Valuation of an early exercise defined benefit underpin hybrid pension

Xiaobai, Zhu∗1, Mary, Hardy1 and David, Saunders1

1Department of Statistics and Actuarial Science, University of Waterloo

Abstract

In this paper we consider three types of embedded options in pension benefit design.

The first is the Florida second election (FSE) option, which has been offered to public employees in the state of Florida since 2002. The state runs both defined contribution (DC) and defined benefit (DB) pension plans. Employees who initially join the DC plan have the option to convert to the (DB) plan at a time of their choosing. The cost of the switch is assessed in terms of the ABO (Accrued Benefit Obligation), which is the expected present value of the accrued DB pension at the time of the switch. If the ABO is greater than the DC account, the employee is required to fund the difference.

The second is the DB Underpin option, also known as a ‘floor offset’ or a ‘Greater-of-benefit’ plan, under which the employee participates in a DC plan, but with a guaranteed minimum benefit based on a traditional DB formula.

The third option can be considered a variation on each of the first two. We remove the requirement from the FSE option for employees to fund any shortfall at the switching date. The resulting plan is similar to the DB underpin, but with the possibility of early exercise.

We adopt an arbitrage-free pricing methodology to value each option. We analyse and value the optimal switching strategy for the employee by constructing an exercise frontier, and we illustrate numerically the difference between the FSE, DB Underpin and Early Exercise DB Underpin options.

*Department of Statistics and Actuarial Science (SAS), University of Waterloo, 200 University Ave. West, Waterloo, ON, Canada, N2L 3G1. Email: x32zhu@uwaterloo.ca
1 Introduction

Over the past two decades, there has been a significant global shift in employer sponsored pension plans from Defined Benefit (DB) to Defined Contribution (DC). According to Watson (2017), DC pension assets in 22 major pension markets have grown from 41.1% in 2006 to 48.4% in 2016. However, there is also growing recognition that DC plans may not always be the best option. In 2013 the OECD\(^1\) identifies six flaws common to DC systems, including excessive volatility of funds in the pre-retirement phase, and decumulation options which are “not fit for purpose”.

Hybrid pension plans, which combine features of DC and the DB plans, may be able to meet employer and employee needs better than DC or DB individually. Numerous hybrid pension structures have been proposed, with different objectives and different balances of employee and employer risk allocation. Buchen et al. (2011) and Turner and Center (2014) present comprehensive surveys on existing hybrid plans; additional plan designs can be found in Khorasanee (2012), Guillén et al. (2006), Goecke (2013) and Blitzstein et al. (2006). Some popular examples include Cash Balance pension plans, Target Benefit pension plans and DB Underpin plans (known as Floor-Offset plans in North America). Studies have demonstrated that the hybrid pension market is expanding in some areas. For example, Kravitz (2016) shows that the number of Cash Balance Plans in the U.S. increased from 1337 in 2001 to 15178 in 2014).

The rising interest in hybrid pension plans suggests a potential for designing new risk-sharing schemes. In this paper, a new hybrid pension design is introduced. The intuition behind the new plan is based on two existing embedded options: the Florida second-election option, and the DB underpin plan.

The Florida second-election option is provided to public employees in Florida. The state sponsors two different pension plans, a DC and a DB. Employees choose either plan at enrollment. Subsequently, they have a one-time opportunity to transfer to the other plan (from DB to DC, or vice versa), prior to their retirement. For an employee switching from the DC to the DB plan, a “buy-in” cost would be calculated, equal to the accrued benefit obligation (ABO) of the DB plan, based on the employee’s current salary and service. The buy-in cost is paid from the DC account balance; if this is insufficient, the employee must cover any difference using his or her own financial resources.

The DB underpin plan is a DC plan that grants a DB type minimum guarantee. At retirement, the

\(^1\)http://www.oecd.org/finance/private-pensions/designingfundedpensionplans.htm
retiree’s pension is the higher of the guaranteed DB benefit and the income from the annuitized DC funds.

The new hybrid design introduced in this paper provides natural links to both of these plans. It is an early exercise DB underpin, under which a DC plan member may elect to lock into the DB plan benefit at some point prior to retirement. This is equivalent to the Florida second election plan but with the transfer of funds at the switch from DC to DB being limited to the DC account balance.

This paper evaluates this new hybrid benefit using market consistent valuation methods (see Hardy et al. (2014), Boyle and Hardy (2003), Marshall et al. (2010), Bacinello (2000) and Chen and Hardy (2009) for the valuation of various pension style products using similar methods) and we compare it with the Florida second election and the DB underpin options. This paper illustrates how the flexible nature of hybrid pension plans allows sponsors to design new risk-sharing schemes based on their risk appetite and objectives. Moreover, we will demonstrate that the creation of the new plan helps in conducting comprehensive and direct comparisons between different existing plans.

The remainder of the paper is structured as follows. Section 2 introduces the notation and assumptions, and provides detailed background information for the second-election option and the DB underpin option. Section 3 develops the new pension plan, as well as presenting some theoretical results in discrete time. Section 4 extends the work into the continuous case and incorporates stochastic salaries. Section 5 displays the numerical results. Section 6 concludes.

2 Notation and Assumptions

In this section we introduce the actuarial and economical notation and assumptions adopted throughout the paper. As is usual in the financial literature, we assume a complete and arbitrage-free market, with all rational agents sharing the same information generated by a filtration. This paper focuses on the investment risks prior to the retirement date, thus we ignore mortality and other demographic considerations.

\( c \) denotes the annual contribution rate (as a constant proportion of salary) into the DC plan. We assume that contributions are paid annually. We also assume that all contributions are paid by the plan sponsor/employer, although this is easily relaxed to allow for employee contributions.
\( T \) denotes the time of retirement of the employee, which is assumed to be known and non-random.

\( r \) denotes the risk free rate of interest, compound continuously, which is assumed to be a constant.

\( b \) denotes the accrual rate in the DB plan, which is assumed to be a constant.

\( \ddot{a}(T) \) denotes the actuarial value at retirement of a pension of 1 per year. Since we ignore longevity changes, and we assume a constant interest rate, we implicitly assume \( \ddot{a}(T) \) is a constant.

\( S_t \) denotes the stochastic price index process of the funds in the DC account. We assume \( S_t \) follows a Geometric Brownian Motion, so that

\[
\frac{dS_t}{S_t} = r dt + \sigma_S dZ^Q_S(t), \quad S_0 > 0
\]

where \( Z^Q_S(t) \) is a standard Brownian Motion under the risk neutral measure \( Q \).

\( L_t \) denotes the salary from \( t \) to \( t + 1 \) for \( t = 0, 1, ..., T - 1 \), where \( t \) denotes years of service. We first assume that salaries increase deterministically at a rate \( \mu_L \) per year, continuously compounded, so that

\[
L_t = L_0 e^{\mu_L t}
\]

We will also consider a stochastic salary process in the continuous setting, which we assume is hedgeable through the financial market (similar to Bacinello (2000)).

\[
\frac{dL_t}{L_t} = r dt + \sigma_L dZ^Q_L(t), \quad L_0 > 0
\]

\[
dZ^Q_L(t) dZ^Q_S(t) = \rho dt
\]

where \( Z^Q_L(t) \) is a standard Brownian Motion under the risk neutral measure \( Q \) and \( \rho \) is the correlation coefficient between \( Z^Q_L(t) \) and \( Z^Q_S(t) \) (correlation between the stock market and the salary increase).

Let \( \mathcal{F} = \{ \mathcal{F}_t, t \geq 0 \} \) be the filtration generated by \( \{ Z^Q_S(t), Z^Q_L(t), t \geq 0 \} \) when the salary is stochastic, or \( \{ Z^Q_S(t), t \geq 0 \} \) when the salary is deterministic.
3 DB Underpin Plan

The DB Underpin pension plan, also known as the floor-offset plan in the USA, provides a guaranteed defined benefit minimum which underpins a DC plan. Plan sponsors make regular specified contributions into the member’s DC account, and separately contribute to an additional fund which covers the extra cost of the guarantee. Employees usually have limited investment options to make the guarantee value more predictable. At the retirement date, after \( T \) years of service, if the member’s DC balance is higher than the value of the guaranteed minimum defined benefit pension, the plan sponsor will not incur any extra cost. However, if the DC benefit is smaller, the plan sponsor will cover the difference.

Using arbitrage-free pricing, we can calculate the present value of the cumulative cost of the DB underpin plan for the pension sponsor at time \( t = 0 \) as follows. We assume (more for clarity than necessity) that DB benefits are based on the final 1-year’s salary, and we ignore all exits before retirement.

\[
C^U = E^Q \left[ \sum_{t=0}^{T-1} e^{-rt} cL_t \right] + E^Q \left[ e^{-rT} \left( bT\bar{a}(T)L_{T-1} - \sum_{t=0}^{T-1} \frac{S_T}{S_t} cL_t \right) \right]
\]

The option value does not have an explicit solution, but can be determined using Monte Carlo simulation. See Chen and Hardy (2009) for details on the valuation and funding of the DB underpin option.

Notice that we have formulated the problem from sponsor’s perspective. In fact, under a risk neutral valuation, the cost function is equivalent as the expected discounted benefit received by the employee. For other pension plans studied in this paper, we will stay with this approach, to be consistent with the pension literature (including, for example, Hardy et al. (2014), Bacinello (2000), Chen and Hardy (2009), Guillén et al. (2006), etc.).

4 Florida Second Election (FSE) Option

Public employees of the State of Florida are given the choice to participate in either the DC or the DB plan at the beginning of their employment. In addition, employees are granted a one-time option to switch to the other plan anytime before their retirement date. This option is called
the second election by the Florida Retirement System. In this paper, we focus on the option to exchange the DC plan for the DB plan, based on the life cycle hypothesis that individuals prefer portanility at younger ages (provided by the DC plan) and stable retirement income at older ages (provided by the DB plan). To exercise the second election option, there is a “buy in” cost equal to the accrued benefit obligation (ABO), which is calculated as the present value of the accrued benefit, based on the current service and the current pensionable salary (ABO = accrual rate \times current salary \times number of years in service \times annuity factor \times discount factor).

We denote the ABO of the DB benefit for an employee with \( t \) years of service as \( K_t \), so that

\[
K_t = b L_{t-1} t \bar{u}(T) e^{-r(T-t)}
\]

Under the FSE hybrid plan, let \( \tau \) denote a stopping time with respect to \( \mathcal{F} \), which represents the time that an employee chooses to switch from the DC plan to the DB plan. If the ABO at \( \tau \) is more than the DC account balance, the employee needs to fund the difference from her own financial resources. If the DC account is more than the ABO at transition, then the employee retains the difference in a separate DC top-up account which can be withdrawn at retirement.

Mathematically, assuming that the employee exercises the option at the time which provides most financial benefit, the present value of the total DC and DB benefit cost at inception forms an optimal stopping problem:

\[
C^{se} = \sup_{\tau \in [0,1,\ldots,T]} \mathbb{E}_Q \left[ \sum_{t=0}^{\tau-1} e^{-rt} c L_t + e^{-r\tau} \left( K_T e^{-r(T-\tau)} - K_\tau \right) \right]
\]

where the first term, as in the previous section, is the present value of the DC contributions, and the second term is the additional funding required for the DB benefit, offset by the ABO at transition, which is funded from the DC contributions. The ‘sup’ indicates that the valuation assumes the switch from DC to DB is made at the time to maximize the cost to the employer, which is indeed equivalent to maximizing the expected value of the employee payoff (by the optional sampling theorem):

\[
C^{se} = \sup_{\tau \in [0,1,\ldots,T]} \mathbb{E}_Q \left[ e^{-rT} \left( \left( \sum_{t=0}^{\tau-1} c L_t \frac{S_\tau}{S_t} - K_\tau \right) \frac{S_T}{S_\tau} + K_T \right) \right]
\]

Milevsky and Promislow (2004) studies the price and optimal switching time of the Florida option with deterministic assumptions.
5 Early Exercise DB underpin plan - Discrete Case

The FSE design has the advantage that it provides employees with the flexibility to choose their plan type based on their changing risk appetite. However, employees retain the investment risk through the DC period of membership, and also have the additional risk of a suboptimal choice of switching time. Moreover, when the DC investment falls below the ABO of the DB plan, the employee may be unable to switch, if they do not have sufficient assets to make up the difference.

Inspired by the idea of combining the DB underpin plan with the Florida option, we investigate a new hybrid design, which we call the Early Exercise DB underpin, that adds a guarantee at the time of the switch from DC to DB. If the employee’s DC account is below the ABO when s/he elects to switch, then the plan sponsor will cover the difference.

5.1 Problem Formulation

We assume that the contributions are made annually into the DC account until the employee switches to DB, and that we are valuing the benefits at \( t \). We assume also that the employee has not switched from DC to DB before time \( t \), that the DC account is \( w_t \) at \( t \), and that future account values (up to the switching time) follow the process

\[
W_{t+\tau} = \frac{w_t S_{t+\tau}}{S_t} + \sum_{u=t}^{t+\tau-1} \frac{S_{t+\tau} c L_u}{S_u} \quad \tau = 1, 2, \ldots T - t.
\]

We assume that the salary is deterministic, though we will relax this assumption when we consider the continuous case in the next section.

We assume that the exercise dates are at the beginning of each year (or at the end of year T), before the contribution is made into the DC account, so the admissible exercise dates are \( \tau = 0, 1, \ldots T - t \).

The present value at time \( t \) of the cost of future benefits (past and future service, DC and DB), which is denoted by \( C_1(t, w_t) \), can be expressed as the sum of three terms:

1. The present value of the future contributions into the DC account before the member switches to the DB plan.
2. The present value of the cost of the DB benefit, offset by the ABO at the time of the switch.
3. The difference between the ABO and the DC balance at the switching time, if positive.

We take the maximum value of the sum of these three parts, maximizing over all the possible switching dates, as follows.

\[
C_1(t, W_t = w) = \sup_{\tau \in \{0, 1, \ldots, T-t\}} E_Q \left[ \sum_{u=0}^{\tau-1} e^{-ru} cL_{t+u} + e^{-r\tau} \left( K_T e^{-r(T-t-\tau)} - K_{t+\tau} \right) + e^{-r\tau} (K_{t+\tau} - W_{t+\tau})^+ \right]_{F_t}
\]

Notice that the switching time \( \tau \) is involved in all three parts, which makes the analysis more complex. However, using put-call parity, the problem can be simplified to:

\[
C_1(t, W_t = w) = E_Q \left[ K_T e^{-r(T-t)} \right]_{F_t} + \sup_{\tau \in \{0, 1, \ldots, T-t\}} E_Q \left[ e^{-r\tau} (W_{\tau+t} - K_{t+\tau})^+ \right]_{F_t} - w \tag{1}
\]

Details are given in Appendix A. This new formulation can be interpreted as the expected value of the future discounted benefit, which also consists of three terms:

- The present value of the DB plan benefits at time \( t \).
- A Bermudan type call option, with underlying process \( W_t \) and strike value \( K_t \).
- Minus the funds available, \( w \) from the existing DC balance at \( t \).

The first and third terms do not affect the switching time. To study the optimal exercise strategy, we may omit the first and third terms and define our value function as

\[
v_1(t, W_t = w) = \sup_{\tau \in \{0, 1, \ldots, T-t\}} E_Q \left[ e^{-r\tau} (W_{t+\tau} - K_{t+\tau})^+ \right]_{F_t}
\]

At time \( t \), given that the DC account balance is \( w \), we define the intrinsic value of the option, denoted \( v^c_1(t, w) \), as the value if the member decides to switch at that date, and the continuation value, denoted \( v^h_1(t, w) \) as the value if the member decides not to switch. Then

\[
v^c_1(t, w) = (w - tbL_{t-1} \bar{u}(T)e^{-r(T-t)})^+
\]

\[
v^h_1(t, w) = E_Q \left[ e^{-r} v_1 \left( t + 1, (w + cL_t) \frac{S_{t+1}}{S_t} \right) \right]_{F_t}
\]

The option is not analytically tractable. In the next section, we present some of the properties of the value function, which will enable us to use numerical solution methods.
5.2 The Exercise Frontier

The following proposition gives the general convexity and monotonicity of the value function

**Proposition 1.** At each observation date $0 \leq t \leq T$, the value function $v_1(t, w)$ is a continuous, strictly positive, non-decreasing and convex function of $w$.

The proof is given in Appendix A.2.

Like other Bermudan and American-type options, there exists a continuation region $C$ and a stopping region (or exercising region) $D$. When the time and DC account value pair, $(t, w)$, is in the continuation region, it is optimal for the member to stay in the DC plan. When $(t, w)$ is in the stopping region, it is optimal for the member to switch to the DB plan. The option to switch is exercised when $(t, w)$ moves into the stopping region. Mathematically, the continuation and stopping regions are defined as

- $v_h^1(t, w) > v_e^1(t, w) \Leftrightarrow (t, w) \in C$
- $v_h^1(t, w) \leq v_e^1(t, w) \Leftrightarrow (t, w) \in D$

**Proposition 2.** There exists a function $\varphi(t)$ such that

\[
  v_1(t, w) = \begin{cases} 
    v_e^1(t, w) & \text{if } w \geq \varphi(t) \\
    v_h^1(t, w) & \text{if } w < \varphi(t)
  \end{cases}
\]

The proof is shown in Appendix A.3. The function $\varphi(t)$ is the exercise boundary that separates continuation and exercise regions. Notice, it is possible, under certain parameters, that $\varphi(t) = \infty$ for some $t < T$, which means that it is not optimal to switch regardless of the DC account value. The next proposition will specify the situations in which $\varphi(t) < \infty$.

**Proposition 3.** The behavior of the exercise boundary $\varphi(t)$ depends on the ratio $\frac{c}{b \tilde{a}(T) e^{-rT}}$ as follows.

(i). If $\frac{c}{b \tilde{a}(T) e^{-rT}} < 1$, then $\varphi(t) < \infty$, $\forall t \in [0, T]$

(ii). If $\frac{c}{b \tilde{a}(T) e^{-rT}} \geq 1$, there exists a $t_\ast \in [0, 1, \ldots, T - 1]$, such that $\varphi(t) = \infty$, $\forall t \leq t_\ast$, and $\varphi(t) < \infty$, $\forall t > t_\ast$. 

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(iii). The value of $t^*$ is $\lceil t' \rceil$, where $\lfloor \cdot \rfloor$ represents the floor function, and $t'$ satisfies

$$(t' + 1)bL_{t'} \tilde{a}(T)e^{-r(T-t')} - t'bL_{t'-1} \tilde{a}(T)e^{-r(T-t')} - cL_t = 0$$

(iv). If $c > b \tilde{a}(T)(1 - e^{-\mu L})T + e^{-\mu L}e^{-r}$ then $t^* = T - 1$, which means $\varphi(t) = \infty, \forall t < T$

(v). If $\frac{c}{b \tilde{a}(T)e^{-rT}} = 1$, then $t^* = 0$.

The proof is given in the Appendix A.4. This proposition demonstrates that the ratio between the DC contribution rate and the accrual rate of the DB benefit will determine the shape of the exercise boundary. In the extreme case, when the DC contribution rate is much higher than the DB accrual rate, it is optimal for the employee to wait until the retirement date, and the Early Exercise DB underpin option simplifies to the DB underpin option.

6 Early Exercise DB underpin plan - Continuous Case

By extending the current model to incorporate a stochastic salary process, solving the value of the option becomes a three dimensional problem. The optimal exercise boundary will depend on both the salary and the DC account balance. In this section, we reconstruct the problem in a continuous setting, with a stochastic salary.

The mathematical formulation replaces the summation sign with the integral sign, and extends the admissible stopping time set to $[t, T]$. By the Optional Sampling Theorem, we can define the value function similarly to the discrete case.

$$v_2(t, W_t = w, L_t = l) = \sup_{0 \leq \tau \leq T-t} E^Q \left[ e^{-r\tau} (W_{t+\tau} - K_{t+\tau})^+ \right| \mathcal{F}_t ]$$

where

$$dW_t = (rW_t + cL_t)dt + \sigma_S W_t dZ^Q_S(t), \quad W_0 = 0$$

$$dL_t = rL_t dt + \sigma_L L_t dZ^Q_L(t), \quad L_0 > 0$$

$$K_t = b t \tilde{a}(T) L_t e^{-r(T-t)}$$

Notice now that the value function also depends on the salary at time $t$. In the following sections we show that the problem can be reduced to two dimensional, in the sense that the value function only depends on the wealth-salary ratio process.
6.1 Definition of the Value Function

We define a new variable \( Y_t = \frac{W_t}{L_t} \), which represents the DC wealth-salary ratio. Using Girsanov’s theorem, we can show that the value function \( v_2(t, w, l) = l v_2(t, w/l) \), where \( v_2(t, y) \) is defined as

\[
v_2(t, Y_t = y) = \sup_{0 \leq s \leq T-t} E^{\tilde{P}} \left[ (Y_{t+s} - b(t + s)\tilde{a}(T)e^{-r(T-t-s)})^+ \right] | F_t
\]

with the wealth-salary ratio follows

\[
dY_t = cdY_t + \sigma Y_t dZ^{\tilde{P}}_Y(t), \quad Y_0 = 0
\]

where \( \tilde{P} \) is a pricing measure absolutely continuous with respect to the risk neutral measure \( Q \) but using the salary \( (L_t) \) as the numeraire, and \( Z^{\tilde{P}}_Y(t) \) is standard Brownian Motion under the measure \( \tilde{P} \). This dimension reduction method using change of measure can be quite often found in the mathematical finance literature (for example, see section 27 in Peskir and Shiryaev (2006) for early exercise Asian call option). Details of the derivation is given in Appendix A.5.

6.2 Properties of the Value Function and Exercise Frontier

**Proposition 4.** The value function \( v_2(t, y) \) is a continuous, strictly positive, non-decreasing and convex function of \( y \), and a continuous and non-increasing function of time \( t \).

The proof can be found in Appendix A.6.

As above, we denote the continuation region \( C = \{(t, y) \in (0, T) \times [0, \infty) : v_2(t, y) > v_e(t, y)\} \) and stopping region \( D = \{(t, y) \in (0, T) \times [0, \infty) : v_2(t, y) = v_e(t, y)\} \), and let \( \varphi(t) \) be the exercise frontier which separates the two regions.

**Proposition 5.** There exists a function \( \varphi(t) \) such that

\[
v_2(t, y) = \begin{cases} 
v_2^e(t, y) & \text{if } y \geq \varphi(t) \\
_2^b(t, y) & \text{if } y < \varphi(t)
\end{cases}
\]

The proof can be found in Appendix A.6.
For numerical solution, we formulate the problem as a free-boundary PDE problem.

\[
\frac{\partial v_2}{\partial t} + c \frac{\partial v_2}{\partial y} + \frac{\sigma_y^2 y^2}{2} \frac{\partial^2 v_2}{\partial y^2} = 0 \quad \text{in} \quad C
\]

\[v_2(t, y) = v_2^e(t, y) \quad \text{in} \quad D\]

\[v_2(t, y) > v_2^e(t, y) \quad \text{in} \quad C\]

\[v_2(t, y) = v_2^e(t, y) \quad \text{for} \quad y = \varphi(t) \quad \text{or} \quad t = T\]

In appendix A.6 we illustrate the condition where \(\varphi(t) = \infty\), and we prove that when the DC contribution rate is very high, such that \(c > b(1 + rT)\hat{a}(T)\), then the option becomes the regular DB underpin option. Figure 1 shows an example of the optimal exercise boundary. As mentioned before, the member should switch to the DB plan only when they have already saved certain extra amount over the cost of ABO, and in this graph, this excess amount is relatively stable. When the time comes very close to the retirement date, the plan member is facing less future opportunities to obtain higher DC account balance from the equity market, and thus willing to make the transfer to the DB plan even with lower DC balance, in order to secure their benefit. In this example, the decrease of the optimal exercise boundary starts only at the last few month, and eventually converge to the value of the DB plan.

![Image of Optimal Exercising Boundary](image)

Figure 1: Example of Optimal Exercising Boundary, \(r = 0.06, \sigma_S = 0.15, \sigma_L = 0.04, c = 0.16, b = 0.016\) and \(\hat{a}(T) = 14.75\)
6.3 Different Discount Rates for ABO Calculation

In practice, actuaries often use a discount rate higher than the observed market risk-free rate to determine the ABO. Here we denote the $\gamma$ as the discount rate for the ABO calculation, so the ABO has the following form:

$$K_t = tbL_t\bar{a}(T)e^{-\gamma(T-t)}$$

It is not difficult to show that the new cost function can be expressed in the same form as equation (1) through the Optional Sampling Theorem. Under our stochastic salary assumption, however, the risk free rate $r$ only appears in the ABO calculations, which means that $\gamma$ and $r$ are mathematically indistinguishable. Thus, we only include $\gamma$ in our study of deterministic salary assumptions. The new value function preserves similar properties (with slight modifications) as those stated in Propositions (1), (2), (3) for the discrete case, and (4) and (5) in the continuous case. Notice that using different discount rates for the ABO does not violate our market-consistent valuation principle. Here we use $v_4$ to denote the price with $\gamma$ as discount rate for ABO:

$$v_4(t, W_t = w) = \sup_{0 \leq \tau \leq T-t} E^Q\left[e^{-r\tau} \left(W_{t+\tau} - b(t+\tau)L_{t+\tau}\bar{a}(T)e^{-\gamma(T-t-\tau)}\right)^+ | F_t\right]$$

7 Numerical Examples

In this section we present numerical results for the values of the Early Exercise DB Underpin plan. For the continuous setting, we use the penalty method (see Forsyth and Vetzal (2002)) and for the discrete case, we use the Least Square Method (see Longstaff and Schwartz (2001)). We compare the Early Exercise option with the DB underpin and with the Florida Second Election option. For the DB underpin option, to be consistent with the valuation method for Early Exercise DB underpin option, we use Monte Carlo simulation in the discrete setting and the Crank Nicolson finite difference method in the continuous setting. For the second election option, we are able to derive explicit solutions under both discrete and continuous settings (see Appendix A.7). Furthermore, we include the Early Exercise option with deterministic salary in the continuous setting (denoted by $v_3$) as an intermediate comparison with the discrete case.
7.1 Benchmark Scenario

The initial benchmark parameter set is as follows. Later we perform some sensitivity tests on each of the parameters.

- \( \mu_L = r = 0.04 \). We set \( \mu_L \) to be the same as the risk free rate, to make consistent comparison with the stochastic salary assumption (when salary is assumed to be hedgeable).
- \( \rho = 0 \), assumes no correlation between salary and investment return.
- \( \sigma_S = 0.15, \sigma_L = 0.04, b = 0.016, c = 0.125 \) and \( \bar{a}(T) = 14.75 \).
- \( L_0 = 1, t = 0 \), so that all values are given per unit of starting salary.

7.2 Cost

Table 1 shows the value of each pension plan in the continuous setting. The price for the Early Exercise DB-Underpin (EDBU), the Florida Second Election (FSE) and the DB underpin plans are expressed as an additional cost on top of the base DB plan. We show values for the Early Exercise DB with stochastic salaries \( v_2 \), and with deterministic salaries, \( v_3 \). Table 2 shows the results in the discrete setting.

<table>
<thead>
<tr>
<th>Time to Retirement</th>
<th>DB</th>
<th>DC</th>
<th>FSE</th>
<th>DBU</th>
<th>EEDBU(s) ( v_2 )</th>
<th>EEDBU(d) ( v_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10yr</td>
<td>2.36</td>
<td>1.25</td>
<td>0</td>
<td>0.0023</td>
<td>0.0070</td>
<td>0.0062</td>
</tr>
<tr>
<td>15yr</td>
<td>3.54</td>
<td>1.87</td>
<td>0</td>
<td>0.0126</td>
<td>0.0354</td>
<td>0.0315</td>
</tr>
<tr>
<td>20yr</td>
<td>4.72</td>
<td>2.50</td>
<td>0.0203</td>
<td>0.0348</td>
<td>0.1010</td>
<td>0.0936</td>
</tr>
<tr>
<td>30yr</td>
<td>7.08</td>
<td>3.75</td>
<td>0.2179</td>
<td>0.1199</td>
<td>0.3492</td>
<td>0.3355</td>
</tr>
<tr>
<td>40yr</td>
<td>9.44</td>
<td>5.00</td>
<td>0.5837</td>
<td>0.2594</td>
<td>0.7380</td>
<td>0.7194</td>
</tr>
</tbody>
</table>

Table 1: Cost of each pension plan per unit of starting salary, continuous setting. Hybrid costs are additional to the basic DB cost. FSE is the Florida second election option; DBU is the DB Underpin, EEDBU(s) \( v_2 \) is the Early Exercise DB underpin with stochastic salaries, and EEDBU(d) \( v_3 \) is the Early Exercise DB underpin with deterministic salaries.
<table>
<thead>
<tr>
<th>Time to Retirement $T$</th>
<th>DB</th>
<th>DC</th>
<th>FSE</th>
<th>DBU</th>
<th>EDBU(d) $v_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10yr</td>
<td>2.2675</td>
<td>1.2500</td>
<td>0.0039 (0.0011)</td>
<td>0.0099 (0.0001)</td>
<td></td>
</tr>
<tr>
<td>15yr</td>
<td>3.4012</td>
<td>1.8750</td>
<td>0.0210 (0.0020)</td>
<td>0.0456 (0.0003)</td>
<td></td>
</tr>
<tr>
<td>20yr</td>
<td>4.5349</td>
<td>2.5000</td>
<td>0.0458 (0.0029)</td>
<td>0.1190 (0.0006)</td>
<td></td>
</tr>
<tr>
<td>30yr</td>
<td>6.8024</td>
<td>3.7500</td>
<td>0.1455 (0.0048)</td>
<td>0.3752 (0.0014)</td>
<td></td>
</tr>
<tr>
<td>40yr</td>
<td>9.0699</td>
<td>5.0000</td>
<td>0.3115 (0.0069)</td>
<td>0.7726 (0.0025)</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Cost of each pension plan, discrete setting. Hybrid costs are additional to the basic DB cost. FSE is the Florida second election option; DBU is the DB Underpin, and EEDBU(d) $v_1$ is the Early Exercise DB underpin (with deterministic salaries).

Some observations can be made:

- EDBU $v_2$ is greater than EDBU $v_3$ which reflects the additional costs from stochastic salaries, but the results are fairly close. We would expect that in the discrete case, a stochastic salary would lead to a higher but close cost as in the deterministic salary assumption.

- When the expected retirement date is near, under the benchmark parameters, participants in the Second Election plan should always be in the DB plan ($t^* = 0$), so there will be no extra cost required to fund the second election option.

- Although the Early Exercise DB underpin option is greater than both the DB underpin and the Second Election, none of the values in the benchmark scenario exceed 10% of the cost of a DB plan. For horizons of 30 years or less, the extra cost from the EDBU option costs around 5% more than the basic DB plan.

- As $T$ increases, the cost is increasing, at an increasing rate, for all three options. The underpin and election options become significantly more costly over 40 years compared with the cost for 30 years.

- In the discrete case, the value of the options are generally greater than in the continuous case, and we would expect a larger difference if stochastic salary is incorporated. However, the observations made in the continuous case also apply in the discrete case.
7.3 Sensitivity Tests

In this section, we present the sensitivity tests over all parameters in the continuous setting. We consider $c$, $\mu_L$, $r$, $\sigma_S$, $\gamma$ and $b$ under the deterministic salary assumption and $\sigma_L$ and $\rho$ under the stochastic salary assumption. We also fix the time horizon to be 30 years. Details are displayed in Table 3.

We note the following points.

- All three plans share the same directional trends as the parameters change, but with some very different sensitivities.

- The additional costs of the hybrid plans react in the opposite direction to the underlying DB costs, for parameters which influence the DB cost. For example, increasing the risk free rate $r$ decreases the DB cost, but increases the additional hybrid costs. Increasing the accrual rate $b$ increases the DB costs, but decreases the additional hybrid costs. Hence, the sensitivity of the total costs to the changing parameters is rather more muted than the sensitivity of the additional costs shown in the table.

- The risk-free rate has a significant impact on the Early Exercise DB underpin option value, especially on the relative cost with respect to the DB plan. However, as shown by the sensitivity test on $\gamma$, the impact mostly comes from the value of the ABO. Also, it is interesting to note that when $r$ is high, the cost of the FSE option is quite close to the Early Exercise DB underpin option.

- Decreases in the accrual rate $b$ for the DB plan increase the relative value of the Early Exercise DB option. The extra cost reflects the fact that the funding of the DC plan is higher than the DB plan. When $b$ is high, the fast accumulation of the DB benefit would discourage employees from entering into the DC account, and the option value will be reduced.

- In our model, the cost of the second election option is independent of the market volatility $\sigma_S$.

Although the structure of the Early Exercise DB underpin plan provides more flexibility and protection to the employee than both the second election option and the DB underpin plan, the additional cost does not appear as large as one may expect. In most scenarios, the cost is less than 5% of the DB plan. The relative cost is high when the risk-free rate increases, however, it is interesting to
### Table 3: Sensitivity tests over \((c, \mu_L, r, \sigma_S, \gamma \text{ and } b)\), using deterministic salaries. \(v^{se}\) is the additional FSE cost, \(v_3\) is the additional EEDBU cost, and \(v_U\) is the additional DB underpin cost.
Table 4: Sensitivity tests over \((\sigma_L, \rho)\), stochastic salaries.

<table>
<thead>
<tr>
<th>Value</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
<th>0.04</th>
<th>0.05</th>
<th>0.06</th>
<th>0.07</th>
<th>0.08</th>
<th>0.09</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma_L)</td>
<td>0.3363</td>
<td>0.3389</td>
<td>0.3432</td>
<td>0.3492</td>
<td>0.357</td>
<td>0.3665</td>
<td>0.3778</td>
<td>0.391</td>
<td>0.4058</td>
</tr>
<tr>
<td>(v_3)</td>
<td>0.1209</td>
<td>0.1238</td>
<td>0.1286</td>
<td>0.1354</td>
<td>0.1443</td>
<td>0.1551</td>
<td>0.1681</td>
<td>0.183</td>
<td>0.2001</td>
</tr>
<tr>
<td>(v_U)</td>
<td>0.2552</td>
<td>0.2432</td>
<td>0.195</td>
<td>0.1472</td>
<td>0.1354</td>
<td>0.1238</td>
<td>0.079</td>
<td>0.0396</td>
<td>0.0311</td>
</tr>
<tr>
<td>(\rho)</td>
<td>-1</td>
<td>-0.9</td>
<td>-0.5</td>
<td>-0.1</td>
<td>0</td>
<td>0.1</td>
<td>0.5</td>
<td>0.9</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 4 displays the sensitivity test for the stochastic salary parameters. The risks involved in the stochastic salary process have less impact overall. Larger volatility in the salary process, as well as negative correlation between the salary and equity market, will increase the volatility of wealth-salary ratio process, and thus, increases the option value. These two risks have same effect on the DB underpin option, which has been previously observed by Chen and Hardy (2009). However, under our assumptions, the value of second election option is immunized to these two risks.

8 Conclusion and Future Work

In this paper, we discuss a new pension design, which combines Florida’s second election option and the DB underpin option, to form an Early Exercise type DB underpin plan. We summarize some key characteristics of the option, such as convexity and monotonicity. Also, we provide illustrations of the behavior of the early exercise region, and specifically include the situation where the Early Exercise DB underpin simplifies to the DB underpin plan.

Our numerical illustrations demonstrate that, although the Early Exercise DB underpin option may end up costing more than both the DB underpin option and the second election option combined, it does not cost more than 10% of the DB plan in general. In cases when the relative cost of the option compared with the DB plan is very large, for example when risk-free rate is high or salary growth rate is low, the actual cost of the Early Exercise DB underpin plan is indeed smaller.

The Early Exercise DB underpin plan shifts more risk and cost to the employer compared with a
DB plan, so it may not be an attractive option for pension sponsors, but the overall costs can be managed to some extent by varying the DC contribution rate and the DB accrual rate. Furthermore, it offers an attractive portable benefit for younger employees, which should help with recruitment, and offers a substantial retention benefit for older employees. It offers predictable income in retirement, popular with employees and labour unions. In addition, we have shown how the Early Exercise DB underpin connects the FSE plan design to the DB underpin design.

There are many outstanding questions that we hope to address in future work.

- The assumption of a complete market may be too strong. It may also be interesting to consider the situation when salary can only be partially hedged.
- The sensitivity of the option value of the risk-free rate indicates that the results might be sensitive to stochastic interest rates.
- The annuity factor is highly sensitive to the choice of discount rate. However, as we assume a constant annuity factor, we may underestimate the risk.
- It may also be interesting to explore the option from the employee’s perspective. For example, evaluating the utility gains from the guarantee.

9 Acknowledgments

We are very grateful to Professor Chen Xinfu, particularly for a thoughtful discussion on the structure of the pension plan and the PDE formulation.

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References and Notes


A Appendix

A.1 Cost Function - Discrete Case

Notice that by the optional sampling theorem, for any stopping time $\tau$:

$$E^Q \left[ e^{-r\tau} W_{t+\tau} | F_t \right] = E^Q \left[ \sum_{u=0}^{\tau-1} e^{-ru} c L_{t+u} | F_t \right] + W_t$$

Then our cost function in equation (1) can be rewritten as:

$$C(t, w) = \sup_{\tau \in [0,1,\ldots,T-t]} \left[ \sum_{u=0}^{\tau-1} e^{-ru} c L_{t+u} + e^{-r\tau} (K T e^{-r(T-t)} - K_{t+\tau}) + e^{-r\tau} (K_{t+\tau} - W_{t+\tau})^+ \right] | F_t \right]$$

$$= \sup_{\tau \in [0,1,\ldots,T-t]} \left\{ E^Q \left[ \sum_{u=0}^{\tau-1} e^{-ru} c L_{t+u} | F_t \right] - E^Q \left[ e^{-r\tau} W_{t+\tau} | F_t \right] + E^Q \left[ K T e^{-r(T-t)} \right] 
+ E^Q \left[ e^{-r\tau} (W_{t+\tau} - K_{t+\tau})^+ | F_t \right] \right\}$$

$$= E^Q \left[ K T e^{-r(T-t)} \right] + \sup_{\tau \in [0,1,\ldots,T-t]} E^Q \left[ e^{-r\tau} (W_{t+\tau} - K_{t+\tau})^+ | F_t \right] - \text{Price of the Option}$$


Price of the Option
A.2 Characteristics of Value Function - Discrete Case

Here is the proof of Proposition (1). For the purpose of clarity and simplicity, we denote \( W_{s|w}^t = W_s|W_t = w, s \geq t \) and \( E_t^Q[\cdot] = E^Q[\cdot|\mathcal{F}_t] \)

A.2.1 Value function is non-decreasing in \( w \)

For \( h > 0 \), we have \((x - k)^+ - (x + h - k)^+ \leq 0\), for all \( x \), therefore,

\[
v(t, w) - v(t, w + h) \leq \sup_{\tau \in [0, 1, \ldots, T-t]} E_t^Q \left[ e^{-r\tau} \left( (W_{t+\tau}^t - K_{t+\tau})^+ - (W_{t+\tau}^{t+w} + h\frac{S_{t+\tau}}{S_t} - K_{t+\tau})^+ \right) \right] \leq 0
\]

since \( \frac{S_{t+\tau}}{S_t} \) is strictly positive a.s.. Notice, although the value function is increasing in the initial DC balance, the cost function \( C(t, w) \) is the opposite.

\[
C(t, w + h) - C(t, w) \leq \sup_{\tau \in [0, 1, \ldots, T-t]} E_t^Q \left[ e^{-r\tau} \left( (W_{t+\tau}^{t+w} + h\frac{S_{t+\tau}}{S_t} - K_{t+\tau})^+ - (W_{t+\tau}^{t,w} - K_{t+\tau})^+ \right) \right] - h \\
\leq \sup_{\tau \in [0, 1, \ldots, T-t]} E_t^Q \left[ e^{-r\tau} \left( h\frac{S_{t+\tau}}{S_t} + (W_{t+\tau}^{t,w} - K_{t+\tau})^+ - (W_{t+\tau}^{t,w} - K_{t+\tau})^+ \right) \right] - h \\
= 0
\]

A.2.2 \( v^h(t, w) \) and \( v^e(t, w) \) are non-decreasing in \( w \)

For \( x > y \),

\[
v^h(t, x) - v^h(t, y) = e^{-r} E_t^Q \left[ v \left( t + 1, (x + cL_t)\frac{S_{t+1}}{S_t} \right) - v \left( t + 1, (y + cL_t)\frac{S_{t+1}}{S_t} \right) \right] \geq 0
\]

Thus, \( v^h \) is increasing in the DC account balance. It is also easy to see that

\[
v^e(t, x) - v^e(t, y) = (x - K_t)^+ - (y - K_t)^+ \geq 0
\]
A.2.3 Continuity of value function in \( w \)

For \( x > y \), using the fact that \( \sup[X] - \sup[Y] \leq \sup[X - Y] \) and \( (x - k)^+ - (y - k)^+ \leq (x - y)^+ \), we have

\[
|v(t, x) - v(t, y)| \\
\leq \sup_{\tau \in [0,1,\ldots,T-t]} E_t^Q \left[ e^{-r\tau} \left( x - y \right) \frac{S_{t+\tau}}{S_t} \right] \\
\leq (x - y)
\]

Thus, \( v \) is Lipschitz continuous in \( w \) and clearly for any \( \epsilon > 0 \)

\[
v(t, w + \epsilon) \leq v(t, w) + \epsilon \tag{5}
\]

The similar property for \( v^h \) follows immediately, for any \( \epsilon > 0 \)

\[
v^h(t, w + \epsilon) = E_t^Q \left[ e^{-r} v \left( t + 1, (w + \epsilon + cL_t) \frac{S_{t+1}}{S_t} \right) \right] \\
\leq E_t^Q \left[ e^{-r} v \left( t + 1, (w + cL_t) \frac{S_{t+1}}{S_t} \right) + e^{-r} \epsilon \frac{S_{t+1}}{S_t} \right] \\
= v^h(t, w) + \epsilon
\]

A.2.4 Convexity of the value function

We follow Ben-Ameur et al. (2002), and prove the convexity by induction. For any \( w_1 > 0 \) and \( w_2 > 0 \), and \( 0 \leq \lambda \leq 1 \)

\[
v^h(T - 1, \lambda w_1 + (1 - \lambda) w_2) = E_{T-1}^Q \left[ e^{-r} \left( \lambda w_1 + (1 - \lambda) w_2 + cL_{T-1} \right) \frac{S_T}{S_{T-1}} - bL_{T-1} T \bar{a}(T) \right] \\
\leq \lambda v^h(T - 1, w_1) + (1 - \lambda)v^h(T - 1, w_2)
\]

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Thus, $v^h$ is convex at time $T-1$, and similarly $v^e$ is convex at time $T - 1$. Since $v(T - 1, w) = \max \left( v^e(T - 1, w), v^h(T - 1, w) \right)$, $v$ is also convex at time $T - 1$.

We now assume that result holds for time $t + 1$, where $0 \leq t \leq T - 2$, then the continuation value at time $t$ is

$$v^h(t, \lambda w_1 + (1 - \lambda)w_2) = E_t^Q \left[ e^{-r}v \left( t + 1, \lambda w_1 + (1 - \lambda)w_2 + cL_t \frac{S_{t+1}}{S_t} \right) \right]$$

$$\leq E_t^Q \left[ e^{-r} \left( \lambda v \left( t + 1, (w_1 + cL_t) \frac{S_{t+1}}{S_t} \right) + (1 - \lambda)v \left( t + 1, (w_2 + cL_t) \frac{S_{t+1}}{S_t} \right) \right) \right]$$

$$= \lambda e_t^Q \left[ e^{-r}v \left( t + 1, (w_1 + cL_t) \frac{S_{t+1}}{S_t} \right) \right] + (1 - \lambda)E_t^Q \left[ e^{-r} \left( t + 1, (w_2 + cL_t) \frac{S_{t+1}}{S_t} \right) \right]$$

Thus, $v^h$ is a convex function at time $t$. Since $v^e$ holds the convexity for all $t$, and $v(t, w) = \max \left( v^e(t, w), v^h(t, w) \right)$ holds for all $t$, then $v(t, w)$ is convex function at time $t$. Then, by induction, we have proved the convexity of $v(t, w)$.

### A.3 Properties of C and D - Discrete Case

Consider first the case when $w < K_t$ at time $t$. We have $v^e(t, w) = (w - K_t)^+$, so

$$w < K_t \Rightarrow v^e(t, w) = 0$$

The value function, given in equation (2), is the expected value of a function bounded below by zero, and which has a positive probability of being greater than zero, which means that the expected value is strictly greater than zero. The continuation function is the expected discounted value of the 1-year ahead value function (assuming the option is not exercised immediately), so it must be strictly greater than 0.

So whenever $w \in [0, K_t]$, we have $v^b(t, w) > 0 = v^e(t, w)$. Thus if $w \in [0, K_t]$, it cannot be optimal to exercise. Therefore, in order to explore the exercise frontier, we need only consider cases when $w > K_t$. 

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When \( w > K_t \) we have \( v^e(t, w) = w - K_t \) and

\[
(t, w) \in D \implies v(w, t) = v^e(t, w) \implies v(w, t) = w - K_t
\]

From Appendix A.2.3, \( v(t, w) + \epsilon \geq v(t, w + \epsilon) \) and \( v^h(t, w) + \epsilon \geq v^h(t, w + \epsilon) \) for any \( w \in \mathbb{R} \) and \( \epsilon > 0 \).  

First, assume that \( w > K_t \) and that \((t, w) \in D\),

\[
(t, w) \in D \implies v^e(t, w) \geq v^h(t, w) \implies \forall \epsilon > 0 \quad v^e(t, w + \epsilon) = v^e(t, w) + \epsilon \geq v^h(t, w) + \epsilon \geq v^h(t, w + \epsilon) \implies (t, w + \epsilon) \in D
\]

Next, assume that \( w > K_t \) and that \((t, w) \in C\). Note that \( v^h(t, w) + \epsilon \geq v^h(t, w + \epsilon) \) for all \( \epsilon > 0 \) implies that \( v^h(t, w - \epsilon) \geq v^h(t, w) - \epsilon \)

\[
(t, w) \in C \implies v^h(t, w) > v^e(t, w) \implies \forall \epsilon > 0 \quad v^h(t, w - \epsilon) \geq v^h(t, w) - \epsilon > v^e(t, w) - \epsilon = v^e(t, w - \epsilon) \implies (t, w - \epsilon) \in C
\]

These results show that it is optimal for the employee to switch to the DB plan only when his/her DC account balance is above a certain threshold at each possible switching time.

### A.4 Properties of \( \varphi(t) \) - Discrete Case

This section provides the proof of Proposition 3. First, notice that if \( \varphi(t) < \infty \), then for sufficiently large \( w \)

\[
v(t, w) = v^e(t, w) \geq v^h(t, w) = E^Q_t \left[ e^{-r(v \left( t + 1, (w + cL_t) \frac{S_{t+1}}{S_t} \right)} \right]  
\]

\[
\geq E^Q_t \left[ e^{-r(v \left( t + 1, (w + cL_t) \frac{S_{t+1}}{S_t} \right)} \right]  
\]

\[
= (w + cL_t) N \left( d_{1,t,w} \right) - (t + 1)bL_t \tilde{u}(T)e^{-r(T-t-1)} e^{-rN(d_{2,t,w})}
\]

where

\[
d_{1,t,w} = \frac{1}{\sigma_S} \left[ \ln \left( \frac{w + cL_t}{(t + 1)bL_t \tilde{u}(T)e^{-r(T-t-1)^p}} \right) + \left( r + \frac{\sigma_S^2}{2} \right) \right]
\]

\[
d_{2,t,w} = d_{1,t,w} - \sigma_S
\]
which is the Black-Scholes Formula, with initial stock price \( w + cL_t \) and strike value \( (t + 1)bL_t\tilde{a}(T)e^{-r(T-t-1)} \). Here we define

\[
f(t) = \lim_{w \to \infty} v^c(t, w) - E^Q_t \left[ e^{-r} v^e \left( t + 1, (w + cL_t) \frac{S_{t+1}}{S_t} \right) \right] \\
= \lim_{w \to \infty} v^c(t, w) - \left( (w + cL_t) N(d_{1,t,w}) - (t + 1)bL_t\tilde{a}(T)e^{-r(T-t-1)} e^{-r} N(d_{2,t,w}) \right) \\
= \lim_{w \to \infty} - \left( (t + 1)bL_t\tilde{a}(T)e^{-r(T-t)} N(-d_{2,t,w}) - (w + cL_t) N(-d_{1,t,w}) \right) \\
- cL_t + (t + 1)bL_t\tilde{a}(T)e^{-r(T-t)} - tbL_{t-1}\tilde{a}(T)e^{-r(T-t)} \\
= (t + 1)bL_t\tilde{a}(T)e^{-r(T-t)} - tbL_{t-1}\tilde{a}(T)e^{-r(T-t)} - cL_t
\]

Clearly, whenever \( \varphi(t) < \infty \), we have \( f(t) \geq 0 \).

If \( \frac{c}{b\tilde{a}(T)e^{-rT}} < 1 \), we prove \( \varphi(t) < \infty \), \( \forall t \in [0, T] \) by induction.

**At time** \( T \), \( \varphi(T) = TbL_{T-1}\tilde{a}(T) < \infty \).

**At time** \( t \), assume \( \varphi(t+1) < \infty \). We observe that for sufficiently large \( w < \infty \), we have \( v(t+1, w) = v^e(t+1, w) \). Next, we can show

\[
\lim_{w \to \infty} v^b(t, w) - E^Q_t \left[ e^{-r} v^e \left( t + 1, (w + cL_t) \frac{S_{t+1}}{S_t} \right) \right] \\
= \lim_{w \to \infty} E^Q_t \left[ e^{-r} \left( v \left( t + 1, (w + cL_t) \frac{S_{t+1}}{S_t} \right) - v^e \left( t + 1, (w + cL_t) \frac{S_{t+1}}{S_t} \right) \right) \right] \\
= 0
\]

The last line is due to the fact that if \( w \geq \varphi(t+1) \)

\( v(t+1, w) - v^e(t+1, w) = 0 \leq \varphi(t+1) < \infty \)

and for \( w < \varphi(t+1) \)

\( v(t+1, w) - v^e(t+1, w) \leq v^e(t+1, \varphi(t+1)) \leq \varphi(t+1) < \infty \)

since \( v(t+1, w) \) is an increasing function of \( w \) (Appendix A.2.1). Thus, the difference is bounded.
by $\varphi(t + 1) < \infty$ and we are able to apply the Dominated Convergence Theorem. Next,

$$\lim_{w \to \infty} v^e(t, w) - v^h(t, w) = \lim_{w \to \infty} v^e(t, w) - E^Q_t \left[ e^{-r} v^e \left( (w + cL_t) \frac{S_{t+1}}{S_t} \right) \right]$$

$$= (t + 1)bL_t \bar{a}(T)e^{-r(T-t)} - tb\bar{a}(T)e^{-r(T-t)} L_{t-1} - cL_t$$

$$> (t + 1)bL_t \bar{a}(T)e^{-r(T-t)} - tb\bar{a}(T)e^{-r(T-t)} L_{t-1} - L_t b\bar{a}(T)e^{-rT}$$

$$> (t + 1)bL_t \bar{a}(T)e^{-r(T-t)} - tb\bar{a}(T)e^{-r(T-t)} L_t - L_t b\bar{a}(T)e^{-r(T-t)}$$

$$= 0$$

Which implies $\varphi(t) < \infty$ (otherwise if $\varphi(t) = \infty$, $\lim_{w \to \infty} v^e(t, w) - v^h(t, w) \leq 0$).

Thus $\varphi(t) < \infty$, $\forall t \in [0, T]$.

For $\frac{c}{b\bar{a}(T)e^{-rT}} \geq 1$, we split the proof into three parts.

1. If $\frac{c}{b\bar{a}(T)e^{-rT}} > 1$, we first prove that there exists a $t_*$ such that $\varphi(t) = \infty$, $\forall t \leq t_*$, then prove that $\varphi(t) < \infty$, $\forall t > t_*$ by induction from time $T$ to $t_* + 1$ as above.

   We have $f(0) < 0$ so that

   $$\lim_{w \to \infty} v^e(0, w) - v^h(0, w) \leq f(0) < 0$$

   and thus $\varphi(0) = \infty$. Also, notice we can write $f(t)$ in the form

   $$f(t) = e^{\mu_L t} \left( te^{-r(T-t)} b\bar{a}(T)(1 - e^{-\mu_L}) + \bar{a}(T)e^{-r(T-t)} - c \right) = e^{\mu_L t} h(t)$$

   where $h(t)$ is a strictly increasing function of time $t$, if both $r$ and $\mu_L$ are non-negative, with at least one of them being strictly positive. We have $f(0) < 0$ and

   $$f \left( \frac{1}{r} \log \left( \frac{c}{b\bar{a}(T)e^{-rT}} \right) \right) > 0$$

   where $\frac{1}{r} \log \left( \frac{c}{b\bar{a}(T)e^{-rT}} \right) > 0$ by assumption.

   Then, we can find $t'$ such that

   $$f(t) < 0 \text{ for } t < t'$$

   $$f(t) = 0 \text{ for } t = t'$$

   $$f(t) > 0 \text{ for } t > t'$$

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Here we set $t_*=\lfloor t'\rfloor$, and we have
\[
\lim_{w\to\infty} v^e(t, w) - v^h(t, w) \leq f(t) < 0, \forall t < t_*
\]
and for $t = t_*$, first notice that
\[
f(t_*) \leq 0 \implies c \geq b\bar{u}(T)e^{-rT}(t_* + 1 - t_*e^{-\mu T})e^{rt_*}
\]
Next, for any finite $w > t_*L_{t_*-1}b\bar{u}(T)e^{-r(T-t_*)}$, we have
\[
v^h(t_*, w) = E^Q_t\left[e^{-r}v\left(t_* + 1, (w + cL_{t_*})\frac{S_{t_*+1}}{S_{t_*}}\right)\right]
\]
\[
> E^Q_t\left[e^{-r}v^e\left(t_* + 1, (w + cL_{t_*})\frac{S_{t_*+1}}{S_{t_*}}\right)\right]
\]
\[
= E^Q_t\left[e^{-r}\left((w + cL_{t_*})\frac{S_{t_*+1}}{S_{t_*}} - (t_* + 1)L_{t_*}b\bar{u}(T)e^{-r(T-t_*-1)}\right)^+\right]
\]
\[
\geq \max\left(0, (w + cL_{t_*} - (t_* + 1)L_{t_*}b\bar{u}(T)e^{-r(T-t_*-1)})\right)
\]
\[
\geq \max\left(0, w - t_*L_{t_*-1}b\bar{u}(T)e^{-rT}e^{rt_*}\right)
\]
\[
v^e(t_*, w)
\]
The third to fourth line follows from Jensen’s Inequality:
\[
E^Q_t\left[e^{-r}\left((w + cL_{t_*})\frac{S_{t_*+1}}{S_{t_*}} - (t_* + 1)L_{t_*}b\bar{u}(T)e^{-r(T-t_*-1)}\right)^+\right]
\]
\[
\geq \max\left(0, E^Q_t\left[e^{-r}\left((w + cL_{t_*})\frac{S_{t_*+1}}{S_{t_*}} - (t_* + 1)L_{t_*}b\bar{u}(T)e^{-r(T-t_*-1)}\right)\right]\right)
\]
\[
= \max\left(0, w + cL_{t_*} - (t_* + 1)L_{t_*}b\bar{u}(T)e^{-r(T-t_*-1)}\right)
\]
Thus, we have $v^h(t_*, w) > v^e(t_*, w), \forall w < \infty$, and
\[
\lim_{t\to\infty} v^e(t_*, w) - v^h(t_*, w) \leq f(t_*) \leq 0
\]
which implies $\varphi(t_*) = \infty$.

Repeating the induction:

**At time $T$,** again we have $\varphi(T) < \infty$.

**At time $t > t_*$, assume $\varphi(t+1) < \infty$.**

\[
\lim_{w\to\infty} v^e(t, w) - v^h(t, w) = f(t) > 0, t > t_*
\]
Thus $\varphi(t) < \infty, \forall t > t_*$. 

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(2) If $c > b\bar{u}(T) ((1 - e^{-\mu_L}) T + e^{-\mu_L}) e^{-r}$

\[
c > b\bar{u}(T) e^{-r} (T - (T - 1)e^{-\mu_L}) \\
> b\bar{u}(T) e^{-r} \geq b\bar{u}(T) e^{-rT}
\]

Thus, we know there exists $t_*$ as defined previously, and for time $T - 1$,

\[
\lim_{w \to \infty} v^e(T - 1, w) - v^h(T - 1, w) \\
= TbL_{T-1}\bar{u}(T)e^{-r} - (T - 1)bL_{T-2}\bar{u}(T)e^{-r} - cL_{T-1} \\
< TbL_{T-1}\bar{u}(T)e^{-r} - (T - 1)bL_{T-2}\bar{u}(T)e^{-r} - b\bar{u}(T) \left( (1 - e^{-\mu_L}) T + e^{-\mu_L} \right) e^{-r} L_{T-1} \\
= 0
\]

Thus, $\varphi(t) = \infty, \forall t \leq T - 1$, and the option simplifies to the DB underpin option.

(3) When $\frac{c}{b\bar{u}(T)e^{-rT}} = 1$, we have $f(0) = 0$. Thus $t_* = t' = 0$, immediately we have $\varphi(0) = \infty$.

### A.5 Formulation of Value Function - Continuous Case

Recall $S_t$, $L_t$ represent the stock process and the salary process, and, $W_t$ represents the wealth accumulation process in the DC account. Their stochastic differential equation representations are:

\[
\begin{align*}
\frac{dS_t}{S_t} &= rS_t dt + \sigma_S S_t dZ^Q_S(t) \\
\frac{dL_t}{L_t} &= rL_t dt + \sigma_L L_t dZ^Q_L(t) \\
\frac{dW_t}{W_t} &= rW_t dt + cL_t dt + \sigma_W W_t dZ^Q_S(t) \\
\frac{\rho dt}{\rho} &= dZ^Q_L(t) dZ^Q_S(t)
\end{align*}
\]

Here we denote $Y_t = \frac{W_t}{L_t}$, as the wealth-salary ratio process. Again, for $s \geq t$ we denote $W^s_{t,w}$, $L^s_{t,l}$ and $Y^s_{t,y}$ as the strong solutions to the respective SDEs at time $s$, starting from $w$, $l$, $y$ at time $t$. Then, we can rewrite our value function:
Then our value function becomes

\[ v(t, w, l) = \sup_{0 \leq \tau \leq T-t} E^Q_t \left[ e^{-r\tau} \left( W_{t+\tau}^l - K_{t+\tau} \right)^+ \right] \]

\[ = \sup_{0 \leq \tau \leq T-t} E^Q_t \left[ e^{-r\tau} L_{t+\tau} \left( \frac{w}{L_{t+\tau}} \frac{S_{t+\tau}}{S_t} + c \int_0^\tau \frac{S_{t+\tau}L_u}{S_u L_{t+\tau}} - b(t + \tau) \bar{a}(T) e^{-r(T-t)} \right)^+ \right] \]

\[ = \sup_{0 \leq \tau \leq T-t} E^Q_t \left[ e^{-r\tau} L_{t+\tau} \left( Y_{t+\tau}^{l,w/l} - b(t + \tau) \bar{a}(T) e^{-r(T-t)} \right)^+ \right] \]

\[ = \sup_{0 \leq \tau \leq T-t} E^Q_t \left[ e^{-r\tau} L_{t+\tau} \left( \frac{\sigma^2 \tau}{2} + \sigma_L (Z_{t+\tau}^Q - Z_L^Q(t)) \right) \left( Y_{t+\tau}^{l,w/l} - b(t + \tau) \bar{a}(T) e^{-r(T-t)} \right)^+ \right] \]

We are able to eliminate the discounting term in the expectation through the change of measure method. Let \( d\tilde{P} = \exp \left( \sigma_L Z_L^Q(t) - (\sigma_L^2/2) t \right) dQ \), by Girsanov Theorem, we have

\[
\begin{pmatrix}
Z_S^\tilde{P}(t) \\
Z_L^\tilde{P}(t)
\end{pmatrix} = \begin{pmatrix}
Z_S^Q \\
Z_L^Q
\end{pmatrix} - \begin{pmatrix}
0 \\
\sigma_L
\end{pmatrix} t
\]

as a two-dimensional standard Brownian motion under \( \tilde{P} \). Then, under the new measure, \((S_t, L_t)\) has the following SDE:

\[
\begin{pmatrix}
dS_t \\
L_t
\end{pmatrix} = \begin{pmatrix}
\sigma_S \sigma_L \rho S_t \\
\sigma_L L_t + \sigma_L^2 L_t
\end{pmatrix} dt + \begin{pmatrix}
\sigma_S \sqrt{1 - \rho^2} & \sigma_S S_t \rho \\
0 & \sigma_L L_t
\end{pmatrix} \begin{pmatrix}
dZ_S^\tilde{P}(t) \\
dZ_L^\tilde{P}(t)
\end{pmatrix}
\]

and the SDE for the wealth process is

\[
dW_t = (rW_t + \sigma_S \sigma_L \rho W_t + cL_t)dt + \sigma_S \sqrt{1 - \rho^2} dZ_S^\tilde{P}(t) + \sigma_S W_t \rho dZ_L^\tilde{P}(t), \quad W_0 = 0
\]

then our value function becomes

\[
v(t, w, l) = \sup_{0 \leq \tau \leq T-t} E^\tilde{P}_t \left[ e^{-r\tau} \left( W^{l,w/l}_{t+\tau} - K_{t+\tau} \right)^+ \right] \]

\[
= \sup_{0 \leq \tau \leq T-t} E^\tilde{P}_t \left[ \frac{dQ}{d\tilde{P}} e^{-r\tau} \left( \frac{\sigma^2 \tau}{2} + \sigma_L (Z_{t+\tau}^Q - Z_L^Q(t)) \right) \left( Y_{t+\tau}^{l,w/l} - b(t + \tau) \bar{a}(T) e^{-r(T-t)} \right)^+ \right] \]

\[ \quad = \sup_{0 \leq \tau \leq T-t} E^\tilde{P}_t \left[ \left( Y_{t+\tau}^{l,w/l} - b(\tau + t) \bar{a}(T) e^{-r(T-t)} \right)^+ \right] \]
where $Y_t$ has the SDE as:

$$
\begin{align*}
    dY_t &= \frac{1}{L_t} dW_t - \frac{W_t}{L_t^2} dL_t - \frac{1}{L_t^2} dL_t dW_t + \frac{W_t}{L_t^2} (dL_t)^2 \\
    &= (r + \sigma_s \sigma_L \rho) Y_t dt + \sigma_s \sqrt{1 - \rho^2} Y_t dZ^\tilde{P}_S(t) + \sigma_s \rho Y_t dZ^\tilde{P}_L(t) \\
    &\quad - Y_t \left( (r + \sigma_L^2) dt + \sigma_L dZ^\tilde{P}_L(t) \right) - Y_t \sigma_L \sigma_S \rho dt + Y_t \sigma_S^2 dt \\
    &= cdY_t + \sigma_s \sqrt{1 - \rho^2} Y_t dZ^\tilde{P}_S(t) + Y_t (\sigma_s \rho - \sigma_L) dZ^\tilde{P}_L(t) \\
    &= cd\tau + Y_t \sigma_Y dZ^\tilde{P}_Y(t)
\end{align*}
$$

where $\sigma_Y = \sqrt{\sigma_S^2 + \sigma_L^2 - 2\sigma_S \sigma_L \rho}$ and $Z^\tilde{P}_Y(t)$ is a standard Brownian Motion under measure $\tilde{P}$.

We can define a new function (2-dimensional):

$$
v(t, y) = \sup_{0 \leq \tau \leq T - t} E^\tilde{P} \left[ (Y_{t+\tau} - b(t + \tau)\bar{a}(T)e^{-(T-t-\tau)r})^+ \bigg| \mathcal{F}_t \right]
$$

Thus, $v(t, w/l) = v(t, w, l)/l$, which clearly suggests that the exercise rule depends on the wealth-to-salary ratio.

## A.6 Characteristics of Value Function - Continuous Case

### A.6.1 Non-decreasing in $y$

By writing value function explicitly

$$
\begin{align*}
    v(t, y) &= \sup_{0 \leq \tau \leq T - t} E^\tilde{P} \left[ (Y_{t+\tau} \frac{S(t + \tau)}{L(t + \tau)S(t)} + \int_t^{t+\tau} \frac{S(t + \tau)L(u)}{S(u)L(t + \tau)} du - b(t + \tau)\bar{a}(T)e^{-(T-t-\tau)r})^+ \right]
\end{align*}
$$

we immediately see that $y \to v(t, y)$ is an increasing and convex function on $[0, \infty)$.
A.6.2 Continuity of value function in y

for $x > y$

$$|v(t, x) - v(t, y)|$$

$$\leq \left| \sup_{0 \leq \tau \leq T-t} E_t^\tilde{P} \left[ (Y_{t+\tau}^{t,x} - b(t + \tau)\tilde{a}(T)e^{-(T-t-\tau)r})^+ - (Y_{t+\tau}^{t,y} - b(t + \tau)\tilde{a}(T)e^{-(T-t-\tau)r})^+ \right] \right|$$

$$\leq \left| \sup_{0 \leq \tau \leq T-t} E_t^\tilde{P} \left[ Y_{t+\tau}^{t,x} - Y_{t+\tau}^{t,y} \right] \right|$$

$$\leq \left| \sup_{0 \leq \tau \leq T-t} E_t^\tilde{P} \left[ L_tS_{t+\tau} - yL_tS_{t+\tau} \right] \right|$$

$$\leq (x - y) \left| \sup_{0 \leq \tau \leq T-t} E_t^\tilde{P} \left[ \frac{L_tS_{t+\tau}}{L_{t+\tau}S_t} \right] \right|$$

A.6.3 Non-increasing in time t

Since $Y_{t_2+t}^{t_1,y} \overset{law}{=} Y_{t_2+t}^{t_1,y}$, and the strike function $G_t = tbe^{-(T-t)r}\tilde{a}(T)$ is an increasing function of time t, immediately we have $v(t, y)$ is non-increasing in t.
A.6.4 Continuity of value function in $t$

For $t_2 > t_1$,

$$0 \leq v(t_1, y) - E_{t_1}^{\bar{P}}[v(t_2, Y_{t_2}^{t_1, y})] = \sup_{\tau_1 \leq T-t_1} E_{t_1}^{\bar{P}}[1_{\tau_1 \leq t_2-t_1} v^e(t_1 + \tau_1, Y_{t_1+t_1}^{t_1, y}) + 1_{\tau_1 \geq t_2-t_1} v(t_2, Y_{t_2}^{t_1, y}) - E_{t_1}^{\bar{P}}[v(t_2, Y_{t_2}^{t_1, y})]] \leq \sup_{\tau_1 \leq T-t_1} E_{t_1}^{\bar{P}}[1_{\tau_1 \leq t_2-t_1} v^e(t_1 + \tau_1, Y_{t_1+t_1}^{t_1, y}) - E_{t_1}^{\bar{P}}[v(t_2, Y_{t_2}^{t_1, y})]] \leq \sup_{\tau_1 \leq T-t_1} E_{t_1}^{\bar{P}}[1_{\tau_1 \leq t_2-t_1} (Y_{t_1+t_1}^{t_1, y} - (t_1 + \tau_1)b\bar{u}(T)e^{-(T-t_1-r)} (Y_{t_2}^{t_1, y} - t_2b\bar{u}(T)e^{-(T-t_2-r)}) + ) - (Y_{t_2}^{t_1, y} - t_2b\bar{u}(T)e^{-(T-t_2-r)}) + ] \leq E_{t_1}^{\bar{P}}(\sup_{\tau_1 \in [t_1, t_2]} |Y_{t_1+t_1}^{t_1, y} - Y_{t_2}^{t_1, y}| + (t_2b\bar{u}(T)e^{-(T-t_2-r)} - t_1b\bar{u}(T)e^{-(T-t_1)r}) \leq C_1 \sqrt{t_2-t_1}$$

where the last line comes from Touzi (2013), Theorem 2.4, that there exists a constant $C$ such that

$$E_{t_1}^{\bar{P}}(\sup_{\tau_1 \in [t_1, t_2]} |Y_{t_1+t_1}^{t_1, y} - Y_{t_2}^{t_1, y}|) \leq C(1 + |y|) \sqrt{t_2-t_1}$$

Next, we have

$$|v(t_1, y) - v(t_2, y)| \leq |v(t_1, y) - E_{t_1}^{\bar{P}}[v(t_2, Y_{t_2}^{t_1, y})]| + |E_{t_1}^{\bar{P}}[v(t_2, Y_{t_2}^{t_1, y})] - v(t_2, y)| \leq C_1 \sqrt{t_2-t_1} + E_{t_1}^{\bar{P}}(\sup_{\tau_1 \in [t_1, t_2]} |Y_{t_1+t_1}^{t_1, y} - Y_{t_2}^{t_1, y}|) \leq C_1 \sqrt{t_2-t_1} + C_2 E_{t_1}^{\bar{P}}(\sup_{\tau_1 \in [t_1, t_2]} |Y_{t_1+t_1}^{t_1, y} - y|) \leq C \sqrt{t_2-t_1}$$

where the third line follows from the Lipschitz continuity of the value function in $y$. Thus, we have proved that the value function is Hölder-Continuous in $t$. 

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A.6.5 Exercise Region

Denote the continuation region $C = \{(t,y) \in (0,T) \times [0,\infty) : v(t,y) > v^e(t,y)\}$ and stopping regions $C = \{(t,y) \in (0,T) \times [0,\infty) : v(t,y) = v^e(t,y)\}$. Then, by standard arguments based on the strong Markov Property (see Peskir and Shiryaev (2006) Corollary 2.9), the first hitting time $\tau_D = \inf \{0 \leq s \leq T - t : (t + s, Y_{t+s}) \in D\}$ is optimal, and the value function is $C^{1,2}$ on $C$ and satisfies:

$$v_t + L_Y v = 0 \quad \text{in} \quad C$$

If $(t,y) \in D$,

$$v(t,y + h) \leq v(t,y) + h = v^e(t,y) + h = v^e(t,y + h)$$

thus $(t,y + h) \in D$.

If $(t,y) \in C$,

if $y - h > G_t$, $v(t,y - h) \geq v(t,y) - h > v^e(t,y) - h = v^e(t,y - h)$

if $y - h < G_t$, $v(t,y - h) > 0 = v^e(t,y - h)$, thus $(t,y - h) \in C$

A.6.6 Conditions when $\varphi(t) = \infty$

Define and substitute $g(t,y) = (y - bt\ddot{a}(T)e^{-r(T-t)})$ for $y > G_t$ into the variational inequality, and define

$$f(t) = -\frac{\partial g}{\partial t} - c \frac{\partial g}{\partial y} - \frac{\sigma_Y^2 y^2}{2} \frac{\partial^2 g}{\partial y^2} = b\ddot{a}(T)e^{-r(T-t)} + rb\ddot{a}(T)e^{-r(T-t)} - c$$

If $f(t) < 0$, then there is a contradiction with the variational inequality, which means $v(t,y) \neq g(t,y)$ for all $y$, and thus $(t,y) \in C, \forall y$. Since $f(t)$ is an increasing function of $t$, there exists a $t^*$ satisfying $f(t^*) = 0$. If $t^* \in [0,T]$, then $(t,y) \in C, \forall (t,y) \in [0,t^*] \times \mathbb{R}$. In particular, if $c > b\ddot{a}(T)(1 + rT)$ (when DC contribution rate is extremely high or horizon is short), the option is equivalent to European Option.

A.7 Price of Second Election

Here we provide the pricing formulae for second election option under three scenarios.
• Stochastic Salary in Continuous Setting:

\[ C_s(t, L_t) = (L_t c(t^* - t) - t^* L_t b\tilde{a}(T)e^{-r(T-t^*)}) + Tb\tilde{a}(T)L_t \]

which we can find a \( t^* = \max(\min(T, t'), t) \), that \( t' \) satisfies

\[ c - b\tilde{a}(T)e^{-\gamma(T-t')} - rt'b\tilde{a}(T)e^{-\gamma(T-t')} = 0 \]

• Deterministic Salary in Continuous Setting:

\[ C_s(t, L_t) = \frac{cL_t}{\mu_L - r} \left( e^{(\mu_L - r)(t^* - t)} - 1 \right) - t^* b\tilde{a}(T)L_te^{(\mu_L - r)(t^* - t)}e^{-\gamma(T-t^*)} + TLte^{(\mu_L - r)(T-t)}b\tilde{a}(T) \]

where \( t^* = \min(\max(t, t'), T) \) and \( t' \) satisfies

\[ c - b\tilde{a}(T)e^{-\gamma(T-t')} - t'(\mu_L - r + \gamma)b\tilde{a}(T)e^{-\gamma(T-t')} = 0 \]

• Deterministic Salary in Discrete Setting:

\[ C_s(t, L_t) = \max_{t^* \in \{t, t+1, \ldots, T\}} L_t \left( c \frac{1 - e^{(\mu_L - r)(t^* - t)}}{e^{\mu_L - r} - 1} - t^* b\tilde{a}(T)e^{\mu_L(t^* - t - 1)}e^{-rt^*}e^{-\gamma(T-t^*)} + Tb\tilde{a}(T)e^{\mu_L(T-1)}e^{-r(T-t)} \right) \]