Modeling Trades in the Life Market as Nash Bargaining Problems: Methodology and Insights

Rui Zhou, Johnny Siu-Hang Li, and Ken Seng Tan

Abstract

Longevity risk, that is, the risk of unexpected longevity improvements, poses huge burdens on pension plan sponsors and annuity writers. Since about ten years ago, institutions that are subject to longevity risk have started to consider securitization as a solution to the problem, leading to the formation of the ‘Life Market’, in which securities linked to future mortality are traded. In this paper, we model the trade of a longevity security as a two-player bargaining game, and use Nash’s bargaining solution to determine the outcome of it. Our model can be used by practitioners to estimate the price of a newly introduced longevity security. Relative to the existing pricing methods, the method we propose has two advantages. First, it does not require the competitive market assumption, which is not satisfied by today’s Life Market. Second, it does not require any market price data, which are not readily available from the public domain. We illustrate the theoretical results with a hypothetical trade of a longevity bond between a pension plan sponsor and an investor.

1 Introduction

The population of the developed world is living longer. In the study by Oeppen and Vaupel (2002), it is shown empirically that record female life expectancy since 1840 has been increasing at a steady pace of 2.43 years per decade, and that the trend has no sign of slowing down. The fact that everyone is living longer is a good news and success story, but unanticipated improvements in life expectancies can pose problems to individuals, corporations and governments. This risk, which is commonly referred to as longevity risk, has received considerable attention in recent years.
More specifically, if mortality improves faster than expected, then governments and corporations offering defined-benefit pension plans to their employees will need to pay out more social security and pension benefits. The impact of longevity risk on insurance companies selling life annuities is similar, but could be more immediate, because new regulations such as Solvency II require these insurance companies to hold additional capital to cushion against the risk. According to the International Monetary Fund 2012 Global Financial Stability Report\(^1\), if individuals live three years longer than expected, then the already large costs of aging could increase by another 50 percent, representing an additional cost of 50 percent of 2010 GDP in advanced economies.

Longevity risk is especially significant in the current low interest rate environment, because distant-future pension and annuity liabilities, which are the most dependent on future longevity improvements, are not heavily discounted. Since about ten years ago, pension sponsors and annuity providers have started to consider securitization as a solution to the problem of longevity risk. This has led to the emergence of the ‘Life Market’, the traded market in assets and liabilities that are linked to human mortality. By trading in the Life Market, longevity risk can be transferred to parties who are interested in taking on the risk to earn a premium and to diversify their investment portfolios.\(^2\)

The first longevity security was announced by BNP Paribas and the European Investment Bank in 2004. Since then, the Life Market has witnessed several longevity derivatives transactions, most notably the £3 billion longevity swap that Rolls Royce transacted with Deutsche Bank in 2011 to cover the longevity risk of its pension plan. However, as exemplified by the following three characteristics, the Life Market is still in its early stage of development.

1. Relative to liquid trading financial markets, the number of participants in the Life Market is small. For instance, according to Blake et al. (2013), there were only seven providers among the known longevity swap transactions in the U.K. from 2007 to 2012.\(^3\)

---

\(^1\)Available at http://www.imf.org/external/pubs/ft/gfsr/2012/01/.

\(^2\)Securities in the Life Market have very low correlations to other asset classes, such as foreign exchange, commodities, equities and credit. The low correlations between longevity and other risks have been discussed by Blake et al. (2006) and shown empirically by Ribeiro and di Pietro (2009).

\(^3\)The seven providers include Credit Suisse, Deutsche Bank, Goldman Sachs, J.P. Morgan, Legal & General, RBS and Swiss Re.
2. The products are inhomogeneous. Rather than being standardized, most transactions took place in the Life Market were bespoke deals, customized to the hedgers’ own risk characteristics.

3. We do not have perfect knowledge of the Life Market. In particular, although the sizes of some transactions are publicly-announced, the pricing information is often proprietary. Some transactions in the market are not even made known to the public.

When it comes to trading in the Life Market, both the seller and buyer need to estimate the price of security being traded. There exist a number of methods for pricing longevity securities, but these methods, as we now explain, may not be appropriate in the current stage of the Life Market.

Most of the prevailing pricing methods are based on the principle of no-arbitrage. Generally speaking, to implement these methods, we first estimate a distribution of future mortality rates under the real-world probability measure. Second, we identify a risk-neutral probability measure, under which existing securities in the market are correctly priced, and transform the real-world distribution to its risk-neutral counterpart. Finally, we can estimate the price of a newly introduced longevity security by discounting, at the risk-free interest rate, its expected payoff that is calculated using the risk-neutral distribution of future mortality rates.

The second step in the procedure above requires market price data as input. For instance, the pricing methods of Lin and Cox (2005), Denuit et al. (2007), Dowd et al. (2006) and Milidonis et al. (2011) make use of a distortion operator such as the Wang transform (Wang, 2000) to create a risk-neutral probability measure. In using these methods, there is a need to estimate the parameters in the distortion operator with prices of securities that are actually traded in the Life Market. Hence, in today’s Life Market where market price data are very limited, it is not easy to implement these pricing methods. Other no-arbitrage pricing methods, including the constrained maximization of entropy (Li, 2010; Li and Ng, 2011; Li et al., 2011) and the use of a risk-adjusted two-factor mortality model (Cairns et al., 2006; Deng et al., 2012), are subject to the same problem.

Recently, researchers have considered economic methods for pricing longevity securities. In particular, Zhou et al. (2011, 2013) treat pricing in the Life Market as a Walrasian tâtonnement process, in which the price of a longevity security is determined through a gradual calibration of supply and demand. Other than the trading
price, one can also estimate the trading quantity by using the demand and supply curves resulting from the tâtonnement process. In general, economic pricing methods do not entail the identification of a risk-neutral probability measure. This means that the use of these methods does not require prices of traded securities as input, thereby sparing us from the problems associated with the lack of market price data.

Nevertheless, a significant problem still remains. All previously mentioned pricing methods require the assumption of a perfectly competitive market. In a competitive market, there are a large number of participants, the products sold are identical, and each participant knows the price and quantity of the goods sold by everyone. The aforementioned characteristics of the current Life Market indicate that it is not even close to competitive. In the present stage of market development, prices calculated from the tâtonnement approach could be used as a benchmark in situations when no-arbitrage methods are difficult to implement, but they may not be regarded as accurate estimates.

In this paper, we relax the assumption of a perfectly competitive market by treating the trade of a longevity security as a two-player bargaining game. The competitive market assumption is waived, because rather than being a price taker, each player in the game can influence the price of the security being traded through the bargaining process. In our set-up, it is assumed that one party is a pension plan sponsor or an annuity writer who intends to hedge her longevity risk exposure, and that the other party is an investor who intends to take on the risk for a risk premium. The bargaining outcome is obtained through Nash’s bargaining solution, an axiomatic solution proposed by Nash (1950) to solve the two-person bargaining game.

Nash’s bargaining solution suppresses many details of the decision making process. It explains outcomes by identifying conditions that any outcome arrived at by rational decision makers should satisfy a priori. These conditions are treated as axioms, from which the outcome is deduced by using set-theoretical arguments. It is therefore not computationally difficult to find the outcome of the game, that is, the trading price and quantity, by using Nash’s bargaining solution. Besides being easy to implement, the pricing framework proposed in this paper also preserves many advantages of the tâtonnement pricing approach, including the provision of a unique trading price and quantity.

Nash’s bargaining solution has been previously applied to problems in insurance. Borch (1974) applied Nash’s bargaining solution to reciprocal reinsurance treaties,
and used it to determine the quota ceded by each player. Kihlstrom and Roth (1982) and Schlesinger (1984) studied how insurance contracts are reached through Nash bargaining, and investigated the effect of risk aversion on the outcome of bargaining about the terms of an insurance contract. Boonen et al. (2012) defined a cooperative non-transferable utility game for the optimal redistribution of longevity risk between pension funds and life insurers. To the best of our knowledge, this paper is the first to model the trade of a longevity security as a two-person bargaining game and to solve it with Nash’s bargaining solution.

The remainder of this paper is organized as follows. Section 2 presents the set-up of the trade of a longevity security between a hedger and an investor. Section 3 details how the trade can be modeled as a two-person bargaining game, and explains how Nash’s bargaining solution can be used to determine the outcome of the trade. Section 4 studies how the modeling of the trade would be different if the Life Market is perfectly competitive. Section 5 applies the theoretical results from the previous two sections to a hypothetical trade. Section 6 investigates the conditions of Pareto optimality, and compares the outcomes arising from Nash’s bargaining solution and the competitive equilibrium. Finally, Section 7 concludes the paper.

2 Setting up the Trade

In this paper, we model the trade of a mortality-linked security between two economic agents, namely Agents A and B.

Agent A could possibly be an annuity provider or a pension sponsor, who has an exposure to longevity risk. Suppose that Agent A has annuity or pension liabilities that are due at times 1, 2, ..., T. The amount due at time \( t \) is \( f_t(Q^L_t) \), which is a deterministic function of \( Q^L_t \), where \( Q^L_t \) is an index that contains information about the mortality of the population associated with Agent A’s annuity or pension liability up to and including time \( t \). At time 0, the values of \( Q^L_t \) for \( t > 0 \) are not known and are governed by an underlying stochastic process.

Agent A can mitigate her longevity risk exposure by purchasing a mortality-linked

---

4Boonen et al. (2012) found that under certain assumptions, proportional risk redistribution is optimal. Proportional risk redistribution may be achieved by means of reinsurance, but not through the trade of standardized securities such as longevity bonds and mortality forwards in the Life Market.
security from Agent B, who is interested in taking on an exposure to longevity risk for a risk premium. It is assumed that the security being traded matures at time $T$, and that at time $t$, $t = 1, \ldots, T$, the security makes a payout of $g_t(Q^H_t)$, which is deterministic function of $Q^H_t$, where $Q^H_t$ is an index that contains information about the mortality of the population associated with the security up to and including time $t$. As with $Q^L_t$, the values of $Q^H_t$ for $t > 0$ are not known at time 0 and are governed by an underlying stochastic process.

We use $P$ to denote the time-0 price of the security and $\theta$ to denote the quantity of the security traded. The cash flows involved in the trade are illustrated diagrammatically in Figure 1.

In real life, the population associated with Agent A’s annuity or pension liability could be different from that associated with the security being traded. In particular, it is likely that the security being traded is linked to a broad population mortality index, which is more transparent to the investor. For instance, the mortality forward contracts traded between J.P. Morgan and Lucida\footnote{Lucida is an insurance company focused on the annuity and longevity risk business, including the defined benefit pension buyout market and the market for bulk annuities.} in January 2008 are linked to the mortality of the general English and Welsh population rather than that of Lucida’s own population. For this reason, in our set-up, we permit $Q^H_t$ to be different from $Q^L_t$. The joint modeling of $Q^H_t$ and $Q^L_t$ will be detailed in Section 5.1.

We assume that each agent’s wealth can only be invested in either the mortality-linked security or a bank account which yields a continuously compounded risk-free interest rate of $r$ per annum. We allow a negative wealth, which means that the agent borrows money from a bank account and pays an interest rate of $r$ to the bank. Other
than the bank account, the mortality-linked security and the life contingent liability, there is no sources of income or payout.

Let $W^A_t$ and $W^B_t$ respectively be the time-$t$ wealth of Agents A and B. The wealth process for Agent A can be expressed as

$$W^A_t(P, \theta) = W^A_{t-1}e^r + \theta g_t(Q^H_t) - f_t(Q^L_t), \quad t = 2, \ldots, T,$$

where $W^A_1(P, \theta) = (W^A_0 - \theta P)e^r + \theta g_1(Q^H_1) - f_1(Q^L_1)$ and $W^A_0$ is a constant. Similarly, the wealth process for Agent B can be expressed as

$$W^B_t(P, \theta) = W^B_{t-1}e^r - \theta g_t(Q^H_t), \quad t = 2, \ldots, T,$$

where $W^B_1(P, \theta) = (W^B_0 + \theta P)e^r - \theta g_1(Q^H_1)$ and $W^B_0$ is a constant.

Let $F = \sum_{t=1}^T f_t(Q_t)e^{r(T-t)}$ and $G = \sum_{t=1}^T g_t(Q_t)e^{r(T-t)}$. These shorthand symbols are used throughout the rest of this paper. It is easy to show that in terms of $F$ and $G$, the terminal wealth of Agent A is given by

$$W^A_T(P, \theta) = W^A_0 e^{rT} + \theta(G - Pe^{rT}) - F,$$

and that of Agent B is given by

$$W^B_T(P, \theta) = W^B_0 e^{rT} - \theta(G - Pe^{rT}).$$

We let $U^A$ and $U^B$ be the utility functions of Agents A and B, respectively. We assume exponential utility functions, that is, $U^A(x) = 1 - e^{-k^A x}$ and $U^B(x) = 1 - e^{-k^B x}$, where $k^A$ and $k^B$ are the absolute risk aversion parameters for the agents. There are two advantages of assuming exponential utility functions. First, if exponential utility functions are assumed, then the resulting estimates of $P$ and $\theta$ would not depend on the agents’ initial wealth. Second, exponential utility functions permit negative wealth. The permission of negative wealth is useful, because in our set-up, an agent’s wealth might become negative at some future time points. These two properties may be lost if other utility functions are assumed.
3 Modeling the Trade in a Non-Competitive Market

3.1 Bargaining Problems

In a non-competitive securities market, participants are not price takers. Each participant has an influence on the price of the security being traded, possibly through bargaining. Here we model the trade described in Section 2 as a Nash bargaining problem (Nash, 1950). Mathematically, a Nash bargaining problem is a pair \( \langle S, d \rangle \), where \( S \subset \mathbb{R}^2 \) is a compact and convex set, \( d = (d_1, d_2) \in S \), and for some \( s = (s_1, s_2) \in S \), \( s_i > d_i \) for \( i = 1, 2 \).

Intuitively, the set \( S \) is the set of all feasible expected utility payoffs to the agents, while the vector \( d = (d_1, d_2) \) represents the disagreement payoff; that is, if the agents do not come to an agreement, then they will receive utility payoffs of \( d_1 \) and \( d_2 \), respectively. We require \( d \in S \), so that the agents can agree to disagree. We also require there exists \( s = (s_1, s_2) \) in \( S \) such that \( s_i > d_i \) for \( i = 1, 2 \), so that the agents have an incentive to reach an agreement (via bargaining).

What we are interested in, of course, is which point in \( S \) the bargaining process will lead to. Nash (1950) modeled the bargaining process by a function \( \zeta : B \to \mathbb{R}^2 \), where \( B \) is the set of all bargaining problems. The function \( \zeta \), which assigns a unique element in \( S \) to each bargaining problem \( \langle S, d \rangle \in B \), is referred to as a bargaining solution.

Rather than explicitly modeling the underlying bargaining procedure, Nash (1950) used a purely axiomatic approach to derive a bargaining solution. In more detail, he specified, as axioms, the following four properties that it would seem natural for a bargaining solution to have.

1. Pareto optimality
   If \( \zeta(S, d) = (z_1, z_2) \) and \( y_i \geq z_i \) for \( i = 1, 2 \), then either \( y_i = z_i \) for \( i = 1, 2 \) or \( (y_1, y_2) \notin S \).

2. Symmetry
   If \( (S, d) \) is a symmetric bargaining problem (i.e., \( d_1 = d_2 \) and \( (x_1, x_2) \in S \Rightarrow (x_2, x_1) \in S \)), then \( \zeta_1(S, d) = \zeta_2(S, d) \), where \( \zeta_i(S, d) \) denotes the \( i \)th entry in \( \zeta(S, d) \).
3. Independence of irrelevant alternatives

If \((S,d)\) and \((T,d)\) are bargaining problems such that \(S \subset T\) and \(\zeta(T,d) \in S\), then \(\zeta(S,d) = \zeta(T,d)\).

4. Independence of equivalent utility representatives

If \((S',d')\) is related to \((S,d)\) in such a way that \(d'_i = a_id_i + b_i\) and \(s'_i = a_is_i + b_i\) for \(i = 1,2\), where \(a_i\) and \(b_i\) are real numbers and \(a_i > 0\), then \(\zeta(S',d') = a_i\zeta_i(S,d) + b_i\) for \(i = 1,2\).

Properties 1 and 3 are particularly related to bargaining. Specifically, Property 1 implies that if the agents agreed on an inferior outcome, then they will renegotiate until the Pareto optimal outcome is reached. Property 3, on the other hand, resembles a gradual elimination of unacceptable outcomes. Eliminated outcomes (i.e., those in \(T\) but not in \(S\)) have no effect on the bargaining solution. We refer readers to Osborne and Rubinstein (1990) for a fuller discussion of these properties.

Nash showed that there is one and only one bargaining solution that satisfies the four axiomatic properties.

**Theorem 1.** There is a unique solution which possesses Properties 1-4. The solution, \(\zeta^N(S,d) : B \to \mathbb{R}^2\), takes the form

\[
\zeta^N(S,d) = \arg \max (s_1 - d_1)(s_2 - d_2),
\]

where the maximization is taken over \((s_1, s_2) \in S\), and is subject to the constraint \(s_i > d_i\) for \(i = 1,2\).

In other words, Nash’s unique bargaining solution selects the utility pair in \(S\) that maximizes the product of the agents’ gain in utility over the disagreement outcome \((d_1, d_2)\). We call \((s_1 - d_1)(s_2 - d_2)\) the Nash product.

### 3.2 The Underlying Bargaining Strategy

Although the derivation of Nash’s bargaining solution does not require knowledge on the details of the underlying bargaining strategy, one may still wonder how the actual bargaining takes place. A possible underlying strategy is the one proposed by Zeuthen (1930), which we now describe.
In the $k$th round of the bargaining process, Player 1 proposes an agreement with a payoff vector $y^{1,k} = (y^{1,k}_1, y^{1,k}_2)$, while Player 2 proposes another agreement with a payoff vector $y^{2,k} = (y^{2,k}_1, y^{2,k}_2)$. If they fail to agree, then they receive the disagreement payoff $d = (d_1, d_2)$.

Assume that $d_i < y^{i,k}_i < y^{i,k}_j$, where $i = 1, 2$, $j = 1, 2$ and $i \neq j$. In round $k + 1$, player $i$ can take one of the following actions:

- accept Player $j$’s offer, leading to an agreement;
- make a concession by counter-proposing $y^{i,k+1}$ such that $y^{i,k}_j < y^{i,k+1}_j < y^{j,k}_j$;
- repeat her last offer.

Let

$$p^{i,k} = \frac{y^{i,k}_i - y^{j,k}_i}{y^{i,k}_i - d_i},$$

where $i = 1, 2$, $j = 1, 2$, and $i \neq j$. Zeuthen’s bargaining strategy can be summarized as follows:

- if $p^{2,k} < p^{1,k}$, then Player 2 makes a concession;
- if $p^{1,k} < p^{2,k}$, then Player 1 makes a concession;
- if $p^{1,k} = p^{2,k}$, then both players make a concession.

This process will continue until the two players agree on a solution. Harsanyi (1956) proved that Zeuthen’s bargaining strategy leads to Nash’s bargaining solution.

### 3.3 Applying Nash’s Bargaining Solution to the Life Market

Now, let us turn our focus back to mortality-linked securities. Following the set-up in Section 2, the expected utility payoffs to the agents are their expected terminal utilities. The utility possible set $S$ is the set of feasible expected utility pairs

$$(\mathbb{E}[U^A(W_T^A(P, \theta))], \mathbb{E}[U^B(W_T^B(P, \theta))]).$$
arising from all possible values of $P$ and $\theta$.\(^6\) Note that the structure of the mortality-linked security is assumed to be fixed. The agents are only allowed to bargain over the price $P$ and the quantity $\theta$. The disagreement utility payoffs are the expected terminal utilities when there is no trade (i.e., $\theta = 0$). It follows that

\[
d = (\mathbb{E}[U^A(W^A_T(0,0))], \mathbb{E}[U^B(W^B_T(0,0))]).
\]

It is obvious that $d \in S$. For now, we assume that there exists $s = (s_1, s_2)$ in $S$ such that $s_i > d_i$ for $i = 1, 2$. In Section 6, we will discuss under what conditions this assumption holds.

We can then use Nash’s bargaining solution (see Theorem 1) to find the price $P$ and the trading quantity $\theta$ upon agreement between the two agents:

\[
\arg\max_{(P, \theta)} \quad (\mathbb{E}[U^A(W^A_T(P,\theta))] - \mathbb{E}[U^A(W^A_T(0,0))]) \\
\times (\mathbb{E}[U^B(W^B_T(P,\theta))] - \mathbb{E}[U^B(W^B_T(0,0))])
\]

subject to

\[
\mathbb{E}[U^A(W^A_T(P,\theta))] - \mathbb{E}[U^A(W^A_T(0,0))] \geq 0
\]
\[
\mathbb{E}[U^B(W^B_T(P,\theta))] - \mathbb{E}[U^B(W^B_T(0,0))] \geq 0
\]
\[
\theta \geq 0
\]
\[
P > 0
\]

The above may be viewed as a non-linear constrained optimization problem. The solution, which we denote by $(\hat{P}, \hat{\theta})$, can be solved numerically by technical software such as MATLAB. The trading of $\hat{\theta}$ units of the security at a price of $\hat{P}$ makes both agents better off relative to the situation when there is no trade, and maximizes the product of the agents’ expected utility gains.

### 4 Modeling the Trade in a Competitive Market

To understand the impact of bargaining on the trade, we need to examine how the trade will turn out to be if the market of mortality-linked securities is perfectly competitive. In what follows, we model the trade described in Section 2 in a perfectly competitive market.

---

\(^6\)We assume exponential utility functions, which are concave. The use of concave utility functions implies that $S$ is a convex set (see, e.g. Kihlstrom and Roth, 1982; Boonen et al., 2012).
In a perfectly competitive market, all participants are price takers. Given a price, each participant determines her supply or demand of the security on the basis of a certain criterion. We assume that the agents will choose a supply or demand of the security that will maximize their expected terminal utilities. This means that, at a given price $P$, Agent A is willing to purchase 

$$\theta^A = \operatorname{argsup}_\theta \mathbb{E}[U^A(W^A_T(P, \theta))]$$

units of the security, while Agent B is willing to sell 

$$\theta^B = \operatorname{argsup}_\theta \mathbb{E}[U^B(W^B_T(P, \theta))]$$

units of the security. At equilibrium when the market clears, we have $\theta^A = \theta^B$.

Zhou et al. (2011, 2013) postulate this trade as a Walrasian auction, and numerically obtain the equilibrium price and trading quantity by gradually adjusting the price until the excess demand or supply becomes zero, that is, $|\theta^A - \theta^B| = 0$. In what follows, we push their results further by deriving an analytic relation between the price and trading quantity at the competitive equilibrium.

**Proposition 1.** Suppose that Agents A and B have exponential utility functions with absolute risk aversion parameters $k^A$ and $k^B$, respectively. The competitive equilibrium $(P^*, \theta^*)$ satisfies 

$$P^* = \frac{\mathbb{E}[e^{-k^A\theta^*G+k^AF}]}{e^{rT} \mathbb{E}[e^{-k^A\theta^*G+k^AF}]} = \frac{\mathbb{E}[e^{k^B\theta^*G}]}{e^{rT} \mathbb{E}[e^{k^B\theta^*G}]}.$$ 

**Proof.** It follows from equation (1) that 

$$\frac{\partial}{\partial \theta|_{\theta = \theta^A}} \mathbb{E}[U^A(W^A_T(P, \theta))] = 0, \quad \frac{\partial^2}{\partial \theta^2|_{\theta = \theta^A}} \mathbb{E}[U^A(W^A_T(P, \theta))] < 0.$$ 

Recall that the terminal wealth of Agent A is $W^A_T(P, \theta) = W^A_0e^{rT} + \theta(G - e^{rT}P) - F$. If Agent A has an exponential utility function with risk aversion parameter $k^A$, then the first order condition can be written as 

$$\mathbb{E}\left[(G - e^{rT}P)e^{-k^A\theta^*G+k^AF}\right] = 0,$$

which implies 

$$P = \frac{\mathbb{E}[e^{-k^A\theta^*G+k^AF}]}{e^{rT} \mathbb{E}[e^{-k^A\theta^*G+k^AF}]}.$$
The second order condition is easy to verify.

On the other hand, it follows from equation (2) that
\[
\frac{\partial}{\partial \theta} \mathbb{E} \left[ U^B(W_T^B(P, \theta)) \right] = 0, \quad \frac{\partial^2}{\partial \theta^2} \mathbb{E} \left[ U^B(W_T^B(P, \theta)) \right] < 0.
\]

Recall that the terminal wealth of Agent B is 
\[
W_T^B(P, \theta) = W_0^B e^{rT} - \theta^B(G - e^{rT}P).
\]
If Agent B has an exponential utility function with risk aversion parameter \( k_B \), then the first order condition can be written as
\[
\mathbb{E} \left[ (G - e^{rT}P)e^{k_B \theta^B G} \right] = 0,
\]
which implies
\[
P = \frac{\mathbb{E}[e^{k_B \theta^B G}]}{e^{rT} \mathbb{E}[e^{k_B \theta^B G}]}.
\]

The second order condition for Agent B is also easy to verify.

At equilibrium, \( \theta^A = \theta^B \). Letting \( \theta^A = \theta^B = \theta^* \), the result then follows.

To find the competitive equilibrium, we first can solve the second part of equation (3) for \( \theta^* \) numerically, and then substitute \( \theta^* \) back to the first part of equation (3) to obtain \( P^* \). Note that equation (3) may not have a solution, which happens when there is no trade between the two agents.

5 A Numerical Illustration

We now illustrate the models in Sections 3 and 4 with a hypothetical trade between two agents. Agent A is a pension plan sponsor, whose liability payouts are higher if the plan members live longer. It is assumed that the plan members’ future mortality experience is the same as that of the U.K. insured lives. On the other hand, Agent B is an investment bank who is interested in selling a mortality-linked security to earn a risk premium. The payoff from the security that Agent B sells is positively related to the survivorship of English and Welsh male population.\(^7\)

\(^7\)We have seen in real life securities that are linked to the future mortality of English and Welsh population. Examples include the q-forward contracts traded between J.P. Morgan and Lucida in 2008 and the longevity bond announced by BNP Paribas and the European Investment Bank in 2004.
In the first subsection, we present a stochastic mortality model, from which we can generate future mortality rates of the two populations involved in the trade. This is followed in the second subsection by a detailed description of the security being traded. The third subsection shows the results that are computed using the baseline assumptions. Finally, the illustration is concluded in the fourth subsection in which some sensitivity tests are presented.

5.1 Mortality Data and Model

The mortality data for English and Welsh male population (the population to which the security is linked) are obtained from the Human Mortality Database (2013), while those for the U.K. insured lives population (a proxy for the hedger’s population) are obtained from the Continuous Mortality Investigation Bureau of the Institute and Faculty of Actuaries. The data comprise of death and exposure counts over the period of 1947 to 2005 and over the age range of 60 to 89. For brevity, in the following discussion, we denote English and Welsh male population by population 1 and the U.K. insured lives population by population 2.

We model future mortality by the two-population mortality model proposed by Cairns et al. (2011). It is built from two classical age-period-cohort models, one for each population:

\[
\ln(m^{(i)}_{x,t}) = \beta^{(i)}_x + \frac{1}{n_a}\kappa^{(i)}_t + \frac{1}{n_a}\gamma^{(i)}_{t-x}, \quad i = 1, 2, \quad (4)
\]

where \(m^{(i)}_{x,t}\) is the central death rate at age \(x\) and in year \(t\) for population \(i\), and \(n_a\) is a constant which equals the total number of ages in the sample age range. In this model, parameter \(\beta^{(i)}_x\) captures population \(i\)'s average level of mortality at age \(x\). For each population, the dynamics of mortality are captured by two indexes: \(\kappa^{(i)}_t\), which measures the variation of mortality over different calendar years, and \(\gamma^{(i)}_{t-x}\), which measures the variation of mortality over different years of birth.\(^8\) Projections of future mortality can be made by extrapolating these two indexes.

Following Cairns et al. (2011), we model \(\kappa^{(1)}_t\) using a random walk with drift,

\[
\kappa^{(1)}_t = \mu_\kappa + \kappa^{(1)}_{t-1} + Z_a(t),
\]

\(^8\)Note that \(t - x\) is the year of birth for an individual who is age \(x\) in year \(t\).
where $Z_\kappa(t)$ is a zero-mean normal random variable and $\mu_\kappa$ is a constant, and model $\gamma_{t-x}^{(1)}$ using a second order autoregressive model with a deterministic trend,

$$\gamma_{t-x}^{(1)} = \mu_\gamma + \phi_{\gamma,1}\gamma_{t-1}^{(1)} + \phi_{\gamma,2}\gamma_{t-2}^{(1)} + \delta_\gamma c + Z_\gamma(c),$$

where $c = t - x$, $Z_\gamma(c)$ is a zero-mean normal random variable, and $\mu_\gamma$, $\phi_{\gamma,1}$, $\phi_{\gamma,2}$ and $\delta_\gamma$ are constants.

Population 1, the larger population, is assumed to be dominant, so that in the long run, $\kappa_t^{(2)}$ and $\gamma_c^{(2)}$ evolve in the same way as $\kappa_t^{(1)}$ and $\gamma_c^{(1)}$ do. We permit short-term random deviations between the indexes for the two populations. Following Cairns et al. (2011), we model $\Delta \kappa(t) = \kappa_t^{(1)} - \kappa_t^{(2)}$ with a first order autoregressive process,

$$\Delta \kappa(t) = \mu_{\Delta \kappa} + \phi_{\Delta \kappa}\Delta \kappa(t-1) + Z_{\Delta \kappa}(t),$$

where $Z_{\Delta \kappa}(t)$ is a zero-mean random variable, and $\mu_{\Delta \kappa}$ and $\phi_{\Delta \kappa}$ are constants. Moreover, we model $\Delta \gamma(c) = \gamma_c^{(1)} - \gamma_c^{(2)}$ with a second order autoregressive process,

$$\Delta \gamma(c) = \mu_{\Delta \gamma} + \phi_{\Delta \gamma,1}\Delta \gamma(c-1) + \phi_{\Delta \gamma,2}\Delta \gamma(c-2) + Z_{\Delta \gamma}(c),$$

where $Z_{\Delta \gamma}(c)$ is a zero-mean random variable, and $\mu_{\Delta \gamma}$, $\phi_{\Delta \gamma,1}$ and $\phi_{\Delta \gamma,2}$ are constants. The stationarity of these two processes implies that $\Delta \kappa(t)$ and $\Delta \gamma(c)$ will revert to their respective long-term means over the long run, thereby ensuring that the long-term dynamics of $\kappa_t^{(2)}$ and $\gamma_c^{(2)}$ will be the same as those for $\kappa_t^{(1)}$ and $\gamma_c^{(1)}$.

As indicated in the empirical results of Coughlan et al. (2011), $\kappa_t^{(1)}$ may be correlated with $\Delta \kappa(t)$, while $\gamma_c^{(1)}$ may be correlated with $\Delta \gamma(c)$. To take these potential correlations into account, $(Z_\kappa(t), Z_{\Delta \kappa}(t))$ and $(Z_\gamma(c), Z_{\Delta \gamma}(c))$ are treated as zero-mean bivariate normal random vectors, with variance-covariance matrices $V_\kappa$ and $V_\gamma$, respectively.

The model is estimated in two stages. In the first stage, we estimate parameters $\beta_{x}^{(i)}, \kappa_{t-x}^{(i)}$ and $\gamma_{t-x}^{(i)}$, $i = 1, 2$, $t = 1947, \ldots, 2005$, $x = 60, \ldots, 89$, in equation (4). The resulting estimates are shown graphically in Figure 2. In the second stage, we estimate parameters in the time-series processes for $\kappa_{t}^{(1)}, \gamma_{t}^{(1)}, \Delta \kappa(t)$ and $\Delta \gamma(c)$. The resulting estimates are displayed in the Table 1. We refer interested readers to Zhou et al. (2011) for details about the procedure for estimating the model parameters.

## 5.2 The Security Being Traded

In our illustration, we set time 0 to the beginning of year 2006, because 2005 is the last year for which the mortality data are available.
Figure 2: Estimates of parameters in equation (4).

Agent A manages a pension plan, which contains 1,500 pensioners at time 0. The age distribution of the pensioners is displayed in Figure 3. For simplicity, we assume that the pension plan is closed, that is, there will be no new entrants to the plan. Agent A has to pay each pensioner an amount of $0.01 at the end of each year until he dies or reaches age 90, whichever is earlier. It follows that the index $Q_L^t$ is the vector of the realized survival rates for the pensioners at time $t$.

To mitigate her longevity risk exposure, Agent A purchases a 25-year annuity bond (a bond without principal repayment) from Agent B at time 0. This bond is highly similar to the longevity bond that was jointly announced by the European Investment Bank and BNP Paribas in November 2004. Specifically, the index to which this bond is linked is $Q^H_t = \prod_{i=1}^t (1 - m_{64+t,2005+t}^{(1)})$, the (approximate) realized survival rate for the cohort of English and Welsh males who are age 65 at time 0 (the beginning of year 2006). At $t = 1, 2, \ldots, 25$, the bond makes a coupon payment of $Q^H_t$.

If the realized survival rates are higher than expected, then the bond will make larger coupon payments, which can then be used to offset the correspondingly higher pension payouts.
<table>
<thead>
<tr>
<th>$\mu_\kappa$</th>
<th>$\mu_{\Delta_\kappa}$</th>
<th>$\phi_{\Delta_\kappa}$</th>
<th>$V_\kappa(1,1)$</th>
<th>$V_\kappa(1,2)$</th>
<th>$V_\kappa(2,2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.3564</td>
<td>-0.0295</td>
<td>0.9692</td>
<td>1.1287</td>
<td>0.2543</td>
<td>0.4053</td>
</tr>
<tr>
<td>$\mu_\gamma$</td>
<td>$\mu_{\Delta_\gamma}$</td>
<td>$\phi_{\Delta_\gamma,1}$</td>
<td>$\phi_{\Delta_\gamma,2}$</td>
<td>$\phi_{\Delta_\gamma,1}$</td>
<td>$\phi_{\Delta_\gamma,2}$</td>
</tr>
<tr>
<td>-0.0658</td>
<td>0.1156</td>
<td>0.6232</td>
<td>0.3430</td>
<td>0.2060</td>
<td>0.7177</td>
</tr>
<tr>
<td>$\delta_\gamma$</td>
<td>$V_\gamma(1,1)$</td>
<td>$V_\gamma(1,2)$</td>
<td>$V_\gamma(2,2)$</td>
<td>$V_\gamma(2,2)$</td>
<td>$V_\gamma(2,2)$</td>
</tr>
<tr>
<td>-0.0014</td>
<td>0.4844</td>
<td>0.1218</td>
<td>0.8280</td>
<td>0.8280</td>
<td>0.8280</td>
</tr>
</tbody>
</table>

Table 1: Estimated parameters in the time-series processes for $\kappa_t^{(1)}$, $\gamma_c^{(1)}$, $\Delta_\kappa(t)$ and $\Delta_\gamma(c)$. (We use $X(i,j)$ to denote the $(i,j)$th element in a matrix $X$.)

5.3 Results Based on the Baseline Assumptions

The baseline assumptions used in this illustration are as follows:

- The continuously compounded risk-free interest rate is $r = 0.01$.

- The absolute risk aversion parameters for Agents A and B are $k^A = 2.0$ and $k^B = 0.1$, respectively. A larger absolute risk aversion parameter means that the agent is more risk averse. It is reasonable to assume that $k^A > k^B$, because Agent A wants to reduce her longevity risk exposure while Agent B is willing to take on the risk in return for a risk premium.

- The initial wealth of Agents A is $W^A_0 = 186$, while that of Agent B is $W^B_0 = 0$. These values are chosen arbitrarily. It can be shown analytically that under the assumption of exponential utility functions, the estimates of the trading price $P$ and quantity $\theta$ do not depend on the agents’ initial wealth.

Table 2 summarizes the outcomes under the two different models for the trade. They are calculated by solving equation (3) and the constrained maximization in Section 3.3, respectively. The strictly positive utility gains indicate that both agents will benefit from the trade of the longevity bond, no matter if the market is competitive or not. In this example, the benefit to Agent A (the hedger) is higher if the market is competitive, whereas the opposite is true for the other agent.

According to the first axiomatic property in Section 3.1, Nash’s bargaining solution is Pareto optimal. It is well known that any outcome resulting from a competitive equilibrium must also be Pareto optimal. An outcome is said to be Pareto optimal if
there is no other outcome that makes every agent at least as well off and at least one agent strictly better off. Equivalently speaking, a Pareto optimal outcome cannot be improved without hurting at least one agent. Therefore, the permission of bargaining can only improve the utility gain of one agent, but not both. In this example, Agent B benefits.

In this example, the trading price under Nash’s bargaining solution is higher than that under the competitive equilibrium. This difference suggests that whether or not market participants are price takers does have an impact on the trading price. Given that the current Life Market is not even close to competitive, practitioners should be cautious when they interpret prices estimated from pricing methods that require the assumption of market competitiveness.

By contrast, in this example, the trading quantities under Nash’s bargaining solution and the competitive equilibrium are the same. This property, as we will revisit in Section 6, is not a coincidence, but always true provided that certain conditions are satisfied.
<table>
<thead>
<tr>
<th>Method</th>
<th>Competitive Equilibrium</th>
<th>Nash’s Bargaining Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trading Price</td>
<td>15.6184</td>
<td>16.2410</td>
</tr>
<tr>
<td>Utility Gain for A</td>
<td>3.4804</td>
<td>3.2627</td>
</tr>
<tr>
<td>Utility Gain for B</td>
<td>0.0614</td>
<td>0.4282</td>
</tr>
<tr>
<td>Nash Product</td>
<td>0.2137</td>
<td>1.3971</td>
</tr>
</tbody>
</table>

Table 2: Trading prices, trading quantities and utility values under Nash’s bargaining solution and the competitive equilibrium.

5.4 Sensitivity Tests

In the baseline calculations, we assumed specific values for the absolute risk aversion parameters \((k^A, k^B)\) and the continuously compounded risk-free interest rate \((r)\). We now examine how changes to these assumed values may affect the trading price and quantity.

In the first sensitivity test, we keep our baseline assumption on \(r\), but vary the values of \(k^A\) and \(k^B\). The estimated values of \(P\) and \(\theta\) under different assumptions on \(k^A\) and \(k^B\) are displayed in Table 3. For both models, if \(k^A\) is kept constant while \(k^B\) is raised, then the estimated value of \(P\) increases. This is because if Agent B is more risk adverse (i.e., has a higher risk aversion parameter), then she will demand a higher premium for accepting the same amount of risk, resulting in a higher trading price. Using similar arguments, we can explain the trend in \(P\) as \(k^A\) is reduced while \(k^B\) is kept constant.

In the second sensitivity test, we retain our baseline assumptions on \(k^A\) and \(k^B\), but vary the value of \(r\). The estimates of \(P\) and \(\theta\) under different assumed values of \(r\) are shown in Table 4. It is not surprising that the estimated trading price reduces as \(r\) increases, because at a higher interest rate, the present value of the cash flows arising from the longevity bonds are lower. The observed trend may also be attributed to the forces exerted by the agents. Specifically, if the interest rate is higher, then later year coupons, which are the most uncertain, will carry a smaller weight. Therefore, the longevity bond will contain less risk, and consequently Agent B will be willing to sell it at a lower price.
Table 3: Estimated trading prices and quantities when different absolute risk aversion parameters are assumed.

6 Conditions for Pareto Optimality

In this section, we discuss the conditions under which the trade between the two agents will be Pareto optimal. Since both Nash’s bargaining solution and the competitive equilibrium are Pareto optimal, knowing the conditions for Pareto optimality may give us some insights about the trading prices and quantities under the two market models.

Let us first introduce the following lemma, which is involved in the conditions for Pareto optimality.

**Lemma 1.** The equation

\[
\mathcal{H}(\theta) = \frac{\mathbb{E}[e^{k_B G}]}{\mathbb{E}[e^{k_B}]} - \frac{\mathbb{E}[e^{-k_A G}]}{\mathbb{E}[e^{-k_A + k_A F}]} = 0.
\]

(5)

has a unique non-zero solution if and only if \(\text{cov}(e^{k_A F}, G) > 0\).

We can then present the conditions for Pareto optimality.
Table 4: Estimated trading prices and quantities when different continuously compounded risk-free interest rates are assumed.

<table>
<thead>
<tr>
<th>r</th>
<th>Competitive Equilibrium</th>
<th>Nash’s Bargaining Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Price</td>
<td>Quantity</td>
</tr>
<tr>
<td>0.03</td>
<td>12.8130</td>
<td>6.7783</td>
</tr>
<tr>
<td>0.05</td>
<td>10.6890</td>
<td>7.3345</td>
</tr>
<tr>
<td>0.07</td>
<td>9.0559</td>
<td>7.8635</td>
</tr>
</tbody>
</table>

Proposition 2. Assume that Agents A and B have exponential utility functions with risk aversion parameters $k^A$ and $k^B$, respectively. When $\text{cov}(e^{k^AF}, G) \leq 0$, the outcome $(\tilde{P}, \tilde{\theta})$ is Pareto optimal if and only if $\tilde{\theta} = 0$. When $\text{cov}(e^{k^AF}, G) > 0$, the outcome $(\tilde{P}, \tilde{\theta})$ is Pareto optimal if and only if $H(\tilde{\theta}) = 0$.

Proofs for Lemma 1 and Proposition 2 are provided in the Appendix.

Proposition 2 states explicitly the conditions under which an outcome $(P, \theta)$ is Pareto optimal. Depending on the sign of $\text{cov}(e^{k^AF}, G)$, Pareto optimality is attained either when there is no trade or when $\tilde{\theta}$ units of the security is traded, where $\tilde{\theta}$ is the solution to the equation $H(\tilde{\theta}) = 0$.

Suppose that a trade between the two agents occurs. Then, according to Lemma 1, there is one and only one trading quantity that would lead to Pareto optimality. Because both Nash’s bargaining solution and the competitive equilibrium are Pareto optimal, they must yield the same trading quantity. This fact explains why the estimated trading quantities in Table 2 are identical. Under our set-up, however, the Pareto optimality conditions do not depend on the trading price $P$. This means that even though the trading quantities in Nash’s bargaining solution and the competitive equilibrium must be the same, the trading prices may not.

Proposition 2 says that whether or not a trade will occur depends entirely on the sign of $\text{cov}(e^{k^AF}, G)$, the covariance between the random variables $e^{k^AF}$ and $G$, where $F$ and $G$ are the accumulated values of the cash flows arising from the liability being hedged and the security under consideration, respectively. This condition is highly intuitive. In particular, if $\text{cov}(e^{k^AF}, G) \leq 0$, which implies $e^{k^AF}$ and $G$ are not positively correlated with each other, then Agent A has no reason to purchase the security, as holding the security will increase but not reduce her risk exposure. This
is completely in line with Proposition 2, which says that if \( \text{cov}(e^{k_A F}, G) \leq 0 \), then no trade will occur.

Proposition 2 also provides with us an alternative way to obtain Nash’s bargaining solution for the trade under consideration. Specifically, we can first find \( \tilde{\theta} \) that solves the equation \( \mathcal{H}(\theta) = 0 \). Because \( \tilde{\theta} \) is the only trading quantity that leads to Pareto optimality, the trading quantity in Nash’s bargaining solution must be \( \tilde{\theta} \). We can then obtain the trading price in Nash’s bargaining solution by maximizing the Nash product evaluated at \( \theta = \tilde{\theta} \). This boils down to solving the first order condition,

\[
-k^A \mathbb{E}[e^{-k^A \tilde{\theta}(G-e^{rT}P)+k^A F}] + k^B \mathbb{E}[e^{k^A F}] \mathbb{E}[e^{k^B \tilde{\theta}(G-e^{rT}P)}] \\
+ (k^A - k^B) \mathbb{E}[e^{-k^A \tilde{\theta}(G-e^{rT}P)+k^A F}] \mathbb{E}[e^{k^B \tilde{\theta}(G-e^{rT}P)}] = 0,
\]

for \( P \).

Nash’s bargaining solution requires the assumption that there exists \( s = (s_1, s_2) \) in the utility possible set \( S \) such that \( s_i > d_i \) for \( i = 1, 2 \). In Section 3.3, we made this assumption without specifying when it is satisfied and when it is not. We now show that the satisfaction of this assumption is dependent on the sign of \( \text{cov}(e^{k_A F}, G) \).

**Proposition 3.** Assume that Agents A and B have exponential utility functions with risk aversion parameters \( k^A \) and \( k^B \), respectively. A necessary and sufficient condition for satisfying the assumption that there exists \( s = (s_1, s_2) \) in \( S \) such that \( s_i > d_i \) for \( i = 1, 2 \) is \( \text{cov}(e^{k_A F}, G) > 0 \).

**Proof.** First, we prove the necessity. The existence of \( s = (s_1, s_2) \) in \( S \) such that \( s_i > d_i \) for \( i = 1, 2 \) implies that \( d = (d_1, d_2) \) is not a Pareto optimal outcome, or equivalently speaking, \( \theta = 0 \) does not lead to Pareto optimality. According to Proposition 2, we must have \( \text{cov}(e^{k_A F}, G) > 0 \). Therefore, \( \text{cov}(e^{k_A F}, G) > 0 \) is a necessary condition.

Next we prove the sufficiency. If \( \text{cov}(e^{k_A F}, G) > 0 \), then according to Lemma 1 and Proposition 2, any outcome with \( \theta = 0 \) is not Pareto optimal. It follows that we can always find a price, say \( P_0 \), and a trading quantity, say \( \theta_0 \), that leads to expected utility payoffs \( (y_1, y_2) \in S \), where \( y_1 \geq d_1 \), \( y_2 \geq d_2 \) and at least one of the two equalities does not hold.

If \( y_1 > d_1 \) and \( y_2 > d_2 \), then obviously there exists \( s \in S \) such that \( s_i > d_i \) for \( i = 1, 2 \).
If \( y_1 > d_1 \) and \( y_2 = d_2 \), we can write \( y_1 = d_1 + \epsilon \) for some \( \epsilon > 0 \). For a fixed \( \theta \), Agent A (the buyer of the security) has a higher expected utility payoff if she purchases the security for a lower price, while Agent B (the seller of the security) has a higher expected utility payoff if she sells the security at a higher price. More specifically, assuming exponential utility functions, \( s_1 \) is a continuously decreasing function of \( P \), while \( s_2 \) is a continuously increasing function of \( P \). It follows that we can always find another price \( P_1 > P_0 \) that leads to expected utility payoffs \( (y_1', y_2') \), where \( d_1 < y_1' < d_1 + \epsilon \) and \( y_2' > d_2 \). Therefore, there exists \( s \in S \) such that \( s_i > d_i \) for \( i = 1, 2 \). Using a similar argument, we can verify the existence of such a point in \( S \) when \( y_1 = d_1 \) and \( y_2 > d_2 \).

\[ \Box \]

Proposition 3 has significant implications. It says that as long as the security under consideration is an effective hedging instrument (in the sense that \( \text{cov}(e^{kA_F}, G) \) takes the desired sign), the bargain between the two agents will always lead to a trade of the security. The trade will benefit both agents, as it will bring expected utility payoffs that are strictly greater than those when there is no trade.

7 Concluding Remarks

At present, there is no liquid secondary market for longevity securities. Most trades in today’s Life Market are outcomes of one-on-one negotiations between hedgers and investors. The model we proposed, which is largely based on a two-player bargaining game, appears to have a strong resemblance to the reality of the current market.

By solving the game with Nash’s bargaining solution, we can obtain estimates of the trading price and quantity. Participants in the Life Market can therefore use our model for pricing purposes. Relative to the existing pricing methods, this new pricing method has two distinct advantages. First, it does not require the competitive market assumption, which is not satisfied during the current stage of market development. Second, it does not need any market price data, which are not readily available from the public domain.

To obtain a deeper understanding about how bargaining may affect the trading of longevity securities, we compared the outcomes resulting from Nash’s bargaining solution and the competitive equilibrium. Assuming the hedger and investor have
exponential utility functions, we derived the following two conclusions. First, under both set-ups, a trade would occur if the longevity security is an effective hedging instrument, in the sense that $\text{cov}(e^{kAF}, G)$ takes the desired sign. Second, provided that a trade occurs, the two set-ups would result in the same trading quantity but different trading prices.

Participants of the Life Market see standardization as a long-term goal. In 2010, they established the Life and Longevity Markets Association (LLMA), whose primary objective is to promote the development of a liquid traded market in longevity risk. Given their efforts, the Life Market could be closer to a competitive market in future. The comparison between the outcomes resulting from Nash’s bargaining solution and the competitive equilibrium may therefore help practitioners to better predict how trades in the Life Market may change as the market continues to develop.

Blake et al. (2013) pointed out that one major challenge to the development of the Life Market is increased regulations, an aftermath of the Global Financial Crisis. Due to the increased regulations restricting their risk-taking activities, investment banks may now find it less attractive to take on longevity risk. The results of the sensitivity test performed in Section 5.4 echoes this point. Specifically, it was found that as $k^B$, the investor’s absolute risk aversion, increases, the trading quantity becomes smaller and the longevity security becomes more costly.

In our illustration, we assumed arbitrary values for the risk aversion parameters. In real life, when the identities of the hedger and investor are known, we can estimate their risk aversion parameters, using methods such as the one proposed by Cox et al. (2010). Other information required may also be found from, for example, the annual reports of the parties involved in the trade. In future research, it would be interesting to conduct a case study of a publicly known deal, and compare the results from our model with the actual outcome.

Acknowledgments

The authors acknowledge the financial support from the Global Risk Institute, the Center of Actuarial Excellence Program of the Society of Actuaries and the Natural Science and Engineering Research Council of Canada.
References


Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute of Demographic Research (Germany). Available at www.mortality.org or www.humanmortality.de (data downloaded on 1 February 2013).


Appendix – Deriving the Conditions for Pareto Optimality

Let us first prove Lemma 1.

Proof. We first show that $\mathcal{H}(\theta)$ is a strictly increasing function of $\theta$. Differentiating the first term of $\mathcal{H}(\theta)$ with respect to $\theta$, we have

$$
\frac{\partial}{\partial \theta} \left( \frac{\mathbb{E}[e^{kB\theta G}G]}{\mathbb{E}[e^{kB\theta G}]} \right) = k^B \frac{\mathbb{E}[e^{kB\theta G}G^2] \mathbb{E}[e^{kB\theta G}] - \left( \mathbb{E}[e^{kB\theta G}] \right)^2}{\left( \mathbb{E}[e^{kB\theta G}] \right)^2}.
$$

Using Hölder’s inequality, we have

$$
\mathbb{E}[e^{kB\theta G}G^2] \mathbb{E}[e^{kB\theta G}] - \left( \mathbb{E}[e^{kB\theta G}] \right)^2 \geq 0.
$$

The equality holds if and only if $e^{kB\theta G}G^2 = e^{kB\theta G}$ almost everywhere. This condition, which is equivalent to $G^2 = 1$ almost everywhere, is obviously not satisfied here. As a result, 

$$
\frac{\partial}{\partial \theta} \left( \frac{\mathbb{E}[e^{kB\theta G}G]}{\mathbb{E}[e^{kB\theta G}]} \right) > 0.
$$

Similarly, we can prove that

$$
\frac{\partial}{\partial \theta} \left( \frac{-\mathbb{E}[e^{-kA\theta G+kA F}G]}{\mathbb{E}[e^{-kA\theta G+kA F}]} \right) > 0.
$$

As a result, $\frac{\partial}{\partial \theta} \mathcal{H}(\theta) > 0$ for all $\theta$. Since $\mathcal{H}(\theta)$ is a strictly increasing function of $\theta$, the solution to equation (5) is unique if it exists.

Since $\theta \in [0, +\infty)$, equation (5) has a unique non-zero solution if and only if
1. $\mathcal{H}(\theta) < 0$, when $\theta = 0$;

2. $\mathcal{H}(\theta) \geq 0$, when $\theta \to +\infty$.

When $\theta = 0$,

$$
\mathcal{H}(0) = \frac{\mathbb{E}[e^{kAF}]\mathbb{E}[G] - \mathbb{E}[e^{kAF}G]}{\mathbb{E}[e^{kAF}]},
$$

which is negative if and only if $\text{cov}(e^{kAF}, G) = \mathbb{E}[e^{kAF}G] - \mathbb{E}[e^{kAF}]\mathbb{E}[G] > 0$.

Condition 2 is satisfied if

$$
\lim_{\theta \to +\infty} \frac{\mathbb{E}[e^{-kA\theta G + kAF}G]}{\mathbb{E}[e^{-kA\theta G + kAF}]} = \inf \{G\},
$$

and

$$
\lim_{\theta \to +\infty} \frac{\mathbb{E}[e^{kB\theta G}]}{\mathbb{E}[e^{kB\theta}]} = \sup \{G\}.
$$

For brevity, we let $M = \sup \{G\}$ and $N = \inf \{G\}$. Since $G \geq 0$, we have $0 \leq N < +\infty$. 

28
We now prove equation (6). For any $\epsilon > 0$, fix $0 < \delta < \frac{\epsilon}{2}$. We have

\[
\left| \mathbb{E} \left[ e^{-kA\theta G + kAF} G \right] - \mathcal{N} \right| = \frac{\mathbb{E} \left[ e^{-kA\theta G + kAF} | G - \mathcal{N} \|_{\mathbb{P}(G-N)\leq\delta} \right] + \mathbb{E} \left[ e^{-kA\theta G + kAF} | G - \mathcal{N} \|_{\mathbb{P}(G-N)>\delta} \right]}{\mathbb{E} \left[ e^{-kA\theta G + kAF} \right]}
\]

\[
\leq \delta + \frac{\mathbb{E} \left[ e^{-kA\theta G + kAF} | G - \mathcal{N} \|_{\mathbb{P}(G-N)>\delta} \right]}{\mathbb{E} \left[ e^{-kA\theta G + kAF} \|_{\mathbb{P}(G-N)\leq\delta} \right] + \mathbb{E} \left[ e^{-kA\theta G + kAF} \|_{\mathbb{P}(G-N)>\delta} \right]}
\]

\[
\leq \delta + \frac{\mathbb{E} \left[ e^{-kA\theta (N+\delta) + kAF} | G - \mathcal{N} \|_{\mathbb{P}(G-N)\leq\delta} \right]}{\mathbb{E} \left[ e^{-kA\theta (N+\delta) + kAF} \|_{\mathbb{P}(G-N)\leq\delta} \right] + \mathbb{E} \left[ e^{-kA\theta (N+\delta) + kAF} \|_{\mathbb{P}(G-N)>\delta} \right]}
\]

\[
< \delta + \frac{\epsilon^{kAF} \|_{\mathbb{P}(G-N)\leq\delta}}{\mathbb{E} \left[ e^{kAF} \|_{\mathbb{P}(G-N)\leq\delta} \right]}
\]

where $\mathbb{I}_A$ is the indicator function for event $A$. When $\theta \geq -\frac{2}{k\alpha} \ln \mathbb{E} \left[ e^{kAF} \|_{\mathbb{P}(G-N)<\delta} \right]$, we have

\[
\left| \mathbb{E} \left[ e^{-kA\theta G + kAF} G \right] - \mathcal{N} \right| < \epsilon
\]

for any $\epsilon > 0$. Therefore, equation (6) holds.

We then prove equation (7). Suppose that $M < +\infty$. For any $\epsilon > 0$, fix $0 < \delta < \frac{\epsilon}{2}$.
We have
\[
\left| \frac{\mathbb{E}\left[ e^{k \theta G} G \right]}{\mathbb{E}\left[ e^{k \theta G} \right]} - M \right| = \mathbb{E}\left[ e^{k \theta G} \right] - M
\]
\[
= \mathbb{E}\left[ e^{k \theta G} |G - M| \right]
\]
\[
\leq \delta + \frac{\mathbb{E}\left[ e^{k \theta G} |G - M| \right]}{\mathbb{E}\left[ e^{k \theta G} \right]}
\]
\[
= \delta + \frac{\mathbb{E}\left[ e^{k \theta G} |G - M| \right]}{\mathbb{E}\left[ e^{k \theta G} \right]}
\]
\[
\leq \delta + \frac{\mathbb{E}\left[ e^{k \theta (M - \delta)} |G - M| \right]}{\mathbb{E}\left[ e^{k \theta (M - \frac{\delta}{2})} \right]}
\]
\[
< \delta + \frac{\epsilon}{2} + \frac{\mathbb{E}\left[ e^{-k \theta \frac{\delta}{2}} \right]}{\mathbb{E}\left[ [G - M] \right]}.
\]

When \( \theta \geq -\frac{2}{k^2 \delta} \ln \frac{\epsilon}{2\mathbb{E}[|G - M|]} \), we have
\[
\left| \frac{\mathbb{E}\left[ e^{k \theta G} G \right]}{\mathbb{E}\left[ e^{k \theta G} \right]} - M \right| < \epsilon
\]
for all \( \epsilon > 0 \). Therefore, equation (7) holds when \( M < +\infty \).
Suppose that $M = +\infty$. For any $\delta > 0$, we have
\[
\mathbb{E}\left[e^{k_B \theta G}(G - \delta)\right] = \mathbb{E}\left[e^{k_B \theta G}(G - \delta)\mathbb{I}_{G \leq \delta}\right] + \mathbb{E}\left[e^{k_B \theta G}(G - \delta)\mathbb{I}_{G > \delta}\right] \\
> \mathbb{E}\left[e^{k_B \theta G}(G - \delta)\mathbb{I}_{G \leq \delta}\right] + \mathbb{E}\left[e^{k_B \theta G}(G - \delta)\mathbb{I}_{G > \delta}\right] \\
> -\delta \mathbb{E}\left[e^{k_B \theta \delta} \mathbb{I}_{G \leq \delta}\right] + \delta \mathbb{E}\left[e^{2k_B \theta \delta} \mathbb{I}_{G > \delta}\right] \\
= \delta e^{k_B \theta \delta} \left(-\mathbb{E}\left[\mathbb{I}_{G \leq \delta}\right] + e^{k_B \theta \delta} \mathbb{E}\left[\mathbb{I}_{G > \delta}\right]\right).
\]
When $\theta \geq \frac{\ln \mathbb{E}[\mathbb{I}_{G \leq \delta}] - \ln \mathbb{E}[\mathbb{I}_{G > \delta}]}{k_B \delta}$, we have $\mathbb{E}\left[e^{k_B \theta G}(G - \delta)\right] > 0$ and hence
\[
\frac{\mathbb{E}\left[e^{k_B \theta G} G\right]}{\mathbb{E}\left[e^{k_B \theta G}\right]} - \delta > 0
\]
for any $\delta > 0$. Therefore, equation (7) also holds when $M = +\infty$. We conclude that equation (7) holds in general.

We then prove Proposition 2, which states the conditions for Pareto optimality.

**Proof.** In the trade of the security under consideration, an outcome $(\hat{P}, \hat{\theta})$ is Pareto optimal if there does not exist any pair of $(P', \theta')$ that satisfy the following conditions:

1. $\mathbb{E}[U^A(W_T^A(P', \theta'))] \geq \mathbb{E}[U^A(W_T^A(\hat{P}, \hat{\theta}))]$;
2. $\mathbb{E}[U^B(W_T^B(P', \theta'))] \geq \mathbb{E}[U^B(W_T^B(\hat{P}, \hat{\theta}))]$;
3. one of the above two inequalities is strict.

Condition (1) can be rewritten as follows:
\[
\frac{\mathbb{E}[U^A(W_T^A(P', \theta'))]}{\mathbb{E}[e^{-k_A \theta' G + k_A F}]} \geq \frac{\mathbb{E}[U^A(W_T^A(\hat{P}, \hat{\theta}))]}{\mathbb{E}[e^{-k_A \hat{\theta} G + k_A F}]} \\
\frac{e^{k_A e^{T}(P' \theta' - \hat{P} \hat{\theta})}}{\mathbb{E}[e^{-k_A \hat{\theta} G + k_A F}]} \leq \frac{\mathbb{E}[e^{k_A \theta' G + k_A F}]}{\mathbb{E}[e^{-k_A \theta' G + k_A F}]} \\
P' \theta' - \hat{P} \hat{\theta} \leq \frac{\ln \mathbb{E}[e^{-k_A \hat{\theta} G + k_A F}] - \ln \mathbb{E}[e^{-k_A \theta' G + k_A F}]}{k_A e^{T}}.
\]
Condition (2) can be rewritten as follows:

\[
E[U^B(W^B_T(P', \theta'))] \geq E[U^B(W^B_T(\tilde{P}, \tilde{\theta}))] \\
E[e^{k^B \theta' G} e^{-k^B P' \theta' e^{rT}}] \leq \frac{E[e^{k^B \tilde{\theta} G}]}{E[e^{k^B \theta' G}]} \\
e^{-k^B e^{rT}(P' \theta' - \tilde{P} \tilde{\theta})} \geq \frac{\ln E[e^{k^B \theta' G}]}{k^B e^{rT}}.
\]

There does not exist any \((P', \theta')\) meeting the three conditions above if and only if \((\tilde{P}, \tilde{\theta})\) satisfies one of the following conditions:

i. \[\frac{\ln E[e^{k^B \theta' G}]}{k^B e^{rT}} - \frac{\ln E[e^{k^B \tilde{\theta} G}]}{k^B e^{rT}} \geq \ln \frac{E[e^{-k^A \tilde{\theta} G + k^A F}]}{E[e^{-k^A \theta' G + k^A F}]} \text{ for any } (P', \theta');\]

ii. \[P' \theta' - \tilde{P} \tilde{\theta} > \ln \frac{E[e^{-k^A \tilde{\theta} G + k^A F}]}{E[e^{-k^A \theta' G + k^A F}]} \text{ for any } (P', \theta');\]

iii. \[P' \theta' - \tilde{P} \tilde{\theta} < \ln \frac{E[e^{-k^A \theta' G + k^A F}]}{E[e^{-k^A \tilde{\theta} G + k^A F}]} \text{ for any } (P', \theta').\]

Condition (ii) cannot be satisfied, because we can always choose an arbitrary \(\theta'\) and then pick a value of \(P'\) from the interval

\[
\left[\frac{\ln E[e^{-k^A \tilde{\theta} G + k^A F}]}{k^A e^{rT} \theta'} - \frac{\ln E[e^{-k^A \theta' G + k^A F}]}{k^A e^{rT} \theta'} + \tilde{P} \tilde{\theta}, +\infty\right).
\]

By using a similar argument, we can easily see that Condition (iii) cannot be satisfied, too. All then that remains is Condition (i).

Condition (i) is equivalent to

\[
\left(\ln \frac{E[e^{-k^A \theta' G + k^A F}]}{k^A} + \ln \frac{E[e^{k^B \theta' G}]}{k^B}\right) - \left(\ln \frac{E[e^{-k^A \tilde{\theta} G + k^A F}]}{k^A} + \ln \frac{E[e^{k^B \tilde{\theta} G}]}{k^B}\right) \geq 0,
\]

for all \(\theta'\), which is satisfied if and only if the function

\[
V(\theta) = \ln \frac{E[e^{-k^A \theta G + k^A F}]}{k^A} + \ln \frac{E[e^{k^B \theta G}]}{k^B}
\]

attains its minimum value at \(\tilde{\theta}\).

It is easy to show that \(\frac{\partial}{\partial \theta} V(\theta) = \mathcal{H}(\theta)\). Also, in the proof for Lemma 1, we showed that \(\frac{\partial^2}{\partial \theta^2} V(\theta) = \frac{\partial}{\partial \theta} \mathcal{H}(\theta) > 0\) for all \(\theta\).
When cov($e^{kAF}, G) > 0$, according to Lemma 1, $\frac{\partial}{\partial \theta} \mathcal{V}(\theta) = \mathcal{H}(\theta) = 0$ has a unique non-zero solution. It follows that $\mathcal{V}(\theta)$ is minimized at $\theta = \tilde{\theta}$ such that $\mathcal{H}(\tilde{\theta}) = 0$.

When cov($e^{kAF}, G) \leq 0$, we have

$$\frac{\partial}{\partial \theta} \mathcal{V}(\theta) = \frac{\mathbb{E}[e^{kAF}]\mathbb{E}[G] - \mathbb{E}[e^{kAF}G]}{\mathbb{E}[e^{kAF}]} = \frac{-\text{cov}(e^{kAF}, G)}{\mathbb{E}[e^{kAF}]} \geq 0.$$

Also, because $\frac{\partial}{\partial \theta_{\theta=0}} \mathcal{V}(\theta) \geq 0$ and $\frac{\partial^2}{\partial \theta^2} \mathcal{V}(\theta) > 0$ for all $\theta$, we have $\frac{\partial}{\partial \theta} \mathcal{V}(\theta) > 0$ for all $\theta \in (0, +\infty)$. It follows that $\mathcal{V}(\theta)$ is a strictly increasing function of $\theta$ for $\theta \in (0, +\infty)$, and that it attains minimum at $\theta = 0$. 

$\square$