Optimal Reinsurance with General Premium Principles

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Abstract

In this paper, we study two classes of optimal reinsurance models from the perspective of an insurer by minimizing its total risk exposure under the criteria of value at risk (VaR) and conditional value at risk (CVaR), assuming that the reinsurance premium principles satisfy three basic axioms: distribution invariance, risk loading and stop-loss ordering preserving. The proposed class of premium principles is quite general in the sense that it encompasses eight of the eleven commonly used premium principles listed in Young (2004). Under the additional assumption that both the insurer and the reinsurer are obligated to pay more for larger loss, we show that the layer reinsurance is quite robust in the sense that it is always optimal over our assumed risk measures and prescribed premium principles. We further use the Wang’s and Dutch premium principles to illustrate the applicability of our results by deriving explicitly the optimal parameters of the layer reinsurance. These two premium principles are chosen since in addition to satisfying the above three axioms, they exhibit increasing relative loading, a desirable property that is consistent with the market convention on some reinsurance pricing.

Key-words: Conditional value at risk; Value at risk; Layer reinsurance; Wang’s premium principle; Dutch premium principle.

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1 Introduction

Since the groundbreaking work by Borch (1960), the study of optimal reinsurance has drawn significant interest from both practicing actuaries and academics. From a practical point of view, reinsurance is an effective risk management tool for the insurers and hence they are constantly seeking for better reinsurance strategies. From a theoretical point of view, the quest for optimal reinsurance involves formulating an optimization problem and deriving its optimal solutions. These reasons have prompted academics and practitioners to propose many interesting and innovative reinsurance models.

Using variance of the retained loss as the criterion, Borch (1960) demonstrates that the stop-loss reinsurance is optimal under the assumption that the reinsurance premium is calculated according to the expected value premium principle. Arrow (1963), who also assumes the expected value premium principle, shows that the same stop-loss reinsurance maximizes the expected utility of the terminal wealth of a risk-averse insurer. Both of these results have been extended in a number of important directions. Just to name a few, Kaluszka (2001) generalizes Borch’s result by considering mean-variance premium principles. Young (1999) generalizes Arrow’s result by assuming Wang’s premium principle. By imposing the assumption that the reinsurance premium is calculated according to the maximal possible claims principle, Kaluszka and Okolewski (2008) show that the limited stop-loss and the truncated stop-loss are the optimal contracts under a number of criteria including the maximization of the expected utility, the stability and the survival probability of the cedent. Cai and Tan (2007), Cai et al. (2008) and Tan et al. (2011), still assuming the expected value premium principle, introduce two classes of optimal reinsurance models by minimizing the value at risk (VaR) and the conditional value at risk (CVaR) of the insurer’s total risk exposure. See also Chi and Tan (2010). Cheung (2010) extends the VaR-minimization reinsurance model in Cai et al. (2008) by considering Wang’s premium principle.

While the afore-mentioned papers have made significant contributions to the study of optimal reinsurance, most of them can be criticized for being too narrow and too restrictive in the sense that they address the optimal purchase of reinsurance under a very specific criterion and a particular prescribed premium principle. The chosen criterion can be one that maximizes the expected utility of the terminal wealth (as in Arrow (1963)) or one that minimizes an appropriate risk measure such as the variance of the retained loss, ruin probability, VaR, CVaR, and many others. In addition, there is also a proliferation of premium principles that one can choose from. See for example Young (2004) who catalogues eleven such premium principles. Because of the abundance of the reinsurance models, a cedent is even more concerned on how the optimal reinsurance would change from one reinsurance model to another. By the nature of the problem, the cedent needs to determine at least the following two assumptions when formulating an optimal reinsurance model: (i) the risk measure that is used to determine the optimality; (ii) and the premium principle that is used to calculate the reinsurance premium. Any change in either one or both of these assumptions could potentially lead to a very different optimal solution. Broadly speaking, if an optimal reinsurance is relatively stable with respect to the changes of either the risk measure or the premium principle or both, the reinsurance strategy is said to be robust. Naturally from a risk management point of view, it is prudent for the cedent to adopt a robust reinsurance strategy.
Some attempts have been made in addressing the robustness of optimal reinsurance. For example, Gajek and Zagrodny (2004) consider general symmetric and even asymmetric risk measures like the absolute deviation and the truncated variance of the retained loss and show that the limited stop-loss and change stop-stop treaties are optimal. In their study, the premium principle is assumed to be the standard deviation. To extend the work of Kaluszka (2001) which only deals with mean-variance premium principles, Kaluszka (2005) studies the optimal reinsurance problems under a class of convex principles of premium calculation (e.g. exponential, semi-deviation and semi-variance, Dutch, Wang’s, etc.). He concludes that the optimal reinsurance could be a combination of stop loss and quota share, limited stop loss, stop loss or more complicated forms. On the other hand, by using more general risk measures and also assuming convex premium principles, Balbás et al. (2009) show that the quota share reinsurance is barely optimal regardless of the objective functions, while a stop loss reinsurance treaty is much more easier to satisfy the optimality conditions. More recently, the same authors provide a further analysis on the stability of the optimal reinsurance (see Balbás et al. (2010)). They investigate the reinsurance models involving coherent and expectation bounded risk measures and under convex reinsurance premium principles, and conclude that the stop-loss contract is a very robust reinsurance treaty.

In this paper, we complement the study on robust reinsurance by considering two specific risk-measure based optimal reinsurance models and analyzing the robustness of the optimal reinsurance over a prescribed class of premium principles. In particular, we choose the reinsurance models under the criteria of VaR and CVaR risk measures as proposed by Cai and Tan (2007). The reason for taking these risk measures as the criteria for optimal reinsurance is due to their popularity among banks and insurance companies for quantifying risks and determining capital requirement. The class of premium principles, on the other hand, is assumed to satisfy the following three basic axioms: distribution invariance, risk loading and stop-loss ordering preserving. These axioms are weak but necessary. The first axiom is an implicit assumption in actuarial science; the second axiom is applied to guarantee the safety of the reinsurer according to the Strong Law of Large Number; and finally the last axiom is consistent with the utility framework for a risk-averse reinsurer and its importance is highlighted in Van Heerwaarden and Kaas (1992). We emphasize that there are many premium principles which satisfy these three axioms. In fact, among the eleven premium principles that are listed in Young (2004), eight of them belong to this particular class. These eight premium principles are net, expected value, exponential, proportional hazards, principle of equivalent utility, Wang’s, Swiss, and Dutch. In addition to the assumed risk measures and the class of premium principles, our study also confines both the ceded and retained loss functions to be increasing\(^1\), i.e. both the insurer and reinsurer are obligated to pay more for larger loss. This assumption is important as it partially precludes the moral hazard.

To proceed, it is useful to define the layer reinsurance treaty. The layer reinsurance treaty is of the form

\[
\min\{(x - a)_+, b\} = (x - a)_+ - (x - (a + b))_+, \quad a, b \geq 0,
\]

where \((x)_+ = \max\{x, 0\}\). Clearly, such a treaty is completely specified by two parameters: the deductible \(a\) and the upper limit \(b\). In particular, when \(a > 0\) and \(b = \infty\), we recover the classical

\(^1\)Throughout this paper, the terms “increasing” and “decreasing” mean “non-decreasing” and “non-increasing” respectively.
stop-loss reinsurance. If \( a > 0 \) and \( 0 < b < \infty \), then the above layer reinsurance is known as the limited stop-loss.

We now summarize the key contributions of this paper. First, our analysis reveals that the layer reinsurance treaty of the form (1.1) is always optimal. This result applies to both VaR and CVaR risk measures and the premium principles belonging to the above prescribed class. For this reason, the layer reinsurance (1.1) is considered quite robust. Furthermore, the optimal reinsurance problems simplify to finding an optimal deductible and an upper limit of the layer reinsurance. However, the explicit solutions of these parameters depend on the specific choice of reinsurance premium principles. We then elaborate our results by resorting to two premium principles known as Wang’s and Dutch. A key reason for considering these two premium principles is that in addition to satisfying the three basic axioms discussed above, they also exhibit the increasing relative loading property. This property is consistent with the market practice on property and casualty reinsurance pricing and its importance is highlighted in Venter (1991).

Second, our findings generalize some of the existing results. In particular, most of the VaR and CVaR based optimal reinsurance models have a common assumption that the reinsurance premium is calculated according to the expected value principle. See for example Cai et al. (2008), Chi and Tan (2010), and Tan et al. (2011). A notable exception is Cheung (2010) who extends the results of Cai et al. (2008) by proving that the quota-share reinsurance is optimal when the premium principle is changed from the expected value to the Wang’s. His result, however, is restrictive in the sense that it holds only under VaR-based reinsurance model with admissible set of increasing convex ceded loss functions. In contrast, the results obtained in this paper apply to not only both VaR and CVaR risk measures but also to the premium principles satisfying the above three basic axioms.

Third, an attempt has been made in Tan et al. (2009) to extend the study of the VaR and CVaR based optimal reinsurance models to other reinsurance premium principles. As a matter of fact, as many as seventeen of the premium principles are investigated in Tan et al. (2009). While sufficient and necessary conditions of the existence of non-trivial optimal reinsurance are established for many of these principles, they have to impose a very stringent assumption on the ceded loss function, under which it is either a quota share or a stop loss. In this paper, we complement their results by relaxing the severe restriction on ceded loss functions. By considering a wider admissible set of ceded loss functions, our results suggest that the quota-share reinsurance is hardly optimal, a finding that is consistent with Balbás et al. (2009). It is important to point out that even though the optimal reinsurance models studied in Balbás et al. (2009) are quite general, there are some distinctive differences between their study and ours. One is that the class of risk measure considered by Balbás et al. (2009) does not encompass VaR, which is one of the risk measures analyzed in this paper. The other is that a critical assumption in Balbás et al. (2009) is the convexity of the premium principles. While there are some overlaps between the convex premium principles and our proposed class of premium principles, it can be shown that under some additional assumptions, the premium principles such as the Swiss belong to our prescribed class of premium principles but they are not convex. For details, see Laeven and Goovaerts (2008).

Finally, we provide a new way of analyzing the optimal reinsurance problems. A common approach to solving an optimal reinsurance model includes two steps. First, given a fixed rein-
urance premium, an attempt is made to derive an optimal reinsurance strategy that meets the
optimality criterion. Next, the effect of the premium on the optimal reinsurance is further studied
to obtain the desired solution. See e.g. Kaluszka (2005), Bernard and Tian (2009), Chi and Tan
(2010) for details. Our approach in this paper, however, relies on a dual method. More precisely,
we first construct a layer reinsurance in such a way that it has the same VaR or CVaR of the
retained loss as that for a given admissible ceded loss function. We then verify that the layer
reinsurance is optimal by proving that the insurer has to pay less reinsurance premium when
choosing such a layer reinsurance.

The rest of this paper is organized as follows. Section 2 introduces the risk measure based
optimal reinsurance models. Section 3 shows that the layer reinsurance is always optimal un-
der both VaR and CVaR criteria and hence simplifies the question to solving two-parameter
optimization problems. Section 4 solves these two-parameter optimization problems when the
reinsurance premium is calculated according to Wang’s and Dutch principles. Finally, we provide
some concluding remarks in Section 5.

2 Preliminaries

Let \( X \) be the loss initially assumed by an insurer (i.e. in the absence of reinsurance). We assume \( X \)
is a non-negative random variable with cumulative distribution function (c.d.f.) \( F_X(x) = \mathbb{P}(X \leq x) \) and \( \mathbb{E}[X] < \infty \). The problem of optimal reinsurance is concerned with the optimal partitioning
of \( X \) into \( f(X) \) and \( R_f(X) \) where \( X = f(X) + R_f(X) \). Here \( f(X) \), satisfying \( 0 \leq f(X) \leq X \),
captures the portion of loss that is ceded to a reinsurer while \( R_f(X) \) is the residual loss retained
by the insurer (cedent). Consequently, \( f(x) \) is known as the ceded loss function while \( R_f(x) \) is
denoted as the retained loss function.

To partially exclude the moral hazard, we assume in this paper that both the insurer and
reinsurer are obligated to pay more for larger loss \( X \). In other words, both the ceded and
retained loss functions are assumed to be increasing. As a result, the set of admissible ceded loss
functions is defined as

\[
\mathcal{C} \triangleq \{0 \leq f(x) \leq x : \text{both } R_f(x) \text{ and } f(x) \text{ are increasing functions}\}. 
\] (2.1)

The ceded loss functions in \( \mathcal{C} \) have some nice properties. For example, as shown in Chi and Tan
(2010), the ceded loss function \( f(x) \in \mathcal{C} \) is increasing and Lipschitz continuous, i.e.

\[
0 \leq f(x_2) - f(x_1) \leq x_2 - x_1, \quad \forall 0 \leq x_1 \leq x_2,
\] (2.2)

and layer reinsurance treaty \( f(x) \) of the form (1.1) is in \( \mathcal{C} \). Moreover, \( \mathcal{C} \) strictly contains the set
of increasing convex ceded loss functions over which the optimal reinsurance problems are studied

When an insurer cedes part of its risk to a reinsurer under a typical reinsurance arrangement,
the insurer incurs an additional cost in the form of reinsurance premium which is payable to the
reinsurer. We use \( \pi \) to represent the reinsurance premium principle. Obviously, \( \pi \) is a non-negative
function, and the reinsurance premium is a function of the loss ceded to the reinsurer such that it
is given by \( \pi(f(X)) \). As noted in the Introduction, we consider a class of premium principles that
satisfies the following three axioms: distribution invariance, risk loading and stop-loss ordering preserving. More specifically, these three axioms are defined as

1. Distribution invariance: \( \pi(X) \) depends only on the cumulative distribution function \( F_X(x) \);

2. Risk loading: \( \pi(X) \geq E[X] \);

3. Stop-loss ordering preserving: if \( X \leq_{st} Y \), i.e.

\[
E[(X - d)_+] \leq E[(Y - d)_+], \quad \forall d \in \mathbb{R},
\]  

then \( \pi(X) \leq \pi(Y) \).

In the presence of the reinsurance, the risk exposure of the insurer is no longer captured by \( X \). The total risk exposure of the insurer is now given by the sum of the retained loss and the incurred reinsurance premium. Using \( T_f(X) \) to denote the total risk exposure of the insurer, we have

\[
T_f(X) = R_f(X) + \pi(f(X)).
\]  

Consequently, a reasonable criterion in determining an optimal ceded loss function can be formulated as one that minimizes an appropriately chosen risk measure on \( T_f(X) \). This is precisely the motivation that underlies the optimal reinsurance models proposed in Cai and Tan (2007).

Prompted by the popularity of the VaR and CVaR risk measures among banks and insurance companies for risk management and for setting regulatory capital, Cai and Tan (2007) exploit these risk measures explicitly in their formulation of the reinsurance models. The VaR and CVaR are defined as follows:

**Definition 2.1.** The VaR of a non-negative random variable \( X \) at a confidence level \( 1 - \alpha \) where \( 0 < \alpha < 1 \) is defined as

\[
VaR_\alpha(X) \triangleq \inf\{x \geq 0 : \mathbb{P}(X > x) \leq \alpha\}. \tag{2.5}
\]

Based upon the definition of VaR, the CVaR of \( X \) at a confidence level \( 1 - \alpha \) is defined as

\[
CVaR_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha VaR_s(X)ds. \tag{2.6}
\]

Note that CVaR is also known as the “average value at risk” and “expected shortfall”. VaR is more appropriately referred to as the quantile risk measure since \( VaR_\alpha(X) \) is exactly a \((1 - \alpha)\)-quantile of the random variable \( X \). It follows from the definition of \( VaR_\alpha(X) \) that

\[
VaR_\alpha(X) \leq x \Leftrightarrow S_X(x) \leq \alpha, \tag{2.7}
\]

where \( S_X(x) = 1 - F_X(x) \). Therefore, \( VaR_\alpha(X) = 0 \) for \( \alpha \geq S_X(0) \). For this reason, we assume that the parameter \( \alpha \) satisfies \( 0 < \alpha < S_X(0) \) to avoid the discussion of trivial cases. Another important property associated with \( VaR_\alpha(X) \) is that for any continuous increasing function \( \phi \), we have (see Theorem 1 in Dhaene et al. (2002))

\[
VaR_\alpha(\phi(X)) = \phi(VaR_\alpha(X)). \tag{2.8}
\]

A key advantage of \( CVaR_\alpha(X) \) over \( VaR_\alpha(X) \) is that \( CVaR \) is a coherent risk measure while VaR is not as it fails to satisfy subadditivity property. More detailed discussions on their properties can be found in Artzner et al. (1999) and Föllmer and Schied (2004).
Based upon these two risk measures, the risk measure based optimal reinsurance models proposed in Cai and Tan (2007) can be summarized succinctly as follows:

**VaR-optimization:**

\[
\text{VaR-optimization: } \, \text{VaR}_\alpha(T_f^*(X)) = \min_{f \in \mathcal{C}} \text{VaR}_\alpha(T_f(X)) \quad (2.9)
\]

and

**CVaR-optimization:**

\[
\text{CVaR-optimization: } \, \text{CVaR}_\alpha(T_f^*(X)) = \min_{f \in \mathcal{C}} \text{CVaR}_\alpha(T_f(X)), \quad (2.10)
\]

where \( f^* \) is the resulting optimal ceded loss function.

### 3 Optimal reinsurance: the VaR and CVaR criteria

In this section, we proceed to solve the optimal reinsurance models (2.9) and (2.10) under the feasible set \( \mathcal{C} \) of ceded loss functions defined in (2.1). We choose the following approach: for any given ceded loss function \( f \in \mathcal{C} \), we construct a layer reinsurance treaty such that it is better than \( f \) in the sense of minimizing the total risk exposure of an insurer under VaR or CVaR criterion. In this sense, we say that the layer reinsurance is optimal.

#### 3.1 VaR minimization model

In this subsection, we focus on the VaR-based optimal reinsurance model (2.9). For any ceded loss function \( f \in \mathcal{C} \), we construct \( h_f(x) \) with the following representation:

\[
\begin{align*}
    h_f(x) & \triangleq \min \left\{ (x - (\text{VaR}_\alpha(X) - f(\text{VaR}_\alpha(X))))_+, f(\text{VaR}_\alpha(X)) \right\}, \\
    & = \min_{0 \leq a \leq \text{VaR}_\alpha(X)} \{ a + \pi(\min\{(X - a)_+, \text{VaR}_\alpha(X) - a\}) \}, \quad (3.1)
\end{align*}
\]

Note that \( h_f(x) \) is a layer reinsurance of the form (1.1) with deductible \( a = \text{VaR}_\alpha(X) - f(\text{VaR}_\alpha(X)) \) and upper limit \( b = f(\text{VaR}_\alpha(X)) \). Furthermore,

\[
    h_f(\text{VaR}_\alpha(X)) = f(\text{VaR}_\alpha(X)) \quad \text{and} \quad h_f(x) \in \mathcal{C}.
\]

Similar to (2.4), we use \( T_{h_f}(X) \) to denote the total risk exposure of the insurer with corresponding ceded loss function \( h_f \). By comparing \( \text{VaR}_\alpha(T_{h_f}(X)) \) and \( \text{VaR}_\alpha(T_f(X)) \), we obtain the following key result of this subsection.

**Theorem 3.1.** For the VaR-based optimal reinsurance model (2.9), the layer reinsurance of the form (3.1) is optimal in the sense that

\[
\text{VaR}_\alpha(T_{h_f}(X)) \leq \text{VaR}_\alpha(T_f(X)), \quad \forall f \in \mathcal{C}.
\]

Moreover, we have

\[
\begin{align*}
    \min_{f \in \mathcal{C}} \text{VaR}_\alpha(T_f(X)) & = \min_{f \in \mathcal{C}} \text{VaR}_\alpha(T_f(X)) \\\n    & = \min_{0 \leq a \leq \text{VaR}_\alpha(X)} \{ a + \pi(\min\{(X - a)_+, \text{VaR}_\alpha(X) - a\}) \}, \quad (3.2)
\end{align*}
\]

where

\[
\mathcal{C}_1 \triangleq \{ \min\{(x - a)_+, \text{VaR}_\alpha(X) - a\} : 0 \leq a \leq \text{VaR}_\alpha(X) \}. \quad (3.3)
\]
Proof. We begin the proof by first demonstrating that for any ceded loss function \( f \in \mathcal{C} \), the following inequality

\[ h_f(x) \leq f(x), \quad \forall x \geq 0 \]

holds, where \( h_f(x) \) is defined in (3.1). To see this, let us recall that the ceded loss function \( f \in \mathcal{C} \) is non-negative and Lipschitz-continuous and hence (2.2) implies

\[ f(x) \geq (x + f(VaR_\alpha(X)) - VaR_\alpha(X))_+ = h_f(x), \quad \forall 0 \leq x \leq VaR_\alpha(X). \quad (3.4) \]

On the other hand, the increasing property of \( f(x) \) leads to

\[ h_f(x) = f(VaR_\alpha(X)) \leq f(x), \quad \forall x \geq VaR_\alpha(X). \]

As a result, we have \( h_f(X) \leq_{sl} f(X) \) according to the definition of stop-loss order in (2.3), then \( \pi(h_f(X)) \leq \pi(f(X)) \) due to the property of stop-loss ordering preserving for \( \pi \). Consequently, the property of translation invariance of \( VaR \) implies

\[
VaR_\alpha(T_f(X)) = VaR_\alpha(R_f(X)) + \pi(f(X)) = R_f(VaR_\alpha(X)) + \pi(f(X))
\]

\[
= VaR_\alpha(X) + \pi(f(X)) - f(VaR_\alpha(X))
\]

\[
= VaR_\alpha(X) + \pi(f(X)) - h_f(VaR_\alpha(X)) \geq VaR_\alpha(T_{h_f}(X)),
\]

where the second equality is derived by the Lipschitz-continuous, increasing properties of \( R_f(X) \), and (2.8). Hence the layer reinsurance of the form (3.1) is optimal.

Finally, set \( a = VaR_\alpha(X) - f(VaR_\alpha(X)) \), then \( 0 \leq a \leq VaR_\alpha(X) \) and all the layer reinsurance treaties of \( h_f(x) \) compose of the set \( \mathcal{C}_1 \) in (3.3). The proof is therefore complete. \( \square \)

Remark 3.1. In addition to the distribution invariance property, the proof of the above theorem only requires that the premium principle \( \pi \) preserves the first stochastic dominance, i.e.

\[ \pi(X) \leq \pi(Y) \quad \text{if} \quad S_X(t) \leq S_Y(t), \quad \forall t \in \mathbb{R}. \]

Consequently, the above result can be naturally generalized by assuming that the reinsurance premiums satisfy the properties of distribution invariance and the first stochastic dominance preserving.

Remark 3.2. The above theorem implies that solving the VaR-based reinsurance model (2.9) boils down to the determination of \( a \) in (3.2). The optimal layer reinsurance treaty is then specified by the deductible \( a \) and the upper limit \( VaR_\alpha(X) - a \). Consequently, the optimal reinsurance model collapses to a one-parameter optimization problem.

3.2 CVaR minimization model

In this subsection, we focus on the CVaR-based optimal reinsurance model (2.10). To proceed, we need to use the following results (see Theorem 3.2.4 in Rolski et al. (1999)): Provided that the expectations of random variables \( Y_1 \) and \( Y_2 \) are finite, if \( Y_1 \) and \( Y_2 \) satisfy

\[
\mathbb{E}[Y_1] \leq \mathbb{E}[Y_2], \quad F_{Y_1}(t) \leq F_{Y_2}(t), \quad t < t_0 \quad \text{and} \quad S_{Y_1}(t) \leq S_{Y_2}(t), \quad t \geq t_0 \quad (3.5)
\]

for some \( t_0 \in \mathbb{R} \), then \( Y_1 \leq_{st} Y_2 \).
For any \( f \in \mathcal{C} \), we can construct a layer reinsurance in the form of

\[
k_f(x) \triangleq \min \left\{ (x - \text{VaR}_\alpha(X) + f(\text{VaR}_\alpha(X)))_+, \mathcal{M} \right\},
\]

where \( \mathcal{M} \geq f(\text{VaR}_\alpha(X)) \) is determined by \( \text{CVaR}_\alpha(f(X)) = \text{CVaR}_\alpha(k_f(X)) \).

By comparing \( \text{CVaR}_\alpha(T_f(X)) \) and \( \text{CVaR}_\alpha(T_{k_f}(X)) \), we obtain the following theorem:

**Theorem 3.2.** For the CVaR-based optimal reinsurance model (2.10), the layer reinsurance of the form (3.6) is optimal in the sense that

\[
\text{CVaR}_\alpha(T_{k_f}(X)) \leq \text{CVaR}_\alpha(T_f(X)), \ \forall f \in \mathcal{C}.
\]

Therefore, we have

\[
\min_{f \in \mathcal{C}} \text{CVaR}_\alpha(T_f(X)) = \min_{f \in \mathcal{C}_2} \text{CVaR}_\alpha(T_f(X)) = \min_{0 \leq a \leq \text{VaR}_\alpha(X)} \{ a + 1/\alpha \mathbb{E}[(X - (a + b))^+] + \pi(\min\{(X - a)^+, b\}) \}, \quad (3.7)
\]

where

\[
\mathcal{C}_2 \triangleq \{ \min\{(x - a)^+, b\} : 0 \leq a \leq \text{VaR}_\alpha(X) \text{ and } a + b \geq \text{VaR}_\alpha(X) \}.
\]

**Proof.** First, we show that for any \( f \in \mathcal{C} \), \( k_f(x) \) defined in (3.6) exists. Specifically, (2.2) implies

\[
f(x) \leq x - \text{VaR}_\alpha(X) + f(\text{VaR}_\alpha(X)), \ \forall x \geq \text{VaR}_\alpha(X)
\]

so that

\[
\text{CVaR}_\alpha(f(X)) = \frac{1}{\alpha} \int_0^\alpha \text{VaR}_s(f(X))ds = \frac{1}{\alpha} \int_0^\alpha f(\text{VaR}_s(X))ds \\
\leq \frac{1}{\alpha} \int_0^\alpha (\text{VaR}_s(X) - \text{VaR}_\alpha(X) + f(\text{VaR}_\alpha(X)))ds,
\]

where the second equality is derived by (2.8). Thus, there exists a \( \mathcal{M} \geq 0 \) such that

\[
\text{CVaR}_\alpha(f(X)) = \frac{1}{\alpha} \int_0^\alpha \min\{\text{VaR}_s(X) - \text{VaR}_\alpha(X) + f(\text{VaR}_\alpha(X)), \mathcal{M}\}ds = \text{CVaR}_\alpha(k_f(X)).
\]

Furthermore, the inequality \( \mathcal{M} \geq f(\text{VaR}_\alpha(X)) \) holds; otherwise, the above equation implies

\[
\text{CVaR}_\alpha(f(X)) = \mathcal{M} < f(\text{VaR}_\alpha(X)) \leq \text{CVaR}_\alpha(f(X)).
\]

Second, (3.5), together with (3.9) and (3.10), leads to

\[
k_f(\text{VaR}_{U_\alpha}(X)) \leq s_l f(\text{VaR}_{U_\alpha}(X)),
\]

where \( U_\alpha \) is a random variable uniformly distributed on \([0, \alpha]\). Moreover, using the similar argument as in (3.4) establishes the following relation:

\[
k_f(x) \leq f(x), \ \forall 0 \leq x \leq \text{VaR}_\alpha(X).
\]

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Consequently, by introducing $U$ as a random variable uniformly distributed on $[0,1]$, we have

$$
E [(k_f(X) - d)_+] = E [(k_f(VaR_U(X)) - d)_+]
$$

$$
= \int_0^\alpha (k_f(VaR_u(X)) - d)_+ du + \int_\alpha^1 (k_f(VaR_u(X)) - d)_+ du
$$

$$
= \alpha E [(k_f(VaR_u(X)) - d)_+] + \int_\alpha^1 (k_f(VaR_u(X)) - d)_+ du
$$

$$
\leq \alpha E [(f(VaR_U(X)) - d)_+] + \int_\alpha^1 (f(VaR_u(X)) - d)_+ du
$$

$$
= E [(f(VaR_U(X)) - d)_+] = E [(f(X) - d)_+], \ \forall d \in \mathbb{R}.
$$

The first equality is derived by $X \sim VaR_U(X)$ and the inequality is implied by (3.11) and (3.12).

As a result, the definition of stop-loss order in (2.3) leads to

$$
k_f(X) \leq_{sl} f(X).
$$

(3.13)

Note that CVaR risk measure is translation invariant, then we have

$$
CVaR_\alpha(T_{k_f}(X)) = CVaR_\alpha(R_{k_f}(X)) + \pi(k_f(X))
$$

$$
= \frac{1}{\alpha} \int_0^\alpha R_{k_f}(VaR_u(X)) ds + \pi(k_f(X))
$$

$$
= CVaR_\alpha(X) - CVaR_\alpha(k_f(X)) + \pi(k_f(X))
$$

$$
= CVaR_\alpha(X) - CVaR_\alpha(f(X)) + \pi(k_f(X))
$$

$$
\leq CVaR_\alpha(X) - CVaR_\alpha(f(X)) + \pi(f(X)) = CVaR_\alpha(T_f(X)),
$$

where the inequality is implied by (3.13) and the stop-loss ordering preserving property of the premium principle $\pi$. As a result, the layer reinsurance $k_f$ is better than $f$ in the sense that it minimizes the total risk exposure of the insurer under the CVaR criterion. The proof is complete by defining

$$
a \triangleq VaR_\alpha(X) - f(VaR_\alpha(X)) \quad \text{and} \quad b \triangleq \mathcal{M} \geq f(VaR_\alpha(X))
$$

and using the fact $\int_0^\alpha (VaR_u(X) - a - b)_+ ds = \int_\alpha^1 (VaR_u(X) - a - b)_+ ds = E [(X - a - b)_+]$. \qed

To conclude this section, we demonstrate that the layer insurance is always optimal under both VaR and CVaR risk measures and over a class of premium principles satisfying certain axioms. Our findings further suggest that deriving optimal solutions to the reinsurance models (2.9) and (2.10) boils down to solving the one-parameter and two-parameter minimization problems as formulated in (3.2) and (3.7), respectively. In order to further solve the parameter-dependent optimization problems, it is then necessary to give the explicit form of the premium principle, in addition to the loss distribution $F_X(x)$. In the next section, we provide additional illustrations by resorting to two premium principles, namely the Wang’s and the Dutch premium principles.

4 Examples

Recall that our prescribed class of premium principles covers as many as eight of the eleven premium principles listed in Young (2004). Specifically, except variance, standard deviation and
Esscher premium principles that fail to satisfy the third axiom, net, expected value, exponential, proportional hazards, Wang’s, Swiss and Dutch premium principles and the principle of equivalent utility are included in our study. Moreover, an empirical phenomenon of reinsurance premium observed by Venter (1991) is expressed as “a premium calculation principle should produce a higher loading, relative to expected losses, for an excess of loss cover than for a primary cover on the same risk”. In other words, the empirically observed phenomenon of reinsurance premium implies the following property of increasing relative risk loading:

\[
\frac{\pi((X - d)_+)}{\mathbb{E}[(X - d)_+]} \quad \text{for } d \geq 0 \text{ is increasing in } d, \tag{4.1}
\]

provided that \(\mathbb{E}[(X - d)_+] > 0\). Note that when the above ratio is required to be strictly increasing in \(d\), the expected value premium principle is excluded. However, we relax such an assumption to simplify the analysis below.

To proceed, we discuss two premium principles—Wang’s and Dutch principles. The Wang’s premium principle (see Wang (1995)) is considered in Wang et al. (1997) where they take an axiomatic approach to characterize the insurance price. Furthermore, such a premium principle has been used to study the optimal reinsurance problems by Young (1999), Kaluszka (2005) and Cheung (2010), among others. Dutch premium principle is proposed by Van Heerwaarden and Kaas (1992) and some nice properties are derived. The following gives the respective definitions of the Wang’s and the Dutch premium principles:

**Definition 4.1.** The Wang’s premium principle is defined as

\[
\pi_w(X) \triangleq (1 + \rho) \int_0^\infty g(S_X(x)) dx, \tag{4.2}
\]

where \(\rho \geq 0\) and \(g : [0, 1] \rightarrow [0, 1]\) is an increasing concave function with \(g(x) \geq x\), \(g(0) = 0\) and \(g(1) = 1\).

Note that \(\rho\) represents the administrative or other expenses related to such a provision of reinsurance or the relative safety loading by the reinsurer, and the function \(g\) is usually called a distortion. Specially, when \(g(x) = x\), the Wang’s premium principle recovers the expected value premium principle, i.e. \(\pi_w(X) = (1 + \rho)\mathbb{E}[X]\).

**Definition 4.2.** The Dutch premium principle is given by

\[
\pi_D(X) \triangleq \mathbb{E}[X] + \theta \mathbb{E}[(X - \gamma \mathbb{E}[X])_+], \quad \text{where } \gamma \geq 1 \text{ and } 0 < \theta \leq 1. \tag{4.3}
\]

**Proposition 4.1.** Both premium principles of Wang’s (4.2) and Dutch (4.3) satisfy the increasing relative loading property (4.1).

**Proof.** (1) For the distortion \(g\) in Wang’s principle, we demonstrate that \(g(x)/x\) decreases in \(x\). Specifically, for any \(0 < x_1 < x_2 \leq 1\), the concavity of the distortion implies

\[
g(x_1) \geq \frac{x_1}{x_2} \times g(x_2) + \left(1 - \frac{x_1}{x_2}\right) \times g(0) = \frac{x_1}{x_2} \times g(x_2).
\]

The remaining proof is similar to that of Theorem 3 in Wang (1995) and hence the details are omitted.
(2) According to the definition of Dutch premium principle (4.3), we have

\[ J(d) \triangleq \frac{\pi_D((X - d)_+)}{E[(X - d)_+]} = 1 + \frac{\int_d^{\infty} \gamma S_x(t) dt}{\int_d^\infty S_x(t) dt}, \quad d \geq 0, \]

provided that \( E[(X - d)_+] > 0 \). Taking the derivatives of \( J(d) \) with respect to (w.r.t.) \( d \), we have

\[ J'(d) = \frac{\theta S_x(d)}{(E[(X - d)_+])^2} \left( \int_d^{\infty} S_x(t) dt - (1 - \gamma S_x(d)) \frac{S_x(d + \gamma E[(X - d)_+] \int_d^\infty S_x(t) dt}{S_x(d)} \right). \]

It is trivial that \( J'(d) \geq 0 \) for the case \( 1 - \gamma S_x(d) \leq 0 \). On the other hand, when \( 1 - \gamma S_x(d) > 0 \), since \( S_x(d + \gamma E[(X - d)_+]) \leq S_X(d) \), then we have

\[ J'(d)(E[(X - d)_+])^2 \geq \theta S_x(d) \left( \gamma S_x(d)E[(X - d)_+] - \int_d^{d + \gamma E[(X - d)_+] S_x(t) dt} \right) \]

\[ \geq \theta S_x(d) \left( \gamma S_x(d)E[(X - d)_+] - S_x(d) \int_d^{d + \gamma E[(X - d)_+] 1 dt} \right) = 0. \]

Collecting all the above results leads to \( J'(d) \geq 0 \) for any \( d \geq 0 \) with \( E[(X - d)_+] > 0 \), then \( \pi_D \) satisfies the axiom of increasing relative loading in (4.1). The proof is therefore complete. \( \square \)

**Remark 4.1.** The proportional hazards premium principle, as a special case of Wang’s premium principle when \( g(x) = x^\epsilon \) for some \( 0 < \epsilon < 1 \) and \( \rho = 0 \), has been shown to satisfy the axiom of increasing relative loading in Wang (1995).

Motivated by the above additional desirable property fulfilled by both Wang’s and Dutch premium principles, these two premium principles are used to demonstrate the applicability of the results established in Theorems 3.1 and 3.2. In particular, explicit closed-form expressions for the optimal deductible and the upper limit in layer reinsurance that solve (3.2) and (3.7) are obtained. These results are presented in the following two subsections.

### 4.1 Wang’s premium principle

In this subsection, we study the optimal reinsurance problems when the reinsurance premium is calculated according to the Wang’s principle (4.2). Due to the property of the distortion \( g(x) \), \( g(S_x(t)) \) is a survival distribution function. Without loss of generality, we let \( Z \) be a random variable satisfying

\[ S_z(t) = g(S_x(t)), \quad \forall t \in \mathbb{R} \]

and denote the VaR of \( Z \) at the confidence level \( \rho/(1 + \rho) \) by

\[ \beta^* \triangleq \text{VaR}_{\rho/(1+\rho)}(Z). \quad (4.4) \]

**Proposition 4.2.** When the reinsurance premium is calculated by the Wang’s premium principle (4.2), \( f^* \) that solves the VaR-based optimal reinsurance model (2.9) is given by

\[ f^*(x) \triangleq \begin{cases} \min\{ (x - \beta^*)_+, \text{VaR}_\alpha(X) - \beta^* \}, & \text{VaR}_\alpha(X) \geq \beta^*; \\ 0, & \text{otherwise.} \end{cases} \quad (4.5) \]
Moreover, we have
\[ \text{VaR}_\alpha(T_f(X)) = (1 + \rho) \int_{\min(\text{VaR}_\alpha(X), \beta^*)}^{\text{VaR}_\alpha(X)} g(S_X(t))dt + \min\{\text{VaR}_\alpha(X), \beta^*\}. \] (4.6)

Proof. When \( \pi \) follows the Wang’s premium principle (4.2), (3.2) implies that for any \( f \in \mathcal{G}_1 \), \( \text{VaR}_\alpha(T_f(X)) \) can be written as
\[ \text{VaR}_\alpha(T_f(X)) = (1 + \rho) \int_a^{\text{VaR}_\alpha(X)} g(S_X(t))dt + a \] (4.7)
for some \( 0 \leq a \leq \text{VaR}_\alpha(X) \), then taking the derivatives of \( \text{VaR}_\alpha(T_f(X)) \) w.r.t. \( a \) yields
\[ \frac{\partial \text{VaR}_\alpha(T_f(X))}{\partial a} = 1 - (1 + \rho)S_Z(a) \leq 1 - (1 + \rho)S_Z(\text{VaR}_\alpha(X)). \]
Consequently, \( \text{VaR}_\alpha(T_f(X)) \) is a convex function of \( a \), since \( \frac{\partial \text{VaR}_\alpha(T_f(X))}{\partial a} \) increases in \( a \).

We demonstrate the minimum value of \( \text{VaR}_\alpha(T_f(X)) \) is attainable at \( a = \text{VaR}_\alpha(X) \wedge \beta^* \). Specifically, if \( \text{VaR}_\alpha(X) < \beta^* \), then \( S_Z(\text{VaR}_\alpha(X)) > \frac{1}{1 + \rho} \) according to (2.7). In this case, we have \( \frac{\partial \text{VaR}_\alpha(T_f(X))}{\partial a} < 0 \) for any \( 0 \leq a \leq \text{VaR}_\alpha(X) \), then \( \text{VaR}_\alpha(T_f(X)) \) attains its minimum at \( a = \text{VaR}_\alpha(X) \) with value \( \text{VaR}_\alpha(T_f(X)) = \text{VaR}_\alpha(X) \); Otherwise, the minimum of \( \text{VaR}_\alpha(T_f(X)) \) is obtained at \( a = \beta^* \), due to the property
\[ \frac{\partial \text{VaR}_\alpha(T_f(X))}{\partial a} \gtrless 0 \quad \text{if} \quad a \gtrless \beta^*. \]

Last, the final result is derived by Theorem 3.1. The proof is therefore complete. \( \square \)

Now, we proceed to study the CVaR-based optimal reinsurance model (2.10). Since the proof of Proposition 4.1 shows that \( g(x)/x \) is a decreasing and continuous function, then
\[ K(t) \triangleq \frac{g(S_X(t))}{S_X(t)} \] (4.8)
is increasing and right-continuous with \( K(t) \geq 1 \), provided that \( S_X(t) > 0 \). Thus, the generalized inverse function of \( K(t) \) could be defined as
\[ K^{-1}(y) \triangleq \inf\{t \geq 0 : K(t) \geq y\}, \] (4.9)
where \( \inf \emptyset = \infty \). Consequently, similar to (2.7), we have
\[ K^{-1}(y) \leq t \iff y \leq K(t). \] (4.10)
Moreover, denote the essential supremum of random variable \( X \) by
\[ \text{ess sup} \, X \triangleq \sup\{x \in \mathbb{R} : F_X(x) < 1\}. \]

**Proposition 4.3.** When \( \pi \) follows the Wang’s premium principle (4.2), \( f^* \) that solves the CVaR-based optimal reinsurance model (2.10) is given by
\[
\frac{\partial \text{VaR}_\alpha(T_f(X))}{\partial a} = \begin{cases} (x - d_1^*)_+ - (x - d_2^*_+), & \forall d_2 \geq \text{VaR}_\alpha(X), \\ (x - d_1^*_+ - (x - d_2^*_+), & \text{otherwise}, \\ \text{VaR}_\alpha(X) = \text{ess sup} \, X; \end{cases} \] (4.11)
where
\[
d_1^* = \min \{ VaR_\alpha(X), \beta^* \} \quad \text{and} \quad d_2^* = \min \left\{ \max \left\{ K^{-1} \left( \frac{1}{\alpha(1 + \rho)} \right), VaR_\alpha(X) \right\}, \text{ess sup} X \right\}.
\]
In particular, when \( 1 + \rho \geq \frac{1}{\alpha^2} \), \( d_1^* = d_2^* = VaR_\alpha(X) \) such that \( f^*(x) = 0 \).

Proof. First, we show that \( \pi_w(f(X)) \) is a linear functional of \( f \in \mathcal{C} \). Specifically, according to the definition of the Wang’s premium principle (4.2), we have
\[
\pi_w(f(X)) = (1 + \rho) \int_0^1 VaR_x(f(x))g(dx) = (1 + \rho) \int_0^1 f(VaR_x(X))g(dx), \quad \forall f \in \mathcal{C}, \quad (4.12)
\]
where the first equality is derived by the proof of Theorem 22 in Kaluszka (2005) and the second equality is implied by (2.8).

Further, Theorem 3.2 shows that the optimal reinsurance model (2.10) simplifies to a two-parameter optimization problem. When \( \pi \) follows the Wang’s premium principle (4.2), the (4.12), together with (1.1), implies that \( CVaR_\alpha(T_f(X)) \) in (3.7) for any \( f \in \mathcal{C} \) can be written as
\[
CVaR_\alpha(T_f(X)) = CVaR_\alpha(X) + Q(a + b) - Q(a)
\]
for some \( 0 \leq a \leq VaR_\alpha(X) \) and \( a + b \geq VaR_\alpha(X) \), where \( Q(l) \) is defined as
\[
Q(l) \equiv \frac{1}{\alpha} \int_0^l (VaR_x(X) - l)_+ ds - (1 + \rho) \int_0^l (VaR_x(X) - l)_+ g(ds), \quad l \geq 0.
\]
As a result, solving the two-parameter minimization problem (3.7) is equivalent to solving the following two one-parameter optimization problems:
\[
\max_{0 \leq a \leq VaR_\alpha(X)} Q(a) \quad \text{and} \quad \min_{a + b \geq VaR_\alpha(X)} Q(c).
\]

We first focus on solving the maximization problem in (4.14). Since \( 0 \leq a \leq VaR_\alpha(X) \), then \( Q(a) \) can be simplified as
\[
Q(a) = CVaR_\alpha(X) - a - \pi_w((X - a)_+)
\]
\[
= CVaR_\alpha(X) - a - (1 + \rho) \int_a^\infty g(S_X(t))dt,
\]
where the first equality is derived by (4.12). Taking the derivatives of \( Q(a) \) w.r.t. \( a \), we have
\[
Q'(a) = (1 + \rho)g(S_X(a)) - 1 = (1 + \rho) (S_Z(a) - 1/(1 + \rho)).
\]
Therefore, \( Q(a) \) is a concave function. By taking the similar proof as that of Proposition 4.2, we obtain that the maximum value of \( Q(a) \) is achieved at \( d_1^* = \min \{ \beta^*, VaR_\alpha(X) \} \).

Next, we proceed to solve the minimization problem in (4.14). Since \( c \geq VaR_\alpha(X) \), then \( Q(c) \) can be represented by
\[
Q(c) = \frac{1}{\alpha} \int_0^1 (VaR_x(X) - c)_+ ds - (1 + \rho) \int_0^1 (VaR_x(X) - c)_+ g(ds)
\]
\[
= \frac{1}{\alpha} \mathbb{E}[(X - c)_+] - \pi_w((X - c)_+)
\]
\[
= \frac{1}{\alpha} \int_c^\infty S_X(t)dt - (1 + \rho) \int_c^\infty g(S_X(t))dt,
\]
where the second equality is derived by (4.12) and $X \overset{d}{=} \text{Var}_U(X)$.

If $\text{VaR}_\alpha(X) = \text{ess sup} X$, then we have $Q(c) = 0$ for any $c \geq \text{VaR}_\alpha(X)$. Thus, any $c \geq \text{VaR}_\alpha(X)$ is the optimal solution to this minimization problem.

Otherwise, we only focus on the interval $[\text{VaR}_\alpha(X), \text{ess sup} X)$, due to $Q(c) = 0$ for any $c \geq \text{ess sup} X$. Taking the derivatives of $Q(c)$ w.r.t. $c$, we have

$$Q'(c) = (1 + \rho) g(S_X(c)) - \frac{1}{\alpha} S_X(c) = (1 + \rho) S_X(c) \left( K(c) - \frac{1}{\alpha(1 + \rho)} \right),$$

where $K(x)$ is defined in (4.8). The following analysis is divided into three cases:

- If $K^{-1}\left(\frac{1}{\alpha(1 + \rho)}\right) \leq \text{VaR}_\alpha(X)$, then (4.10) implies that $K(\text{VaR}_\alpha(X)) \geq \frac{1}{\alpha(1 + \rho)}$. Consequently, we have $Q'(c) \geq 0$ for any $c \in [\text{VaR}_\alpha(X), \text{ess sup} X)$ due to the increasing property of $K(x)$. As a result, the minimum value of $Q(c)$ is attained at $\text{VaR}_\alpha(X)$;

- Else if $K^{-1}\left(\frac{1}{\alpha(1 + \rho)}\right) > \text{ess sup} X$, then (4.10) implies $Q'(c) < 0$ for any $\text{VaR}_\alpha(X) \leq c < \text{ess sup} X$. In this case, $Q(c)$ attains its minimum value at $\text{ess sup} X$.

- Otherwise, $Q(c)$ attains its minimum value at $K^{-1}\left(\frac{1}{\alpha(1 + \rho)}\right)$, since

$$Q'(c) \geq 0 \quad \text{if} \quad c \geq K^{-1}\left(\frac{1}{\alpha(1 + \rho)}\right).$$

Collecting all the above arguments, the optimal ceded loss function $f^*$ is derived by straightforward simplification.

Last, when $1 + \rho \geq \frac{1}{\alpha}$, $K^{-1}\left(\frac{1}{\alpha(1 + \rho)}\right) = 0$ according to the definition of generalized inverse function of $K(x)$ and

$$\beta^* = \text{Var}_{1+\rho}^{-1}(Z) = \text{VaR}_\alpha(Z) = \text{VaR}_\alpha(X),$$

where the last inequality is derived by $g(x) \geq x$. Thus, $d^*_1 = d^*_2 = \text{VaR}_\alpha(X)$ such that $f^*(x) = 0$. The proof is complete.

**Remark 4.2.** When $d^*_2 = \text{ess sup} X$ or $\text{VaR}_\alpha(X) = \text{ess sup} X$, $f^*(X) = (X - d^*_1)_+$, a.s. according to the optimal ceded function $f^*(x)$ defined in (4.11). Thus, it is equivalent to choosing the stop-loss reinsurance with $f^*(x) = (x - d^*_1)_+$.

**Remark 4.3.** When $\pi$ follows the expected value premium principle, i.e. $g(x) = x$, Proposition 4.2 shows that the optimal reinsurance under VaR criterion is given by

$$f^*(x) \triangleq \begin{cases} 
(x - \text{Var}_{1/(1+\rho)}(X))_+ - (x - \text{VaR}_\alpha(X))_+, & \text{VaR}_\alpha(X) \geq \text{Var}_{1/(1+\rho)}(X); \\
0, & \text{otherwise}.
\end{cases}$$

Furthermore, $K^{-1}(y)$ equals to 0 for $y \leq 1$ and $\infty$ for $y > 1$, and hence $d^*_1 = \text{Var}_{1+\rho}^{-1}(X)$, $d^*_2 = \text{ess sup} X$ for $\frac{1}{\alpha} > (1 + \rho)$. Proposition 4.3, together with Remark 4.2, implies that the optimal reinsurance treaty under CVaR criterion is

$$f^*(x) \triangleq \begin{cases} 
(x - \text{Var}_{1+\rho}^{-1}(X))_+, & \alpha < 1/(1 + \rho); \\
0, & \text{otherwise}.
\end{cases}$$
It is important to point out that the above results are also obtained by Theorems 3.2 and 4.1 in Chi and Tan (2010). They similarly study the VaR-based and CVaR-based optimal reinsurance problems except that their discussion is confined to the expected value premium principle. Thus, we generalize their results by confirming that the form of optimal reinsurance treaty remains unchanged even for other premium principles that we considered in this paper.

According to the definition of \(d^*_{2}\) in the above proposition, we know \(d^*_{2}\) may equal to \(\text{ess sup } X\). In this case, the optimal reinsurance is stop-loss according to Remark 4.2. Instead, the following example is applied to illustrate the case the limited stop-loss reinsurance with \(d^*_{2} < \infty\) is optimal.

Example 4.1. When the distortion is given by \(g(x) = \sqrt{x}\) and \(X\) follows Pareto distribution with density function

\[ f_X(x) = \frac{px^p}{(x + x_0)^{p+1}}, \quad x \geq 0, \]

where \(p > 1\) and \(x_0 > 0\), we have \(\text{ess sup } X = \infty\), and \(K(t) = \frac{1}{\sqrt{S_X(t)}}\) such that \(K^{-1}(y) = \text{VaR}_{1/y^2}(X) = x_0(y^{\frac{1}{p}} - 1)\) for \(y \geq 1\). Moreover, \(\beta^* = \text{VaR}_{1/(1 + \rho)^2}(X) = x_0((1 + \rho)^\frac{2}{p} - 1).\) Consequently, we have

\[ d^*_1 = \beta^* = x_0 \left( (1 + \rho)^\frac{2}{p} - 1 \right) \quad \text{and} \quad d^*_2 = K^{-1}(\frac{1}{\alpha(1 + \rho)}) = x_0 \left( (\alpha(1 + \rho))^{-\frac{2}{p}} - 1 \right) < \infty \]

for \(\alpha(1 + \rho)^2 < 1\); otherwise, the optimal ceded loss function \(f^*(x) = 0\).

4.2 Dutch premium principle

In this subsection, we study the optimal reinsurance models (2.9) and (2.10) when the reinsurance premium is calculated according to the Dutch principle (4.3).

Proposition 4.4. When \(\pi\) follows the Dutch premium principle (4.3), \(f^*\) that solves the VaR-based optimal reinsurance model (2.9) is given by

\[ f^*(x) \triangleq \min\{x, \text{VaR}_\alpha(X)\}. \quad (4.15) \]

Proof. When the reinsurance premium is calculated according to the Dutch principle, (3.2) implies that \(\text{VaR}_\alpha(T_f(X))\) for any \(f \in \mathcal{C}_1\) can be written as

\[ \text{VaR}_\alpha(T_f(X)) = a + \mathbb{E}[(X - a)_+] - \mathbb{E}[(X - \text{VaR}_\alpha(X))_+] + \theta \int_{\min\{w(a), \text{VaR}_\alpha(X)\}}^{\text{VaR}_\alpha(X)} S_X(t)dt \]

for some \(0 \leq a \leq \text{VaR}_\alpha(X)\), where

\[ w(a) \triangleq a + \gamma \mathbb{E}[f(X)] = a + \gamma \left( \mathbb{E}[(X - a)_+] - \mathbb{E}[(X - \text{VaR}_\alpha(X))_+] \right). \]

If \(w(a) \geq \text{VaR}_\alpha(X)\), then

\[ \text{VaR}_\alpha(T_f(X)) = \mathbb{E}[\max\{X, a\}] - \mathbb{E}[(X - \text{VaR}_\alpha(X))_+], \]

that increases in \(a\).
Otherwise, taking the derivatives of $VaR_\alpha(T_f(X))$ w.r.t. $a$, we have

$$\frac{\partial VaR_\alpha(T_f(X))}{\partial a} = 1 - S_X(a) - \theta S_X(w(a))(1 - \gamma S_X(a)).$$

We demonstrate $\frac{\partial VaR_\alpha(T_f(X))}{\partial a} \geq 0$. Specifically, the result is trivial for the case: $1 - \gamma S_X(a) \leq 0$; on the other hand, if $1 - \gamma S_X(a) > 0$, we have

$$\frac{\partial VaR_\alpha(T_f(X))}{\partial a} \geq 1 - S_X(a) - (1 - \gamma S_X(a)) = (\gamma - 1)S_X(a) \geq 0$$

because of $0 < \theta \leq 1$ and $\gamma \geq 1$.

To summarize, $VaR_\alpha(T_f(X))$ increases in $a$ and hence attains its minimum at $a = 0$. Thus, the final result is derived by Theorem 3.1 and the proof is complete.

Before stating the results pertaining to the CVaR-based optimal reinsurance model, it is useful to introduce the following additional notations. Let

$A(c) \equiv c - \gamma \mathbb{E}[\min\{X, c\}], \quad c \geq 0,$

then $A(c)$ is a convex function with $A(0) = 0$ and attains its minimum value at $c = VaR_\frac{1}{\alpha}(X) \geq 0$. Moreover, let

$$c_0 \equiv \max \left\{ \sup \left\{ c \geq VaR_\alpha(X) : A(c) = 0 \right\}, VaR_\alpha(X) \right\}, \quad (4.16)$$

where $\sup \emptyset = -\infty$. Furthermore, denote

$$m(c) \equiv -S_X\left(\gamma \mathbb{E}[\min\{X, c\}]\right), \quad c \geq 0,$$

then $m(c)$ is an increasing and right continuous function. Thus, just like (4.9), the generalized inverse function of $m(c)$ is denoted by $m^{-1}(t)$.

**Proposition 4.5.** When $\pi$ follows the Dutch premium principle (4.3), $f^*$ that solves the CVaR-based optimal reinsurance model (2.10) is given by

$$f^*(x) \equiv \begin{cases} \min\{x, \mathcal{M}^*\}, & 1 + \theta > \frac{1}{\alpha} \text{ and } c_0 < \text{ess sup } X; \\ x, & \text{otherwise}, \end{cases} \quad (4.17)$$

where

$$\mathcal{M}^* = \min \left\{ \max \left\{ m^{-1}\left(\frac{1/\alpha - (1 + \theta)}{\theta \gamma}\right), c_0 \right\}, \text{ess sup } X \right\}. \quad (4.18)$$

**Proof.** When $\pi$ follows the Dutch premium principle, (3.7) implies that $CVaR_\alpha(T_f(X))$ for any $f \in \mathcal{C}_2$ can be written as

$$CVaR_\alpha(T_f(X)) = a + \mathbb{E}[(X - a)_+] + \left(\frac{1}{\alpha} - 1\right)\mathbb{E}[(X - c)_+] + \theta \int_{\min\{v(a, c), c\}}^{c} S_X(y)dy \quad (4.19)$$

for some $0 \leq a \leq VaR_\alpha(X) \leq c$, where

$$v(a, c) = a + \gamma \left(\mathbb{E}[(X - a)_+] - \mathbb{E}[(X - c)_+]\right).$$
First, given a fixed \( c \geq VaR_\alpha(X) \), using the proof similar to that of Proposition 4.4, we have \( CVaR_{\alpha}(T_f(X)) \) attains its minimum at \( a = 0 \) with the value

\[
T(c) = \mathbb{E}[X] + (1/\alpha - 1)\mathbb{E}[(X - c)_+] + \theta \int_{\min\{v(0,c),c\}}^{c} S_X(y)dy, \quad c \geq VaR_\alpha(X). \tag{4.20}
\]

Consequently, by virtue of Theorem 3.2, the optimal reinsurance problem (2.10) simplifies to

\[
\min_{c \geq VaR_\alpha(X)} T(c).
\]

More precisely, we only need to focus on the interval \( [VaR_\alpha(X), ess\sup X] \), since

\[
T(c) = \mathbb{E}[X] + \theta \int_{\min\{\gamma\mathbb{E}[X],c\}}^{c} S_X(y)dy = T(ess\sup X), \quad \forall c \geq ess\sup X.
\]

If \( VaR_\alpha(X) = ess\sup X \), then the above equation shows that \( T(c) \) is a constant function. Thus, we could choose the optimal reinsurance strategy \( f^*(x) = x \) according to Remark 4.2.

Else if \( c_0 \geq ess\sup X \) where \( c_0 \) is defined in (4.16), then \( A(c) < 0 \) for any \( c \in [VaR_\alpha(X), ess\sup X) \). In other words, \( c < v(0,c) \), then

\[
T(c) = \mathbb{E}[X] + (1/\alpha - 1)\mathbb{E}[(X - c)_+] \tag{4.21}
\]
decreases in \( c \) and attains its minimum at \( c = ess\sup X \).

Else if \( 1 + \theta \leq \frac{1}{\alpha} \), then taking the derivatives of \( T(c) \) w.r.t. \( c \), we have

\[
T'(c) = \begin{cases} 
S_X(c) [(1 + \theta) - 1/\alpha - \theta\gamma S_X(v(0,c))] , & v(0,c) < c; \\
-(\frac{1}{\alpha} - 1)S_X(c), & \text{otherwise},
\end{cases} \leq 0. \tag{4.22}
\]

Thus, \( T(c) \) attains its minimum at \( c = ess\sup X \);

Else, \( A(c) \) is non-positive for \( c \in [VaR_\alpha(X), c_0) \) and non-negative for \( c \in [c_0, ess\sup X) \). For \( c \in [VaR_\alpha(X), c_0) \), \( T(c) \) can be written as in (4.21), then its minimum is attainable at \( c_0 \). Thus, the minimum value of \( T(c) \) appears on \([c_0, ess\sup X)\]. On this interval, \( T'(c) \) is given by the upper equation in (4.22). Since \( (1 + \theta) - 1/\alpha - \theta\gamma S_X(v(0,c)) \) increases in \( c \), then (4.10) implies

\[
T'(c) \geq 0 \quad \text{if} \quad c \geq m^{-1} \left( \frac{1/\alpha - (1 + \theta)}{\theta\gamma} \right).
\]

Consequently, \( T(c) \) attains its minimum value at \( c = \mathcal{M}^* \), where \( \mathcal{M}^* \) is defined in (4.18).

Collecting all the above arguments, we conclude that \( f^* \) defined in (4.17) is a solution of the optimal reinsurance model (2.10) with Dutch premium principle. This completes the proof.

\[\square\]

**Remark 4.4.** Under the assumption of Dutch premium principle, while Proposition 4.5 seems to suggest that the optimal reinsurance strategy for an insurer with CVaR risk measure is to cede either all the risk to the reinsurer or up to a limit stipulated by \( \mathcal{M}^* \), we emphasize that in practice the optimal strategy for the insurer is likely to cede all the risk. This can be argued by noting \( 0 < \theta \leq 1 \) and that \( \alpha \) tends to be a small value such as less than 5%. Hence we have \( 1/\alpha \gg 1 + \theta \) so that the first condition in (4.17) fails to be satisfied. On the other hand, under the VaR criterion, the optimal strategy is to cede all the lower risk to the reinsurer (see Proposition 4.4).
5 Concluding remarks

The quest for optimal reinsurance has remained a fascinating topic in actuarial science, and many interesting optimal reinsurance models have been proposed in the last few decades. In this paper, we have provided some new findings and new insights to the VaR-based and CVaR-based optimal reinsurance models proposed by Cai and Tan (2007). In contrast to the existing literature, we have investigated the optimality of the reinsurance under a much wider class of premium principles satisfying the three basic axioms: distribution invariance, risk loading and stop-loss ordering preserving. Under the additional assumption that both the insurer and the reinsurer are obligated to pay more for larger loss, we have shown that the layer reinsurance is quite robust in the sense that it is always optimal over our assumed risk measures (i.e. VaR and CVaR) and our prescribed class of premium principles. We have also demonstrated the applicability of our results by resorting to the Wang’s and the Dutch premium principles, and explicit solutions to the optimal parameters of layer reinsurance have been derived. These two premium principles are chosen because they exhibit a desirable property of increasing relative loading that is consistent with the market convention on reinsurance pricing.

References


