Portfolios with Positive Weights on the Efficient Frontier

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Abstract

One of the fundamental insights of the CAPM is that the market portfolio is mean variance efficient. Since the market portfolio has positive weights on all assets, the conditions under which frontier portfolios have this property are of interest. This paper derives a simple explicit solution for an efficient portfolio with positive weights. Assuming the covariance matrix is given, we obtain an expected return vector such that there is a compatible frontier portfolio. This portfolio is derived from the dominant eigenvector of the correlation matrix and provides a proxy for the market portfolio. Examples are provided to illustrate the basic idea.

Keywords: Efficient frontier, Positive weights, Compatible expected returns.

JEL Codes G11, G12.

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1 Introduction

Markowitz (1952) provided the first scientific analysis of the portfolio selection problem. This led to the development of the Capital Asset Pricing Model. A key insight of the CAPM is that, in equilibrium, all agents invest a fraction of their wealth in the market portfolio which has to lie on the efficient frontier. Roll (1977) provides a clear discussion of this point. The weights of the assets in the market portfolio are of necessity all positive. There has been considerable research devoted to finding conditions under which such a portfolio exists. For typical sets of expected return vectors and covariance matrices it is very unlikely that all the weights on a frontier portfolio will be positive. As the number of assets increases the probability of finding such a portfolio tends to zero.

The conditions under which frontier portfolios have positive weights is therefore of interest given its relevance for one of the most basic models in finance. This topic has been examined by several authors. For example, Best and Grauer (1985) derive restrictions on the return vector, given the weights on a frontier portfolio and a known covariance matrix. Green (1986) derives conditions for the existence of positively weighted frontier portfolios. Best and Grauer (1992) derive algebraic conditions on the expected return vector and covariance matrix for a frontier portfolio to have all positive weights. The consensus of these papers is that the fraction of frontier portfolios with positive weights is small and decreases as the number of assets increases.

Brennan and Lo (2010) coin the term impossible frontiers for those cases where every portfolio on the frontier has at least one short position. They show analytically that under certain plausible assumptions, impossible frontiers are the rule rather than the exception. For any arbitrary set of expected returns and a randomly chosen
covariance matrix, the probability that the resulting frontier is impossible approaches one as the number of assets becomes large. They show empirically that frontiers based on standard empirical estimates have significant amounts of short selling. This updates the earlier studies by Best and Grauer and Green which reached the same conclusions.

Levy and Roll (2010) use a different approach to address this issue. Specifically they look for an expected return vector and a covariance matrix that make a given market portfolio mean variance efficient with the proviso that the expected return vector and the covariance matrix are as close as possible to some prior estimates. They hold the correlation matrix constant and vary the standard deviation vector and the return vector. Relatively small perturbations in these parameters result in the market portfolio being mean variance efficient. The Levy Roll approach shows that even if under a representative return vector and covariance matrix, a given market portfolio is inefficient, there is a neighboring expected return vector and covariance matrix that makes the given market portfolio efficient. Essentially their result demonstrates that a set of measure zero need not be empty. The current paper is in the spirit of Levy and Roll in that we obtain an explicit solution.

Specifically we derive an efficient frontier portfolio with positive weights that is consistent with a given covariance matrix. We exploit the fact that the dominant factor in a principal component analysis is a good proxy for the market. This factor typically has positive weights. However, the portfolio, $P$, associated with this dominant factor will typically not lie on the frontier. We use the eigenvectors of the correlation matrix to construct a complete set of orthogonal portfolios where portfolio $P$ associated with the dominant eigenvector has positive weights. As Roll (1980) demonstrates the expected returns on portfolios orthogonal to $P$ will in general differ

\[1\]We address the issue of negative weights in Section Two.
when \( P \) is not on the frontier. Roll also shows that if \( P \) is on the frontier, the expected returns on these orthogonal portfolios have a very simple structure. If \( P \) is on the frontier, the expected return on all portfolios that are orthogonal to \( P \) have the same expected return or zero beta rate. Our paper exploits this idea using a well defined set of orthogonal portfolios.

Hence, this paper assumes that portfolio \( P \) is on the efficient frontier. We fix the expected return of portfolio \( P \) and the corresponding zero beta rate. This determines the expected returns on all the \( n \) orthogonal portfolios. Consequently we have \( n \) linear equations for the \( n \) components of the expected return vector \( \mu \). Hence for a given covariance matrix \( V \) we obtain a frontier portfolio with positive weights and a compatible expected return vector.

The orthogonal portfolios we use are derived from the eigenvectors of the correlation matrix as in Avellaneda and Lee (2010). The weighting on a given stock is inversely proportional to its volatility. As noted by Avellaneda and Lee, when the weights are positive this is consistent with a market capitalization weighting since larger stocks tend to be less volatile. Hence our frontier portfolio is a reasonable proxy for the market portfolio.

We discuss the conditions under which the dominant eigenvector of the correlation matrix has all its components positive. We know from the classical Perron-Frobenius theorem that a sufficient condition is that all the correlations are positive. However this condition is too strong and we draw attention to some extensions to the Perron-Frobenius theorem where some of the correlations can be negative and the dominant eigenvector still has all its components positive. Empirical studies show that some of the correlations can be negative and in some cases the pervasiveness of negative correlations produces negative weights in the dominant eigenvector. We explain how these negative weights can be eliminated by replacing the historical correlation matrix.
with the shrinkage estimate proposed by Ledoit and Wolf (2004).

The remainder of the paper is as follows. Section Two sets up our notation and derives the main result. Section Three shows that our structure provides an intuitive illustration of one of Green’s results relating to the conditions for positively weighted portfolios on the frontier. Section Four analyzes an instructive three asset example examined by Brennan and Lo (2010). The structure of this example permits us to derive simple closed form expressions for the frontier portfolio weights and the compatible expected return vector. Section Five discusses an empirical application using the 30 stocks in the Dow Jones Industrial Average. We compute the portfolio weights and the expected return vector for this example. Section Six shows that these weights are robust in terms of their sensitivity to estimation errors in $V$. Section Seven compares the out of sample performance of our positively weighted portfolios to that of the equally weighted strategy for samples of S & P 500 stocks. Finally, the Appendix summarizes the conditions under which the correlation matrix has a dominant eigenvector where all the elements are positive.

## 2 Derivation of the Solution

In this section we set out our notation and describe the approach. Assume there are $n$ risky securities with a positive definite covariance matrix $V$. We describe how to construct an efficient portfolio with positive weights and an expected return vector $\mu$, so that this efficient portfolio is compatible with $\mu$ and $V$. We do this by first constructing $n$ orthogonal portfolios and then identifying the frontier portfolio with the principal eigenvector. This portfolio is derived from the principal eigenvector of the correlation matrix and in general\(^2\) it has (or can be modified to have) positive

\(^2\)We discuss the theoretical conditions under which all the weights will be positive in the Appendix.
weights. From the mean variance properties of frontier portfolios all the portfolios that are orthogonal to a frontier portfolio have the same expected return. This provides a natural way to select the expected return vector.

Suppose we have a given covariance matrix $V$. If we select any $n$ uncorrelated factor portfolios they will normally lie inside the Markowitz mean variance frontier. The expected returns of the factor portfolios depend on the expected return vector $\mu$. However for any given set of portfolios, all the variances and all the covariances are fixed since $V$ is fixed. Thus as we change the input expected return vector $\mu$ the position of a given portfolio in mean variance space moves vertically up or down along a line of constant variance.

We now explain how to select the $n$ orthogonal portfolios. We factorize the covariance matrix as follows:

$$V = SCS$$

where

$$S = \begin{bmatrix}
\sigma_1 & 0 & \cdots & 0 \\
0 & \sigma_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & 0 & \cdots & \sigma_n
\end{bmatrix}$$

and $C$ is the correlation matrix and $\sigma_i^2$ is the variance of asset $i$.

The matrix $C$ has $n$ positive\footnote{Because it is positive definite.} eigenvalues

$$\lambda_1 > \lambda_2 > \cdots \lambda_n$$
and $n$ associated eigenvectors

$$v^{(1)}, v^{(2)} \ldots v^{(n)}.$$ 

We have

$$Cv^{(i)} = \lambda_i v^{(i)}, \quad 1 \leq i \leq n.$$ 

Any scalar multiple of an eigenvector of $C$ is also an eigenvector of $C$. These eigenvectors are pairwise orthogonal in the linear algebra sense.

$$\left(v^{(i)}\right)^T v^{(j)} = 0, \quad i \neq j$$

The principal or dominant eigenvector $v^{(1)}$ is associated with the largest eigenvalue $\lambda_1$.

There are two steps in converting these eigenvectors into portfolios. First we divide each component by the volatility of the corresponding asset. This means that all the first components are divided by $\sigma_1$, all the second components are divided by $\sigma_2$ and so on. This adjustment ensures that the corresponding portfolios will be orthogonal with respect to the covariance matrix $V$. The second step is to normalize these adjusted vectors so that their weights sum to one so that each one corresponds to a portfolio.

In the first step we form $n$ new vectors

$$w^{(1)}, w^{(2)} \ldots w^{(n)}$$

from the eigenvectors by dividing by the standard deviations. In matrix notation we have

$$w^{(i)} = S^{-1}v^{(i)}, \quad i = 1, 2 \ldots n.$$
The second step is to normalize the \( \mathbf{w} \) so that their components add up to one to convert these \( n \) vectors into \( n \) bona fide portfolios.

\[
y^{(i)} = \frac{w^{(i)}}{e^T w^{(i)}} = \frac{w^{(i)}}{k_i}, \quad i = 1, 2 \cdots n.
\]

where \( e \) is the \( n \) by one vector of ones and \( k_i = e^T w^{(i)} \).

The \( y \) vectors correspond to the factor portfolios. Our construction ensures that these portfolios are pairwise orthogonal with respect to the covariance matrix \( V \). For \( i \neq j \) we have

\[
(y^{(i)})^T V y^{(j)} = \frac{1}{k_i k_j} (w^{(i)})^T V w^{(j)} = \frac{1}{k_i k_j} (S^{-1} v^{(i)})^T V S^{-1} v^{(j)} = \frac{1}{k_i k_j} (v^{(i)})^T S S^{-1} v^{(j)} = \frac{1}{k_i k_j} (v^{(i)})^T C v^{(j)} = \frac{1}{k_i k_j} \lambda_j (v^{(i)})^T v^{(j)} = 0
\]

We now derive the expected return vector \( \mathbf{\mu} \) that is compatible with \( V \) and consistent with \( y^{(1)} \) being on the efficient frontier. We assume that \( y^{(1)} \) is on the frontier with expected return \( \mu_m \). In this case, the expected return on each of the other \((n-1)\) orthogonal portfolios must be equal. Furthermore this expected return corresponds to the expected return on the zero beta frontier portfolio that is orthogonal to the frontier portfolio with expected return \( \mu_m \). Denote the expected return on this zero beta portfolio by \( \mu_z \). Then all the other \((n-1)\) factor portfolios have an expected
return equal to $\mu_z$. From this we obtain $n$ equations for the elements of $\mu$ in terms of $\mu_m, \mu_z$ and the weights on the $n$ orthogonal portfolios. These equations can be summarized in matrix form.

$$Y\mu = q$$

where

$$Y = \begin{bmatrix}
y^{(1)}_1 & y^{(1)}_2 & \cdots & y^{(1)}_n \\
y^{(2)}_1 & y^{(2)}_2 & \cdots & y^{(2)}_n \\
\vdots & \vdots & \ddots & \vdots \\
y^{(n)}_1 & y^{(n)}_2 & \cdots & y^{(n)}_n
\end{bmatrix}, \quad \mu = \begin{bmatrix}
\mu_1 \\
\mu_2 \\
\vdots \\
\mu_n
\end{bmatrix}, \quad q = \begin{bmatrix}
\mu_m \\
\mu_z \\
\vdots \\
\mu_z
\end{bmatrix}$$

An alternative way to obtain the expected return vector uses the fact that there is a CAPM type relation between every asset and a frontier portfolio. In our case $y^{(1)}$ is on the frontier with expected return $\mu_m$. See Roll (1977). Let $x^{(i)}$ denote the portfolio that consists of one hundred percent in the $i^{th}$ asset and zero in the each of the remaining assets. We have

$$\mu_i - \mu_z = \beta_i(\mu_m - \mu_z), \quad 1 \leq i \leq n.$$  

where $\beta_i$ is given by

$$\beta_i = \frac{(x^{(i)})^T V y^{(1)}}{(y^{(1)})^T V y^{(1)}} = \frac{(x^{(i)})^T (SCS) S^{-1}v^{(1)}}{(y^{(1)})^T (SCS) S^{-1}v^{(1)}}$$
\[
\left( \frac{(x^{(1)})^T S y^{(1)}}{(y^{(1)})^T S y^{(1)}} \right)
\]

These different expressions for \( \beta_i \) follow from our previous results. It is clear from the middle expression that each \( \beta_i \) is positive so that \( \mu_i > \mu_z \) for \( 1 \leq i \leq n \).

It may be helpful at this stage to summarize the steps in the derivation. We started with the \( n \) eigenvectors of the correlation matrix. These eigenvectors were converted into \( n \) portfolios by scaling them with the volatilities and normalizing them to sum to one. By construction every pair of these portfolios has zero covariance. The portfolio corresponding to the principal eigenvector is identified with the market portfolio. For a given pair \((\mu_m, \mu_z)\) we can solve for the expected return vector that is compatible with these parameters and the matrix \( V \).

If we consider these \( n \) orthogonal portfolios as \( n \) new assets that span the same mean variance as the original assets, we see that these new assets have a diagonal covariance matrix whose \( j^{th} \) term is the variance of \( y^{(j)} \). This variance is

\[
(y^{(j)})^T V y^{(j)} = \frac{1}{c_j}
\]

where \( c_j \) is the reciprocal of the variance of portfolio \( y^{(j)} \). The zero beta (frontier) portfolio is a linear combination of the \((n - 1)\) portfolios \( y^{(2)}, y^{(3)}, \ldots, y^{(n)} \) that have expected return \( \mu_z \). Its weights are

\[
x_z = \sum_{j=2}^{n} \left( \frac{c_j}{\sum_{k=2}^{n} c_k} \right) y^{(j)}
= \sum_{j=2}^{n} \alpha_j y^{(j)}
\]

\(^4\)To go from the second line to the third line we recognize that \( v^{(1)} \) an eigenvector of \( C \).
We note that the $\alpha_j$ are all positive for $2 \leq j \leq n$. Hence if we express the efficient frontier in terms of these new assets, all the frontier portfolios with an expected return $\mu_p$ in the range $(\mu_m > \mu_p > \mu_z)$ will have positive weights in terms of these new assets. The weights on the frontier portfolio corresponding to $\mu_p$ are

$$\begin{pmatrix} \mu_p - \mu_z \\ \mu_m - \mu_z \end{pmatrix} y^{(1)} + \begin{pmatrix} \mu_m - \mu_p \\ \mu_m - \mu_z \end{pmatrix} x_p$$

It is clear that these frontier portfolios are positive linear combinations of the $n$ portfolios $y^{(1)}, y^{(2)}, \cdots y^{(n)}$. Of course in terms of the original assets the weights on any frontier portfolios need not be positive apart from the frontier portfolio with expected return $\mu_m$ and a small segment containing this portfolio.

In order for the dominant eigenvector to produce a viable market portfolio, each of its components has to be positive. A sufficient condition for the existence of a dominant eigenvector with positive entries is that all the pairwise correlations are positive so that the correlation matrix has positive entries. This is a consequence of the Perron-Frobenius theorem. This classic result has been extended to include the case of some negative correlations. We summarize this work in the Appendix. These results are of interest since they provide conditions under which the dominant eigenvector has positive weights. However in any actual empirical application we can compute the dominant eigenvector directly and check its entries.

We know empirically there can be some negative correlations among stock returns. If the incidence of negative correlation terms is sufficiently pervasive to produce negative weights in the dominant eigenvector, we can restore positivity through the use of a shrinkage estimate. Ledoit and Wolf (2004) show that using a covariance matrix that is a convex combination of the sample covariance matrix $\Sigma$ and a shrink-
age target matrix, $F$ outperforms the stand alone sample matrix in terms of portfolio performance. Their shrinkage target is based on a matrix with a common constant correlation equal to the average of the sample correlations. The revised estimate is given by

$$\delta^* F + (1 - \delta^*) \Sigma$$

where optimal shrinkage intensity, $\delta^*$, has an explicit form given in Ledoit and Wolf (2004). We show empirically in Section 7 that using the shrinkage estimate as the input covariance matrix generally ensures that the all the elements of the dominant eigenvector of the correlation matrix are positive.

Even if this procedure, based on the optimal $\delta^*$, still results in one or more negative elements we can always increase the shrinkage intensity, $\delta$ to ensure that all the elements of the dominant eigenvector of the correlation matrix are positive. The target matrix $F$ will clearly have a dominant eigenvector with all positive weights. Hence by continuity, there exists a $1 > \delta_0 > 0$ such that the shrinkage estimator

$$\delta F + (1 - \delta) \Sigma$$

produces a correlation matrix with a strictly positive dominant eigenvector for $\delta \geq \delta_0$.

### 3 Connection with Green’s Conditions

Our framework permits an intuitive interpretation of Green’s conditions related to the existence of frontier portfolios with positive weights. One of Green’s results shows that a necessary and sufficient condition for a frontier portfolio to have strictly positive weights on all assets is that there must exist no nontrivial portfolio, with expected return equal to the zero beta rate, that is either non-positively or non-
negatively correlated with all assets. Our structure of orthogonal portfolios enables us to provide a very direct interpretation of this result. We are able to specify this portfolio when it exists. When this portfolio does not exist we provide a direct proof of its non existence.

First we recall Green’s result.

*Theorem 1 of Green’s paper (1986) states that a necessary and sufficient condition for a frontier portfolio to have strictly positive weights on all assets is that there must exist no nontrivial

(i) hedge positions with expected payoffs equal to zero and non-negative correlation with all assets

or

(ii) portfolios that are either non-negatively or non-positively correlated with all assets and have expected returns equal to the zero-beta rate $\mu_z$.

We now illustrate case (ii) of Green’s Theorem. We assume that one of the $n$ orthogonal portfolios is on the frontier and that its expected return is $\mu_m$. The other $(n - 1)$ orthogonal portfolios have the same expected return, $\mu_z$. There are two situations. The first is when the frontier portfolio corresponds to the dominant eigenvector. In this case frontier portfolio has positive weights an all the assets. The second is when the frontier portfolio does not correspond to the dominant eigenvector. In this case the frontier portfolio will have some negative weights.
3.1 Case when frontier portfolio corresponds to the dominant eigenvector.

Assume the frontier portfolio corresponds to the dominant eigenvector. In this case the weights on this portfolio are all positive. By Green’s result there cannot exist a portfolio \( h \) which is either non-negatively or non-positively correlated with all assets and have expected returns equal to the zero-beta rate \( \mu_z \). We assume that \( h \) exists and derive a contradiction.

First note that \( h \) is orthogonal to \( y^{(1)} \). Hence

\[
(y^{(1)})^T V h = 0
\]  

(4)

Now the covariance between \( h \) and the primitive assets is \( Vh \). If \( Vh \geq 0 \), at least one of its components must be positive. If this were not so and all the components of \( Vh \) were zero, the variance of \( h \) would be zero, since

\[
h^T V h = 0.
\]

Since all the elements of \( y^{(1)} \) are positive

\[
(y^{(1)})^T V h > 0
\]

This contradicts equation (4).
In the same way, if $Vh \leq 0$, then since all the elements of $y^{(1)}$ are positive.

$$(y^{(1)})^T Vh < 0$$

This again contradicts equation (4). Hence we have shown that the portfolio $h$ cannot exist.

3.2 Case when frontier portfolio does not correspond to the dominant eigenvector.

The second case is when the frontier portfolio does not correspond to the dominant eigenvector. In this case we know that the frontier portfolio has negative weights. Hence by Green’s result there must exist a portfolio with non negative weights whose expected return is equal to the zero beta rate and whose covariances with the primitive assets all either all non positive or all non negative. In our case this portfolio is $y^{(1)}$. Its weights are all positive and

$$V y^{(1)} = \lambda_1 S^2$$

so the covariances between $y^{(1)}$ and all the $n$ primitive assets are all positive.

4 Three Asset Example

It is convenient to illustrate our approach with a three asset example based on one in Brennan and Lo (2010). The advantage of this example is that all the expressions of interest are available in closed form. The covariance matrix is parametrized using $d$ where $0 < d < 1$. The standard deviation vector, the correlation matrix and the
covariance matrix are given by

\[
\sigma = \tilde{\sigma} \begin{bmatrix} d & 1 \\ 1 & d^{-1} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & d & d^4 \\ d & 1 & d \\ d^4 & d & 1 \end{bmatrix}, \quad V = \tilde{\sigma}^2 \begin{bmatrix} d^2 & d^2 & d^4 \\ d^2 & 1 & 1 \\ d^4 & 1 & d^{-2} \end{bmatrix},
\]

where \( \tilde{\sigma} \) is a constant.

Brennan and Lo prove that for all \( d \in (0, 1) \) this correlation matrix, \( C \) is positive definite.

First we find the eigenvalues of the correlation matrix \( C \). Straightforward calculations show that the eigenvalues of \( C \) are

\[
\lambda_1 = 1 + \frac{dh}{2}, \quad \lambda_2 = 1 - d^4, \quad \lambda_3 = 1 + \frac{dg}{2},
\]

where

\[
h = d^3 + \sqrt{d^6 + 8}, \quad g = d^3 - \sqrt{d^6 + 8}.
\]

It is readily shown that \( \lambda_1 > \lambda_2 > \lambda_3 > 0 \) so that \( \lambda_1 \) is the dominant eigenvector.

The corresponding eigenvectors of \( C \) are

\[
v^{(1)} = \begin{bmatrix} 1 \\ \frac{4}{h} \\ 1 \end{bmatrix}, \quad v^{(2)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad v^{(3)} = \begin{bmatrix} 1 \\ \frac{4}{g} \\ 1 \end{bmatrix},
\]

In each case we have scaled the eigenvector so that its first component is equal to one. We note that the dominant eigenvector \( v^{(1)} \) has all its components positive but that \( v^{(2)} \) and \( v^{(3)} \) have one negative component.
As described in the previous section the unnormalized portfolio weights $w$ are obtained by multiplying the eigenvectors by $S^{-1}$ where

$$S^{-1} = \frac{1}{\sigma} \begin{bmatrix} d^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{bmatrix}$$

Hence we have

$$w^{(1)} = \frac{1}{\sigma} \begin{bmatrix} d^{-1} \\ 4h^{-1} \\ d \end{bmatrix}, \quad w^{(2)} = \frac{1}{\sigma} \begin{bmatrix} d^{-1} \\ 0 \\ -d \end{bmatrix}, \quad w^{(3)} = \frac{1}{\sigma} \begin{bmatrix} d^{-1} \\ 4g^{-1} \end{bmatrix},$$

The final step is to scale the $w$ vectors so that the weights sum to one. The resulting portfolios are the $y$ vectors.

$$y^{(1)} = \frac{1}{(1 + d^2)h + 4d} \begin{bmatrix} h \\ 4d \\ d^2h \end{bmatrix}, \quad y^{(2)} = \frac{1}{1 - d^2} \begin{bmatrix} 1 \\ 0 \\ -d^2 \end{bmatrix}, \quad y^{(3)} = \frac{1}{(1 + d^2)g + 4d} \begin{bmatrix} g \\ 4d \\ d^2g \end{bmatrix}$$

We note that the components of $y^{(1)}$, our frontier portfolio, are all positive for all $d \in (0, 1)$. The other two portfolios, $y^{(2)}$ and $y^{(3)}$ have negative weights.

We are now in a position to solve for the expected return vector $\mu$. We obtain

$$\mu_1 = \mu_z + \left( \frac{(1 + d^2)h + 4d}{4\sqrt{d^6} + 8} \right) (\mu_m - \mu_z) \quad (7)$$
\[
\mu_2 = \mu_z + \left(\frac{8(1 + d^2) - 4dg}{8d\sqrt{d^8} + 8}\right) (\mu_m - \mu_z)
\] (8)

\[
\mu_3 = \mu_z + \left(\frac{(1 + d^2)h + 4d}{4d^2\sqrt{d^8} + 8}\right) (\mu_m - \mu_z)
\] (9)

We note that the coefficients of the \((\mu_m - \mu_z)\) term are all positive. Recall that \(g < 0\). Our analysis shows that for all values of \(d \in (0, 1)\) the frontier portfolio with expected return \(\mu_m\) that is compatible with \((\mu, V)\) has positive weights. Here \(\mu\) is given by equations (6), (7) and (8) and \(V\) is given by (4).

This situation contrasts with the Brennan and Lo specification which leads to impossible frontiers. They use the same covariance matrix but they use a different specification for the expected return vector. They assume

\[
\mu = \bar{\mu} \begin{bmatrix} d \\ 1 \\ d^{-1} \end{bmatrix}
\] (10)

where \(\bar{\mu}\) is a constant.

For their \((\mu, V)\) combination they show that every frontier portfolio will always contain at least one short position for all \(d > .683\). We have shown that with our specification of \(\mu\) in equations (7), (8) and (9) and the same covariance matrix there will always exist a frontier portfolio with positive weights. Our solution is consistent with the Levy Roll finding. They showed that even if a particular set of parameters leads to impossible frontiers, there can exist nearby parameters that produce frontiers that are not impossible.
4.1 Numerical Example for Three Asset Case

We illustrate our solution for a specific numerical example for this three asset case. This will make it easier to see what is going on and assist with the intuition. We use the same numerical parameters as Brennan and Lo except that we let our model determine the expected return vector $\mu$ whereas they specify it up front. We assume that $d = 0.80$, $\mu_m = .12$, $\mu_z = 0$, $\bar{\sigma} = .2$. For these parameters we have

$$V = \begin{bmatrix}
0.0256 & 0.0256 & 0.016384 \\
0.0256 & 0.0400 & 0.0400 \\
0.016384 & 0.0400 & 0.0625
\end{bmatrix}$$

The three eigenvalues of the correlation matrix are

$$\lambda_1 = 2.3546, \quad \lambda_2 = 0.5904, \quad \lambda_3 = 0.0550.$$ 

We use the dominant eigenvector to compute the weights in the frontier portfolio $v^{(1)}$. The weights in the three orthogonal portfolios are as follows

$$y^{(1)} = \begin{bmatrix} 0.3868 \\ 0.3656 \\ 0.2476 \end{bmatrix} \quad y^{(2)} = \begin{bmatrix} 2.7778 \\ 0.0000 \\ -1.7778 \end{bmatrix} \quad y^{(3)} = \begin{bmatrix} 3.5033 \\ -4.7455 \\ 2.2421 \end{bmatrix}$$

These portfolio weights do not all add up to one because of rounding. From
equations (6), (7) and (8) the expected return vector is

\[
\begin{bmatrix}
\mu_1 &= 0.0914 \\
\mu_2 &= 0.1349 \\
\mu_3 &= 0.1428
\end{bmatrix}
\]

We plot the frontier based on this expected return vector and the covariance matrix \( V \) in Figure 1. We display the mean variance coordinates of the three orthogonal portfolios. Note that \( y^{(1)} \) is on the frontier but that the other two portfolios are not. They both have the same expected return as the zero beta portfolio and thus lie on a straight line inside the frontier. Figure 1 illustrates the geometric properties of the orthogonal portfolios in mean variance space.
Portfolio $y_1$ is on frontier
Portfolios $y_2$ and $y_3$ lie on the zero beta line

Figure 1: Efficient frontier and the three orthogonal factor portfolios with $d = .8$
5 Empirical Application

We illustrate our approach using the thirty stocks in the Dow Jones Industrial Average as at February 27 2012. We used weekly price data from June 14 2001 to February 27 2012 in our estimation of the correlation matrix. It turns out that all the elements of the estimated correlation matrix are positive. Hence from the Perron Frobenius theorem all the elements of the dominant eigenvector will be positive. All the other eigenvectors will have at least one negative sign. A direct computation of the eigenvectors matrix confirms these predictions.

Instead of writing out the entire $30 \times 30$ matrix of eigenvectors, we summarized the sign variation in each eigenvector as follows. First, replace each element in this matrix by its sign. If the entry is positive, we replace it by $+1$ and if it is negative, we replace it by $-1$. Then, we sum all the signs for each vector. The results are in Table 1. The elements of the first eigenvector are all positive so the sum of the signs is 30. The second eigenvector has 14 positive elements and 16 negative elements so the sum of its signed elements is negative two as shown.

The portfolio weights in the orthogonal portfolios are obtained by rescaling the eigenvectors by the standard deviations and normalizing so that the weights sum to one. The portfolio corresponding to the dominant eigenvector is the only one with positive weights as Table 1 makes clear. The remaining 29 portfolios involve short selling and some involve very extreme positions.
Table 1: Sum of signed eigenvectors based on empirical correlation matrix for 30 DJIA stocks. Empirical correlation matrix estimated using 558 weekly returns from June 14 2001 to February 27 2012.

<table>
<thead>
<tr>
<th>Eigenvector</th>
<th>Sum of signs</th>
<th>Eigenvector</th>
<th>Sum of signs</th>
<th>Eigenvector</th>
<th>Sum of signs</th>
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<td>-6</td>
<td>21</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>-2</td>
<td>12</td>
<td>2</td>
<td>22</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>13</td>
<td>2</td>
<td>23</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>14</td>
<td>0</td>
<td>24</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>15</td>
<td>-4</td>
<td>25</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>16</td>
<td>4</td>
<td>26</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>17</td>
<td>-4</td>
<td>27</td>
<td>8</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>18</td>
<td>-2</td>
<td>28</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>19</td>
<td>4</td>
<td>29</td>
<td>4</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>20</td>
<td>0</td>
<td>30</td>
<td>2</td>
</tr>
</tbody>
</table>

One way to measure the extent of short selling is to use a gross leverage factor. We define the gross leverage of a portfolio as

$$\frac{\text{Sum of longs} + |\text{Sum of shorts}|}{\text{Net equity}} - 1$$

We subtract one so that a long only portfolio has a leverage of zero. As an example consider the third portfolio in the numerical example in Section Five. The portfolio weights in this case were [3.5, -4.7, 2.2] and so in this case the gross leverage is

$$\frac{3.5 + 2.2 + 4.7} {3.5 + 2.2 - 4.7} - 1 = 9.4$$

Table 2 shows the gross leverage for each of the thirty orthogonal portfolios for our 30 Dow Jones stocks. This table shows that apart from the first one these portfolios tend to be very highly levered. The average gross leverage for these 29 portfolios is 103. The most extremely levered portfolio is portfolio number 27, with a gross leverage of 944.8.
Table 2: Gross Leverage factors of the 30 orthogonal portfolios constructed from the 30 DJIA stocks. Empirical correlation matrix estimated using 558 weekly returns from June 14 2001 to February 27 Feb 2012.

<table>
<thead>
<tr>
<th>Portfolio number</th>
<th>Gross leverage</th>
<th>Portfolio number</th>
<th>Gross leverage</th>
<th>Portfolio number</th>
<th>Gross leverage</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td>11</td>
<td>94.1</td>
<td>21</td>
<td>36.8</td>
</tr>
<tr>
<td>2</td>
<td>2.4</td>
<td>12</td>
<td>46.4</td>
<td>22</td>
<td>217.3</td>
</tr>
<tr>
<td>3</td>
<td>3.7</td>
<td>13</td>
<td>55.0</td>
<td>23</td>
<td>523.0</td>
</tr>
<tr>
<td>4</td>
<td>31.3</td>
<td>14</td>
<td>15.4</td>
<td>24</td>
<td>50.5</td>
</tr>
<tr>
<td>5</td>
<td>9.4</td>
<td>15</td>
<td>65.8</td>
<td>25</td>
<td>75.8</td>
</tr>
<tr>
<td>6</td>
<td>15.8</td>
<td>16</td>
<td>38.3</td>
<td>26</td>
<td>27.3</td>
</tr>
<tr>
<td>7</td>
<td>198.5</td>
<td>17</td>
<td>35.6</td>
<td>27</td>
<td>944.8</td>
</tr>
<tr>
<td>8</td>
<td>55.1</td>
<td>18</td>
<td>34.3</td>
<td>28</td>
<td>44.4</td>
</tr>
<tr>
<td>9</td>
<td>16.2</td>
<td>19</td>
<td>97.0</td>
<td>29</td>
<td>99.4</td>
</tr>
<tr>
<td>10</td>
<td>55.8</td>
<td>20</td>
<td>24.2</td>
<td>30</td>
<td>75.9</td>
</tr>
</tbody>
</table>

We use these orthogonal portfolios in conjunction with equation (3) to compute the expected returns. For illustrative purposes, we have set $\mu_m = 0.08$ and $\mu_z = 0$. Table 3 gives the computed expected return for each stock as well as its weight in the frontier portfolio and its standard deviation. The weight of a given stock in the frontier portfolio is negatively correlated with its standard deviation with a correlation coefficient of negative .87. The stocks with the highest expected returns correspond to those with the highest standard deviation. In this case the correlation is .95.

The three stocks with the lowest weights in the efficient portfolio- Bank of America, Alcoa and JPMorgan- have the highest standard deviations. The six stocks with the highest weights (exceeding four percent) are Chevron, Coca Cola, Exxon, Johnson and Johnson, Procter and Gamble and United Technologies. These companies have low standard deviations with an average of 21.6%.

The estimation period we used includes the 2007-2009 global financial crisis when volatilities and correlations were higher than normal. Our method will work just as well if we base it on forward looking estimates of the input parameters. Kempf, Korn and Saßning (2012) propose a method to estimate the covariance of stock returns
using option prices and demonstrate its advantages over historical based estimates. It's possible to incorporate their approach with this method.

Table 3: Results for 30 DJIA stocks. Portfolio weights, expected returns and standard deviations. The expected returns are based on $\mu_m = .08$ and $\mu_z = 0$. Empirical covariance matrix estimated using weekly returns from June 14 2001 to February 27 Feb 2012.

<table>
<thead>
<tr>
<th>Stock ticker symbol</th>
<th>Weight in frontier portfolio</th>
<th>Expected return on stock</th>
<th>Standard Deviation of stock</th>
</tr>
</thead>
<tbody>
<tr>
<td>MMM</td>
<td>0.0460</td>
<td>0.0750</td>
<td>0.2320</td>
</tr>
<tr>
<td>AA</td>
<td>0.0242</td>
<td>0.1445</td>
<td>0.4437</td>
</tr>
<tr>
<td>AXP</td>
<td>0.0289</td>
<td>0.1269</td>
<td>0.3804</td>
</tr>
<tr>
<td>T</td>
<td>0.0328</td>
<td>0.0638</td>
<td>0.2535</td>
</tr>
<tr>
<td>BAC</td>
<td>0.0167</td>
<td>0.1533</td>
<td>0.5509</td>
</tr>
<tr>
<td>BA</td>
<td>0.0295</td>
<td>0.1029</td>
<td>0.3391</td>
</tr>
<tr>
<td>CAT</td>
<td>0.0300</td>
<td>0.1140</td>
<td>0.3540</td>
</tr>
<tr>
<td>CVX</td>
<td>0.0404</td>
<td>0.0752</td>
<td>0.2479</td>
</tr>
<tr>
<td>CSCO</td>
<td>0.0244</td>
<td>0.0957</td>
<td>0.3596</td>
</tr>
<tr>
<td>KO</td>
<td>0.0415</td>
<td>0.0465</td>
<td>0.1922</td>
</tr>
<tr>
<td>DD</td>
<td>0.0390</td>
<td>0.0948</td>
<td>0.2832</td>
</tr>
<tr>
<td>XOM</td>
<td>0.0431</td>
<td>0.0679</td>
<td>0.2280</td>
</tr>
<tr>
<td>GE</td>
<td>0.0315</td>
<td>0.1035</td>
<td>0.3291</td>
</tr>
<tr>
<td>HPQ</td>
<td>0.0248</td>
<td>0.0983</td>
<td>0.3615</td>
</tr>
<tr>
<td>HD</td>
<td>0.0294</td>
<td>0.0977</td>
<td>0.3312</td>
</tr>
<tr>
<td>INTC</td>
<td>0.0269</td>
<td>0.1001</td>
<td>0.3504</td>
</tr>
<tr>
<td>IBM</td>
<td>0.0382</td>
<td>0.0742</td>
<td>0.2533</td>
</tr>
<tr>
<td>JNJ</td>
<td>0.0412</td>
<td>0.0454</td>
<td>0.1907</td>
</tr>
<tr>
<td>JPM</td>
<td>0.0232</td>
<td>0.1313</td>
<td>0.4321</td>
</tr>
<tr>
<td>KFT</td>
<td>0.0357</td>
<td>0.0471</td>
<td>0.2087</td>
</tr>
<tr>
<td>MCD</td>
<td>0.0343</td>
<td>0.0520</td>
<td>0.2237</td>
</tr>
<tr>
<td>MRK</td>
<td>0.0254</td>
<td>0.0666</td>
<td>0.2942</td>
</tr>
<tr>
<td>MSFT</td>
<td>0.0326</td>
<td>0.0742</td>
<td>0.2739</td>
</tr>
<tr>
<td>PFE</td>
<td>0.0330</td>
<td>0.0673</td>
<td>0.2594</td>
</tr>
<tr>
<td>PG</td>
<td>0.0454</td>
<td>0.0430</td>
<td>0.1769</td>
</tr>
<tr>
<td>TRV</td>
<td>0.0296</td>
<td>0.0766</td>
<td>0.2920</td>
</tr>
<tr>
<td>UTX</td>
<td>0.0416</td>
<td>0.0883</td>
<td>0.2648</td>
</tr>
<tr>
<td>VZ</td>
<td>0.0351</td>
<td>0.0592</td>
<td>0.2361</td>
</tr>
<tr>
<td>WMT</td>
<td>0.0398</td>
<td>0.0535</td>
<td>0.2105</td>
</tr>
<tr>
<td>DIS</td>
<td>0.0359</td>
<td>0.0959</td>
<td>0.2969</td>
</tr>
</tbody>
</table>
6 Robustness to Estimation Error

It is well known that the frontier portfolios that are obtained by an optimization procedure are highly sensitive to estimation errors. Their weights are very sensitive to the input expected return vector and also to the input covariance matrix. There is a perverse interaction between estimation risk and the optimization procedure. Kan and Smith (2008) show that if the traditional sample estimates are used that the resulting frontier produces a biased and misleading estimate of the true population frontier. Indeed they show that the benefits of the optimization procedure are swamped by estimation errors for the typical windows used in practice.

Given the sensitivity of standard portfolio selection procedures to estimation risk, it is of interest to examine the sensitivity of our procedure to estimation error in the inputs. In our case the sole input is the covariance matrix $V$. In this section we use a simple Monte Carlo experiment to address this issue. Since we do not make explicit use of optimization in our approach there is the prospect that estimation error will be less of an issue here than usual in portfolio problems. This turns out to be the case: our procedure appears$^7$ to be robust.

To explore this issue we assume that we have the true population matrix $V$. We assume that the return generating process for the $n$ securities is multivariate normal. This enables us to generate $N$ sample covariance matrices where each matrix is based on an estimation period with $M$ observations. For each covariance matrix we compute the weights of the efficient frontier portfolio and the corresponding expected return vectors. In this way we obtain a distribution of the weights in the efficient portfolio and the associated distribution of the expected return vector.

We use the example in Section Five to illustrate this. We assume the estimated

$^7$At least for the example considered here.
$V$ is the true population covariance matrix. We set $N = 10,000$. For each simulation run, we generate ten years of returns from which we estimate 10,000 covariance matrices. From each covariance matrix we obtain an efficient portfolio and an associated set of weights. The distribution of the portfolio weights is summarized in Figure Two. The 30 vertical lines on the graph contain information on the average portfolio weight for each stock and two standard deviations on each side of the average. The distribution in every case is compact indicating that our portfolio weights are robust to estimation error. Figure Three contains similar information on the distribution of the expected return vectors for each of the 30 stocks. These expected returns are also very robust as well.
7 Performance of Positively Weighted Portfolios

In this section we examine the out of sample performance of our positively weighted portfolios over a wider data set and a longer historical period. In this case we encounter negative correlations and situations where the dominant eigenvector has some
negative elements. We illustrate the effectiveness of the Ledoit Wolf shrinkage approach in restoring positivity to the dominant eigenvector.

We estimate the empirical covariance matrix on a rolling basis, adjust it using the shrinkage procedure and use our approach to obtain the weights on the frontier portfolio. Then we form this portfolio and record its performance over the next period. At the end of this period we update the covariance matrix by including the most recent returns and dropping the earliest returns in the series and make the shrinkage adjustment. Based on the updated covariance matrix, we recompute the weights on the frontier portfolio and rebalance our existing portfolio to reflect these new weights. By repeating this procedure we are able to implement a feasible portfolio strategy. To implement this strategy all we require are the weights on the frontier portfolio and we do not require the expected return vector. Hence we do not need to make assumptions about the return on the market nor the zero beta return.

It is instructive to compare this strategy with the corresponding equally weighted strategy. De Miguel, Garlappi and Uppal (2009) show that the equally weighted strategy outperforms a value weighted strategy and is no worse than many other popular portfolio strategies. More recently Plyaka, Uppal and Vilkov (2012) identify the sources of the superior performance of the equally weighted strategy. One of their main findings is that the rebalancing provides a natural contrarian strategy that contributes to its superior alpha. In light of its documented superior performance the equally weighted strategy is a natural benchmark for us to use. This comparison is designed to check if the returns on the positively weighted portfolios are reasonable.

To compare the two strategies we used daily returns from January 1, 1990 to December 31, 2011 using all stocks that were in the S&P 500 at any time during this period. We used a five year period to estimate the covariance matrix. Any stock in our sample that was delisted from the S&P was replaced with a random unselected
stock for the rest of the period. We used random samples of 50 stocks each from this universe and repeated the analysis for 1000 such samples.

As we discussed earlier, it is not necessary for all the pairwise correlations to be positive to ensure that the dominant eigenvector contains only positive elements. Some negative correlations are permitted but if the negative\(^8\) correlations become too pervasive some of the elements of the dominant eigenvectors become negative. Using the Ledoit Wolf shrinkage procedure eliminates almost all the situations with negative elements in the dominant eigenvector. If we just use the unadjusted historical matrix the average value of the short positions over our 1000 samples is 0.20\% of total portfolio value. When we use the Ledoit Wolf optimal shrinkage adjustments, the average value of the short positions over the 1000 samples reduces to 0.01\% of total portfolio value which is essentially negligible. For our 1000 samples 90 \% of the optimal shrinkage intensities lie in the range [0.10, 0.25]. The average optimal shrinkage intensity over the 1000 samples was 0.18 with a standard deviation of 0.05.

The results are summarized in Table 4. These results are based on the averages taken over 1000 samples of 50 stocks each for the period 1990 to 2011. The first five years were used to obtain the initial estimates of the historical covariance matrices. We find that for all rebalancing frequencies the equally weighted portfolio strategy outperforms the positively weighted portfolios in terms of total returns and Sharpe ratios.

These empirical investigations show that our approach can be used to obtain positively weighted portfolios and that the returns generated by these positively weighted portfolios are quite plausible. The returns obtained under our approach are reasonably close to those obtained by following the equally weighted strategy.

\(^8\)Mining stocks, for example tend to have negative correlations with most other stocks during certain periods.
Table 4. Comparison of equally weighted portfolio strategy with the positively weighted portfolio strategy. Results based on daily data for the period 1990 to 2011. The numbers below represent the averages of 1000 random samples of 50 S & P 500 stocks.

<table>
<thead>
<tr>
<th>Rebalancing Frequency</th>
<th>Equally Weighted Annualized Return</th>
<th>Positively Weighted Annualized Return</th>
<th>Equally Weighted Sharpe Ratio</th>
<th>Positively Weighted Sharpe Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Six months</td>
<td>13.44%</td>
<td>12.52%</td>
<td>0.607</td>
<td>0.570</td>
</tr>
<tr>
<td>Three months</td>
<td>13.50%</td>
<td>12.62%</td>
<td>0.605</td>
<td>0.574</td>
</tr>
<tr>
<td>One month</td>
<td>13.47%</td>
<td>12.64%</td>
<td>0.600</td>
<td>0.574</td>
</tr>
</tbody>
</table>

8 Summary

This paper uses a simple approach to obtain a mean variance efficient portfolio with positive weights. This portfolio is a proxy for the market portfolio. We showed how to derive this portfolio from the eigenvectors of the correlation matrix. Our method takes the covariance matrix as fixed and does not use the sample expected return vector. This may be an attractive alternative in some situations since the expected return vector is so difficult to estimate. Our approach is similar in spirit to the Levy Roll method but they need to specify an input expected return vector. Our method also has some of the flavor of the Black Litterman model, which also ignores the sample expected returns and uses an equilibrium based approach to impute the expected returns. In our case we also take the covariance matrix as given and then derive a market proxy and use it to impute the expected returns. We showed that it can be applied in an empirical setting and that it produces reasonable results.
References


Perron, O, (1907), ”Zur theorie der matrizen”, *Mathematische Annalen* 64, 248-263.


Tarazaga, P. M. Raydon and A. Hurman (2011), ”Perron-Frobenius theorem for matrices with some negative entries”, *Linear Algebra and its Applications* 328, 57-68.
Appendix: Conditions for Principal Eigenvector to have Positive Weights

This appendix provides a brief summary of recent extensions to the Perron Frobenius result. These extensions derive general conditions for principal eigenvector to have positive weights. In this connection two papers are relevant to our case. The first is Tarazaga et al. (2001) and the second is Noutsos (2006).

Tarazaga et al. (2001) extend the result to a set of matrices with some negative entries. They obtained a sufficient condition that ensures a symmetric matrix has a nonnegative dominant eigenvector.

**Theorem (Tarazaga et al.)**

Let $S_n$ denote the space of $n \times n$ symmetric matrices. Let $e$ be the column vector of 1’s. Given $A \in S_n$ satisfying

$$e' Ae \geq \sqrt{(n-1)^2 + 1\|A\|_F},$$

where

$$\|A\|_F = \text{trace}(A^T A)^{\frac{1}{2}},$$

then $A$ has a nonnegative dominant eigenvector regardless of the sign of its entries.

This theorem is framed in terms of an inequality between the sum of elements of $A$ and the sum of squares of its elements. When this inequality is satisfied the matrix $A$ is guaranteed to have a nonnegative dominant eigenvector. However even if this condition is violated the dominant eigenvector may still have strictly positive elements.
Here is an example of a $(3 \times 3)$ symmetric matrix with negative elements and a strictly positive dominant eigenvector that does not satisfy the above conditions.

\[
A = \begin{bmatrix}
1.00 & 0.58 & 0.45 \\
0.58 & 1.00 & -0.35 \\
0.45 & -0.35 & 1.00
\end{bmatrix}
\]

The eigenvalues are

\[\{1.5955, 1.3311, 0.0734\}\]

so $A$ is positive definite. The dominant eigenvector of $A$ is

\[
v^{(1)} = \begin{bmatrix}
0.7595 \\
0.6148 \\
0.2126
\end{bmatrix}
\]

However the matrix $A$ does not satisfy the Tarazaga inequality since

\[
e^T A e = 4.36 \geq 4.6491 = \sqrt{(n-1)^2 + 1} \|A\|_F,
\]

The Tarazaga inequality only provides sufficient conditions.

Noutos (2006) provides stronger conditions that are both necessary and sufficient. To present his results we need two definitions.

**Definition One**

An $n \times n$ matrix $A$ is said to possess the *strong Perron-Frobenius property* if its dominant eigenvalue $\lambda_1$ is positive and each element of the corresponding eigenvector $v^{(1)}$ is positive.
Definition Two

An $n \times n$ matrix $A$ is said to be \textit{eventually positive} if there exists a positive integer $k_0$ such that $A^k > 0$ for all $k > k_0$. Noutsos (2006) proves the following theorem which is relevant to our case.

Theorem (Noutsos)

For any symmetric $n \times n$ matrix $A$ the following properties are equivalent.

1. $A$ possesses the strong Frobenius-Perron property.

2. $A$ is eventually positive.

The intuition behind this result is straightforward. The matrices $A$ and $A^k$ have the same eigenvectors. If $A^k$ is positive then we know from the Perron Frobenius theorem that its dominant eigenvector has all its components positive. Hence the corresponding dominant eigenvector of $A$ has all its elements positive.

In terms of our previous example, it is easy to check that the matrix $A$ is eventually positive since $A^k > 0$ for all $k \geq 7$.

This theorem gives conditions under which the dominant eigenvector has positive weights but in any actual empirical application we can compute the dominant eigenvector directly and check its entries.