_t) \geq (1 - \kappa_t)F_t, \quad \forall F_t \geq 0, \quad \forall t \in [0, T).
\]  

That is, surrender penalties must be sufficiently high so that the continuation value of the contract, \( V^*(t, F_t) \), is always at least as large as the surrender benefit, \( (1 - \kappa_t)F_t \), for any given time \( t \). Proposition 3.1 provides the minimal surrender charge function that is needed to satisfy condition (6). Its proof is given in Appendix C.

**Proposition 3.1.** Using notation from Section 2 in the constant fee case \( (\beta = \infty) \), the minimal value of \( \kappa_u \) at each time \( u \in [t, T) \) such that it is never optimal to surrender the policy before \( T \) is equal to

\[
\kappa_u = 1 - e^{-c(T-u)}, \quad t \leq u < T.
\]  

The result given by Proposition 3.1 is essentially model-free: it holds for any arbitrage-free complete market model, and not just for the Black-Scholes model. It shows that if the surrender charge function is chosen according to (7), then it cannot be optimal to surrender the contract. However, the condition \( \kappa_t \geq 1 - e^{-c(T-t)} \) is also sufficient to guarantee that it is not optimal to lapse at time \( t \), regardless of the form assumed for \( \kappa_u \), for \( t < u < T \).
To understand why, observe that

\[
V^*(t, F_t) \geq \frac{E_Q[e^{-r(T-t)} \max(F_T, G)\mid F_t]}{F_t(1 - \kappa_t)} \geq \frac{e^{-c(T-t)}}{1 - \kappa_t}, \quad \forall F_t \geq 0.
\]

It is clear that whenever \( \kappa_t \geq 1 - e^{-c(T-t)} \), the continuation value, \( V^*(t, F_t) \), must be greater than the surrender value, \( F_t(1 - \kappa_t) \), for any values of \( F_t \) at time \( t \), making surrender sub-optimal. Note that the converse of this result does not necessarily hold, that is, the condition \( \kappa_t \geq 1 - e^{-c(T-t)} \) is not necessary for surrender not to be optimal at time \( t \), \( \forall F_t \geq 0 \). In other words, there may be a value of \( \kappa_t \in (0, 1 - e^{-c(T-t)}) \) which makes lapsation not optimal at time \( t \). In fact, \( \kappa_t = 1 - e^{-c(T-t)} \) is a strict lower bound for the surrender charge at time \( t \) if and only if it is never optimal to surrender the contract after \( t \).

Proposition 3.1 allows us to determine a fairly priced VA product design that does not give rise to a surrender incentive. For example, under the same assumptions as in Section 3.1 \((F_0 = G = 100, T = 10, r = 0.03, \text{and } \sigma = 0.165)\), consider the following product design

\[
c = 0.0106 \quad \text{and} \quad \kappa_t = 1 - e^{-0.0106(T-t)}, \quad 0 \leq t \leq T.
\]

As a result of Proposition 3.1, we know that given \( c = 0.0106 \) this product design includes the smallest surrender penalties not giving rise to a surrender incentive. Moreover, this VA contract is fairly priced because \( c = 0.0106 \) corresponds to the value of the fair fee when the policyholder behaves optimally in this case, i.e., does not surrender (see Table 1). However, such high surrender values could significantly impact the attractiveness of the product. In fact, the surrender charge starts at approximately 10% and is still larger than 5% after five years. Such penalties are significantly higher than what is typically observed on the market.

## 4 VA contract in the state-dependent fee case

In Section 3, we have shown that in the constant fee case (the typical fee structure in the industry), it is possible to eliminate the surrender incentive by setting the surrender charge above a certain level. However, under reasonable market assumptions this minimal surrender charge is too high for the contract to be marketable. For this reason, we explore
the impact of a state-dependent fee on surrender incentives. Since fees are only paid when the fund value is below a certain threshold, the incentive to surrender the contract for high account values is significantly reduced. We explore how the combination of a state-dependent fee and surrender penalties impact the optimal surrender region. In particular, we show that the minimal surrender charge that eliminates the surrender incentive can be significantly lowered by a state-dependent fee.

4.1 Numerical illustrations

The VA contract and assumptions considered in this section are the same as the ones analyzed in Section 3, but fees are now paid only when the account value is below a certain threshold \( \beta < \infty \).

Fair fee

Table 2 presents the fair fees associated with different thresholds \( \beta \) and surrender charge functions \( \kappa_t \), under the assumptions of no surrenders and optimal surrender behavior. We consider fee barrier thresholds of \( \beta = 120 \) and 150. Note that Bernard, Hardy, and MacKay (2014) studied the case \( \beta = F_0^{(\beta)} \) without any surrender charges. This design leads to an unrealistically high fair fee. In this section, we focus on more practical contract designs, and also incorporate surrender charges.

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>No Surrender</th>
<th>Optimal Surrender</th>
</tr>
</thead>
<tbody>
<tr>
<td>120</td>
<td>0.0236</td>
<td>0.0236</td>
</tr>
<tr>
<td>150</td>
<td>0.0155</td>
<td>0.0159</td>
</tr>
</tbody>
</table>

Table 2: Fair fee for different VA contracts with \( T = 10 \), \( r = 0.03 \), and \( \sigma = 0.165 \).

In general, the fair fee is lower for higher \( \beta \) because the fee is expected to be paid for a longer period of time. However, when the policyholder is assumed to surrender optimally, and when \( \kappa_t = 0 \), the fair fee remains the same. This is due to the shape of the surrender boundary, and is further explained in the next paragraph.
Optimal surrender region when there are no surrender charges

The fact that the fair fee is the same for $\beta = 120, 150, \text{ and } \infty$ when $\kappa_t = 0$ (see Tables 1 and 2) suggests that the state-dependent fee structure may not always lead to a decrease of the surrender incentive when a policyholder behaves optimally. In other words, this policyholder is not able to profit from the state-dependent fee, since it is rational to lapse before the account value reaches the fee barrier threshold $\beta$. Figure 4 shows that the optimal surrender boundaries for both of $\beta = 120$ and 150 lie below 120, and are identical to the optimal surrender boundary when $\beta = \infty$ (constant fee case, illustrated in Figure 2). Therefore, a policyholder behaving optimally will never wait for the fund to reach the fee barrier threshold of $\beta = 120$ or 150, and from her perspective, product designs with $\beta = 120, 150, \text{ or } \infty$ are equivalent. This explains why the fair fee and the optimal surrender regions are the same for these three designs.

![Figure 4: Optimal surrender region for $\beta = 120, 150$ or $\infty$ and $\kappa_t = 0$, priced assuming optimal surrenders ($c = 0.0350$).](image)

Impact of surrender charges on the optimal surrender region

Figure 5 presents the optimal surrender regions for $\beta = 150$ under the same two surrender charge functions considered in Section 2.2: $\kappa_t = 1 - e^{0.005(T-t)}$, and $\kappa_t = 0.05(1 - t/T)^3$. 

18
In contrast to the case $\kappa_t = 0$ analyzed previously, the state-dependent fee structure now significantly impacts the surrender region (for example, compare Figures 4 and 5). Three important observations can be made.

(i) The optimal surrender strategy is no longer threshold-type.

(ii) It is not optimal for the policyholder to lapse the contract when the account value is close to or above the fee barrier threshold $\beta$.

(iii) The surrender incentive is eliminated in the early years of the contract term. The high surrender penalties early in the contract incentivizes the policyholder to wait for the account value to grow above the fee barrier threshold.

Figure 5: Left panel: Optimal surrender region for $\beta = 150$, $c = 0.0159$, and $\kappa_t = 1 - e^{-0.005(10-t)}$. Right panel: Optimal surrender region for $\beta = 150$, $c = 0.0176$, and $\kappa_t = 0.05(1 - t/10)^3$.

Observation (ii) can be proved to hold in general and is formalized in Proposition 4.1 (the proof is given in Appendix D).

**Proposition 4.1.** Let $F_t^{(\beta)}$ and $\kappa_t$ be defined as in Section 2 and assume $\beta < \infty$. Then, for any $t \in [0, T]$,

$$V^*(t, F_t^{(\beta)}) \geq F_t^{(\beta)}(1 - \kappa_t), \quad \forall F_t^{(\beta)} \geq \beta. \quad (8)$$
If \( \kappa_t > 0 \) at time \( t \), the inequality in (8) is strict.

Proposition 4.1 simply states that when the fee is state-dependent and the account value is above the fee barrier threshold \( \beta \), the contract is always worth at least as much as the surrender benefit \( F_t^{(\beta)}(1 - \kappa_t) \). The intuition for this result is the following. If the account value is above \( \beta \), the policyholder does not have a clear incentive to surrender because the VA product offers her a maturity guarantee for which she is not required to pay for at the moment (the guarantee is offered for free). As a result, a rational policyholder will wait for the account value to reach \( \beta \), before even considering surrendering the contract.

Remark 4.1. When the surrender charge at time \( t \) is greater than 0, that is, \( \kappa_t > 0 \), Proposition 4.1 implies that the optimal surrender region cannot include fund values above or equal to \( \beta \). However, as explained in the proof of Proposition 4.1 (see Appendix D), when \( \kappa_t = 0 \), we could have,

\[
V^*(t, F_t^{(\beta)}) = F_t^{(\beta)}, \quad \forall F_t^{(\beta)} \geq \beta.
\] (9)

By the definition of the optimal surrender region given in Section 2.2 (the policyholder is assumed to lapse when the contract is worth at least as much as the surrender benefit), the equality in equation (9) actually induces surrender. This argument explains why the optimal surrender region includes fund values above \( \beta \) in Figure 4. However, strictly speaking, a rational policyholder would actually be indifferent to lapse in this situation because the surrender benefit is exactly equal to the continuation value of the contract.

4.2 Minimal surrender charge to eliminate surrender incentives

In Section 3, we explicitly derived the minimal surrender charge function that eliminates the surrender incentive in the constant fee case \( (\beta = \infty) \), and showed that the resulting surrender penalties may be too high to be of practical value. We now explain how to obtain a minimal surrender charge function in the state-dependent fee case \( (\beta < \infty) \) and show an example where the resulting contract design appears marketable under reasonable assumptions.

Obtaining a simple closed-form expression for the minimal surrender charge function that eliminates the surrender incentive when \( \beta < \infty \) is challenging due to the following rea-
sons. First, although there exists an integral representation for the value of the maturity benefit (see, Bernard, Hardy, and MacKay (2014)), or for the discounted expectation of the account value at future times, the expressions involved are generally complex and depend on the current account value $F_t^{(\beta)}$ in more than one way. Second, the function $E_Q[e^{-r(T-t)} \max(F_T^{(\beta)}, G) | F_t]/F_t^{(\beta)}$ is generally not monotone in $F_t^{(\beta)}$ because the expected future fees are themselves not necessarily monotone in $F_t^{(\beta)}$ when we consider a state-dependent fee — the fee rate at $t$ increases with $F_t^{(\beta)}$ while $F_t^{(\beta)} < \beta$, but drops to 0 as soon as $F_t^{(\beta)} \geq \beta$. To solve numerically for the minimal surrender schedule that eliminates the surrender incentive, we use the following procedure.

Step 1: Find the fair value of the fee rate $c$ for the European contract, i.e., assuming that the contract is held until maturity.

Step 2: For each $t \in [0, T)$, numerically obtain the account value $F_t^*$ at which the ratio of the value of the maturity benefit to the fund value is the smallest: \[ F_t^* = \arg \inf_{F_t^{(\beta)}>0} \left\{ E_Q[e^{-r(T-t)} \max(F_T^{(\beta)}, G) | F_t]/F_t^{(\beta)} \right\}. \]

Step 3: Finally, set \[ \kappa_t = \max \left( 1 - \frac{E_Q[e^{-r(T-t)} \max(F_T^{(\beta)}, G) | F_t^{(\beta)}]}{F_t^*}, 0 \right). \]

This procedure generates the minimal surrender charge function, $\kappa_t$, that eliminates the surrender incentive in the state-dependent fee case.

The fair fee in Step 1 can be obtained by the PDE approach, or by Proposition 3.1 of Bernard, Hardy, and MacKay (2014). In Step 2, $F_t^*$ is easily obtained after setting up a finite difference grid for the value of the European contract because this grid provides us

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\(^5\)This step implicitly assumes that the continuation value of the VA contract is equal to the value of the maturity benefit, i.e., $V^*(t, F_t^{(\beta)}) = E_Q[e^{-r(T-t)} \max(F_T^{(\beta)}, G) | F_t]$, which is the correct assumption to make because $\kappa_t$ is chosen in such a way that surrender is not optimal during the entire length of the contract.
with values of $E_Q[e^{-r(T-t)} \max(F_T^{(\beta)}, G)|\mathcal{F}_t]$ at discrete values of $t$ and $F_t^{(\beta)}$ in the grid.\footnote{Alternatively, $F_t^*$ can be solved numerically using Proposition 3.1 of Bernard, Hardy, and MacKay (2014).}

### 4.3 Marketable VA product with no surrender incentive

In this section, we use the same assumptions as in Section 3.1 for the market model ($F_0^{(\beta)} = G = 100$, $T = 10$, $r = 0.03$, and $\sigma = 0.165$), and we consider a fee barrier threshold of $\beta = 150$. Using the algorithm outlined in Section 4.2, we obtain a fair fee of $c = 0.0155$, and the surrender charge function illustrated on the left of Figure 6. The values $F_t^*$ used in the calculation of the surrender charge are given on the right of Figure 6. Observe that $V(t, F_t^*) = (1 - \kappa_t)F_t^*$ along this boundary, and that surrender is never optimal on either side of the boundary. It is therefore never clearly optimal for the policyholder to lapse her contract.

![Figure 6: Left panel: Minimal surrender charge function not giving rise to an optimal lapsation boundary when $c = 0.0155$ and $\beta = 150$. Right panel: Values of $F_t^{(150)}$ at which the infima of the function $E_Q[e^{-r(T-t)} \max(F_T^{(150)}, G)|\mathcal{F}_t]/F_t^{(150)}$ were computed.](image)

This contract design includes surrender charges and fees which appear to be practical. The surrender penalties start below 3.5% and decrease to 0 at maturity. This is significantly
lower than the minimal surrender charge schedule required to eliminate the surrender incentive in the constant fee case (see Section 3.2).

We can further compare this product design to a typical constant fee product design with $\beta = \infty$ and the exact same schedule of surrender charges. It turns out that when pricing this contract under optimal policyholder behavior, the fair value of $c$ when $\beta = \infty$ is once again 0.0155, and the optimal lapsation boundary corresponds to the curve on the right of Figure 6. This result, which may seem surprising at first, has an intuitive explanation which is detailed in the next section. Since both contracts charge the same fee rate and have the same surrender penalties, the contract design with $\beta = 150$ is more attractive to a policyholder than the one with $\beta = \infty$ due to the presence of a threshold above which the fee is not paid. However, from a risk management standpoint, it can be argued that this design is also preferable for the insurer because it does not give rise to an optimal surrender region. As a consequence, the VA product can be managed assuming no surrenders (as this is the optimal behavior when $\beta = 150$), which simplifies the construction of the hedging portfolio and reduces the importance of modeling lapses for pricing and hedging purposes. In this context, early surrenders can only be sub-optimal generating additional revenue for the insurer.

4.3.1 Intuitive explanation for the fair value of $c$

In this section, we explain why the fair value of $c$ is 0.0155 when $\beta = \infty$ and the schedule of surrender charges is given by the function on the right of Figure 6. This surrender charge function corresponds to the minimal function which eliminates the surrender incentive for the state-dependent fee case of $\beta = 150$ and $c = 0.0155$. In other words, we explain why contracts with $\beta = 150$ and $\beta = \infty$ have the same price when these surrender penalties are considered.

First, note that for a given surrender charge function, and assuming that the policyholder lapses optimally, the state-dependent fair fee ($\beta < \infty$) is always at least as much as the constant fair fee ($\beta = \infty$). This is due to the fact that under the state-dependent fee design, the fee might be paid over a period of time shorter than in the constant fee case. Consequently, assuming optimal policyholder behavior, the state-dependent fee is an upper bound for the fair fee when $\beta = \infty$. In our specific example, this implies that the fair fee
when $\beta = \infty$ is at most 0.0155.

Second, to see why 0.0155 is also a lower bound for the constant fair fee, consider a policyholder who lapses as soon as the account value hits the curve on the right of Figure 6. At this exact moment, the surrender benefit is equal to the value of the VA contract in the state-dependent fee case ($\beta = 150$). This is simply because the (minimal) surrender charge schedule was established to satisfy the following condition along the curve on the right of Figure 6:

$$E_Q[e^{-r(T-t)} \max(F_T^{(150)}, G)|F_t] = (1 - \kappa_t)F_t^{(150)}.$$  \hspace{1cm} (10)

This strategy (holding a contract with $\beta = \infty$ and $c = 0.0155$, and surrendering as soon as the account value hits the curve on the right of Figure 6) can be replicated by holding the state-dependent fee contract with $\beta = 150$ and $c = 0.0155$, and surrendering it as soon as condition (10) is satisfied. The surrender boundary for both cases is the same, because it was defined through equation (10). Since that surrender boundary is always under $\beta = 150$, the policyholder will pay the exact same fees until surrender or maturity under both contracts. We know that the state-dependent fee contract is priced fairly at $c = 0.0155$. Thus, since under this particular surrender strategy the policyholder receives the same payoff from holding the constant fee or the state-dependent fee contract, they should both have the same price. This entails that $c = 0.0155$ must be a lower bound for the fair fee when $\beta = \infty$, as it is the fair $c$ under one possible surrender strategy.

Finally, the arguments presented in this paragraph imply that (i) the fair fee for the constant fee case $\beta = \infty$ must be exactly 0.0155 because it is bounded above and below by this value, and (ii) the curve on the right of Figure 6 must be the associated optimal lapsation boundary. Note that this result about the equivalence of the fair fee when $\beta < \infty$ and $\beta = \infty$ will hold when (i) the surrender charge function is chosen as the minimal one not giving rise to a surrender incentive for the case $\beta < \infty$, and (ii) the value of $F_t$ at the infimum of the function $E_Q[e^{-r(T-t)} \max(F_T^{(\beta)}, G)|F_t] / F_t^{(\beta)}$ is below $\beta$, for $0 \leq t \leq T$.

### 5 Dynamic hedging

This section illustrates why eliminating the surrender incentive in the VA product design can simplify the insurer’s hedging strategy and make it more effective. Before presenting
our results on dynamic hedging, we review some concepts with respect to hedging VAs, and explain how we calculate the insurer’s hedged loss.

5.1 Calculation of the net hedged loss at maturity

Assume that we have a path of stock values, \( \{S_t\}_{0 \leq t \leq T} \), and corresponding account values, \( \{F_t^{(\beta)}\}_{0 \leq t \leq T} \), sampled at discrete time intervals \( h \), where, for example, \( h = 1/52 \) entails weekly observations. Following Hardy (2000), for example, we define the net hedged loss at maturity as \( L - H \), where,

\[
L = \text{Net unhedged loss at maturity}, \\
H = \text{Cumulative mark-to-market gain on the hedge}.
\]

When the insurer does not use a hedging strategy, its net loss at maturity is \( L \). When it employs a hedging strategy, its net loss is \( L - H \). The losses are net because they take into account the fee income and surrender charges received by the insurer.

If the policyholder does not surrender her contract, the net unhedged loss at maturity \( T \) is

\[
L = \text{payoff to the policyholder} - \text{accumulated value of fees} \\
= \max(0, G - F_T^{(\beta)}) - \sum_{i=0}^{T/h-1} F_{ih}^{(\beta)}(1 - e^{-ch})e^{r(T-ih)}1_{\{F_{ih}^{(\beta)} < \beta\}}.
\]

In the event of surrender at time \( t = \tau \), the net unhedged loss at maturity \( T \) is

\[
L = -(\text{accumulated value of fees and surrender charges}) \\
= -\sum_{i=0}^{\tau/h-1} F_{ih}^{(\beta)}(1 - e^{-ch})e^{r(T-ih)}1_{\{F_{ih}^{(\beta)} < \beta\}} - F_\tau^{(\beta)} \kappa_\tau e^{r(T-\tau)}.
\]

To calculate the net hedged loss at maturity, the cumulative mark-to-market gain on the hedge must be subtracted from the net unhedged loss. Assuming that the hedging strategy consists of a delta hedge, the mark-to-market gain at time \( t + h \) of the hedge established
at time $t$ is

$$\Delta_t(S_{t+h} - S_t e^{rh}),$$

where $\Delta_t$ is the delta used in the hedge (the hedging ratio), and is defined in Appendix E. The cumulative mark-to-market gain on the hedge corresponds to the accumulated value of these gains to maturity:

$$H = \sum_{i=0}^{\tau/h-1} \Delta_{ih}(S_{(i+1)h} - S_{ih} e^{rh}) e^{r(T-(i+1)h)},$$

where $\tau$ represents the time at which the hedging strategy is stopped (surrender or maturity). Finally, the net hedged loss at maturity is simply $L - H$.

If the pricing of the VA contract and its hedging are both performed assuming optimal policyholder behavior, the insurer is theoretically super-hedging. In other words, the hedge will always yield enough money for the insurer to cover the payoff of the VA as well as the surrender benefit. If the policyholder adopts a sub-optimal behavior, then the insurer will also gain from the hedge. Unfortunately, these statements are only valid under the rather stringent assumptions of the Black-Scholes model. In practice, even if the insurer implements the optimal hedge, the presence of both discretization and model errors can expose the insurer to potential losses.

### 5.2 Modeling policyholder behavior

Given that the insurer establishes its hedging strategy assuming a particular form of policyholder behavior, it is important to verify that the effectiveness of this strategy is robust to a wide range of dynamic lapse behavior observed in practice. For example, there is empirical evidence (e.g., Knoller, Kraut, and Schoenmaekers, 2013; Milliman, 2011) that the moneyness of the guarantee is a key driver of lapse behavior among policyholders. The Canadian Institute of Actuaries (2002) and the American Academy of Actuaries (2005) both recommended the use of lapse rate assumptions which depend on the moneyness of the guarantee. According to a Society of Actuaries (2012) research report, approximately 60% of insurers follow this practice.

Therefore, we use the following stopping time function to model different forms of policy-
holder behavior in our analysis of hedging effectiveness:

\[ \tau_M = \inf_{0 < t < T} \left\{ \frac{F_t(\beta)(1 - \kappa_t)}{G} \geq M_t \right\}, \] (11)

where \( F_t(\beta)(1 - \kappa_t)/G \) denotes what we call the moneyness ratio at time \( t \), and \( M_t \) is a moneyness threshold, at which the VA is surrendered. If the moneyness threshold is never attained, then we set \( \tau_M = T \). When \( M_t = \infty \) for all \( t \), then \( \tau_M = T \) a.s., which means that the contract is held until maturity. Moreover, since we can rewrite the condition \( F_t(\beta)(1 - \kappa_t)/G \geq M_t \) as \( F_t(\beta) \geq M_t G/(1 - \kappa_t) \), this stopping time encompasses all threshold-type strategies, and, therefore all optimal strategies for the case \( \beta = \infty \). For example, if we choose \( M_t = M_t^{\text{opt}} \), so that \( M_t^{\text{opt}} G/(1 - \kappa_t) \) matches the optimal lapsation fund threshold for \( 0 \leq t \leq T \), then this stopping time is the optimal one. The stopping time in equation (11) therefore allows us to consider two extreme cases of lapse modeling, (no surrenders and optimal surrenders) and in addition, it can be used to specify realistic sub-optimal lapse behavior. For example, if \( M_t \) is constant \( \forall t \), say \( M_t = 1.5 \), then the policyholder will surrender her contract when the surrender benefit, \( F_t(\beta)(1 - \kappa_t) \), is (at least) 50% larger than the guarantee. The rationale behind this type of surrender behavior is to avoid paying fees when the guarantee has little value. We will consider such surrender strategies based on a fixed moneyness ratio in our hedging analysis.

5.3 Results

To illustrate why eliminating the surrender incentive in the VA product design can simplify the insurer’s hedging strategy and make it more effective, we use the assumptions presented in Section 3.1 \((F_0(\beta) = G = 100, T = 10, r = 0.03, \text{ and } \sigma = 0.165)\) and revisit the two product designs analyzed in Section 4.3. The first is a fair-price constant fee design \((\beta = \infty)\) with \( c = 0.0155 \) and the surrender charge schedule given on the left-hand side of Figure 6. The optimal hedging strategy for this design is to hedge assuming the lapsation boundary is given by the curve on the right-hand side of Figure 6. The second design has the same surrender charge schedule and the same fair fee rate, but this fee is now paid only when the account value is below \( \beta = 150 \). The optimal hedging strategy for this design is to hedge assuming the policyholder will hold on to her contract until maturity.
Table 3 shows the statistics of the insurer’s net delta hedging loss at maturity \((H - L)\) for the first product design with \(\beta = \infty\) based on 500,000 stock paths projected on a weekly frequency \(h = 1/52\) over \(T = 10\) years, assuming prices (real-world) follow a geometric Brownian motion, with \(\mu = 0.07\) and \(\sigma = 0.165\) (as assumed for the risk neutral assumptions). The hedging portfolio is rebalanced weekly, and is established assuming either optimal behavior (Opt) or no surrenders (NS). We also consider five possible types of surrender behaviors based on the stopping time in (11) with \(M_t = M_t^{\text{opt}}, 1.3, 1.5, 1.7,\) or \(\infty\) (see Section 5.2 for more details).

<table>
<thead>
<tr>
<th>Behavior</th>
<th>(M_t = M_t^{\text{opt}})</th>
<th>(M_t = 1.3)</th>
<th>(M_t = 1.5)</th>
<th>(M_t = 1.7)</th>
<th>(M_t = \infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hedge</td>
<td>Opt</td>
<td>NS</td>
<td>Opt</td>
<td>NS</td>
<td>Opt</td>
</tr>
<tr>
<td>Mean</td>
<td>0.0</td>
<td>2.5</td>
<td>0.0</td>
<td>2.5</td>
<td>-1.0</td>
</tr>
<tr>
<td>StDev</td>
<td>0.7</td>
<td>4.1</td>
<td>0.7</td>
<td>4.3</td>
<td>1.3</td>
</tr>
<tr>
<td>95% CTE</td>
<td>1.6</td>
<td>7.7</td>
<td>1.6</td>
<td>8.5</td>
<td>1.5</td>
</tr>
<tr>
<td>99% VaR</td>
<td>1.9</td>
<td>8.0</td>
<td>1.9</td>
<td>8.8</td>
<td>1.8</td>
</tr>
</tbody>
</table>

First, observe that on average the optimal hedging strategy never results in a loss, regardless of the policyholder behavior assumed. This is consistent with a super-hedge, but note that the insurer is still exposed to hedging risk as the 95% CTE is close to a loss of 1.5 for all scenarios. Nonetheless, hedging assuming optimal policyholder behavior gives good results because it corresponds to hedging the worst-case scenario. However, given that the insurer...
sells many different VA products, implementing this optimal hedge for each product may be impractical or even impossible. For this reason, insurers may use a simplified hedging strategy, such as a delta hedge that ignores the probability of surrenders. The results in Table 3 show that this simplification significantly impairs hedging effectiveness when the policyholder can surrender her contract before maturity. In every case where surrender is possible, the hedge which ignores surrender generates a loss, on average, for the insurer, and the tail risk measures are significant.

We now turn our attention to the second product design with $\beta = 150$. The second part of Table 3 shows the statistics of the insurer’s net delta hedging loss at maturity ($H - L$) for this product design based on the same 500,000 simulated weekly stock paths. We again analyze the same five surrender behaviors as in the first part of Table 3, but now consider only a delta hedge of the maturity benefit (no surrenders), as this strategy is also optimal for this design.

We observe that hedging effectiveness for the scenarios with $M_t = M_t^{\text{opt}}$ and $M_t = 1.3$, in the second part Table 3 is comparable with what was obtained in the first part (for the product design with $\beta = \infty$). However, the risk measures for the net hedging loss are a bit higher for the other scenarios. This increase is due to the fact that for the product design with $\beta = 150$, the insurer does not receive any fee income when the account value is above 150, but it is still exposed to hedging errors.

From a risk management standpoint, a product design which does not give rise to a surrender incentive seems preferable. First, the VA product can be hedged conservatively assuming no surrenders which simplifies the construction of the hedging portfolio. Second, the hedging strategy can be implemented in a uniform manner across the portfolio of VAs because the optimal lapsation boundary does not have to be taken into consideration for each of the different product designs. Third, early surrenders can only be sub-optimal and generate additional revenue for the insurer. This additional revenue can compensate the insurer for the liquidity strain that arises with early surrenders, or for the need to adjust its hedging portfolio after a lapse.
6 Concluding Remarks

In this paper, we provide some insights to answer this very practical question: how can an insurer use product design to mitigate lapse risk, and to simplify risk management (hedging) in VAs? To answer this question, we examined the interplay between the fee structure of a VA with a guaranteed minimum accumulation benefit (GMAB) and the schedule of surrender charges. We show that by adjusting the fee and surrender charge design, an insurer can create a contract which will be rarely (or never) optimal to lapse, while still being marketable. This creates a more robust risk management strategy for the insurer, as it eliminates the need to model surrender behavior for pricing and hedging purposes, and the hedge effectiveness is no longer highly sensitive to surrender behavior. Through the analysis of hedging errors, we demonstrated that such a hedging strategy performs well under optimal and sub-optimal lapse behavior, making the state-dependent fee an attractive design from a risk management perspective.

Our focus on optimal surrender behavior can also be justified by the possibility of secondary markets for equity-linked life insurance. In fact, Gatzert, Hoermann, and Schmeiser (2009) explain how both consumers and insurers can benefit from a secondary market for life insurance contracts. On the one hand, consumers get a better price (lower surrender charges) and on the other hand, it makes the life insurance market more attractive and thus potentially increases the demand for the insurer’s products. Hilpert, Li, and Szimayer (2014) further discuss the impact of the surrender option and the existence of a secondary market for equity-linked life insurance, and show that the introduction of sophisticated investors by the secondary market may lead to higher premiums that account for a higher proportion of optimal surrenders. This result is somewhat consistent with Gatzert, Hoermann, and Schmeiser (2009) who explain that life insurers need to abandon lapse-supported pricing (i.e. pricing under the assumption of suboptimal lapses that benefit the insurer). In other words, policyholders with access to a secondary market would tend to act optimally, as contract arbitrages would be suppressed through the secondary market mechanism.

Further research should investigate the robustness of product design and dynamic hedging strategies under various market models. The product design analysis could also be extended to VA contracts offering other types of financial guarantees.
References


31


A Proof of Equation (5)

To prove (5) we need the two following lemmas.

**Lemma A.1.** Let \( F_t^{(\beta)} \), \( 0 \leq t \leq T \), be as defined in Section 2 and let \( \beta < \infty \). Then,

\[
\lim_{x \to \infty} \frac{E_Q[e^{-r(T-t)}F_T^{(\beta)}|F_t^{(\beta)} = x]}{x} = 1.
\]  

**Proof.** Let \( m_F(t,u) = \inf_{t \leq s \leq u} F_s^{(\beta)} \) and \( m_S(t,u) = \inf_{t \leq s \leq u} S_s \) be the minimum values attained by the account and the index, respectively, between times \( t \) and \( u \). Then,

\[
\frac{E_Q[e^{-r(T-t)}F_T^{(\beta)}|F_t^{(\beta)} = x]}{x} = \frac{E_Q[e^{-r(T-t)}F_T^{(\beta)}1_{\{m_F(t,T) > \beta\}}|F_t^{(\beta)} = x]}{x} + \frac{E_Q[e^{-r(T-t)}F_T^{(\beta)}1_{\{m_F(t,T) \leq \beta\}}|F_t^{(\beta)} = x]}{x}.
\]  

To prove that \( \lim_{x \to \infty} \frac{V(t,x)}{x} = 1 \), we show that the first term of expression (13) goes to 1 as \( x \to \infty \), and then show that the second term goes to 0 as \( x \to \infty \).

Let \( C_t = e^{-c \int_0^t 1_{\{F_s^{(\beta)} > \beta\}} ds} \) and note that \( C_t \) is \( F_t \)-measurable. Observe that if \( F_t^{(\beta)} = C_t S_t > \beta \), then

\[
F_u^{(\beta)}1_{\{m_F(t,u) > \beta\}} = C_t S_u1_{\{m_S(t,u) > \beta/C_t\}} \quad \text{a.s. for} \quad t < u \leq T,
\]  

since the fee is not paid when the account value is above \( \beta \). It follows that

\[
E_Q[e^{-r(u-t)}F_u^{(\beta)}1_{\{m_F(t,u) > \beta\}}|F_t^{(\beta)} = x] = C_tE_Q\left[ e^{-r(u-t)}S_u1_{\{m_S(t,u) > \beta/C_t\}} | S_t = \frac{x}{C_t} \right].
\]  

The expectation on the right-hand side of equation (15) is the price of a down-and-out contract on the underlying stock with barrier \( \beta/C_t \) and maturity \( u \). Under the Black-Scholes model, the price of this option has a closed-form solution (see, for example, Chapter 18 of Björk (2004)), and we can write

\[
C_tE_Q\left[ e^{-r(u-t)}S_u1_{\{m_S(t,u) > \beta/C_t\}} | S_t = \frac{x}{C_t} \right] = \frac{\ln \frac{x}{\beta} + \left( r + \frac{\sigma^2}{2} \right) (u-t)}{\sigma \sqrt{u-t}} N \left( \frac{\ln \frac{\beta}{x} + \left( r + \frac{\sigma^2}{2} \right) (u-t)}{\sigma \sqrt{u-t}} \right) - \beta \left( \frac{\beta}{x} \right)^{\frac{\sigma}{2}} N \left( \frac{\ln \frac{\beta}{x} + \left( r + \frac{\sigma^2}{2} \right) (u-t)}{\sigma \sqrt{u-t}} \right),
\]  

34
where $\mathcal{N}(\cdot)$ denotes the standard normal cumulative distribution function. Thus,

$$
C_tE_Q[e^{-r(u-t)}F_T^{(\beta)}1_{\{m_F(t,u)\leq\beta\}}|F_t^{(\beta)} = x]
$$

$$
= \mathcal{N}\left(\frac{\ln \left(\frac{x}{\beta}\right) + \left(r + \frac{\sigma^2}{2}\right)(u-t)}{\sigma \sqrt{u-t}}\right) - \left(\frac{\beta}{x}\right)^{\frac{2\beta+1}{2}} \mathcal{N}\left(\frac{\ln \left(\frac{\beta}{x}\right) + \left(r + \frac{\sigma^2}{2}\right)(u-t)}{\sigma \sqrt{u-t}}\right).
$$

The result follows since $\lim_{y \to \infty} \mathcal{N}(y) = 1$ and $\lim_{y \to -\infty} \mathcal{N}(y) = 0$.

To show that the second term of (13) vanishes for large values of $x$, we first note that

$$
E_Q\left[\frac{e^{-r(T-t)}F_T^{(\beta)}1_{\{m_F(t,T)\leq\beta\}}|F_t^{(\beta)} = x}}{x}\right] \leq E_Q\left[\frac{e^{-r(T-t)}S_T1_{\{m_S(t,T)\leq x/C_t\}}|S_t = \frac{x}{C_t}}{x}\right], \quad (16)
$$

since for any $0 \leq t \leq T$, $F_t = S_tC_t \leq S_t$, a.s. The right-hand side of equation (16) is the price of a down-and-in contract on the underlying stock with barrier $\beta/C_t$. The price of this contract also has a closed-form solution (again, see Chapter 18 of Björk (2004)), which allows us to write

$$
\frac{1}{C_t}\left\{ \mathcal{N}\left(\frac{\ln \left(\frac{\beta}{x} - (\tilde{r} + \sigma^2)(u-t)}{\sigma \sqrt{u-t}}\right) + \left(\frac{\beta}{x}\right)^{\frac{2\beta+2}{2}} \mathcal{N}\left(\frac{\ln \left(\frac{\beta}{x} + (\tilde{r} + \sigma^2)(u-t))}{\sigma \sqrt{u-t}}\right)\right)\right\},
$$

Since $\lim_{y \to -\infty} \mathcal{N}(y) = 0$, $\lim_{x \to \infty} E_Q[e^{-r(T-t)}F_T^{(\beta)}1_{\{m_F(t,T)\leq\beta\}}|F_t^{(\beta)} = x] = 0$. \qed

**Lemma A.2.** Let $F_t^{(\beta)}$, $0 \leq t \leq T$, be as defined in Section 2. Then,

$$
\lim_{x \to \infty} x + E_Q[e^{-r(T-t)}(G - F_T^{(\beta)})^+|F_t^{(\beta)} = x] = 1,
$$

where $(G - F_T^{(\beta)})^+ = \max(G - F_T^{(\beta)}, 0)$.

**Proof.** Denote by $p_t,s_t(T,G,\delta)$ the price at time $t$ of a European put option with strike $G$ and maturity $T$ on a stock $S_t$ paying dividends at a continuous rate $\delta$. Using

$$
\frac{S_T e^{-r(T-t)}}{S_t} < \frac{F_T^{(\beta)}}{F_t^{(\beta)}} < \frac{S_T}{S_t}, \quad \text{a.s.,}
$$

35
it is easy to show that

\[ p_{t,x}(T, G, 0) \leq E_Q[e^{-r(u-t)}(G - F_u^{(\beta)})^+|F_t^{(\beta)} = x] \leq p_{t,x}(T, G, c). \]

Since \( \forall \delta \geq 0, \lim_{x \to \infty} p_{t,x}(T, G, \delta) = 0 \), the desired result follows from

\[ \lim_{x \to \infty} E_Q[e^{-r(T-t)}(G - F_T)^+|F_t^{(\beta)} = x] = 0. \quad \square \]

Using Lemmas A.1 and A.2, we can now prove (5) by showing

\[ \lim_{x \to \infty} V(t, x) = 1. \quad (17) \]

where \( V(t, x) \) is the price at \( t \) of the VA contract with \( \beta < \infty \), when \( F_t^{(\beta)} = x \). First, we show that

\[ E_Q[e^{-r(T-t)}F_T^{(\beta)}|F_t] \leq V(t, F_t^{(\beta)}) \leq F_t + E_Q[e^{-r(T-t)}(G - F_T^{(\beta)})^+|F_t]. \quad (18) \]

The first inequality stems from the fact that the price of the contract with surrender option, \( V(t, F_t^{(\beta)}) \) is worth at least as much as the present value of the maturity benefit, which is itself at least equal to the expectation of the account value at maturity. To show the second inequality, recall that the payoff of the contract is either \((1 - \kappa_u)F_u^{(\beta)} \) if the contract is surrendered at time \( u < T \), or \( F_T^{(\beta)} + (G - F_T^{(\beta)})^+ \) at time \( T \) if the contract is kept until then. Note also that the present value of the surrender benefit is at most \( F_t^{(\beta)} \) since for any \( u < t < T \),

\[ E_Q[e^{-r(u-t)}(1 - \kappa_u)F_u^{(\beta)}|F_t] \leq E_Q[e^{-r(u-t)}F_u^{(\beta)}|F_t] \leq F_t^{(\beta)}. \quad (19) \]

Thus, the value of the VA contract is bounded above by an amount greater than the expected value of either payoff, and it follows that

\[ V(t, F_t^{(\beta)}) \leq F_t + E_Q[e^{-r(T-t)}(G - F_T^{(\beta)})^+|F_t]. \]

From (18),

\[ \frac{E_Q[e^{-r(T-t)}F_T^{(\beta)}|F_t^{(\beta)} = x]}{x} \leq \frac{V(t, x)}{x} \leq \frac{F_t^{(\beta)} + E_Q[e^{-r(T-t)}(G - F_T^{(\beta)})^+|F_t^{(\beta)} = x]}{x}. \quad (20) \]

To complete the proof of (17), it suffices to take the limit of (20) as \( x \to \infty \). The result follows from Lemma A.1 and Lemma A.2, since the first and the third terms of (20) both go to 1 in the limit. \( \square \)
B Additional details on the PDE pricing approach

To solve the PDE in (3), we use finite difference methods. The equation is first expressed in terms of $x_t = \ln F_t^{(\beta)}$ and discretized over a rectangular grid representing the truncated, discretized domain of $(t, x_t)$. For small account values at time $t$, the contract price is well approximated by $Ge^{-r(T-t)}$ and we do not need to include fund values which are very close to zero. The upper truncation point of the grid in the $x_t$ dimension must be large enough so that the asymptotic results derived in Section 2 can be used reliably to approximate the contract price at the highest fund values in the grid. Moreover, it is preferable to choose this maximal value such that the probability that it is reached by the process $x_t$ is very small. In our numerical illustrations, we use a grid for $x_t$ which spans from $\ln 20$ to $\ln 400$ with steps of $dx = 0.0005$. Under the assumptions stated in Section 3 ($\mu = 0.07$, $\sigma = 0.165$, and $T = 10$), $E^Q[S_T|S_0 = 100] = 201.38$ and $\sqrt{Var^Q[S_T|S_0 = 100]} = 112.65$, so our grid covers the most likely paths of $F_t^{(\beta)}$ since $F_t^{(\beta)} \leq S_t$ for any $\beta \geq 0$ and $t \in [0, T]$.

When an optimal surrender boundary exists for all $t \in [0, T]$, it is not necessary to consider values that are above this boundary, because the price of the contract is known exactly in this region (and equal to the value of the surrender benefit). In these cases, we use a lower maximal value to decrease computational time.

We use an explicit method with time steps $dt = (dx/\sigma)^2/3$ to ensure stability of our numerical scheme (see, for example, Racicot and Théoret (2006)). Implicit methods were also explored for validation purposes and to examine stability, but the explicit scheme proved to be the most efficient as we were able to implement it in C++. Central differences were used to approximate the first order term. Again, other methods were explored. In particular, we also used forward differences to make sure that all the coefficients were positive (for more details, see Chapter 9 of Duffy (2006)), but the precision of the results obtained using central differences was very similar.

C Proof of Proposition 3.1

Proof. First, suppose that the surrender charge $\kappa_u$, for $t \leq u < T$, is sufficiently high to eliminate the optimal lapsation boundary for $t \leq u < T$. This situation is possible because we can consider the extreme case where $\kappa_u = 1$, for $t \leq u < T$. Then, the value of the contract at time $u$ must simply be the risk-neutral discounted expectation of the payoff at maturity, and be greater or equal to the surrender benefit:

$$V^*(u, F_u^{(\beta)}) = E_Q[e^{-r(T-u)} \max(F_T^{(\beta)}, G)|\mathcal{F}_u] \geq F_u^{(\beta)}(1 - \kappa_u), \quad \forall F_u^{(\beta)} > 0.$$
The previous inequality can be rewritten as
\[ \kappa_u \geq 1 - \frac{E_Q[e^{-r(T-u)} \max(F_T^{(\beta)}, G)|\mathcal{F}_u]}{F_u^{(\beta)}}, \quad \forall F_u^{(\beta)} \geq 0. \]

Therefore, the minimal surrender penalty that can be charged at time \( u \) while the inequality above is satisfied corresponds to
\[ \kappa^*_u = \max \left( 1 - \inf_{F_u^{(\beta)} > 0} \left\{ \frac{E_Q[e^{-r(T-u)} \max(F_T^{(\beta)}, G)|\mathcal{F}_u]}{F_u^{(\beta)}} \right\}, 0 \right). \]

Since for all \( F_u^{(\beta)} > 0, \)
\[ \frac{E_Q[e^{-r(T-u)} \max(F_T^{(\beta)}, G)|\mathcal{F}_u]}{F_u^{(\beta)}} = \frac{F_u^{(\beta)} e^{-c(T-u)} + E_Q[e^{-r(T-u)} \max(G - F_T^{(\beta)}, 0)|\mathcal{F}_u]}{F_u^{(\beta)}} > e^{-c(T-u)}, \]
and,
\[ \lim_{F_u^{(\beta)} \to \infty} \frac{E_Q[e^{-r(T-u)} \max(F_T^{(\beta)}, G)|\mathcal{F}_u]}{F_u^{(\beta)}} = e^{-c(T-u)}, \]
then, we must have \( \kappa^*_u = 1 - e^{-c(T-u)}, \quad t \leq u < T. \) \( \square \)

**D Proof of Proposition 4.1**

**Proof.** Suppose that there are no surrender charges, i.e., \( \kappa_t = 0, \forall t, \) that the fee is only paid below \( \beta, \) and that \( F_t^{(\beta)} \geq \beta \) at time \( t. \) Consider the stopping time
\[ \tau_\beta = \inf \{ t < u < T : F_u^{(\beta)} < \beta \}, \]
with the convention that \( \tau_\beta = T, \) if the barrier \( \beta \) is never reached. Then, we can write,
\[ V^*(t, F_t^{(\beta)}) = \sup_{\tau \in T_t} E_Q[e^{-r(T-t)} \psi(\tau, F_\tau^{(\beta)})|\mathcal{F}_t] \]
\[ \geq E_Q[e^{-r(\tau_\beta-t)} \psi(\tau_\beta, F_{\tau_\beta}^{(\beta)})|\mathcal{F}_t] \]
\[ = E_Q[e^{-r(\tau_\beta-t)} F_{\tau_\beta}^{(\beta)} \mathbb{1}_{\{\tau_\beta \in (t,T)\}}|\mathcal{F}_t] + E_Q[e^{-r(\tau_\beta-t)} F_{\tau_\beta}^{(\beta)} \mathbb{1}_{\{\tau_\beta = T\}}|\mathcal{F}_t] \]
\[ = \beta E_Q[e^{-r(\tau_\beta-t)} \mathbb{1}_{\{\tau_\beta \in (t,T)\}}|\mathcal{F}_t] + E_Q[e^{-r(T-t)} F_T^{(\beta)} \mathbb{1}_{\{\tau_\beta = T\}}|\mathcal{F}_t], \]
where the first term is the payoff of a down rebate option which pays \( \beta \) if the fund \( F_t^{(\beta)} \) reaches \( \beta \) before maturity \( T \) and zero otherwise, and the second term is the payoff of
a down-and-out European call option with zero strike which pays $F_T^{(β)}$ at maturity $T$, provided that $F_u^{(β)} ≥ β$, for $t ≤ u ≤ T$. Since the combined payoff of these two options can be replicated by holding the fund $F_t^{(β)}$ and selling it as soon as $F_t^{(β)} = β$, simple no-arbitrage arguments imply that the total price of these two options is exactly $F_t^{(β)}$, which gives

$$V^*(t, F_t^{(β)}) ≥ F_t^{(β)}.$$  \hspace{1cm} (21)

The inequality in (21) becomes an equality if and only if $τ_β$ is the optimal strategy that can be undertaken, i.e.,

$$\sup_{τ \in T} E_Q[e^{-r(τ - t)}\psi(τ, F_{τ}^{(β)})|F_t] = E_Q[e^{-r(τ_β - t)}\psi(τ_β, F_{τ_β}^{(β)})|F_t].$$

The equality, $V^*(t, F_t^{(β)}) = F_t^{(β)}$, therefore holds if and only if it is optimal to surrender the VA contract just below $β$, for all $t < u < T$. If this occurs, by definition of the optimal surrender region given in Section 2.2 ($V^*(t, F_t^{(β)}) ≤ F_t^{(β)}$ induces surrender), the policyholder surrenders at time $t ∀F_t^{(β)} ≥ β$. However, strictly speaking, the policyholder is actually indifferent to lapse because the surrender benefit is exactly equal to the continuation value of the contract.

Now, consider the case where the surrender charge function $κ_u$, $t ≤ u ≤ T$, is a decreasing function of $u$, and is strictly positive at $t$:

$$V^*(t, F_t^{(β)}) = \sup_{τ \in T} E_Q[e^{-r(τ - t)}\psi(τ, F_{τ}^{(β)})|F_t]$$

$$≥ E_Q[e^{-r(τ_β - t)}\psi(τ_β, F_{τ_β}^{(β)})|F_t]$$

$$= E_Q[e^{-r(τ_β - t)} F_{τ_β}^{(β)}(1 - κ_{τ_β})1_{\{τ_β ∈ (t, T]\}}|F_t] + E_Q[e^{-r(τ_β - t)} F_{τ_β}^{(β)} 1_{\{τ_β = T\}}|F_t]$$

$$= β E_Q[e^{-r(τ_β - t)} (1 - κ_{τ_β})1_{\{τ_β ∈ (t, T]\}}|F_t] + E_Q[e^{-r(T - t)} F_T^{(β)} 1_{\{τ_β = T\}}|F_t]$$

$$> (1 - κ_t) \left\{β E_Q[e^{-r(τ_β - t)} 1_{\{τ_β ∈ (t, T]\}}|F_t] + E_Q[e^{-r(T - t)} F_T^{(β)} 1_{\{τ_β = T\}}|F_t]\right\}$$

$$= (1 - κ_t) F_t^{(β)},$$

as the term inside the braces in equation (22) was shown to be exactly $F_t^{(β)}$. This result implies that in the presence of surrender charges, it is never optimal to surrender the VA contract when the fund is above or equal to the fee threshold barrier $β$. \hspace{1cm} □

### E Calculation of $Δ_t$

To hedge, the insurer must first specify the objective function and the assumptions. For example, suppose that $β = ∞$, and that the insurer wants to set-up a delta hedge of its
net liability assuming no surrenders occur. In this context, the net liability of the insurer towards the policyholder at time \( t \), denoted by \( \Psi_t \), corresponds to the fair value of the maturity benefit minus the account value:

\[
\Psi_t = E_Q[e^{-r(T-t)} \max(F_T^{(\infty)}, G) | F_t] - F_t^{(\infty)}
\]

\[
= E_Q[e^{-r(T-t)} \max(G - F_t^{(\infty)}, 0) | F_t] - F_t^{(\infty)}(1 - e^{-c(T-t))}).
\]

Equation (23) offers an alternative interpretation of the net liability, as the value of the underlying European put option minus the fair value of the fees that will be collected by the insurer until maturity.\(^7\) For this particular case, the delta of the net liability is available in closed form as

\[
\frac{\partial}{\partial S_t} \Psi_t = \frac{\partial \Psi_t}{\partial F_t^{(\infty)}} \frac{\partial F_t^{(\infty)}}{\partial S_t} = -e^{-cT}N(-d_1) - (e^{-cT} - e^{-ct}),
\]

where

\[
d_1 = \frac{\log(F_t^{(\infty)}/G) + (r - c + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}.
\]

In more general situations, where we want to hedge assuming optimal policyholder behavior, we cannot obtain the delta analytically. However, we can write,

\[
\Psi_t = V(t, F_t^{(\beta)}) - F_t^{(\beta)},
\]

where \( V(t, F_t^{(\beta)}) \) represents the fair value of the VA contract with surrender option, as defined in equation (2.3). The delta is then obtained with,

\[
\frac{\partial}{\partial S_t} \Psi_t = \frac{\partial \Psi_t}{\partial F_t^{(\beta)}} \frac{\partial F_t^{(\beta)}}{\partial S_t} = \left[ \frac{\partial V(t, F_t^{(\beta)})}{\partial F_t^{(\beta)}} - 1 \right] \frac{\partial F_t^{(\beta)}}{\partial S_t},
\]

where, \( \partial V(t, F_t^{(\beta)})/\partial F_t^{(\beta)} \) must be estimated numerically based on a finite difference grid and \( \partial F_t^{(\beta)}/\partial S_t = F_t^{(\beta)}/S_t \), which is implicitly obtained in the derivation of (3) in Appendix A.

\(^7\)The fair value of the fees that will be collected by the insurer after time \( t \) corresponds to \( F_t^{(\infty)}(1 - e^{-c(T-t)}) \). If we interpret the fee \( c \) as a dividend rate, the fair value of dividends to be received between times \( t \) and \( T \) is the difference between the fund value at time \( t \), \( F_t^{(\infty)} \), and the prepaid forward price at \( t \) for a claim paying \( F_T^{(\infty)}e^{-c(T-t)} \) at time \( T \), \( (F_t^{(\infty)}e^{-c(T-t)}) \).