Abstract

This report focuses on hedging financial risks in variable annuities with guarantees. We start by a review of existing standard hedging strategies. We then discuss the pros and cons of each of them and identify the important properties. Using standard dynamic hedging techniques, we show that best hedging performance is obtained when the hedging strategy is rebalanced at the dates when the variable annuity fees are collected. Thus, our results suggest that the insurer should incorporate the specificity of the periodic payment of variable annuities fees as a percentage of the underlying to construct the hedging portfolio and focus on hedging the net liability instead of the liability. Since standard dynamic hedging is costly in practice because of the large number of rebalancing dates, we propose a new hedging strategy based on a semi-static hedging technique and thus with fewer rebalancing dates. We confirm that this new strategy outperforms standard dynamic hedging as well as traditional semi-static hedging strategies that do not consider the specificity of the payments of fees in their optimization. We also find that short-selling is necessary to construct an efficient hedging portfolio and adding put options as hedging instruments gives better hedging performance.

Key-words: Hedging variable annuities, semi-static hedging
1 Introduction

Life expectancies are steadily increasing and the post-retirement life is becoming longer. The population faces the need to guarantee a sustainable income after retirement. Traditional sources of retirement income such as social security supported by government and defined benefit pension plans are becoming unsustainable, or are disappearing. As a consequence, there has been a growing demand for variable annuities, which offer guaranteed income for post-retirement life. Variable annuities are also called unit-linked products in Europe, segregated funds in Canada or more generally equity-linked insurance (Hardy (2003)). A recent state-of-the-art on the Variable Annuities market can be found in Haeffeli (2013) with figures and charts illustrating the Variable Annuities market in the U.S., Canada, Europe and Japan between 1990 and 2011.

In a typical variable annuity contract, there are two phases. During the accumulation phase, the policyholder invests an initial premium and possibly subsequent premiums into a basket of invested funds (including typically subaccounts invested in stocks, bonds and money market accounts). In a second phase starting typically at the end of the accumulation phase, the policyholder may receive a lump sum or may annuitize it to provide retirement income. Variable annuities contain both investment and insurance features. They improve upon traditional fixed annuity contracts that offer a stream of fixed retirement income, as policyholders of variable annuities could expect higher returns in a bull market. Moreover, variable annuities also provide some protection against downside risk with several kinds of optional benefit riders. Traditional annuities could simply be hedged by an appropriate portfolio of bonds. But the presence of protection benefits require sophisticated hedging strategies. Variable annuities turn out to be exposed to a variety of risks (mortality risk, market risks and policyholder behavioral risks) as we will discuss in full details.

A variable annuity can be described as follows. A policyholder has a separate account for investment where she invests her premiums. This separate account is protected from downside risks of investing in the financial market through additional options that are sold with the variable annuity and also called benefit riders. Here are some popular examples of such benefits. More details on each of these guarantees can be found in Hardy (2003).

- **Guaranteed Minimum Accumulation Benefit (GMAB).** The GMAB is the simplest benefit: it guarantees a specified lump sum on a specific future date or anniversary. But it is not as popular as the following more complex benefits, GMIBs and GMWBs.

- **Guaranteed Minimum Income Benefit (GMIB):** GMIB guarantees a stream of lifetime annuity income after policyholder’s annuitization decision is made. The annuitization decision is irreversible. The annuity income is then determined based on the policyholder’s account value and some minimum specified benefit base linked to the policyholder’s age at annuitization.
• Guaranteed Minimum Withdrawal Benefit (GMWB): GMWB guarantees the ability to withdraw a specified percentage of the benefit base during a specified number of years or it could be a lifetime benefit (Kling, Ruez, and Russ (2011)).

• Guaranteed Minimum Death Benefit (GMDB): GMDB guarantees a specified lump sum benefit at the time of death of the policyholder.

According to LIMRA (2012), 65% of VAs sold in 2012 include living benefits (GMWB, GMIB, GMAB) where the most popular option is a lifetime GMWB as it provides protection against longevity risk and market risk. The withdrawal decision is made by the policyholder. These guarantees are often associated with a minimum guaranteed rate that determines the level of benefits and that has additional features, such as roll-ups and ratchets (or resets). Roll-ups are typically compounded minimum guaranteed rates whereas ratchets allow to reset the benefits level at some prespecified dates based on the highest value attained by the fund or respectively on the current fund value.

There are lots of studies on the pricing of variable annuities and how to find the fair fee: Milevsky and Posner (2001) and Bacinello (2003) investigate the valuation of GMDB in variable annuities using risk-neutral pricing. They consider various death benefits such as return on principal, rising floors and ratchets and compute the fair fee needed to fund the promised benefits. Lin, Tan, and Yang (2009) price simple guarantees in VAs in a regime switching model. Marshall, Hardy, and Saunders (2010) study the value of a GMIB in a complete market, and the sensitivity of the value of GMIB to the financial variables is examined. They suggest that the fee rate charged by insurance companies for GMIB may be too low. GMWB is studied intensively by several authors including Milevsky and Salisbury (2006), Chen, Vetzal, and Forsyth (2008), Dai, Kwok, and Zong (2008), Kolkiewicz and Liu (2012) and Liu (2010). The main result of Milevsky and Salisbury (2006) is that GMWB fees charged in the market are too low and not sustainable. They argue that the fees have to increase or the product design should change to avoid arbitrage. Chen, Vetzal, and Forsyth (2008) also conclude that normally charged GMWB fees are not enough to cover the cost of hedging. See Sherris (1995) for an overview. Dai, Kwok, and Zong (2008) investigate the optimal withdrawal strategy of a rational policyholder that maximizes the expected discounted value of the cash flows generated from a variable annuity contract with GMWB. Stochastic control is used to model the optimal behavior of the policyholder. Bacinello, Millossovich, Olivieri, and Pitacco (2011) suggest a general framework for the valuation of guarantees under optimal behavior of policyholder. One of their main results is that GMIB in the market are underpriced and the charged fees are too small. Bauer, Kling, and Russ (2008) propose a unifying framework for variable annuities valuation. Chi and Lin (2012) consider flexible premium variable annuities (FPVA) instead of single premium variable annuities (SPVA). Their result shows that the mortality and expense fee is significantly higher for FPVAs than that for SPVAs. The recent study of Huang, Milevsky, and Salisbury (2014) deals with the optimal initiation of withdrawals.

There are fewer papers on hedging, although hedging embedded guarantees of variable
annuities is a challenging and crucial problem for insurers. There are three main sources of risks, financial risks, policyholders' behavioral risks and mortality risks.

When mortality risk is fully diversifiable, it is straightforward to hedge mortality risks by selling independent policies to a group of policyholders with similar risks of death. However, mortality risks cannot always be diversified and VAs are exposed to longevity risk. Longevity risk refers to a systematic change in mortality risk affecting simultaneously all the population. It is a topic of research by itself and a thorough analysis of longevity risk in VAs can be found in Ngai and Sherris (2011), Hanewald, Piggott and Sherris (2012), Gatzert and Wesker (2012), Fung, Ignatieva, and Sherris (2013) for example.

Behavioral risks in variable annuities come from the uncertainty faced by the insurer about the policyholders' decisions (choice of surrender, partial withdrawal, annuitization, reallocation, additional contributions, and so on). In general, they are hard to hedge. Under the assumption that investors do not act optimally and base their decisions on non-financial variables, behavioral risk can be diversified similarly as mortality risk. A typical assumption to hedge behavioral risks is then to use a deterministic decision making process using historical statistics which state for instance that $x\%$ of the policyholders follow a given behaviour in a specific situation. Hedging is done by pooling such as these risks are diversified away similarly as mortality risks. However, there is empirical evidence that policyholders may act optimally, or at least that their decision is correlated with some market factors and depends on the moneyness of the guarantee, so that all behavioral risks may not be diversified away (see for instance the empirical study of Knoller, Kraut, and Schoenmaekers (2013)). Kling, Ruez, and Russ (2014) study the impact of behavioral risks on the pricing as well as on the effectiveness of the hedging of VAs. They consider various assumptions on behaviors (from deterministic to optimal decision making). They also investigate the impact of stochastic volatility combined with behavioral risks. Interestingly, the impact of model misspecification on policyholders' behaviors depends highly on the design of the policy. For instance, the effect of the surrender decision is more important in VAs without ratchets (but ratchets are highly sensitive to volatility and are difficult to hedge for financial risks). See also Augustyniak (2013) for a study of the effect of lapsations on the hedging effectiveness of the guaranteed minimum maturity benefit (GMMB).

In this report, we ignore mortality and longevity risks, policyholder behavioral risks and focus on the hedging of financial risks. It can be done via delta hedging, semi-static hedging or static hedging. In practice, delta hedging requires to rebalance at a discrete set of dates the portfolio and to purchase and sell the underlying. It can be interpreted as a semi static hedging strategy for which there are no options available and trading involves only a risk-free asset and the underlying risky asset. We will start in Section 2 with an extensive review of the different methods proposed in the literature for hedging financial risks and their pros and cons. In particular, we will explain how the focus of many papers in the literature is on the hedge of the liability ignoring the periodicity of the fees paid. However, the guarantees in VAs differ from a put option in that no certain premium is paid upfront so that uncertainty and risk are inherent in the payoff and in
the premiums. Therefore, both components should be hedged. Kolkiewicz and Liu (2012) and Augustyniak (2013)\(^1\) point out the importance of hedging the net liability taking into account the periodic fees that are paid as a percentage of the underlying fund. In Section 3, we describe a simple GMMB as the toy example to illustrate and compare the different hedging strategies. We then use a standard delta-hedging strategy to show the importance of hedging the net liability in Section 4. In Section 5, we show how to improve delta-hedging strategies by developing a new semi-static hedging strategy. Section 5.3 compares the hedging effectiveness of the optimal semi-static strategy to other hedging strategies. Section 6 concludes.

\section{Hedging Variable Annuities}

The guarantees in VAs are similar to options (financial derivatives) on the fund value and the insurer plays the role of option writer. However, they are also very different from standard derivatives. A crucial difference is that the costs of these options are not paid upfront like initial premiums of options. On the opposite, fees are paid periodically as a percentage of the fund value throughout the life of the contract. The fees collected should then be invested to hedge the provided guarantees. The first issue consists of finding the suitable level of fees to cover the guarantees (fair pricing of VAs). The second issue consists of hedging the guarantees by investing these collected fees in an appropriate way so that at the time the guarantees must be paid the investment matches the guarantees.

We review three standard approaches consisting of dynamic delta hedging, mean variance dynamic hedging (Papageorgiou, Rémillard, and Hocquard (2008), Hocquard, Papageorgiou, and Remillard (2012a,b)) and static hedging. We then propose an improved hedging strategy called semi-static hedging, which is partially dynamic in the sense that it involves some rebalancing dates for the hedge, but fewer than one would typically have in a dynamic delta-hedging strategy. The goal is to take the advantages of the delta-hedging strategies (i.e. to replicate the final benefit by rebalancing the hedging portfolio to match the value of the guarantees at all time) with the advantages of static strategies (which have lower costs given that transaction fees are paid only at inception of the hedging strategy).

\subsection{Specificity of hedging VAs}

The main particularity of hedging variable annuities (instead of standard financial derivatives) is also the main difficulty. It comes from the mismatch of the (random) value of fees collected from the policyholder’s account and the hedging cost. In general, we observe that the value of collected fees and the cost of hedging move in opposite directions. Typically, a

\(^1\)We refer to Chapter 5 of Augustyniak (2013) on “Measuring the effectiveness of dynamic hedges for long-term investment guarantees”.
predetermined percentage of policyholder’s account value is withdrawn periodically as the fee to match the value of the guarantees. When the policyholder’s account value is high, the value of the fee is also high while the value of the embedded option in the guarantee is low. If the account value is low, the value of the fee is low but the insurer needs more money for the guaranteed benefits because the value of embedded option is high. This mismatch represents a challenge for hedging guarantees in VAs. It is also exactly the reason why hedging guarantees in VAs differ from hedging standard options (for which fees are paid at inception only). One focus in this report is to explain how to use the periodic fees paid for the guarantees to develop an efficient hedge.

When hedging a guarantee in a VA, we distinguish between the cash-flow $V_T$ of the policy (which can be also called “unhedged liability” for the insurer) at the payment date $T$, and the “net liability” which is equal to $V_T - Z_T$ where $Z_T$ is the value at $T$ of the accumulated fees that have been collected throughout the life of the contract until time $T$. It is a “net liability” as it incorporates the income of the insurer related to the sale of the policy. Recall that a contract is fairly priced (at $t = 0$) if, under the risk neutral probability, the expected value of the discounted benefit $V_T$ is equal to the expected value of the discounted collected fees $Z_T$ during the life of the contract at inception ($t = 0$).

We will show how our newly proposed semi-static hedging strategy can utilize collected fees (at the exact time they are paid) for rebalancing the hedging portfolio. In particular, such hedging strategy outperforms traditional hedging strategies, which do not take intermediate fees into account to construct the hedging portfolio.

### 2.2 Available Methods

The most common approach to hedge financial risks is to perform a dynamic delta-hedging approach to replicate the embedded financial guarantees. “Dynamic hedging programs are not set up with the aim of making speculative gains but are designed and applied according to strict risk management rules to mitigate exposures to various market movements stemming from the guarantees provided to policyholders” (Haefeli (2013)). The idea is simple, when hedging a guarantee that depends on a tradable fund $F_t$ (or at least a replicable fund $F_t$), the hedger makes sure that he holds at any time a number of shares of fund equal to the delta of the guarantee (Boyle and Emanuel (1980)). The delta is the sensitivity of the value of the guarantee to a change in the underlying price. Assume that the market is complete, there are no transaction costs and the hedge is continuously rebalanced over time in a self-financing manner. Then a dynamic delta hedging strategy theoretically achieves a perfect hedge of the guarantees at the time they must be paid out. The self-financing condition means that at each rebalancing date, the amount of money available is equal to the amount of money reinvested so that there are no withdrawals or intermediate investment between the inception and the payout of the guarantee. However, a lot of assumptions underlying the theoretical effectiveness of delta hedging are not realistic. For instance,
continuous dynamic hedging is not feasible in practice and frequent rebalancing can be too costly because of transaction costs. Dynamic delta hedging is also highly prone to model risk on the underlying given that the delta of the guarantee needs to be computed at each date in some chosen financial model. Delta-hedging and delta-vega hedging are studied in Kling, Ruez, and Russ (2014) and Kling, Ruez, and Russ (2011). They find that hedging effectiveness is very sensitive to model misspecification (much more than pricing).

An alternative popular dynamic hedging is based on a mean variance criteria and is sometimes referred as “optimal dynamic hedging” (Papageorgiou, Rémillard, and Hocquard (2008), Hocquard, Papageorgiou, and Remillard (2012a,b)). See also Coleman, Li, and Patron (2006) and Coleman, Kim, Li, and Patron (2007) for examples of dynamic hedging in variable annuities. The dynamic optimal hedging strategy can be described as follows. It has an initial value and then a sequence of “weights” representing the number of shares of the underlying to be bought at any time. These weights are such that the self-financing conditions is satisfied at any time and such that the hedging error at maturity is as small as possible. The measurement of the hedging error can be done in many ways and is typically done using the expected square hedging error as a measure of quality of replication.

Instead of dynamic hedging, Hardy (2003, 2000) and Marshall, Hardy, and Saunders (2010, 2012) investigate static hedging and suggest replicating maturity guarantees with a static position in put options. Static hedging consists of taking positions at inception in a portfolio of financial instruments that are traded in the market (at least over-the-counter) so that the benefits at a future date of the VA matches the hedge as well as possible. Static hedging strategies have a strong advantage in terms of cost, as there are no intermediary costs between the inception and the maturity of the benefits. Static strategies tend also to be highly robust to model risk because no rebalancing is involved. There are several issues with this approach, in particular the non-liquidity (and non-availability) of the long-term options needed to match the long-term guarantees. Often, they are only sold over the counter and thus subject to liquidity risk as well as counterparty risk. Also, most of guarantee benefits are path-dependent and therefore are hard to hedge with static hedging of available European path-independent options available in the market. Finally, static hedging of VAs tends to forget about the specificity of the options embedded in VAs. Their premiums are paid in a periodic way and a more natural hedging strategy should account for these premiums and focus on the hedging of the net liability. In a static strategy, the insurer must borrow a large amount of money at the inception of the contract to purchase the hedge. This borrowed money will then potentially be offset by the future fees collected as a percentage of the fund. The insurer is subject to the risk that the fees collected in the future do not match the amount borrowed at the beginning to purchase the hedge. Typically, if the contract is fairly priced, only the expected value of the discounted future fees will indeed match the initial cost needed to hedge the guarantees.

Several authors including Coleman, Li, and Patron (2006), Coleman, Kim, Li, and Patron (2007), and Kolkiewicz and Liu (2012) have studied semi-static hedging for variable annuities including options as hedging instruments. In semi-static hedging, the hedging
portfolio is constructed at each rebalancing date by following an optimal hedging strategy for some optimality criterion. The hedging portfolio is not altered until the next rebalancing date. Coleman, Li, and Patron (2006) and Coleman, Kim, Li, and Patron (2007) investigated hedging of embedded options in GMDB with ratchet features. By assuming that mortality risk can be diversified away, they reduce the problem of hedging of variable annuities to the hedging of lookback options with fixed maturity. They show that semi-static hedging with local risk minimization is significantly better than delta hedging. It is also proved that hedging using standard options is superior to hedging using underlying.

Kolkiewicz and Liu (2012) develop efficient hedging strategies of GMWB using local risk minimization as the optimality criterion for each hedging date. Their result shows that semi-static requires fewer portfolio adjustments than delta hedge for the same hedging performance. Thus, semi-static hedging outperforms delta hedge, especially when there are random jumps in the underlying price. In this paper, we provide additional evidence of the superiority of semi-static hedging strategies over dynamic delta hedging in the context of hedging guarantees in VAs. We propose a new version of semi-static hedging which utilizes the collected fees right away to construct the hedging portfolio.

We propose here a new semi-static hedging strategy, which consists of rebalancing the hedging portfolio at the exact dates when the fees are paid. We assume that put options written on the underlying market index related to policyholder’s separate investment account as well as the underlying market index itself can be used as hedging instruments for semi-static hedging. In practice, there are several variable annuity products which allow policyholders to decide their own investment decision. In this case, the insurer faces basis risk arising from the fact that the fund where premiums are invested is not directly traded. It is thus not directly possible to take long or short positions in the fund and neither to buy options on the fund. For the ease of exposition, we assume that the premium paid by policyholder is invested in a market index, which is actively traded in the market, e.g. S&P 500, or highly correlated to the market index. This assumption gives us a liquid underlying index market and option market for our hedging strategies.

2.3 Key Factors in hedge effectiveness

We provide evidence of the following properties of the hedge of variable annuities.

#1 The effectiveness of the hedge is improved by focusing on the liability net of collected fees.

#2 The insurer should start initially to borrow money to hedge long term guarantees from time 0 and anticipate that collected fees will cover this borrowed money in the future.

#3 The insurer should be able to short sell the underlying to perform an efficient hedge of long term guarantees.

8
The insurer should make use of put options\textsuperscript{2} in the hedge portfolio.

To study \#1, we first implement a standard delta hedging strategy where investment is in stocks and bonds only. We observe that the strategy performs poorly unless it is frequently rebalanced, with a frequency higher than the payment of fees. The performance is greatly improved by hedging the net liability instead of the liability. Hedging is also improved significantly by including put options as hedging instruments in a semi-static hedge which is rebalanced at the dates fees are collected only. Finally, we find that semi-static hedging is well adapted to VAs due to the periodicity of fee payments. Semi-static hedging is able to utilize short term options that are more liquid and traded at a wider range of strikes. Finally semi-static hedging of VAs, when accounting for future collected fees, allows to decrease the initial borrowing. We study the semi-static hedging strategies in three environments, first without any constraints when all positions in stocks, bonds and options can be taken, then with short selling constraints and finally when put options are not available.

We find results that are consistent with Coleman, Li, and Patron (2006), Coleman, Kim, Li, and Patron (2007), and Kolkiewicz and Liu (2012). For example, similarly to these studies, we find that an efficient semi-static strategy must include put options in the hedge: it is not optimal to solely invest in stocks and bonds (delta hedging strategy). Companies need to be proactive and start to hedge the full guarantee from time 0 by borrowing to buy the proper hedge. Most of the time, the cost of borrowing at time 0 will be offset by future fees received. In fact, there is a point (when the underlying fund is high enough) from which any additional fees correspond to income for the company and are not needed anymore to hedge the guarantees. It can thus be used to increase reserves or pay bonuses to shareholders as they bear some risk at the inception when guarantees are sold as there is some risk that the company will never receive enough fees to cover the full cost needed to hedge.

Our improved semi-static strategy outperforms the traditional semi-static strategies with local risk minimizing. Especially, if short-selling is allowed, our strategy gives much better hedging performance than the other strategies with and without put options as hedging instruments. Moreover, we observe that less borrowing is needed at time 0 with our new strategy. Since the fees are not considered in the optimality criteria of traditional semi-static hedging strategy, it costs more to construct the initial hedging portfolio at time 0. With our new strategy, the fee collected at time 0 is enough to construct the optimal

\textsuperscript{2}Haefeli (2013) explains that “Annual National Association of Insurance Commissioners’ (NAIC) surveys and various investment bank derivative market reports show that the total U.S. insurance industry derivative positions (comprising the totality of life and savings activities) represent less than 1 per cent of total volumes worldwide. Variable annuity hedging programs typically operate using the most liquid exchange-traded and over the counter (OTC) derivative instruments that exist in the world (e.g. U.S. bond treasury futures, S&P500 futures, US$, EUR and JPY swaps, swaptions and equity put options). The use of derivatives to manage any liability driven business is a sound risk practice that does not pose any form of systemic risk irrespective of the portfolios size or interconnectedness.”
hedging portfolio at time 0 and almost no borrowing is necessary.

We also confirm that short-selling is necessary to hedge the options embedded in guarantees because the hedging targets are decreasing functions of the underlying index. Especially, if the range of strike prices of put options that can be used for hedging is limited, short-selling is crucial to have good hedging performance. The larger the set of strike prices of the put options used for hedging, the better the hedging performance. If there are short-selling constraints, the use of put options should be increased. Therefore, put options become more important hedging instruments when short-selling is not allowed or is limited.

In practice, there is also another way to improve the effectiveness of the hedge that we do not study in this report. It is possible to mitigate risks across various variable annuities products and guarantees and thus to improve hedging programs by combining various products together to produce natural hedges by choosing timing of sales, product designs and embedded guarantees so that risks may offset each other. Natural hedges against mortality risks are very standard (Cox and Lin (2007)) but it can also be useful to hedge financial risks (as shown for example by Bernard and Boyle (2011) for volatility risk).

### 3 Description of a Variable Annuity Guarantee

To illustrate the hedging of variable annuities, we consider a single premium variable annuity contract sold at time 0. Let $F_0$ be the single premium paid by the policyholder at time 0 and $F_T$ be the account value of the policyholder’s fund at time $T > 0$. We further assume that the variable annuity contract guarantees $K$ at time $T$, that is, the policyholder is guaranteed the payoff $G_T := \max\{F_T, K\}$ at time $T$. Our model can be used to model the following situations:

- A GMAB, which guarantees a lump sum benefit $K$ at time $T$.
- A variable annuity contract with GMIB rider, which provides flat life annuity payment $b = \eta \max\{F_T, K\}$ with annuitization rate $\eta > 0$ (see Bacinello, Millossovich, Olivieri, and Pitacco (2011)).
- A variable annuity contract with GMDB rider which guarantees $K$ with assumption that the mortality risk can be diversified away (see Coleman, Li, and Patron (2006) and Coleman, Kim, Li, and Patron (2007)).

Notice that the payoff $G_T$ of the variable annuity contract can be written as

$$G_T = \max\{F_T, K\} = F_T + (K - F_T)^+,$$
with \( x^+ := \max\{x, 0\} \). Therefore, the payoff \( G_T \) corresponds to the policyholder’s account value at time \( T \) plus an embedded option with payoff \( V_T := (K - F_T)^+ \) at time \( T \), where \( T \) can be interpreted as the maturity of a put option written on the fund.

### 3.1 Guarantee put option

The embedded option payoff \( V_T \) must be funded by the fees collected from the policyholder’s account. Let us consider the time step size \( \Delta t := T/N \) corresponding to a number of periods \( N > 0 \) and time steps \( t_k := k\Delta t \) for \( k = 0, 1, \ldots, N \), that is,

\[
0 = t_0, t_1, \ldots, t_N = N\Delta t = T.
\]

Then we assume that the fees \( \varepsilon F_{t_k} \) are withdrawn periodically at each time \( t_k \), for \( k = 0, 1, \ldots, N - 1 \), from the policyholder’s account. For \( k = 0, \ldots, N - 1 \), the fee at time \( t_k \) is determined by \( \varepsilon F_{t_k} \) where \( F_{t_k} \) is the account value of the policyholder at time \( t_k \) right before the withdrawal of the fee, and \( \varepsilon \) is the fee rate. At time \( t_{k-1} \), \( k = 1, \ldots, N - 1 \), the remaining account value \( (1 - \varepsilon)F_{t_{k-1}} \) after the fee withdrawal stays invested in the fund whose value at time \( t \geq 0 \) is \( S_t \). Therefore, at time \( t_k \), the fund value satisfies the following relationship.

\[
F_{t_k} = (1 - \varepsilon)F_{t_{k-1}} \frac{S_{t_k}}{S_{t_{k-1}}}, \quad k = 1, \ldots, N - 1.
\]

At time \( T = t_N \), \( F_T \) is given by

\[
F_T = F_{t_N} = (1 - \varepsilon)F_{t_{N-1}} \frac{S_{t_N}}{S_{t_{N-1}}}.
\]

Consequently, \( F_{t_k} \) for \( k = 0, 1, \ldots, N \) is given by

\[
F_{t_k} = (1 - \varepsilon)^k F_0 \frac{S_{t_k}}{S_0}.
\]

To simplify the exposition, we neglect the basis risk, which arises from the mismatch between the policyholder’s investment account and the available hedging instruments and assume that the money is fully invested in a traded index \( S_t \) and such that put options written on \( S_t \) can also be used as hedging instruments. For instance we can assume that \( S_t \) is a market index, e.g., S&P 500 with a liquid market of options written on this index. This assumption gives us a liquid underlying index market\(^3\) and options market to implement our hedging strategies.

\(^3\)In practice, liquid index futures are used for hedging, however, we use the index directly for simplicity.
3.2 Fair valuation of the VA

Assume a constant risk-free interest rate \( r \) and let \( Z_T \) be the value at time \( T \) of all accumulated fees taken from inception to time \( T \), then \( Z_T \) is given by

\[
Z_T = \varepsilon F_0 e^{r N \Delta t} + \varepsilon F_{t_1} e^{r(N-1)\Delta t} + \cdots + \varepsilon F_{t_{N-1}} e^{r \Delta t}.
\] (1)

The value of the payoff of the embedded option at time \( T \) is

\[ V_T = (K - F_T)^+. \]

Since the collected fees are used to hedge the embedded option, the fair value of the fee rate is determined by solving the following equation

\[ \mathbb{E}^Q [Z_T] = \mathbb{E}^Q [V_T], \]

where \( Q \) is the risk-neutral measure. Here using the risk-neutral measure is essential because we assume that financial risks can be hedged using the financial market and that a risk-neutral probability gives rise to prices that are consistent with an arbitrage free financial market.

3.3 Liability and net liability

The VA terminal payoff with GMMB is equal to

\[ G_T = F_T + (K - F_T)^+. \]

In a VA, the policyholder typically chooses the fund allocation. We thus assume that the fund is invested according to the policyholder’s choice. Its value will fluctuate but \( F_T \) is perfectly replicated in that the investment can be liquidated and the fund value transferred to the policyholder at \( T \). Therefore, it does not need to be hedged per se. However, the remaining liability is the option payoff

\[ (K - F_T)^+ \] (2)

and the liability net of collected fees is then equal to

\[ (K - F_T)^+ - Z_T, \] (3)

where \( Z_T \) refers to the accumulated fees defined in (1). Given that the insurer should provide the guarantee \( (K - F_T)^+ \) and receive a random amount of fees at discrete dates, considering the net liability plays a crucial role in the efficiency of the hedge as described in what follows.
4  Delta Hedging

We consider the VA contract described in the previous section. This section is dedicated to
delta hedging and to show how a delta hedging strategy of the net liability (3) outperforms
a naive delta hedging strategy of the liability (2).

Recall that in the absence of market frictions such as transaction costs, and when the
underlying asset is tradable in a complete market, the option can be hedged perfectly by
rebalancing the hedging portfolio continuously with a dynamic delta hedging strategy. In
practice, we cannot rebalance the hedging portfolio continuously and frequent rebalancing
causes high transaction costs. Therefore, perfect hedging is not feasible and the hedging
portfolio can be rebalanced only at a discrete set of times. In the case of a VA, it is
natural to assume that the portfolio is rebalanced at the dates when the fees are collected:
t_0 = 0, t_1, t_2, ..., t_{N-1} and potentially at intermediary dates.

4.1 Profit and loss of the hedge

We denote the only part that needs to be hedged (the option payoff) by
V_T := (K - F_T)^+. The collected fees taken periodically from the fund are assumed to be sufficient to cover
this guarantee as discussed above in Sections 3.2 and 3.3.

To compare the performance of various hedging strategies, we consider the profit and
loss of the hedge at time T (surplus if it is positive and hedging error if it is negative). We
denote it by Π_{t_N}. It is equal to the difference between the hedging portfolio of the insurer
at T, X_T, and the guarantee, V_T, that needs to be paid to the policyholder at time T

Π_{t_N} = X_T - V_T.

4.2 No hedge case

As a benchmark, we consider the case when there is no hedge at all from the insurance
company and that all collected fees are directly invested in the bank account. Then, the
profit and loss at time T is given by

Π_{t_N} = \sum_{i=0}^{N-1} \varepsilon F_{t_i} e^{r(T-t_i)} - (K - F_T)^+,
and the expected value of the profit and loss can be computed explicitly as follows

\[
\mathbb{E}^p[\Pi_{t,N}] = \mathbb{E}^p \left[ \sum_{i=0}^{N-1} \varepsilon F_{t_i} e^{r(T-t_i)} - (K - F_T)^+ \right] \\
= \sum_{i=0}^{N-1} \varepsilon e^{r(T-t_i)} \mathbb{E}^p[F_{t_i}] - \mathbb{E}^p[(K - F_T)^+] \\
= \sum_{i=0}^{N-1} \varepsilon e^{r(T-t_i)} (1 - \varepsilon) F_0 \mathbb{E}^p[S_{t_i}] - e^{\mu T} \mathbb{E}^p[e^{-\mu T}(K - F_T)^+] \\
= \sum_{i=0}^{N-1} \varepsilon e^{r(T-t_i)} (1 - \varepsilon) F_0 e^{\mu t_i} - \Phi(-\bar{d}_2)K + e^{\mu T} \Phi(-\bar{d}_1)(1 - \varepsilon)^N F_0.
\]

which is easily derived from the Black-Scholes formula (using \( \mu \) instead of \( r \)) with

\[
\bar{d}_1 = \frac{\ln \left( \frac{(1-\varepsilon)^N F_0}{K} \right) + T(\mu + \frac{\sigma^2}{2})}{\sigma \sqrt{T}}, \quad \bar{d}_2 = \bar{d}_1 - \sigma \sqrt{T}.
\]

### 4.3 Delta Hedge of the Liability

A delta-hedge (\( \Delta \)-hedge) is a self financing portfolio consisting of the underlying index, \( S \), and a bond to replicate a target payoff. Our goal is to implement a \( \Delta \)-hedge with rebalancing at each dates \( t_k \) for \( k = 0, 1, \ldots, N - 1 \). We start from \( X_0 = \varepsilon F_0 \) at time 0, and at each \( t_k \), we construct a hedging portfolio consisting of \( \pi_{t_k} \) units of underlying index and \( X_{t_k} - \pi_{t_k} S_{t_k} \) risk-free bonds. Then, for time \( t_k \), we can represent our self-financing replication relation as follows

\[
X_{t_{k+1}} = (X_{t_k} - \pi_{t_k} S_{t_k}) e^{r\Delta t} + \pi_{t_k} S_{t_{k+1}} + \varepsilon F_{t_{k+1}} \quad k = 0, 1, \ldots, N - 2
\]

with \( X_0 = \varepsilon F_0 \), and define the profit and loss at time \( T \) as

\[
\Pi_{t,N} = X_{t,N} - (K - F_T)^+
\]

to investigate the hedging performance. Let us also define \( \Pi_0 = \varepsilon F_0 - \pi_0 \). If \( \Pi_0 \) is negative, then \(-\Pi_0\) is the amount we should borrow at time 0, and if it is positive, then \( \Pi_0 \) is the amount invested in risk-free bonds at time 0.

To replicate the payoff \((K - F_T)^+\) at each \( t_k \) using \( \Delta \)-hedge, we compute \( V_{t_k} \) the no-arbitrage value (risk-neutral expectation of discounted payoff conditional on \( F_{t_k} \)) of \( V_T \) at time \( t_k \). We have

\[
V_{t_k} = \mathbb{E}^Q \left[ e^{-r(T-t_k)} (K - F_T)^+ | F_{t_k} \right] \\
= \mathbb{E}^Q \left[ e^{-r(T-t_k)} \left( K - (1 - \varepsilon)^N F_0 \frac{S_T}{S_0} \right)^+ | F_{t_k} \right].
\]
Let $\alpha = (1 - \varepsilon)^N \frac{F_0}{S_0}$. Then, 

$$V_{tk} = \alpha \mathbb{E}^Q \left[ e^{-r(T-t_k)} \left( \frac{K}{\alpha} - S_T \right)^+ \left| \mathcal{F}_{tk} \right. \right] = \Phi(-d_2(k)) Ke^{-r(T-t_k)} - \alpha \Phi(-d_1(k)) S_{tk}$$

which is also derived from the Black-Scholes formula with 

$$d_1(k) = \frac{\ln(\frac{\alpha S_{tk}}{K}) + (T-t_k)(r + \frac{\sigma^2}{2})}{\sigma \sqrt{T-t_k}}$$

and thus, it is then known that the number of shares to invest in shares to hedge the liability is 

$$\pi_{tk} = -\alpha \Phi(-d_1(k)).$$

### 4.4 Delta-Hedge of the Net Liability

The no-arbitrage value of net liability (4) at time $t_k$ is 

$$V_{tk} = \sum_{i=0}^{N-1} \varepsilon (1 - \varepsilon)^i \frac{F_0}{S_0} \mathbb{E}^Q \left[ S_{ti} \left| \mathcal{F}_{tk} \right. \right] e^{-r(t_i-t_k)}$$

$$= V_{tk} - \sum_{i=0}^{k-1} \varepsilon (1 - \varepsilon)^i \frac{F_0}{S_0} S_{tk} e^{r(t_k-t_i)} - \sum_{i=k}^{N-1} \varepsilon (1 - \varepsilon)^i \frac{F_0}{S_0} S_{tk}. \quad (6)$$

By differentiating with respect to $S_{tk}$, the new number of shares is 

$$\bar{\pi}_{tk} := \pi_{tk} - \frac{F_0}{S_0} (1 - \varepsilon)^k (1 - (1 - \varepsilon)^{N-k})$$

where $\pi_{tk}$ indicates the number (5) of shares that is needed to hedge the terminal liability ignoring the intermediary payments of fees. The self-financing replication relation that corresponds to $\bar{\pi}_{tk}$ can be obtained by replacing $\pi_{tk}$ (defined in (4)) by $\bar{\pi}_{tk}$ (defined in (7)).

**Remark 1.** It is obvious from (5) and (7) that both $\pi_{tk}$ and $\bar{\pi}_{tk}$ are negative for all $k = 0, 1, \ldots, N - 1$. In other words, we always need a short position in the underlying index when we use delta-hedge for VAs. Consequently, if there is any limitation on short-selling or the cost of short-selling is high even though it is not modeled in our setting, we may not get a good hedging performance with delta-hedge.

Moreover, it can be seen from (7) that 

$$\bar{\pi}_{tk} < \pi_{tk} < 0, \quad \text{for all } k = 0, 1, \ldots, N - 1.$$ 

This implies that delta-hedge of the net liability requires more short-selling than delta-hedge of the liability.
4.5 Numerical Illustration

We now illustrate and compare the hedging performance of the two proposed hedging strategies with numerical examples in the Black-Scholes framework. Let us assume that the index follows a geometric Brownian motion

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dW_t
\]

with constant \( \mu > 0 \) and \( \sigma > 0 \) and standard Brownian motion \( W_t \) under the real-world probability measure \( \mathbb{P} \). Without loss of generality, we may assume that \( S_0 = 1 \). We then generate the sample paths of the underlying index from time 0 to \( T \). Notice that we only need the values of the underlying index at the rebalancing dates, i.e. \( S_t \) for \( k = 0, 1, \ldots, N \). Since we assume (8), we have

\[
S_{tk+1} = S_{tk}e^{(\mu - \frac{\sigma^2}{2})\Delta t + \sigma(W_{tk+\Delta t} - W_{tk})} \sim S_{tk}e^{(\mu - \frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t}z},
\]

where \( z \sim N(0,1) \) and we can easily generate the sample paths of the underlying index. We generate 10,000 paths for the simulation. The benchmark parameters for the financial market and the variable annuity contract are given by

\[
T = 10 \text{ (years)}, \quad N = 20, \quad \Delta t = 0.5 \text{ (years)}, \\
r = 2\% \text{ (per year)}, \quad \sigma = 20\% \text{ (per year)}, \quad \mu = 6\% \text{ (per year)} \\
F_0 = 100, \quad K = 120.
\]

Under these parameters, the fair fee rate is \( \varepsilon = 0.0415 \), which is a bit higher than the usual fee rate charged in practice. However, if we consider a longer maturity \( T \), a smaller volatility \( \sigma \) or a lower guaranteed value \( K \), the fee rate becomes much lower than this example. Moreover, our findings do not hinge on the level of fee rate so we use this set of parameters for our numerical experiments.

We illustrate the performance of a delta-hedging strategy using the computation of the shares as in (5) and as in (7). Results are given in Table 1 and Figure 1.

From Table 1 and the two panels of Figure 1, we find the following properties of \( \Delta \)-hedging:

- We observe a significant reduction in the standard deviation and hedge improvement when \( \Delta \)-hedging the net liability instead of the liability. The \( \Delta \)-hedging strategy performs best when we account for the future fees in the computation of the delta and it performs very well with a very high frequency of rebalancing (20 times per year).

- **Hedging is necessary.** In Table 1, the column “no hedge” means that the insurer does not use the collected fees for trading the underlying index or put options. Instead
he invests all fees in the risk-free asset as if the guarantees of the VA were fully diversifiable. All digits for the performance of “no hedge” are significant. Note that the no hedge performance is very poor: $\Pi_{t_N}$ has a very high standard deviation and the Value-at-Risk $\text{VaR}_{95\%}$ and $\text{VaR}_{90\%}$ are also very high. This implies that the insurer may encounter significant loss in unfavorable market conditions if no hedging program is implemented. Thus a good hedging strategy is necessary for the insurer to avoid catastrophic losses.

- The $\Delta$-hedging strategy does not require borrowing at time 0 but significant short-selling of the underlying index is needed.

- When rebalancing more often, the performance of $\Delta$-hedging must improve. However, we observe that the uncertainty of the collected fees that is not taken into account in the $\Delta$-hedge of the liability prevents any significant improvement in performance of the $\Delta$-hedge. On the other hand, we observe a significant improvement in the case of $\Delta$-hedging the net liability.

Table 1: Comparison of Hedging Performance by Monte Carlo simulations with $10^5$ simulations. The column corresponding to “No hedge” contains explicit computations (when available) or is obtained with 1,000,000 simulations so that all digits are significant. $\Pi_0 > 0$ represents the amount invested in risk-free bonds.

<table>
<thead>
<tr>
<th>Characteristics of $\Pi_{t_N}$</th>
<th>No hedge</th>
<th>$\Delta$-hedge with $N$ rebalancing dates</th>
<th>$\Delta$-hedge with $10N$ rebalancing dates</th>
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</thead>
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<tr>
<td></td>
<td>$\Delta$-hedge of Liability</td>
<td>$\Delta$-hedge of Net Liability</td>
<td>$\Delta$-hedge of Liability</td>
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<tr>
<td>Mean</td>
<td>32.9007</td>
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<td>Median</td>
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<td>Std</td>
<td>57.8022</td>
<td>28.27</td>
<td>5.88</td>
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<td>$\text{VaR}_{95%}$</td>
<td>47.2143</td>
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<td>14.78</td>
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<td>$\text{CVaR}_{95%}$</td>
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<td>$\text{VaR}_{90%}$</td>
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<tr>
<td>$\text{CVaR}_{90%}$</td>
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<td>23.66</td>
<td>19.51</td>
</tr>
<tr>
<td>$\Pi_0$</td>
<td>0</td>
<td>40.15</td>
<td>102.24</td>
</tr>
</tbody>
</table>
5 Semi-Static Hedging

In this section, we develop semi-static hedging techniques that minimize the variance of the hedge at a set of discrete dates by investing in stocks, risk-free bonds and some put options. For the ease of exposition, we assume that the hedging strategies utilize the collected fees at the time they are paid to construct the hedging portfolio. Thus we assume that the rebalancing dates at which the portfolio is hedged coincide exactly with the dates \( t_k \) for \( k = 0, 1, \ldots, N - 1 \) (time steps at which fees are taken from the policyholder’s account). It is possible to include other time steps, but this assumption simplifies the model and does not change our conclusions.

5.1 Two semi-static strategies

At time \( t_k, k = 0, 1, \ldots, N - 1 \), we construct a hedging portfolio \( X_{t_k} \) consisting of \( \beta_{t_k} \) risk-free bonds (in other words invested in a money market account), \( \pi_{t_k} \) units of underlying index (whose value at time \( t_k \) is \( S_{t_k} \)), and \( \alpha_{t_k} \) put options with strike price \( K_{t_k} \) and maturity \( t_{k+1} \). Then the value of the hedging portfolio at time \( t_k \) becomes

\[
X_{t_k} := \pi_{t_k} S_{t_k} + \beta_{t_k} + \alpha_{t_k} P(S_{t_k}, K_{t_k}, \Delta t), \quad k = 0, 1, \ldots, N - 1,
\]

where \( P(S, K, \tau) \) denotes the price of a put option with current underlying asset value \( S \), strike price \( K \), and time to maturity \( \tau \). It is possible to add more put options but the exposition would be more complicated. This simplest situation with only put options with given strike and given number of that particular put option is already of interest.
At time $t_{k+1}$ before any rebalancing, the change in the value of hedging portfolio over $\Delta t$ due to the movement of the underlying index is

$$\pi_{t_k}(S_{t_{k+1}} - S_{t_k}) + \beta_{t_k}(e^{\rho\Delta t} - 1) + \alpha_{t_k}((K_{t_k} - S_{t_{k+1}})^+ - P(S_{t_k}, K_{t_k}, \Delta t)).$$

Then we define the accumulated gain (possibly negative when it is a loss) at time $t_k$ as

$$Y_{t_k} := \sum_{n=0}^{k-1} \pi_{t_n}(S_{t_{n+1}} - S_{t_n}) + \beta_{t_n}(e^{\rho\Delta t} - 1) + \alpha_{t_n}((K_{t_n} - S_{t_{n+1}})^+ - P(S_{t_n}, K_{t_n}, \Delta t))$$

for $k = 1, 2, \ldots, N$ with $Y_0 = 0$, and the cumulative cost at time $t_k$ as

$$C_{t_k} := X_{t_k} - Y_{t_k}, \quad k = 0, 1, \ldots, N,$$

with $X_{t_N} := V_T = (K - F_T)^+.$

**Remark 2.** The cost increment $C_{t_{k+1}} - C_{t_k}$ can be interpreted as the variation in the value of the hedging portfolio between $t_k$ and $t_{k+1}$ adjusted by potential gains and losses. It can be computed as

$$C_{t_{k+1}} - C_{t_k} = X_{t_{k+1}} - (X_{t_k} + Y_{t_{k+1}} - Y_{t_k})$$

with

$$X_{t_{k+1}} = \begin{cases} 
\pi_{t_{k+1}}S_{t_{k+1}} + \beta_{t_{k+1}} + \alpha_{t_{k+1}}P(S_{t_{k+1}}, K_{t_{k+1}}, \Delta t), & k = 0, 1, \ldots, N - 2, \\
(K - F_T)^+ = (K - (1 - \varepsilon)^N F_0 \frac{S_{t_N}}{S_0})^+, & k = N - 1, 
\end{cases} \quad (9)$$

$$X_{t_k} + Y_{t_{k+1}} - Y_{t_k} = \pi_{t_k}S_{t_{k+1}} + \beta_{t_k}e^{\rho\Delta t} + \alpha_{t_k}(K_{t_k} - S_{t_{k+1}})^+, \quad k = 0, 1, \ldots, N - 1. \quad (10)$$

Ultimately, we are interested in the profit and loss of the hedging portfolio at the date $T$ when the payoff is paid. To do so, we need to take into account the collected fees and define the profit and loss $\Pi_t$ at any time $t$. At time 0, the insurer receives the fee $\varepsilon F_0$ and constructs a hedging portfolio whose value is $C_0 = X_0$. Therefore, the profit and loss at time 0 is

$$\Pi_0 = \varepsilon F_0 - C_0.$$ 

At time $t_1$, the initial profit and loss (computed at time 0) becomes $\Pi_0 e^{\rho\Delta t}$ and the insurer receives the fee $\varepsilon F_{t_1}$. However, additional cost $C_{t_1} - C_0$ is needed to construct the hedging portfolio at time $t_1$. As a result, the profit and loss at time $t_1$ is given by

$$\Pi_{t_1} = \Pi_0 e^{\rho\Delta t} + \varepsilon F_{t_1} - (C_{t_1} - C_0) = (\varepsilon F_0 - C_0)e^{\rho\Delta t} + \varepsilon F_{t_1} - (C_{t_1} - C_0).$$

A similar argument gives the following equation for the profit and loss at time $T$.

$$\Pi_T = \Pi_{t_N} = \sum_{k=0}^{N-1} \varepsilon F_{t_k} e^{\rho(N-k)\Delta t} - C_0e^{\rho\Delta t} - \sum_{k=1}^{N} (C_{t_k} - C_{t_{k-1}})e^{\rho(N-k)\Delta t}. \quad (11)$$
If $\Pi_{t_N}$ is positive, the insurer has positive surplus at $T$ after paying $(K - F_T)^+$ to the policyholder. On the other hand, if $\Pi_{t_N}$ is negative, the insurer is short of money to pay the guaranteed value at time $T$ to the policyholder. Thus $\Pi_{t_N}$ can be considered as the profit and loss and the hedging performance of strategies can be investigated in terms of $\Pi_{t_N}$. 

**Strategy 1: Traditional semi-static hedging**

Consider a traditional semi-static hedging strategy (referred as Strategy 1). Since perfect hedging is not feasible with semi-static hedging, some optimality criterion is necessary at each rebalancing date. In Strategy 1, the hedging strategy at each rebalancing date is determined to minimize local risk which is defined as the conditional second moment of the cost increment during each hedging period. At each time step $t_k$ for $k = N-1, N-2, \ldots, 0$, starting from the last period, one solves the following optimization recursively

$$
\min_{(\pi_{t_k}, \beta_{t_k}, \alpha_{t_k}, K_{t_k}) \in \mathcal{A}_{t_k}} L^\otimes_k(\pi_{t_k}, \beta_{t_k}, \alpha_{t_k}, K_{t_k}; s),
$$

with

$$
L^\otimes_k(\pi_{t_k}, \beta_{t_k}, \alpha_{t_k}, K_{t_k}; s) := \mathbb{E}_{t_k}\left[ \left( C_{t_{k+1}} - C_{t_k} \right)^2 \mid S_{t_k} = s \right]
= \mathbb{E}_{t_k}\left[ \left( X_{t_{k+1}} - \left( X_{t_k} + Y_{t_{k+1}} - Y_{t_k} \right) \right)^2 \mid S_{t_k} = s \right],
$$

(12)

where $\mathcal{A}_{t_k}$ is some admissible set of controls at time $t_k$ and $\mathbb{E}_{t_k} [\cdot] := \mathbb{E}_{t_k} [\cdot | \mathcal{F}_{t_k}]$ is the conditional expectation on the information at time $t_k$. $\mathcal{A}_{t_k}$ may be defined differently depending on the constraints we consider. The equation (12) implies that, at time $t_k$ with Strategy 1, $X_{t_{k+1}}$ is the hedging target and our goal is to minimize the conditional second moment of increment of cost.

This optimality criterion was applied to semi-static hedging of variable annuities by Coleman, Li, and Patron (2006), Coleman, Kim, Li, and Patron (2007), and Kolkiewicz and Liu (2012). In these papers, the authors use several standard options with different strike prices in the hedging strategy, but as described above, we only consider a single put option and the strike price is determined by the optimization. Our model gives insights about the moneyness of the options needed for hedging variable annuities.

In (12), note that $X_{t_{k+1}} + Y_{t_{k+1}} - Y_{t_k}$ is the amount available at time $t_{k+1}$ from the result of hedging portfolio $X_{t_k}$. However, the insurer receives the fee $\varepsilon F_{t_k}$ at time $t_k$ for $k = 0, 1, \ldots, N-1$. In Strategy 1, we only use $(X_{t_k} + Y_{t_{k+1}} - Y_{t_k})$ to construct the hedging portfolio at time $t_{k+1}$ and the fee is not considered in the hedging strategy. The following Strategy 2 shows how to take this fee into account in the design of the semi-static hedging strategy.
Strategy 2: Improved semi-static hedging strategy

For \( k = N - 1, N - 2, \ldots, 0 \), solve the following optimization recursively

\[
\min_{(\pi_t^{tk}, \beta_t^{tk}, \alpha_t^{tk}, K_t^{tk}) \in A_t^{tk}} L_k^{\ominus}(\pi_t^{tk}, \beta_t^{tk}, \alpha_t^{tk}, K_t^{tk}; s),
\]

with

\[
L_k^{\ominus}(\pi_t^{tk}, \beta_t^{tk}, \alpha_t^{tk}, K_t^{tk}; s) := \mathbb{E}_{t^{tk}} \left[ \left\{ C_{t^{tk+1}} - (C_{t^{tk}} + \varepsilon F_{t^{tk+1}} \mathbb{1}_{\{0 \leq k \leq N-2\}}) \right\}^2 \bigg| S_t^{tk} = s \right]
\]

\[
= \mathbb{E}_{t^{tk}} \left[ \left\{ X_{t^{tk+1}} - (X_{t^{tk}} + Y_{t^{tk+1}} - Y_{t^{tk}} + \varepsilon F_{t^{tk+1}} \mathbb{1}_{\{0 \leq k \leq N-2\}}) \right\}^2 \bigg| S_t^{tk} = s \right].
\]

The optimality criterion for Strategy 2 enables us to utilize the collected fees right away for the hedging strategy. Notice that \( \Pi_{t^{N}} \) in (11) can be written as

\[
\Pi_{t^{N}} = (\varepsilon F_0 - C_0)e^{rN\Delta t} + \sum_{k=1}^{N-1} (\varepsilon F_t + C_{t^{k-1}} - C_{t^{k}})e^{r(N-k)\Delta t} + (C_{t^{N-1}} - C_{t^{N}}),
\]

and Strategy 2 minimizes conditional second moments of every term of \( \Pi_{t^{N}} \) except the first term in (13) whereas Strategy 1 does not consider the collected fees. We find that Strategy 2 gives a better hedge performance than Strategy 1.

We now illustrate the hedging performance of these two strategies under various market environments. The effect of fees on hedging performance under different assumptions on the market environment is also discussed.

5.2 Description

We describe the semi-static hedging procedure based on Strategy 2. But similar explanations hold for Strategy 1 because the only difference between Strategy 2 and Strategy 1 is that the fees do not appear in the optimality criterion of Strategy 1. Therefore, we can obtain the solution for Strategy 1 by removing the terms with the fees \( \varepsilon (1 - \varepsilon)^{k+1} F_{t^{tk+1}} S_{t^{tk+1}} / S_0 \mathbb{1}_{\{0 \leq k \leq N-2\}} \) wherever they appear for Strategy 2.

One of the variables included in the optimization is the optimal level of the strike price of the put options used in the hedging procedure. Since the strike prices of put options that are actively traded in the market are limited and related to the underlying index value at \( t_k \), we define a set \( K_{t_k}(S_{t_k}) \) of strike prices available in the market at time \( t_k \) with underlying index value \( S_{t_k} \). Specifically, we describe the available strikes as a percentage of the underlying value, so that a percentage equal to 100% represents an at-the-money
option, a percentage higher (respectively lower) than 100\% corresponds to an out-of-the-money option (respectively in-the-money). Mathematically, it amounts to restricting the set of possible strikes to the following set $\mathcal{K}_{tk}(S_{tk})$:

$$\mathcal{K}_{tk}(S_{tk}) = \{ \xi S_{tk} | \xi \in \{ \xi_1, \xi_2, \ldots, \xi_M \} \}, \quad (14)$$

where $\xi_1 < \xi_2 < \cdots < \xi_M$. Then we may examine the effect of $\xi_1$ and $\xi_M$ on the hedging performance. It is also straightforward to extend our approach with additional constraints imposed on the other control variables $\pi_{tk}$, $\beta_{tk}$, and $\alpha_{tk}$.

We propose to examine separately the effectiveness of the semi-static hedging strategy in three environments that we call “UC”, “SC” and “w/o Put”.

- “UC” refers to the situation where hedging is done without any constraints. Short selling of the underlying index is allowed and put options on the underlying index is used as hedging instruments as well as the underlying index (see details in Appendix A.1).
- “SC” refers to the situation where hedging is done under short-selling constraints on the underlying index. We also use put options as hedging instruments (see details in Appendix A.2).
- “w/o Put” refers to the situation where hedging is done without any constraints on the underlying index but we do not use put options as hedging instruments. Only the money market account and the underlying index can be used for hedging (see details in Appendix A.3).

### 5.3 Hedging Performance

We use the same parameter set as the numerical illustration in Section 4.5. The theoretical results needed for the numerical implementation are given in Appendix A.

In Figures 2 and 3, and in Table 2, we use $\xi_1 = 0.7$ and $\xi_M = 1.1$. This choice is consistent with market data of available strike prices of put options on the S&P 500 index that are actively traded in the market.

Here are some comments on the tables and figures of this section.

- **The improved semi-static hedging strategy, Strategy 2, outperforms Strategy 1.** In Figure 2a, we represent the probability distribution of $\Pi_{t_N}$ of Strategies 1 and 2 without any constraint. It can be directly compared with Figure 2b where the probability distributions of $\Pi_{t_N}$ of Strategies 1 and 2 are displayed when no put options are used in the semi-static hedging strategies. In both cases, we observe that Strategy 2 gives better hedging performance. This also can be observed from the
mean, variance, VaR, CVaR in Table 2. In fact, better hedging performance with Strategy 2 is expected from the equation (13) because Strategy 2 minimizes each term in $\Pi_{t_N}$ while Strategy 1 only focuses on the additional cost part of $\Pi_{t_N}$.

Figure 2: Comparison of Probability Densities of $\Pi_{t_N}$
We compare the probability distribution functions of $\Pi_{t_N}$ for the two strategies described in Section 5 when there are no constraints on the semi-static hedging strategy and when put options are not available.

(a) Unconstrained, $\xi_1 = 0.7$, $\xi_M = 1.1$
(b) Without Put Options, $\xi_1 = 0.7$, $\xi_M = 1.1$

Table 2: Comparison of Hedging Performances of Strategies 1 and 2, $\xi_1 = 0.7$, $\xi_M = 1.1$

<table>
<thead>
<tr>
<th>Characteristics of $\Pi_{t_N}$</th>
<th>No hedge</th>
<th>Strategy 1</th>
<th>Strategy 2</th>
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<tbody>
<tr>
<td>Median</td>
<td>22.1358</td>
<td>7.83</td>
<td>7.50</td>
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<td>Std</td>
<td>57.8022</td>
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<td>22.38</td>
<td>26.20</td>
</tr>
<tr>
<td>CVaR$^{95%}$</td>
<td>56.4590</td>
<td>27.36</td>
<td>30.20</td>
</tr>
<tr>
<td>VaR$^{90%}$</td>
<td>36.5202</td>
<td>17.16</td>
<td>21.54</td>
</tr>
<tr>
<td>CVaR$^{90%}$</td>
<td>51.2175</td>
<td>23.48</td>
<td>27.01</td>
</tr>
<tr>
<td>$\Pi_0$</td>
<td>0</td>
<td>-53.06</td>
<td>-51.63</td>
</tr>
</tbody>
</table>

UC: Unconstrained, SC: Short-selling constraint $D_k = 0$, w/o Put: Without put option
• **Less borrowing is needed at time 0 with Strategy 2.** Another advantage of Strategy 2 over Strategy 1 is that the initial profit and loss $\Pi_0 = \varepsilon F_0 - X_0$ is greater than any other cases as we can see in Table 2. Negative $\Pi_0$ means that the fee collected at time 0 is not enough to construct the hedging portfolio $X_0$ at time 0 so that the insurer needs to borrow to construct the optimal hedging portfolio at time 0. Since the fee rate charged in this example is $\varepsilon = 0.0415$ and the single premium paid by the policyholder at time 0 is $F_0 = 100$, the fee collected at time 0 is $\varepsilon F_0 = 4.15$. The insurer needs to borrow more than 50 which is more than 1000% of $\varepsilon F_0$ to construct the optimal hedging portfolio of Strategy 1 without constraints. On the contrary, the insurer only needs to borrow 0.02 which is less than 0.3% of $\varepsilon F_0$ to construct the optimal hedging portfolio of Strategy 2. This is mainly due to the fact that we account for collected fees to construct the hedging portfolio at each rebalancing date in Strategy 2.

• **Including put options as hedging instruments gives better hedging performance.** In Table 2, the standard deviation, VaR, CVaR of the hedging strategy without put options (w/o Put) are higher than the same risk measures of the hedging strategy involving put options (UC). This means that better hedging performance can be achieved by including put options as hedging instruments. This result is consistent with the findings of Coleman, Li, and Patron (2006), Coleman, Kim, Li, and Patron (2007), and Kolkiewicz and Liu (2012).

• **Short-selling is crucial for hedging.** From Table 2, it can be noticed that the hedging performance under short-selling constraint $D_k = 0$ (SC) is significantly worse than that without short-selling constraint (UC) for each strategy under study. Moreover, the hedging performance under short-selling constraints is even worse than that without put options as hedging instruments (w/o Put). This finding can be understood from the shape of the hedging target as a function of the underlying index. Figure 3 depicts the hedging target at time $t_5$ as an example. Since the shape of hedging targets of Strategies 1 and 2 are decreasing convex functions of the underlying index (Figure 3), a short position of the underlying index is essential to construct an efficient hedging portfolio and the combination of a short position in the underlying index and put options allows to match the hedging target well and thus for a good hedging performance. Therefore, if short-selling is not allowed, we get much worse hedging performance with Strategies 1 and 2.

• **A larger set of strike prices gives a better hedge.** In Figure 4, and Table 3, we examine the hedging performance with $\xi_1 = 0.5$ and $\xi_M = 1.5$. This change gives the insurer wider range of strike prices of put options that can be used to construct the hedging portfolio. It means that put options become more useful hedging instruments when they are offered with a wider range of strike prices. This implies that extreme strikes are needed either deep out-of-the-money options or deep in-the-money options. Obviously it can be expected that the hedging performance of strategies which use
put options as hedging instruments are improved when a wider range of strike prices is available. If we compare Table 2 and Table 3, the hedging performance without short-selling constraint is slightly improved for both Strategy 1 and Strategy 2. Noticeably the hedging performance is significantly improved in the presence of the short-selling constraint. For example, the standard deviation of $\Pi_{t_N}$ with Strategy 2 under short-selling constraint is 21.09 if $\xi_1 = 0.7$ and $\xi_M = 1.1$. On the contrary, if $\xi_1 = 0.5$ and $\xi_M = 1.5$, the standard deviation is reduced dramatically and becomes 2.87. This significant improvement of hedging performance under short-selling constraint can also be observed in Figure 4.

Figure 3: Hedging Targets with Short-selling Constraint $D_k = 0$, $\xi_1 = 0.7$, $\xi_M = 1.1$

Moreover, we can observe from Table 3 that the hedging performance under short-selling constraint (SC) is (slightly) better than the hedging performance without put option (w/o Put) if $\xi_1 = 0.5$ and $\xi_M = 1.5$. This is because the put option is the only hedging instrument to match the shape of hedging target in the presence of short-selling constraints. Therefore, the effect of wider range of strikes prices is stronger in this case than other cases.

Figure 5 shows why the hedging performance under short-selling constraint is improved significantly when a larger set of strike prices is available. In Figure 5b, if $\xi_1 = 0.5$ and $\xi_M = 1.5$, the optimal strike prices without any constraints (UC) are far below the upper bound of available strike prices. However, the optimal strike prices under short-selling constraint (SC) are on the upper bound of available strike prices in Figure 5b. Therefore, in Figure 5a with $\xi_M = 1.1$, the upper bound of available strike prices is much lower and the optimal strike prices under short-selling constraint are far below the optimal levels attained in Figure 5b with $\xi_M = 1.5$. As a result,
the hedging performance under short-selling constraint is affected significantly by the range of strike prices.

Figure 4: Comparison of Probability Densities of $\Pi_{tN}$
We compare the probability distribution functions of $\Pi_{tN}$ for the two strategies described in Section 5 when there are short-selling constraints on the semi-static hedging strategy and when put options are not available for a wider range of strikes.

(a) Short-selling constraint $D_k = 0$, $\xi_1 = 0.5$, $\xi_M = 1.5$

(b) Without Put Option, $\xi_1 = 0.5$, $\xi_M = 1.5$

Table 3: Comparison of Hedging Performances of Strategies 1 and 2
$\xi_1 = 0.5$, $\xi_M = 1.5$

<table>
<thead>
<tr>
<th>Characteristics of $\Pi_{tN}$</th>
<th>Strategy 1</th>
<th>Strategy 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>UC</td>
<td>SC</td>
</tr>
<tr>
<td>Mean</td>
<td>12.91</td>
<td>13.02</td>
</tr>
<tr>
<td>Median</td>
<td>7.51</td>
<td>7.66</td>
</tr>
<tr>
<td>Std</td>
<td>29.29</td>
<td>29.47</td>
</tr>
<tr>
<td>VaR95%</td>
<td>23.05</td>
<td>23.19</td>
</tr>
<tr>
<td>CVaR95%</td>
<td>27.69</td>
<td>27.93</td>
</tr>
<tr>
<td>VaR90%</td>
<td>17.76</td>
<td>17.99</td>
</tr>
<tr>
<td>CVaR90%</td>
<td>23.95</td>
<td>24.16</td>
</tr>
<tr>
<td>$\Pi_0$</td>
<td>-4.00</td>
<td>-3.93</td>
</tr>
</tbody>
</table>

UC: Unconstrained, SC: Short-selling constraint $D_k = 0$, w/o Put: Without put option
Figure 5: Comparison of Optimal Strike Prices at Time $t_1$

We compare the optimal strike prices of strategy 2 at time $t_1$ for different set of strike prices.

(a) $\xi_1 = 0.7, \xi_M = 1.1$

(b) $\xi_1 = 0.5, \xi_M = 1.5$

6 Conclusions

This report focuses on the hedging of financial guarantees in variable annuities. We review existing methods and introduce a new semi-static hedging strategy that outperforms delta hedging. We show that it is crucial to take into account the fees collected periodically in the design of the hedging strategy. Our proposed improved semi-static hedging strategy outperforms existing methods in the literature. It gives better hedging performance than delta hedging and than traditional semi-static strategies ignoring the collected fees in the optimization. Another advantage of our new strategy is that almost no borrowing is needed at the beginning and the cost to construct the initial hedging portfolio can be matched to the fee collected initially.

We also show that short-selling may help in the hedging of guarantees in variable annuities. As other studies already pointed out, adding put options as additional hedging instruments gives better hedging performance of variable annuities. The larger the set of put options available in the market (wider range of strikes), the better the hedging performance with put options.
A Optimal Semi-Static Hedging Strategies

We here provide full details of the implementation of the semi-static strategy. We first consider the unconstrained optimization problem “UC” as a benchmark (Appendix A.1), and then we consider constrained problems “SC” (Appendix A.2) and “w/o Put” (Appendix A.3) to examine the effect of short-selling constraints and the absence of options in the hedging portfolio.

A.1 Unconstrained Semi-Static Hedging Strategy: “UC"

At each time step $t_k$, $k = 1, 2, \ldots, N$, consider a discretization $\{S_{t_k}^i\}_{i=1}^I$ of the underlying index. Then we can find the optimal hedging strategies recursively in backward from $k = N - 1$ to $k = 0$. The following are the steps to find the optimal strategy.

1. At time $t_k$, the hedging target $X_{t_{k+1}} = X_{t_{k+1}}(S_{t_{k+1}})$ is already known as a function of $S_{t_{k+1}}$ from the previous optimization problem at time $t_{k+1}$. For each $S_{t_k}^i$, there is a corresponding set of strike prices $K_{t_k}(S_{t_k}^i)$ of put options. Then do the following steps 2 and 3 for all $i = 1, 2, \ldots, I$.

2. For each strike price $K_{t_k}^{i,j} \in K_{t_k}(S_{t_k}^i)$, $j = 1, 2, \ldots, M$, let us define $(\bar{\pi}_{t_k}^{i,j}, \bar{\beta}_{t_k}^{i,j}, \bar{\alpha}_{t_k}^{i,j})$ as

$$
(\bar{\pi}_{t_k}^{i,j}, \bar{\beta}_{t_k}^{i,j}, \bar{\alpha}_{t_k}^{i,j}) := \arg\min_{(\pi_{t_k}, \beta_{t_k}, \alpha_{t_k}) \in \mathbb{R}^3} L_k^{\otimes}(\pi_{t_k}, \beta_{t_k}, \alpha_{t_k}, K_{t_k}^{i,j}; S_{t_k}^i) \quad (15)
$$

and find $(\bar{\pi}_{t_k}^{i,j}, \bar{\beta}_{t_k}^{i,j}, \bar{\alpha}_{t_k}^{i,j})$ for given $K_{t_k}^{i,j}$. Notice that $(\bar{\pi}_{t_k}^{i,j}, \bar{\beta}_{t_k}^{i,j}, \bar{\alpha}_{t_k}^{i,j})$ is the optimal choice of rebalancing decision for a given value of underlying index $S_{t_k}^i$ and a given strike price $K_{t_k}^{i,j}$ of the put option. If we choose $K_{t_k}^{i,j} \in K_{t_k}(S_{t_k}^i)$ such that it gives the minimum value of $L_k^{\otimes}(\pi_{t_k}^{i,j}, \beta_{t_k}^{i,j}, \alpha_{t_k}^{i,j}, K_{t_k}^{i,j}; S_{t_k}^i)$, it must be the optimal strike price.

3. Let us define $\bar{K}_{t_k}^i$ as

$$
\bar{K}_{t_k}^i := K_{t_k}^{i,j*} = \arg\min_{K_{t_k}^{i,j} \in K_{t_k}(S_{t_k}^i)} L_k^{\otimes}(\pi_{t_k}^{i,j}, \beta_{t_k}^{i,j}, \alpha_{t_k}^{i,j}, K_{t_k}^{i,j}; S_{t_k}^i),
$$

then $\bar{K}_{t_k}^i$ is the optimal strike price of the put option for given time $t_k$ and underlying index $S_{t_k}^i$. The corresponding optimal rebalancing decisions are given by

$$
\bar{\pi}_{t_k}^i := \bar{\pi}_{t_k}^{i,j*}, \quad \bar{\beta}_{t_k}^i := \bar{\beta}_{t_k}^{i,j*}, \quad \bar{\alpha}_{t_k}^i := \bar{\alpha}_{t_k}^{i,j*}.
$$

4. Repeat the above procedures 1, 2, and 3 from $k = N - 1$ to $k = 0$. Then we obtain the optimal rebalancing decisions for Strategy 2 without any constraints.
Proposition 1. From (9) and (10),
\[
L_k^\otimes(p_{tk}, \beta_{tk}, \alpha_{tk}, K^i_{tk}, S^i_{tk}) = \mathbb{E}_k \left\{ X_{tk+1} - \pi_{tk} S_{tk+1} - \beta_{tk} e^{r\Delta t} - \alpha_{tk} (K^i_{tk} - S_{tk+1})^+ + \frac{\varepsilon(1 - \varepsilon)^{k+1} F_0 S_{tk+1}}{S_0} 1_{\{0 \leq k \leq N-2\}} \right\}^2 \mid S_{tk} = S^i_{tk}.
\]
Thus, the optimization problem in (15) is a least squares problem so that we can derive \((\bar{\pi}^i_{tk}, \bar{\beta}^i_{tk}, \bar{\alpha}^i_{tk})\) as follows.
\[
\bar{\pi}^i_{tk} = -\frac{1}{2} \frac{4a_k b_k c_k - 2b_k e_k i_k - 2c_k d_k h_k + d_k f_k i_k + e_k f_k h_k - f_k^2 g_k}{4a_k b_k c_k - a_k f_k^2 - b_k e_k^2 - c_k d_k^2 + d_k e_k f_k},
\]
\[
\bar{\beta}^i_{tk} = -\frac{1}{2} \frac{14a_k c_k h_k - 2a_k f_k i_k - 2c_k d_k g_k + d_k e_k i_k + e_k f_k g_k - e_k^2 h_k}{4a_k b_k c_k - a_k f_k^2 - b_k e_k^2 - c_k d_k^2 + d_k e_k f_k},
\]
\[
\bar{\alpha}^i_{tk} = -\frac{1}{2} \frac{14a_k b_k h_k - 2a_k f_k h_k - 2b_k e_k g_k + d_k e_k h_k + d_k f_k g_k - f_k^2 i_k}{4a_k b_k c_k - a_k f_k^2 - b_k e_k^2 - c_k d_k^2 + d_k e_k f_k},
\]
where
\[
\begin{align*}
a_k & := \mathbb{E}_t \left[ S^2_{tk+1} \mid S_{tk} = S^i_{tk} \right] \\
b_k & := e^{2r\Delta t} \\
c_k & := \mathbb{E}_t \left[ \{(K^i_{tk} - S_{tk+1})^+\}^2 \mid S_{tk} = S^i_{tk} \right] \\
d_k & := 2\mathbb{E}_t \left[ S_{tk+1} S^{r\Delta t} \mid S_{tk} = S^i_{tk} \right] \\
e_k & := 2\mathbb{E}_t \left[ S_{tk+1} (K^i_{tk} - S_{tk+1})^+ \mid S_{tk} = S^i_{tk} \right] \\
f_k & := 2\mathbb{E}_t \left[ e^{r\Delta t} (K^i_{tk} - S_{tk+1})^+ \mid S_{tk} = S^i_{tk} \right] \\
g_k & := 2\mathbb{E}_t \left[ S_{tk+1} (\varepsilon(1 - \varepsilon)^{k+1} F_0 S_{tk+1}/S_0 1_{\{0 \leq k \leq N-2\}} - X_{tk+1}) \mid S_{tk} = S^i_{tk} \right] \\
h_k & := 2\mathbb{E}_t \left[ e^{r\Delta t} (\varepsilon(1 - \varepsilon)^{k+1} F_0 S_{tk+1}/S_0 1_{\{0 \leq k \leq N-2\}} - X_{tk+1}) \mid S_{tk} = S^i_{tk} \right] \\
i_k & := 2\mathbb{E}_t \left[ (K^i_{tk} - S_{tk+1})^+ (\varepsilon(1 - \varepsilon)^{k+1} F_0 S_{tk+1}/S_0 1_{\{0 \leq k \leq N-2\}} - X_{tk+1}) \mid S_{tk} = S^i_{tk} \right] \\
j_k & := \mathbb{E}_t \left[ (\varepsilon(1 - \varepsilon)^{k+1} F_0 S_{tk+1}/S_0 1_{\{0 \leq k \leq N-2\}} - X_{tk+1})^2 \mid S_{tk} = S^i_{tk} \right].
\end{align*}
\]
To calculate the above conditional expectations, we need the hedging target \(X_{tk+1}\) as a function of underlying index \(S_{tk+1}\). However, we only have the values of \(X_{tk+1}\) which correspond to the discretized values \(\{S^i_{tk+1}\}_{i=1}^N\). Therefore, the hedging target \(X_{tk+1}\) corresponding to \(S_{tk+1}\) other than \(\{S^i_{tk+1}\}_{i=1}^N\) must be obtained by interpolation or extrapolation.
A.2 With a Short-selling Constraint: “SC”

Consider the following short-selling constraint

\[ \pi_t k \geq -D_k, \quad D_k \geq 0, \quad k = 0, 1, \ldots, N - 1. \]  

(19)

Then the optimal hedging strategies under short-selling constraint can be obtained by following a similar procedure as we used for the unconstrained problem.

1. At time \( t_k \), suppose that the hedging target \( X_{t_{k+1}} = X_{t_{k+1}}(S_{t_{k+1}}) \) is known as a function of \( S_{t_{k+1}} \) from the previous optimization at time \( t_{k+1} \). For each \( S_t k \), there is a corresponding set of strike prices \( K_k (S_t k) \) of put options. Do the following steps 2 and 3 for all \( i = 1, 2, \ldots, I \).

2. For each \( K_t k \in K_k (S_t k) \), let us define \( (\hat{\pi}_{t k} i j, \hat{\beta}_{t k} i j, \hat{\alpha}_{t k} i j) \) as

\[
(\hat{\pi}_{t k} i j, \hat{\beta}_{t k} i j, \hat{\alpha}_{t k} i j) := \arg\min_{(\pi, \beta, \alpha) \in \mathbb{R}^3} L_k^\oplus(\pi_k, \beta_k, \alpha_k, K_t k, S_t k) \quad \text{subject to} \quad \pi_k \geq -D_k.
\]  

(20)

(21)

3. Let us define \( \hat{K}_t k i \) as

\[ \hat{K}_t k i := \hat{K}_t k i = \arg\min_{K_t k i \in K_k (S_t k)} L_k^\oplus(\hat{\pi}_{t k} i j, \hat{\beta}_{t k} i j, \hat{\alpha}_{t k} i j, K_t k i, S_t k), \]

then \( \hat{K}_t k i \) is the optimal strike price of put option, and corresponding optimal rebalancing decisions are given by

\[ \hat{\pi}_{t k} i := \hat{\pi}_{t k} i, \quad \hat{\beta}_{t k} i := \hat{\beta}_{t k} i, \quad \hat{\alpha}_{t k} i := \hat{\alpha}_{t k} i. \]

4. Repeat the above steps 1, 2, and 3 from \( k = N - 1 \) to \( k = 0 \) to obtain the optimal rebalancing decisions for Strategy 2 under short-selling constraints (19) on \( \pi \).

Proposition 2. Let \( (\bar{\pi}_{t k} i j, \bar{\beta}_{t k} i j, \bar{\alpha}_{t k} i j) \) be the solution to the unconstrained optimization (15). Then the solution \( (\hat{\pi}_{t k} i j, \hat{\beta}_{t k} i j, \hat{\alpha}_{t k} i j) \) of constrained optimization (21) can be determined as follows

If \( \bar{\pi}_{t k} i j \geq -D_k \),

\[ \hat{\pi}_{t k} i j = \bar{\pi}_{t k} i j, \quad \hat{\beta}_{t k} i j = \bar{\beta}_{t k} i j, \quad \hat{\alpha}_{t k} i j = \bar{\alpha}_{t k} i j. \]
If $\bar{\pi}_{tk}^{i,j} < -D_k$,

$$
\bar{\pi}_{tk}^{i,j} = -D_k,
$$

$$
\beta_{tk}^{i,j} = \frac{(-2c_kh_k + f_ki_k) + (2c_kd_k - e_kf_k)D_k}{4b_kc_k - f_k^2},
$$

$$
\alpha_{tk}^{i,j} = \frac{(-2b_ki_k + f_kh_k) + (2b_ke_k - d_kf_k)D_k}{4b_kc_k - f_k^2},
$$

where $b_k, c_k, d_k, e_k, f_k, h_k, i_k$ are obtained in the unconstrained problem “UC”.

### A.3 Without Put Options in the Semi-static Hedging Strategy: “w/o Put”

In the last market environment considered in this document, we assume that put options are not available and not used in the hedging procedure. Then the optimization problem for given $S_t^i$ at time $t_k$ becomes the following simpler optimization problem.

$$
(\tilde{\pi}_{tk}^{i,j}, \tilde{\beta}_{tk}^{i,j}) := \arg\min_{(\pi_{tk}, \beta_{tk}) \in \mathbb{R}^2} L_k^\square (\pi_{tk}, \beta_{tk}, 0, 0; S_t^i),
$$

where

$$
L_k^\square (\pi_{tk}, \beta_{tk}, 0, 0; S_t^i) = \mathbb{E}_k \left\{ X_{tk+1} - \pi_{tk}S_{tk+1} - \beta_{tk}e^{r\Delta t} - \varepsilon(1 - \varepsilon)^{k+1}F_0 \frac{S_{tk+1}}{S_0} 1_{\{0 \leq k \leq N-2\}} \right\}^2 \bigg| S_{tk} = S_t^i.
$$

It can be verified that the optimal rebalancing decision without put option defined by (22) is

$$
\tilde{\pi}_{tk}^{i,j} = \frac{-2b_kg_k + d_kh_k}{4a_kb_k - d_k^2},
$$

$$
\tilde{\beta}_{tk}^{i,j} = \frac{-2a_kh_k + d_kg_k}{4a_kb_k - d_k^2},
$$

where $a_k, b_k, d_k, g_k,$ and $h_k$ are obtained in the unconstrained problem.

Note that the third argument of $L_k^\square$ in (23) is zero, i.e. $\alpha_{tk} = 0$, because we do not use put options in this case. The fourth argument of $L_k^\square$ in (23) corresponds to the strike price of the put option, it does not need to be zero because $L_k^\square$ is no longer affected by the strike price.
References


