Discrete-Time CPPI under Transaction Cost and Regime Switching

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Abstract

This paper studies the performance of discrete-time constant proportional portfolio insurance (CPPI) under proportional trading cost and regime switching. Explicit formulas are developed for a variety of measures for the performance of a CPPI portfolio, and a double-sided Laplace inversion method is developed to compute the Omega measure of a CPPI portfolio. The established formulas can be easily implemented for sensitivity analysis on performance of a CPPI portfolio, and a numerical example with a real data set of S&P 500 index is presented to illustrate the effects the regime switching feature of the financial market and the existence of transaction cost can exert on the performance of a CPPI portfolio.

Keywords: Finance, Investment analysis, Risk management, Markov processes, Constant proportion portfolio insurance, Regime switching

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1. Introduction

Portfolio insurance strategies have been widely used in the financial industry. They typically refer to portfolio management techniques which are designed to protect terminal value of a portfolio from dropping below certain percentage of its initial value. These techniques allow investors to participate in risky investments for potential upside profits and meanwhile control downside risk. The two main standard portfolio insurance methods are Constant Proportion Portfolio Insurance (CPPI) and Option-Based Portfolio Insurance (OBPI). The OBPI combines a long position in a risky asset with a put option on the asset. The strike of the put is predetermined corresponding to the guarantee level in the insurance strategy. The OBPI strategy was first proposed by Leland and Rurbinstein (1988) and consequently studied by, for example, El Karoui et al. (2005) among many others.

The present paper focuses on the CPPI strategy, which involves no option. The CPPI adopts a simplified self-financing strategy and rebalance a portfolio between a risky asset (typically a traded fund or index) and a reserve asset (typically a bond) dynamically over time. In this method, the portfolio manager starts by setting a floor equal to the lowest acceptable value of the portfolio and computes the cushion as the excess of the portfolio value over the floor. Then, an amount of capital proportional to the cushion is invested in the risky asset, called the exposure, and all the remaining capitals are invested in the reserve asset. The proportional factor is called multiplier. The floor and multiplier are predetermined exogenously depending on the investor’s risk tolerance and both play a prominent role in the risk-and-reward profile of the resulting CPPI portfolio. This portfolio strategy implies that, if the cushion value becomes close to zero, the exposure approaches zero too. In a continuous-time setting, this prevents portfolio value from falling below the floor, unless there is a sharp price drop in the market before the portfolio manager can modify the portfolio composition.

The CPPI strategy was initially introduced by Perold (1986) for fixed-income instruments and Black and Jones (1987) for equity instruments. It is shown by Perold and Sharpe (1988) that the CPPI strategies implemented in continuous-time trading on asset prices following geometric Brownian motions are expected utility maximizing for investors with HARA utilities. The optimality of CPPI strategies regarding the HARA expected utility maximizing is further analyzed by Balder and Mahayni (2010) and Branger et. al. (2010). The performance of CPPI is often investigated with a comparison to other portfolio insurance strategies, particularly the OBPI strategy. For example, a comparison between OBPI and CPPI is given by Do (2002); Bertrand and Prigent (2005); Balder and Mahayni (2010) and Pézier and Scheller (2013) among many others. The performance of credit CPPI along with the so-called constant proportion debt obligation structures was studied by Garcia et al. (2008) and Joossens and Schoutens (2010) under a dynamic multivariate jump-driven model for credit spreads. The effect from price jumps on the performance of CPPI strategy was investigated by Cont and Tankov (2009). The existing literature also deals with stochastic volatility models and extreme value approaches on the CPPI method; see Bertrand and Prigent (2002, 2003). A general framework of CPPI for investment and protection strategies is formulated by Dersch (2010) along with a review on some other portfolio insurance techniques as well. The influence of estimation risk on the performance of CPPI strategies as well as the mitigation effect of the estimation risk by the robustification of mean-variance efficient portfolios are studied.
by Schöttle and Werner (2010). The effectiveness of the CPPI under a discrete-time setting with transaction cost is discussed by Balder et al. (2009) and Balder and Mahayni (2010).

The risky asset prices are widely recognized and empirically observed to be non-stationary in the financial industry, and this fact rendered a spur to research on CPPI method with a non-stationary risky asset price process. Weng (2013) recently analyzed the CPPI strategy in a continuous-time setting with the risky asset price modelled by a regime switching exponential Lévy process. Analytical forms of a variety of measures for the risk-and-reward profile of a CPPI portfolio were established, and their implementation was discussed in details for many popular Lévy models including the Merton's jumpdiffusion, Kou's jumpdiffusion, variance gamma and normal inverse Gaussian models. More recently, Ameur and Prigent (2013) proposed a CPPI method with a time-varying multiplier, which are determined, in response of the changes of market conditions, by a quantile and expected shortfall based criterion to control the gap risk.

The present paper can be viewed as a sister paper of Weng (2013) with both in a regime switching setting. In reality, the economic state usually shows an obvious feature of transition between two or among several states, and the financial return shows quite different characteristics under a different economic state. The regime switching framework offers a transparent and intuitive way to capture market behavior through different economic conditions. Markov-modulated regime switching processes have been widely advocated in econometrics as well as other relevant areas since the pioneering work of Hamilton (1989). For its applications in finance and actuarial science, we refer to Buffington and Elliott (2002), Elliott et al. (1995), Elliott et al. (2005), Hardy (2001), Li et al. (2008), Siu (2005) as well as many others.

Compared with Weng (2013), the model analyzed in the present paper is much more realistic. First, the CPPI portfolio in the present paper is rebalanced over a finite set of discrete times, whereas the model in Weng (2013) is under continuous-time trading. In reality, a portfolio can only be revised for a finite number of times over a finite investment horizon. Second, the transaction cost is taken into account in the present paper, while it can hardly be incorporated into a continuous-time model like the one in Weng (2013). In the real market, the transaction cost is often a prominent factor to consider as it may exert substantial effects on the performance of a portfolio.

In the present paper, the log-returns of both the risky asset and the reserve asset are assumed to follow distinct distributions at distinct market states (e.g., bull and bear), and the transition from one market state to another is supposed to follow a hidden Markov process with a finite state space. A variety of measures for the risk-and-reward profile of the CPPI portfolio are established with explicit forms. Moreover, a Laplace inversion methodology is developed for computing the Omega measure of a CPPI portfolio. The results are illustrated with a numerical example based on a real financial dataset.

The rest of the paper is organized as follows. Section 2 is the model setup. In Section 3, the terminal value of the CPPI portfolio is studied and a set of measures are defined for the risk-and-reward profile of the CPPI portfolio. The analytical formalts for various measures are obtained in Section 4, and numerical examples are presented in Section 5.
2. Model setup

Throughout the paper, all the random elements are defined on a common probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The expectation of a random variable \(X\) under the probability measure \(\mathbb{P}\) is denoted by \(\mathbb{E}[X]\). The transpose of a matrix \(\mathbf{A}\) (a vector \(\mathbf{a}\)) is denoted by \(\mathbf{A}^\prime\) (correspondingly \(\mathbf{a}^\prime\)). All the random variables are denoted by capital letters such as \(X, Y\) and \(Z\), possibly with a subscript or subscript attached to signify certain meanings accordingly. Capital letters \(\mathbf{P}\) and \(\mathbf{Q}\) in bold, possibly with a superscript, are exclusively saved to denote matrixes. \(\mathbf{1}\) donates an identity matrix of an appropriate dimension, whereas \(\mathbf{1}\) is a column vector with all elements equal to 1. \(\{H_k, k \geq 0\}\) denotes the regime process exclusively. Greek letters in bold are used to denote a real vector so that \(\boldsymbol{\xi}\) denotes \((\xi_1, \ldots, \xi_r)\) with real numbers \(\xi_i, i = 1, \ldots, r\). The real line are the complex plane are respectively denoted by \(\mathbb{R}\) and \(\mathbb{C}\).

2.1. Regime switching financial market

The CPPI portfolio is allocated between a risky asset \(\{S_t, t \geq 0\}\) and a reserve asset \(\{B_t, t \geq 0\}\) over a finite set of trading times \(\{t_0 < t_1 < \cdots < t_n\}\), where \(t_0 = 0\) denotes the initial investment time, \(t_n = T\) is the terminal trading time, and the other trading times are evenly distributed over the investment horizon so that \(t_k - t_{k-1} = T/n\) for \(k = 1, \ldots, n\).

The state of the financial market is described by a discrete-time finite state Markov chain \(H = \{H_k, k = 0, 1, \ldots, n\}\), where \(H_k\) is the market state over period \((t_{k-1}, t_k]\), \(k = 1, \ldots, n\) and \(H_0\) is the initial market state before entering period \((t_0, t_1]\). In the specific calibration of a regime switching financial market model, two or three states are commonly assumed for the Markov chain \(H\). They respectively represent a bullish and bearish state in a model with two states (or regimes), and a third one, if exists, is typically interpreted as an intermediate (or normal) state between the bullish and bearish states. For presentation convenience, notations from Elliott et al. (1995) will be used to assume that the state space for \(H\) consists of \(r\) unit vectors \(\{h_1, \ldots, h_r\}\), where \(h_j = (0, \cdots, 0, 1, 0, \cdots, 0)\) \(\in \mathbb{R}^r\) with 1 in its \(j\)th coordinate and 0 in all the others. The convenience of such a representation for the states can be seen from the analysis in the sequel.

For each \(j = 1, \ldots, r\), let

\[
(X^{(j)}, R^{(j)}) := \left\{ \left( X_1^{(j)}, R_1^{(j)} \right), \ldots, \left( X_n^{(j)}, R_n^{(j)} \right) \right\}
\]

be a sequence of independent and identically distributed bivariate random vectors, with \(X_k^{(j)}\) and \(R_k^{(j)}\) respectively standing for the log-retuns of the risky asset and the reserve asset over period \((t_{k-1}, t_k]\) given \(H_k = h_j\). In other words, \((X_k^{(j)}, R_k^{(j)})\) is the log-return vector of the investment assets over a period of \((t_{k-1}, t_k]\), provided that the market stays in the \(j\)-th regime. Besides, \((X^{(j)}, R^{(j)})\) as sequences of random vectors are independent to each other among distinct \(j\)’s. The distribution of \((X_k^{(j)}, R_k^{(j)})\) should be assumed different from that of \((X_k^{(i)}, R_k^{(i)})\) for \(j \neq i\); otherwise, it is not necessary to distinguish between the two regimes \(i\) and \(j\). To proceed, put \(X_k = (X_k^{(1)}, \ldots, X_k^{(r)})', k = 1, \ldots, n\), and define

\[
X_k = \langle H_k, X_k \rangle, \quad k = 1, \ldots, n,
\]
where $\langle \mathbf{a}, \mathbf{b} \rangle$ denotes the inner product of two real vectors $\mathbf{a}$ and $\mathbf{b}$ so that $\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}'\mathbf{b} = \sum_j a_j b_j$. The resulting sequence $\{X_1, \ldots, X_n\}$ are the log-return of the risky asset without conditioning on the market state. Similarly, we put $\mathbf{R}_k = (R_k^{(1)}, \ldots, R_k^{(r)})$, $k = 1, \ldots, n$. The unconditional log-return sequence $\{R_1, \ldots, R_n\}$ for the reserve asset is defined similarly as follows

$$R_k = \langle H_k, \mathbf{R}_k \rangle, \; k = 1, \ldots, n. \quad (3)$$

It is worth noting that, although the log-return vectors given in (1) for a fixed regime $j$ are independent and identically distributed, the sequence $\{(X_k, R_k), k = 1, \ldots, n\}$ of the unconditional log-returns is non-stationary due to the existence of the underlying Markov process $H$. Moreover, as commented by (Elliott et al., 1995, p16), since $H_t$ takes values in $\{r_1, \ldots, r_r\}$, any $C^d$-valued function $f(H_t)$ for a positive integer $d$ can be expressed as a linear functional $f(H_t) = \langle \mathbf{f}, H_t \rangle$, where $\mathbf{f} = (f(h_1), \ldots, f(h_r))$. Thus,

$$f(H_t) = \sum_{j=1}^r f(h_j) \langle H_t, h_j \rangle. \quad (4)$$

This fact will be used in the derivation for formulas of various measures in section 4.

2.2. Discrete-time CPPI under proportional transaction cost

As specified in the preceding subsection, the CPPI portfolio is dynamically rebalanced between a risky asset $\{S_t, t \geq 0\}$ and a reserve asset $\{B_t, t \geq 0\}$ over revision times $\{t_0 < t_1 < \cdots < t_n\}$, which are evenly distributed on the investment horizon so that that $\delta := T/n = t_k - t_{k-1}, \; k = 1, \ldots, n$. The framework introduced below is similar to that of Balder et al. (2009).

To distinguish the corresponding quantities between before and after a revision time $t_k$, in what follows a subscript $t_k^+$ will be attached on the right lower corner of a notation to signify that this notation represents a quantity immediately after the portfolio revision at time $t_k$. In contrast, a notation with a subscript $t_k$ means a quantity at time $t_k$ before the portfolio revision. For example, $C_{t_k^+}$ and $C_{t_k}$ respectively denote the cushion values before and after the portfolio revision at time $t_k$, $k = 0, 1, \ldots, n$, and similarly $e_{t_k^+}$ is the exposure in the risky asset immediately after the portfolio revision at time $t_k$ while $e_{t_k}$ is the exposure at time $t_k$ right before the portfolio revision.

Suppose that the floor $\{F_t, t \geq 0\}$ is fully determined by the reserve asset price process $\{B_t, t \geq 0\}$ and without loss of any generality, assume $F_0 = B_0$ and

$$F_t = B_t = F_0 \exp \left\{ \sum_{k=1}^t R_k \right\}, \; t \geq 0,$$

where $F_0$ denotes the initial value of the floor and $R_k$ is the effective yield rate of the reserve asset over period $(t_{k-1}, t_k]$ as defined in (3). Let $V_t$ denote the portfolio value at time $t$ so that the cushion $C_t = V_t - F_t, \; t \geq 0$. To avoid the trivial case where the CPPI portfolio value is below the floor from the very beginning, we assume $V_0 > F_0$. 

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According to the CPPI strategy, if the cushion $C_{t_k}^+ > 0$, the portfolio at time $t_k^+$ is revised to include a position of $mC_{t_k}^+$ in the risky asset. The cushion $C_{t_k}^+$, however, depends on the transaction cost which occurs across the portfolio revision, in addition to the value of $C_{t_k}$. To clarify their relations, note that the main transaction cost over a financial market, such as a stock market, is typically proportional to the absolute change in exposure. Thus, we follow Balder et al. (2009) and assume that the transaction cost arising in the portfolio revision at time $t_k$ is given by $\theta |e_{t_k^+} - e_{t_k}|$, where $\theta$ is the transaction cost proportional factor, and $|e_{t_k^+} - e_{t_k}|$ is the absolute change of exposure across revision time $t_k$. Since the CPPI strategy is self-financing, $\theta |e_{t_k^+} - e_{t_k}|$ is also the deduction in portfolio value at time $t_k$ so that $V_{t_k^+} = V_{t_k} - \theta |e_{t_k^+} - e_{t_k}|$ and $C_{t_k}^+ = C_{t_k} - \theta |e_{t_k^+} - e_{t_k}|$.

On the other hand, if $C_{t_k}^+ \leq 0$, the entire portfolio is invested in the reserve asset, which means that the exposure $e_{t_k}$ in risky asset will be sold at time $t_k$ and the transaction cost will be $\theta e_{t_k}$, leading to $C_{t_k}^+ = C_{t_k} - \theta e_{t_k}$. Combining both scenarios yields

$$C_{t_k}^+ = \begin{cases} 
  C_{t_k} - \theta |mC_{t_k}^+ - e_{t_k}|, & \text{if } C_{t_k}^+ > 0, \\
  C_{t_k} - \theta e_{t_k}, & \text{if } C_{t_k}^+ \leq 0.
\end{cases}$$

Another assumption underlying the relation (5) is that there is no transaction associated with the changes in the position on the reserve asset.

In what follows, the evolution of the CPPI portfolio value will be analyzed. Since the portfolio value is the sum of the floor and the cushion, and the floor is fully determined by the price process of the reserve asset, it is sufficient to focus on the evolution of the cushion process. To proceed, we additionally assume $m\theta < 1$, which is a quite mild condition from a practical point of view. Consequently, some simple algebras on system (5) yields

$$C_{t_k}^+ = \begin{cases} 
  \frac{C_{t_k} + \theta e_{t_k}}{1 + m\theta}, & \text{if } e_{t_k} \leq mC_{t_k}, \\
  \frac{C_{t_k} - \theta e_{t_k}}{1 - m\theta}, & \text{if } m\theta e_{t_k} \leq mC_{t_k} < e_{t_k}, \\
  C_{t_k} - \theta e_{t_k}, & \text{if } mC_{t_k} \leq m\theta e_{t_k}.
\end{cases}$$

As commented previously, according to the spirit of the CPPI strategy, the exposure at time $t_k^+$ is revised to $e_{t_k^+} = \max \left\{ mC_{t_k}^+, 0 \right\}$; thus, equation (6) leads to

$$e_{t_k^+} = \begin{cases} 
  \frac{mC_{t_k} + m\theta e_{t_k}}{1 + m\theta}, & \text{if } e_{t_k} \leq mC_{t_k}, \\
  \frac{mC_{t_k} - m\theta e_{t_k}}{1 - m\theta}, & \text{if } m\theta e_{t_k} \leq mC_{t_k} < e_{t_k}, \\
  0, & \text{if } mC_{t_k} \leq m\theta e_{t_k}.
\end{cases}$$

It is interesting, but not surprising, that if there is no transaction fee, i.e., $\theta = 0$, then the cushion and the exposure in equations (6) and (7) become $C_{t_k^+} = C_{t_k}$ and $e_{t_k^+} = \max \left\{ mC_{t_k}, 0 \right\}$ respectively. Also note that transaction cost occurs immediately at time 0 when the CPPI portfolio is constructed by entering a position of an exposure $e_{t_0^+} = mC_{t_0^+}$ in the risky asset, and it follows from the first case in (6) with $e_{t_0} = 0$ that the initial cushion reduces to $C_{t_0^+} = C_0/(1 + m\theta)$ from $C_0$. 

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Upon (6), we need to show how $C_{t_k}$ and $e_{t_k}$ are related to $C_{t_{k-1}^+}$, in order to establish a recursion between $C_{t_{k-1}^+}$ and $C_{t_{k-1}^+}$. To this end, we need to consider two cases. First, if $C_{t_{k-1}^+} \leq 0$, the entire portfolio is invested in the bond since time $t_{k-1}^+$ so that

$$V_{t_k^+} = \left[ F_{t_{k-1}} + C_{t_{k-1}^+} \right] e^{R_{t_k}} = F_{t_k} + C_{t_{k-1}^+} e^{R_{t_k}},$$

which implies

$$C_{t_{k}^+} = C_{t_{k-1}^+} e^{R_{t_k}} \text{ for } C_{t_{k-1}^+} \leq 0. \tag{8}$$

Second, if $C_{t_{k-1}^+} > 0$ at a revision time $t_{k-1}^+$, an amount of $m C_{t_{k-1}^+}$ is invested in risky asset and the remainder $V_{t_{k-1}} - m C_{t_{k-1}^+} = F_{t_{k-1}} - (m - 1) C_{t_{k-1}^+}$ is allocated in the reserve asset, and such a portfolio composition is maintained until the next revision time $t_k$. Recall from the preceding subsection that $X_k$ is the log-return of the risky asset over period $(t_{k-1}, t_k]$, i.e., $X_k = \ln S_{t_k} - \ln S_{t_{k-1}}$, $k = 1, 2, \ldots, n$. Then, at the moment right before the $k$th revision the portfolio has an exposure of

$$e_{t_k} = m C_{t_{k-1}^+} e^{X_k} \tag{9}$$

in the risky asset and a dollar amount of

$$\left( F_{t_{k-1}^+} - (m - 1) C_{t_{k-1}^+} \right) e^{R_{t_k}} = F_{t_k} - (m - 1) C_{t_{k-1}^+} e^{R_{t_k}}$$

in the reserve asset so that the cushion

$$C_{t_{k}} = m C_{t_{k-1}^+} e^{X_k} - (m - 1) C_{t_{k-1}^+} e^{R_{t_k}} = e^{R_{t_k}} C_{t_{k-1}^+} \left[ m e^{Y_k} - (m - 1) \right], \tag{10}$$

where $Y_k = X_k - R_k$ is the excess return rate over the $k$-th period $(t_{k-1}, t_k]$, $k = 1, \ldots, n$. Substituting (9) and (10) into (6) yields the following recursive formula:

$$C_{t_{k}^+} = C_{t_{k-1}^+} e^{R_{t_k}} \left( \Lambda_k - m \theta \Lambda_k \mathbb{I}_{\{\Lambda_k \leq 0\}} \right) \text{ for } C_{t_{k-1}^+} > 0, \tag{11}$$

where $\mathbb{I}_{\{\}}$ is the indicator function, and for $k = 1, \ldots, n$,

$$\Lambda_k = \begin{cases} 
\frac{m(1 + \theta)e^{Y_k} - (m - 1)}{1 + m\theta}, & \text{if } Y_k \geq 0, \\
\frac{m(1 - \theta)e^{Y_k} - (m - 1)}{1 - m\theta}, & \text{if } Y_k < 0.
\end{cases} \tag{12}$$

Combining (8) and (11) yields

$$C_{t_{k}^+} = \begin{cases} 
C_{t_{k-1}^+} e^{R_{t_k}} \left( \Lambda_k - m \theta \Lambda_k \mathbb{I}_{\{\Lambda_k \leq 0\}} \right), & \text{if } C_{t_{k-1}^+} > 0, \\
C_{t_{k-1}^+} e^{R_{t_k}}, & \text{if } C_{t_{k-1}^+} \leq 0,
\end{cases} \tag{13}$$
Given that $C_{t_{k+1}} > 0$, (11) and (12) together imply that $C_{t_k} \leq 0$ if and only if $\Lambda_k \leq 0$, and in this case, we say shortfall occurs at time $t_k$. Note that once a shortfall occurs at certain revision time $t_k$, $V_{t_k} \leq F_{t_k}$ and the entire portfolio is invested in the reserve asset since then. Therefore, if we define

$$\rho = \inf \{ k \geq 1 : \Lambda_k \leq 0 \} = \inf \left\{ k \geq 1 : Y_k \leq \ln \left( \frac{m-1}{m(1-\theta)} \right) \right\}, \quad (14)$$

$\rho$ is the shortfall time. By definition, if $\rho \leq n$, shortfall occurs within the investment horizon $[0,T]$ and otherwise there is no shortfall over the investment horizon. Moreover,

$$\{ \rho = k \} = \{ \Lambda_j > 0 \quad j = 1, \ldots, k-1, \quad \text{and} \quad \Lambda_k \leq 0 \}, \quad k = 1, \ldots, n$$

and it follows trivially from (13) that

$$C_{t_k} = C_{t_0} e^{\sum_{t=1}^{k} R_t} \prod_{j=1}^{\min(k,\rho)} (\Lambda_j - m\theta\Lambda_j 1_{\{\Lambda_j \leq 0\}}), \quad k = 1, \ldots, n-1. \quad (15)$$

3. Terminal cushion value and measures for risk-and-reward profile

3.1. Terminal cushion value

For the terminal cushion value, over the last period we need to distinguish between two cases: (1) there is a zero position in the risky asset; (2) there is a positive position in the risky asset. In the first case, there will be no transaction cost required to cash the portfolio at the investment terminal time. Nevertheless, in the second case, the portfolio can only be cashed at a value of $V_T - \theta e_T$, where the deduction $\theta e_T$ is the transaction cost in selling the risky asset. We call the cushion after cashing all the risky asset in the portfolio as the net terminal cushion, denoted by $C_{T+}$. We shall focus on the net cushion in this paper, as it accurately determines the final payoff of the CPPI portfolio. From (15), we can obtain an explicit expression for the net terminal cushion as follows.

If $\rho \leq n-1$, according to (15)

$$C_{t_{n-1}} = C_{t_0} e^{\sum_{t=1}^{n-1} R_t} \prod_{j=1}^{\rho} (\Lambda_j - m\theta\Lambda_j 1_{\{\Lambda_j \leq 0\}}) \leq 0,$$

which implies that there is no position in the risky asset and the entire portfolio is in the reserve asset over the last period $[t_{n-1}, t_n]$ to earn a log-return rate of $R_n$. Thus,

$$C_{T+} = C_{t_{n-1}} e^{R_n} = C_{t_0} e^{\sum_{t=1}^{n} R_t} \prod_{j=1}^{\rho} (\Lambda_j - m\theta\Lambda_j 1_{\{\Lambda_j \leq 0\}})$$

$$= -C_{t_0} e^{\sum_{t=1}^{n} R_t} \left( \prod_{j=1}^{\rho-1} \Lambda_j^+ \right) \Xi_{\rho}, \quad \text{if} \quad \rho \leq n-1, \quad (16)$$
where

\[ \Xi_k = m(1 - \theta)e^{Y_k} - (m - 1) \]  

(17)

\[ \Xi_k^- = \max\{0, -\Xi_k\} \text{ and } \Xi_k^+ = \max\{0, \Xi_k\}, \ k = 1, \ldots, n. \]

If \( \varrho \geq n \), equation (15) gives

\[ C_{t_{n-1}^+} = C_{t_0^+} e^{\sum_{t=1}^{n-1} R_t} \prod_{j=1}^{n-1} \Lambda_j > 0, \]  

(18)

so that the portfolio value at the end of the last period

\[ V_T = mC_{t_{n-1}^+} e^{X_n} + \left[ F_{t_{n-1}} - (m - 1)C_{t_{n-1}^+} \right] e^{R_n} = F_t + C_{t_{n-1}^+} \left( me^{X_n} - (m - 1)e^{R_n} \right). \]

In this case the portfolio contains a position of \( mC_{t_{n-1}^+} e^{X_n} \) in the risky asset and therefore the liquidation leads to a trading cost of \( \theta mC_{t_{n-1}^+} e^{X_n} \) so that the net terminal cushion

\[ C_T^+ = \begin{cases} 
- C_{t_0^+} e^{\sum_{t=1}^{n-1} R_t} \left( \prod_{j=1}^{\varrho-1} \Lambda_j \right) \Xi_{\varrho}^- \ni \varrho \leq n, \\
C_{t_0^+} e^{\sum_{t=1}^{n-1} R_t} \left( \prod_{j=1}^{n-1} \Lambda_j^+ \right) \Xi_n^+ \ni \varrho > n. 
\end{cases} \]  

(19)

Combining (16) and (19) yields

\[ C_T^+ = \begin{cases} 
- C_{t_0^+} e^{\sum_{t=1}^{n-1} R_t} \left( \prod_{j=1}^{\varrho-1} \Lambda_j^+ \right) \Xi_{\varrho}^- \ni \varrho \leq n, \\
C_{t_0^+} e^{\sum_{t=1}^{n-1} R_t} \left( \prod_{j=1}^{n-1} \Lambda_j^+ \right) \Xi_n^+ \ni \varrho > n. 
\end{cases} \]  

(20)

To evaluate \( C_{T^+} \), we choose the reserve asset as a numeraire and define

\[ C_{t_k^+}^* = \frac{C_{t_k^+}}{F_{t_k}}, \text{ where } F_{t_k} = F_0 \exp \left( \sum_{t=1}^{k} R_t \right), \ni k = 0, 1, \ldots, n. \]

Then, the evaluation on \( C_{T^+} \) can be conducted via \( C_{T^+}^* \), which by (20) is given by

\[ C_{T^+}^* = \begin{cases} 
- C_{t_0^+}^* \left( \prod_{j=1}^{\varrho-1} \Lambda_j^+ \right) \Xi_{\varrho}^- \ni \varrho \leq n, \\
C_{t_0^+}^* \left( \prod_{j=1}^{n-1} \Lambda_j^+ \right) \Xi_n^+ \ni \varrho > n, 
\end{cases} \]  

(21)

with \( \Lambda_j \) and \( \Xi_j \) respectively given by (12) and (17).
3.2. Measures for risk-and-reward profile

While the CPPI strategy can be directly applied by investors in practice, there are CPPI funds: for a CPPI portfolio invested over a finite time investment horizon \([0, T]\) with \(T > 0\), the investor pays an initial value of \(V_0\) at time 0 and is guaranteed to receive at least a value of \(F_T\) at time \(T\). If the net terminal portfolio value \(V_T\) is smaller than \(G\), a third party will pay to the investor the shortfall amount \(F_T - V_T\). In practice, this guarantee is usually provided by the bank which owns the CPPI portfolio and charge on the investors a premium. Consequently, at the expiration date \(T\), the investors will receive a payoff of

\[
\max\{V_T, F_T\} = F_T + \max\{C_T^+, 0\} = F_T + C_T^+ \mathbb{I}_{\{C_T^+ > 0\}},
\]

and, in exchange for the premium, the CPPI guarantor will be subject to a gap risk of \(C_T^+ \mathbb{I}_{\{C_T^+ \leq 0\}}\). The evaluation on both quantities are interesting. An insightful investigation on the payoff can help the investors to establish a comprehensive understanding on their risk-and-reward profile in investing a CPPI fund, and a thorough analysis on the gap risk is necessary for the CPPI guarantor not only in computing a reasonable premium charged on the investors but also in their internal risk management. Obviously, both the gap risk of the guarantor and the payoff of the investors depend on the performance of reserve asset, in addition to the risky asset. While it is interesting to conduct the investigation by taking into account the performance of both assets, in this paper we focus on the effect from the risky asset, since the randomness associated with a reserve asset is typically much less than a risky asset. Therefore, we measure both quantities at time 0 by discounting them using the reserve asset price process, so that the time 0 value of the payoff for the investors is

\[
1 + \max\left\{\frac{C_T^+}{F_T}, 0\right\} = 1 + \max\{C_T^+, 0\}
\]

and the time 0 value of the gap risk is \(C_T^+ \mathbb{I}_{\{C_T^+ \leq 0\}}\). Note that the three events \(\{\varrho \leq n\}\), \(\{C_T \leq 0\}\) and \(\{C_T^* \leq 0\}\) are eventually the same, all indicating that a gap risk comes up during the investment time horizon \([0, T]\). Therefore, the time 0 value of the gap risk can be equivalently expressed by

\[
C_T^+ \mathbb{I}_{\{C_T^+ \leq 0\}} = C_T^+ \mathbb{I}_{\{C_T^+ \leq 0\}} = C_T^+ \mathbb{I}_{\{\varrho \leq n\}}.
\]

From equations (22) and (23), we note that the time 0 values of both the payoff and the gap risk only depend on the discounted cushion \(C_T^+\). We shall focus on this quantity in our subsequent analysis. To quantify the gap risk and payoff, we shall investigate the following quantities:

i) **Shortfall probability** \(\text{SP}_{H_0} = \Pr(C_T^+ < 0 | H_0)\);

ii) **Unconditional expected shortfall** \(\text{UES}_{H_0} = \mathbb{E}\left(-C_T^+ \mathbb{I}_{\{C_T^+ \leq 0\}} | H_0\right)\);

iii) **Conditional expected shortfall** \(\text{CES}_{H_0} = \mathbb{E}\left(-C_T^+ | C_T^+ \leq 0, H_0\right)\);

iv) **Unconditional expected gain** \(\text{UEG}_{H_0} = \mathbb{E}\left(C_T^+ \mathbb{I}_{\{C_T^+ > 0\}}, H_0\right)\);
v) Conditional expected gain \( \text{CEG}(T) = \mathbb{E} \left( C_T^* \mid C_T^* > 0, H_0 \right) \).

Additionally, we will study the Omega measure (see subsection 4.5 for the precise definition) of the discounted terminal portfolio value \( V_T^* \) by a Laplace transform methodology.

4. CPPI portfolio performance evaluation

4.1. Preliminaries

To ease the presentation in the sequel, some notations are necessary. For \( t \geq 1 \) and \( j = 1, \ldots, r \), define

\[
\Gamma_j(t) = \sum_{k=1}^{t} \langle h_j, H_k \rangle, \tag{24}
\]

which is the total number of periods over which the regime process \( H \) spends in state \( h_j \) over the time horizon \( (0, t] \). Clearly, \( \sum_{j=1}^{r} \Gamma_j(t) = t \) almost surely, \( t \geq 0 \), and moreover, for any vector of complex numbers \( a = (a_1, \ldots, a_r) \in \mathbb{C}^r \),

\[
\sum_{j=1}^{r} a_j \Gamma_j(t) = \sum_{k=1}^{t} \langle a, H_k \rangle. \tag{25}
\]

Let \( P = (p_{ij}) \) be the one-step transition probability matrix of the Markov process \( H \) such that \( p_{ij} = \Pr(\rho_t = h_i \mid \rho_{t-1} = h_j) \), \( t \geq 1, i, j = 1, \ldots, h \). Recall that \( I \) denotes an identity matrix and \( 1 \) is a column vector with all elements equal to 1.

In the rest of the subsection, two lemmas will be developed for the use in the sequel. The proof of the following equation (27) has been given by (Elliott et al., 1995, p.17). The result (26) is developed on its own interest.

**Lemma 4.1.** The regime process \( \{H_t, t \geq 0\} \) admits the following representation

\[
H_t = H_0 + \sum_{i=0}^{t-1} (P - I) H_i + M_t, \quad t \geq 1, \tag{26}
\]

so that

\[
H_t = PH_{t-1} + A_t, \quad t \geq 1, \tag{27}
\]

where \( A_t = M_t - M_{t-1} \) and \( \{M_t, t \geq 0\} \) with \( M_0 = 0 \) is a martingale with respect to \( \{\mathcal{F}_t^H, t \geq 0\} \) the natural filtration generated by process \( \{H_t, t \geq 0\} \).

**Proof.** Let \( M_t = H_t - H_0 - \sum_{i=0}^{t-1} (P - I) H_i, t \geq 1 \). It is sufficient to show that \( \{M_t, t \geq 0\} \) is a martingale with respect to \( \{\mathcal{F}_t^H, t \geq 0\} \). Indeed, trivially \( M_1 = H_1 - PH_0 \) and thus...
\[ \mathbb{E}[M_1 | \mathcal{F}_0^H] = \mathbb{E}[H_1 - PH_0 | H_0] = 0. \] For any \( t \geq 2, \)
\[
\mathbb{E}[M_t | \mathcal{F}_{t-1}^H] = \mathbb{E} \left[ \left( H_t - H_0 - \sum_{i=0}^{t-1} (P - I)H_i \right) \mid \mathcal{F}_{t-1}^H \right] = \mathbb{E}[H_t | \mathcal{F}_{t-1}^H] - H_0 - \sum_{i=0}^{t-1} (P - I)H_i = PH_{t-1} - H_0 - \sum_{i=0}^{t-2} (P - I)H_i = H_{t-1} - H_0 - \sum_{i=0}^{t-2} (P - I)H_i = M_{t-1},
\]
by which the proof is complete. \( \square \)

**Lemma 4.2.** Denote \( D_a = \text{diag} (e^{a_1}, \ldots, e^{a_r}) \) for a vector \( a = (a_1, \ldots, a_r)' \in C^r. \) Then, for any integer \( t \geq 1, \)
\[
(a) \quad \mathbb{E} \left[ \exp \left( \sum_{j=1}^r a_j \Gamma_j(t) \right) H_{t+k} \bigg| H_0 \right] = P^k (D_a P)^t H_0, \quad k = 0, 1, 2, \ldots;
\]
\[
(b) \quad \mathbb{E} \left[ \exp \left( \sum_{j=1}^r a_j \Gamma_j(t) \right) \bigg| H_0 \right] = \langle 1, (D_a P)^t H_0 \rangle.
\]

**Proof.** (a) By the Markov property of \( H \) and the linear representation (4), it is easy to obtain
\[
\mathbb{E} \left[ e^{(a,H_t)} H_t | \mathcal{F}_{t-1}^H \right] = \mathbb{E} \left[ \sum_{j=1}^r e^{(a,H_j)} h_j \langle H_t, h_j \rangle \mid \mathcal{F}_{t-1}^H \right] = \mathbb{E} \left[ \text{diag} (e^{a_1}, \ldots, e^{a_r}) H_t | \mathcal{F}_{t-1}^H \right] = D_a PH_{t-1},
\]
where the last step is due to Lemma 4.2. To proceed, let \( U_t = \exp \left( \sum_{j=1}^r a_j \Gamma_j(t) \right) \) and \( V_t = U_t H_t. \) Then, (28) implies
\[
\mathbb{E} \left[ V_1 | H_0 \right] = \mathbb{E} \left[ e^{\sum_{j=1}^r a_j \Gamma_j(1)} H_1 | H_0 \right] = \mathbb{E} \left[ e^{(a,H_1)} H_1 | H_0 \right] = D_a PH_0,
\]
where the last equality follows in the same way as (28). For \( t \geq 2, \) we apply (25) to obtain
\( U_t = U_{t-1}e^{(a,H_t)} \) and thus
\[
\mathbb{E}[V_t | \mathcal{F}_{t-1}^H] = \mathbb{E}[U_t H_t | \mathcal{F}_{t-1}^H] = \mathbb{E}[U_{t-1}e^{(a,H_t)} H_t | \mathcal{F}_{t-1}^H] = U_{t-1} \mathbb{E}[e^{(a,H_t)} H_t | \mathcal{F}_{t-1}^H] = U_{t-1} D_a PH_{t-1} = D_a PV_{t-1},
\]
which implies \( \mathbb{E}[V_t | H_0] = D_a PH_0 = D_a P \mathbb{E}[V_{t-1} | H_0], \) and thus, combining this with (29) we obtain \( \mathbb{E}[V_t | H_0] = (D_a P)^t H_0. \) Thus, the desired result holds for \( k = 0. \)
By Lemma 4.1, $H_{t+1} = PH_t + A_{t+1}$, and therefore

$$\mathbb{E} \left( e^{\sum_{j=1}^{n} a_j \Gamma_j(t)} H_{t+1} \middle| H_0 \right) = \mathbb{E} \left[ P \cdot e^{\sum_{j=1}^{n} a_j \Gamma_j(t)} H_t \middle| H_0 \right] + \mathbb{E} \left[ e^{\sum_{j=1}^{n} a_j \Gamma_j(t)} A_{t+1} \middle| F_t^H \right] \middle| H_0 \right]
$$

$$= \mathbb{E} \left[ PV_t \middle| H_0 \right] + \mathbb{E} \left[ e^{\sum_{j=1}^{n} a_j \Gamma_j(t)} \mathbb{E} \left( A_{t+1} \middle| F_t^H \right) \middle| H_0 \right]
$$

$$= \mathbb{P} \mathbb{E} \left[ V_t \middle| H_0 \right]
$$

$$= \mathbb{P} \left( D_s P \right)^t H_0,$$

which proves the desired result for $k = 1$. The result for a general $k$ can be obtained by induction principle.

(b) From the definition of $V_t$, we have $\exp \left( \sum_{j=1}^{n} a_j \Gamma_j(t) \right) = \langle 1, V_t \rangle$. Taking expectation on both sides of the equation and using the established (a) yields the desired result. \qed

4.2. Shortfall probability

Recall that the excess return rate $Y_k$ in (12) is defined by $Y_k = X_k - R_k$, where $X_k$ and $R_k$ are given in (2) and (3) respectively. Thus, if we put $Y_k = (Y_k^{(1)}, \ldots, Y_k^{(h)})$ with $Y_k^{(j)} = X_k^{(j)} - R_k^{(j)}$ for $k = 1, \ldots, n$ and $j = 1, \ldots, r$, then the excess return rate $Y_k = \langle H_k, Y_k \rangle$, $k = 1, \ldots, n$, and $\{Y_1, \ldots, Y_n\}$ are conditionally independent given $(H_1, \ldots, H_n)$. To proceed, we denote $s = (s_1, \ldots, s_r)'$ with $s_j = \ln \left[ \Pr \left( Y_k^{(j)} > \ln \left( \frac{m-1}{m(1-\theta)} \right) \right) \right]$, $j = 1, \ldots, r$.

**Proposition 4.1.** The shortfall probability for the CPPI portfolio

$$\text{SP}_{H_0} = \Pr(\rho \leq n \middle| H_0) = \Pr(C^{\tau+}_{\rho} \leq 0 \middle| H_0) = 1 - \langle 1, \left( D_s P \right)^n H_0 \rangle,$$

where $D_s = \text{diag} \left( e^{s_1}, \ldots, e^{s_r} \right)$.

**Proof.** By the definition of $\rho$ in (14), $\{\rho > n\} = \{Y_k > \ln \left( \frac{m-1}{m(1-\theta)} \right), k = 1, \ldots, n\}$ and therefore, it follows from the conditional independence of $\{Y_1, \ldots, Y_n\}$ that

$$\text{SP}_{H_0} = 1 - \mathbb{P} \left( Y_k > \ln \left( \frac{m-1}{m(1-\theta)} \right), k = 1, \ldots, n \middle| H_0 \right)
$$

$$= 1 - \mathbb{E} \left[ \Pr \left( Y_k > \ln \left( \frac{m-1}{m(1-\theta)} \right), k = 1, \ldots, n \middle| H_1, \ldots, H_n \right) \middle| H_0 \right]
$$

$$= 1 - \mathbb{E} \left[ \prod_{k=1}^{n} \Pr \left( Y_k > \ln \left( \frac{m-1}{m(1-\theta)} \right) \middle| H_k \right) \middle| H_0 \right]
$$

$$= 1 - \mathbb{E} \left[ \prod_{k=1}^{n} \langle e^s, H_k \rangle \middle| H_0 \right],
$$

where $e^s$ denotes the vector $(e^{s_1}, \ldots, e^{s_r})$. Consequently, the desired result follows from part (b) of Lemma 4.2 as follows

$$\text{SP}_{H_0} = 1 - \mathbb{E} \left[ \exp \left( \sum_{k=1}^{n} \langle s, H_k \rangle \right) \middle| H_0 \right] = 1 - \mathbb{E} \left[ \exp \left( \sum_{j=1}^{r} s_j \Gamma_j(n) \right) \middle| H_0 \right]
$$

$$= 1 - \langle 1, \left( D_s P \right)^n H_0 \rangle.$$
Remark 4.1.  (a) Note that $s_j$ is decreasing in $m$ since $\ln \left( \frac{m-1}{m(1-\theta)} \right)$ is increasing as a function of $m$. Thus the shortfall probability with an expression in (30) increases with the multiplier $m$. This observation well accords to our intuition on a CPPI portfolio: the higher the multiplier, the more capital invested in the risky asset and thus the faster portfolio approaches the floor when there is a sustained decrease in the risky asset price.

(b) From Proposition 4.1, we can determine an upper bound on the multiplier for the CPPI portfolio to satisfy a quantile condition, following the same idea as in Ameur and Prigent (2013). More specifically, to control the shortfall probability $\text{SP}_{H_0}$ with certain tolerance level $\epsilon$, we need to choose a multiplier to satisfy $\text{SP}_{H_0} \leq \epsilon$, which, by Proposition 4.1 and the fact that $\langle 1, (D_s P)^n H_0 \rangle$ is decreasing in the multiplier $m$, is equivalent to choosing a multiplier no larger than a value for $m$ which satisfies

$$\langle 1, (D_s P)^n H_0 \rangle = 1 - \epsilon.$$  

(c) The proof of Proposition 4.1 essentially provides us with a way to compute the distribution of the minimum statistics $Y_{\min} := \min \{ Y_k, k = 1, \ldots, n \}$. Let $s_j(x) = \ln \left[ \Pr \left( Y_{(j)}^* > x \right) \right], x \in \mathbb{R}, j = 1, \ldots, r$, by a little bit notation abuse. Then, the same lines with $\ln \left( \frac{m-1}{m(1-\theta)} \right)$ replaced by $x$ in the proof of Proposition 4.1 yields

$$F_{Y_{\min}} (x | H_0) := \Pr \left( Y_{\min} \leq x | H_0 \right) = 1 - \langle 1, (D_s P)^n H_0 \rangle, \; x \in \mathbb{R},$$

where $s(x) = (s_1(x), \ldots, s_r(x))'$ and $D_s = \text{diag} \left( e^{s_1(x)}, \ldots, e^{s_r(x)} \right)$. Since

$$\text{SP}_{H_0} = F_{Y_{\min}} \left( \ln \left( \frac{m-1}{m(1-\theta)} \right) | H_0 \right),$$

the quantile condition $\text{SP}_{H_0} \leq \epsilon$ mentioned in part (b) for multiplier $m$ is equivalent to

$$m \leq \frac{1}{1 - (1-\theta) \exp \left( F_{Y_{\min}}^{-1} (\epsilon | H_0) \right)},$$

where $F_{Y_{\min}}^{-1} (\cdot | H_0)$ denotes the inverse of $F_{Y_{\min}} (\cdot | H_0)$.

4.3. Expected shortfall

Recall the definition of random variables $\Lambda_k$ in (12). We similarly define $\Lambda_k^{(j)}$ by replacing $Y_k$ with $Y_k^{(j)}$ in (12) for $k = 1, \ldots, n$ and $j = 1, \ldots, r$. We use $\Lambda_k^{(j)+}$ and $\Lambda_k^{(j)-}$ respectively to denote their positive and negative parts. We similarly define $\Xi_k^{(j)+}$ and their positive part $\Xi_k^{(j)+}$ and negative part $\Xi_k^{(j)-}$ by replacing $Y_k$ by $Y_k^{(j)}$ in (17). Note that $\{ (Y_k^{(j)}, \Lambda_k^{(j)}, \Xi_k^{(j)}) \}$ for $k = 1, \ldots, n$ is a sequence of independent and identically distributed for a fixed $j = 1, \ldots, r$. Furthermore, we denote $\lambda^+ = (\lambda_1^+, \ldots, \lambda_r^+)'$ and $\xi^- = (\xi_1^-, \ldots, \xi_r^-)'$ with $\lambda_j^+ = \ln \mathbb{E} \left[ \Lambda_k^{(j)+} \right]$ and $\xi_j^- = \mathbb{E} \left[ \Xi_k^{(j)-} \right], j = 1, \ldots, r$. 

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Proposition 4.2. The unconditional expected shortfall of the CPPI portfolio

\[ \text{UES}_{H_0} = \mathbb{E} \left[ -C_{T+}^* \mathbb{I}_{\{\varphi \leq \eta \}} \right] = \mathbb{E} \left[ -C_{T+}^* \mathbb{I}_{\{C_{T+}^* \leq 0 \}} \right] \]

\[ = C_{t_0^+}^* \left( \boldsymbol{\xi}^-, \left( I - \mathbf{P} \cdot \mathbf{D}_\Lambda^+ \right)^{-1} \mathbf{P} \left( I - (\mathbf{D}_\Lambda \mathbf{P})^n \right) H_0 \right), \]

provided that \( \left( I - \mathbf{P} \cdot \mathbf{D}_\Lambda^+ \right) \) is invertible, where \( \mathbf{D}_\Lambda = \text{diag} \left( e^{\lambda_1^+, \ldots, e^{\lambda_r^+}} \right) \).

Proof. Let \( B_k = -C_{t_0^+}^* \mathbb{I}_{\{\varphi = k \}} \) for \( k = 1, \ldots, n \). Then, by (21),

\[ B_k = C_{t_0^+}^* \left( \prod_{t=1}^{k-1} \Lambda_t^+ \right) \cdot \Xi_k^-, \quad k = 1, \ldots, n. \]

Further define

\[ J_k^{(j)} = \prod_{t=1}^{k} \left( \Lambda_t^{(j)^+} \right) = \exp \left( \sum_{t=1}^{k} \ln \Lambda_t^{(j)^+} \right), \quad j = 1, \ldots, r \quad \text{and} \quad k = 1, \ldots, n. \]

Since \( \{\Lambda_t^{(j)^+}, t = 1, \ldots, n\} \) are independent and identically distributed for a fixed \( j \), \( \ln J_k^{(j)} \) has independent and identically distributed increments as a stochastic process indexed by \( k \).

For \( k = 1 \), trivially we have

\[ \mathbb{E}[B_1|H_0] = C_{t_0^+}^* \mathbb{E}[\Xi_1|H_0] = C_{t_0^+}^* \cdot \langle \boldsymbol{\xi}^-, \mathbf{P} H_0 \rangle. \]

Next, we consider \( k \geq 2 \), and let \( M \) be the number of transitions which the regime switching process \( H \) experiences over horizon \([0, k-1]\). Note that the regime transition over \([0, k-1]\) can only possibly occur at times \( t_0, t_1, \ldots, t_{k-2} \). Denote \( \tau_0 = 0 \) and \( \tau_{M+1} = k - 1 \), and let \( \tau_i = l \) if the \( i \)th transition occurs at time \( t_l \) for \( l = 0, 1, \ldots, k - 2 \) and \( i = 1, \ldots, M \).

Note that \( \tau_1 \) could be equal to 0, which means that the first transition occurs at time 0 so that \( H_1 \neq H_0 \). Further let \( \eta_j \) denote the corresponding inter-transition times such that \( \eta_j = \tau_j - \tau_{j-1} \) for \( j = 1, \ldots, M \) and write \( \eta_{M+1} = \tau_{M+1} - \tau_M \). Let \( h_{K_{j+1}} \) denote the state in which the process \( H \) stays over period \((\tau_j, \tau_{j+1}]\), i.e., \( H \) stays in the \( K_{j+1} \)-th state over period \((\tau_j, \tau_{j+1}]\), \( j = 0, 1, \ldots, M \). Then,

\[ \prod_{j=1}^{k-1} \Lambda_j^+ = \prod_{j=0}^{M} \left( \frac{J_{\eta_{j+1}}^{(K_{j+1})}}{J_{\tau_j}^{(K_{j+1})}} \right), \quad k \geq 2. \]

Noticing that \( \{(\Lambda_1, \Xi_1), \ldots, (\Lambda_k, \Xi_k)\} \) are conditionally independent given \( \mathcal{G}_k^H := \sigma \{H_1, \ldots, H_k\} \), we obtain

\[ \mathbb{E} \left[ \left( \prod_{j=1}^{k-1} \Lambda_j^+ \right) | \mathcal{G}_k^H \right] = \mathbb{E} \left[ \prod_{j=0}^{M} \left( \frac{J_{\eta_{j+1}}^{(K_{j+1})}}{J_{\tau_j}^{(K_{j+1})}} \right) | \mathcal{G}_k^H \right] = \prod_{j=0}^{M} \mathbb{E} \left[ J_{\eta_{j+1}}^{(K_{j+1})} | K_{j+1} \right] \]

\[ = \prod_{j=0}^{M} \mathbb{E} \left[ \Lambda_1^{(K_{j+1})} | K_{j+1} \right]^{\eta_{j+1}} = \prod_{j=0}^{M} \mathbb{E} \left[\eta_{j+1} \lambda_{K_{j+1}}^+ \right] \]

\[ = \exp \left( \sum_{j=0}^{M} \eta_{j+1} \lambda_{K_{j+1}}^+ \right), \]

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and hence it follows from (32) that
\[
\mathbb{E} \left[ B_k | G_k^H \right] = C_{t_0}^* \cdot \mathbb{E} \left[ \left( \prod_{j=1}^{k-1} \Lambda_j^+ \right) | G_k^H \right] \cdot \mathbb{E} \left( \Xi_k^- | G_k^H \right) 
\]
\[= C_{t_0}^* \cdot \exp \left( \sum_{j=0}^{M} \eta_{j+1} \lambda_{K_{j+1}}^+ \right) \cdot \langle H_k, \xi^- \rangle. \]

Note that \( \sum_{j=0}^{M} \eta_{j+1} \lambda_{K_{j+1}}^+ = \sum_{j=1}^{r} (\lambda_j^+ \Gamma_j (k - 1)) \), where \( \Gamma_j (k - 1) \) is defined in (24). Therefore, by part (a) of Lemma 4.2, we obtain
\[
\mathbb{E} [ B_k | H_0 ] = \mathbb{E} \left[ C_{t_0}^* \cdot \exp \left( \sum_{j=1}^{r} (\Gamma_j (k - 1) \cdot \lambda_j^+) \right) \cdot \langle H_k, \xi^- \rangle | H_0 \right] 
\]
\[= C_{t_0}^* \cdot \left\langle \xi^-, \mathbb{E} \left[ \exp \left( \sum_{j=1}^{r} (\lambda_j^+ \Gamma_j (k - 1)) \right) | H_k \right] H_0 \right\rangle 
\]
\[= C_{t_0}^* \cdot \left\langle \xi^-, \mathbb{P} \left( D_{\lambda}^+ P \right)^{k-1} H_0 \right\rangle, \quad k \geq 2. \] (34)

Combining (33) and (34), we obtain the unconditional expected shortfall as follows
\[
UES_{H_0} = \mathbb{E} \left[ -C_{T^+}^* \mathbb{I}_{\{\varrho \leq n\}} | H_0 \right] = \sum_{k=1}^{n} \mathbb{E} \left[ -C_{T^+}^* \mathbb{I}_{\{\varrho = k\}} | H_0 \right] 
\]
\[= C_{t_0}^* \cdot \sum_{k=1}^{n} \left\langle \xi^-, \mathbb{P} \left( D_{\lambda}^+ P \right)^{k-1} H_0 \right\rangle 
\]
\[= C_{t_0}^* \cdot \left\langle \xi^-, \left( I - P \cdot D_{\lambda}^+ \right)^{-1} \mathbb{P} \left( I - \left( D_{\lambda}^+ P \right)^n \right) H_0 \right\rangle, \quad \text{provided that } \left( I - P \cdot D_{\lambda}^+ \right) \text{ is invertible.} \]

**Remark 4.2.** Combining Propositions 4.1 and 4.2 yields the following formula for the conditional expected shortfall:

\[
\mathbb{E} \left[ -C_{T^+}^* | \varrho \leq n, H_0 \right] \equiv \mathbb{E} \left[ -C_{T^+}^* | C_{T^+}^* \leq 0, H_0 \right] 
\]
\[= C_{t_0}^* \left\langle \xi^-, \left( I - P \cdot D_{\lambda}^+ \right)^{-1} \mathbb{P} \left( I - \left( D_{\lambda}^+ P \right)^n \right) H_0 \right\rangle \left/ 1 - \langle 1, \left( D_{\lambda}^+ P \right)^n H_0 \rangle \right. \]

Moreover, the proof of Proposition 4.2 also implies that the \( \nu \)-th moment of the shortfall of the CPPI portfolio, if exists, can be computed by the following formula
\[
\mathbb{E} \left[ (-C_{T^+}^*)^\nu \mathbb{I}_{\{\varrho \leq n\}} \right] 
\]
\[= \mathbb{E} \left[ (-C_{T^+}^*)^\nu \mathbb{I}_{\{C_{T^+}^* \leq 0\}} \right] 
\]
\[= \left( C_{t_0}^* \right)^\nu \cdot \left\langle \xi(\nu)^-, \left( I - P \cdot D_{\lambda(\nu)}^+ \right)^{-1} \mathbb{P} \left( I - \left( D_{\lambda(\nu)}^+ P \right)^n \right) H_0 \right\rangle, \]

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provided that \((1 - P \cdot D_{\lambda})^+\) is invertible, where

\[
D_{\lambda\nu}^- = \text{diag} \left[ \mathbb{E} \left[ (\Lambda_1^{(j)})^\nu \right], \ldots, \mathbb{E} \left[ (\Lambda_k^{(j)})^\nu \right] \right] \tag{35}
\]

\[
\lambda\nu = \left( \ln \mathbb{E} \left[ (\Lambda_1^{(j)})^\nu \right], \ldots, \ln \mathbb{E} \left[ (\Lambda_k^{(j)})^\nu \right] \right) \tag{36}
\]

\[
\xi\nu = \left( \ln \mathbb{E} \left[ (\Xi_1^{(j)})^\nu \right], \ldots, \ln \mathbb{E} \left[ (\Xi_k^{(j)})^\nu \right] \right) \tag{37}
\]

### 4.4. Expected gain

An analytical expression for the (unconditional and conditional) expected gain can be obtained in a similar way as for the expected shortfall in the preceding subsection. To proceed, define \(\xi^+ = (\xi^+_1, \ldots, \xi^+_r)'\) with \(\xi^+_j = E \left[ \Xi_1^{(j)} \right], j = 1, \ldots, r\).

**Proposition 4.3.** The unconditional expected gain is given by

\[
\text{UEG}_{H_0} = E \left[ C_{\lambda T}^* \sum_{\varphi > n} \right] = E \left[ C_{\lambda T}^* \sum_{C_{\lambda T} > 0} \right] = C_{t_0}^* \cdot \langle \xi^+, \ P \left( D_{\lambda} \cdot P \right)^{n-1} H_0 \rangle,
\]

where \(D_{\lambda}^+\) is a diagonal matrix defined in Proposition 4.2.

**Proof.** From (21), \(C_{\lambda T}^* = C_{t_0}^* \left( \prod_{j=1}^{n-1} \Lambda_j^+ \right) \cdot \Xi^+\) for \(\varphi > n\). Similar to (34), it follows from part (b) of Lemma 4.2 that

\[
E \left[ C_{\lambda T}^* \sum_{\varphi > n} \right] = E \left[ C_{t_0}^* \cdot \exp \left( \sum_{j=1}^{r} (\Gamma_j (n-1) \cdot \lambda_j^+) \right) \cdot H_n, \xi^+ \right] = C_{t_0}^* \cdot \langle \xi^+, \ E \left[ \exp \left( \sum_{j=1}^{r} (\Gamma_j (n-1) \cdot \lambda_j^+) \right) \cdot H_n \right] \rangle
\]

by which the proof is complete. \(\Box\)

**Remark 4.3.** Similar to the comments for the conditional expected shortfall in Remark 4.2, the conditional expected gain can be computed by using Propositions 4.1 and 4.3 as follows

\[
E \left[ C_{\lambda T}^* \sum_{\varphi > n} \right] = E \left[ C_{\lambda T}^* \sum_{C_{\lambda T} > 0} \right] = C_{t_0}^* \cdot \langle \xi^+, \ P \left( D_{\lambda} \cdot P \right)^{n-1} H_0 \rangle, \tag{38}
\]

by which the proof is complete.
Moreover, by combining Propositions 4.2 and 4.3, the expected terminal cushion can be obtained as follows

\[ E[C^*_{T+} | H_0] = E[C^*_{T+} \mathbb{I}_{\{\varphi > n\}} | H_0] + E[C^*_{T+} \mathbb{I}_{\{\varphi \leq n\}} | H_0] \]

\[ = C^*_t \cdot \left\langle \xi^+, \ P \left( D_{\lambda^+} \mathbb{P} \right)^{n-1} H_0 \right\rangle \]

\[ - C^*_t \cdot \left\langle \xi^-, \ (I - P \cdot D_{\lambda^+})^{-1} P \left( I - \left( D_{\lambda^+} \mathbb{P} \right)^n \right) H_0 \right\rangle , \]

provided that \((I - P \cdot D_{\lambda})^+\) is invertible. Finally, the \(\nu\)-th moment of the shortfall can be calculated as follows

\[ E\left( (C^*_{T+})^\nu \mathbb{I}_{\{\varphi > n\}} | H_0 \right] = E\left( (C^*_{T+})^\nu \mathbb{I}_{\{C^*_{T+} > 0\}} | H_0 \right] \]

\[ = \left( C^*_t \right)^\nu \cdot \left\langle \xi(\nu)^+, \ P \left( D_{\lambda(\nu)} \mathbb{P} \right)^{n-1} H_0 \right\rangle , \]

where \(\lambda(\nu)^+\) is a vector defined in (38) and

\[ \xi(\nu)^+ = \left( \ln E \left[ \left( \Xi_{i(j)}^+ \right)^\nu \right], \ldots, \ln E \left[ \left( \Xi_{k(j)}^+ \right)^\nu \right] \right) . \]

4.5. Omega measures

The payoffs of portfolio insurance strategies are typically non-linear with respect to the risky reference asset, which induces asymmetric return distributions. Traditional performance measures such as mean-variance, mean-lower partial moment (for instance the mean-CVaR), and Sharpe’s ratio are believed inadequate to evaluate the performance of a CPPI portfolio. Bertrand and Prigent (2011) introduced Kappa performance measures and especially the Omega measure to take account of the entire return distribution of the CPPI terminal portfolio value. It is designed to overcome the shortcomings of performance measures based only on the mean and certain particular (partial) moment of the distribution of the terminal portfolio value.

The Omega measure was first introduced by Keating and Shadwick (2002); Cascon, et al. (2003). This measure splits the return into two sub-parts according to a threshold which corresponds to a minimum acceptable return. More precisely, the Omega measure for a profit-loss random variable \(Z\) is defined as the probability weighted ratio of gains to losses relative to a return threshold as follows

\[ \Omega(Z; L) = \frac{E[(Z - L)^+] \cdot \ln E \left[ \left( \Xi_{i(j)}^+ \right)^\nu \right]}{E[(L - Z)^+]}. \]

Usually the threshold is chosen lower than the expected portfolio returns. In the portfolio insurance framework, additional constraints must be taken into account, for instance the threshold must be higher than the guaranteed amount.

This subsection is dedicated to computing the Omega measure of the CPPI terminal portfolio value under the transaction cost and regime switching which has already been
specified in the previous sections. To proceed, recall that the terminal portfolio value \( V_{T^+} = C_{T^+} + F_{T} \) and thus the discounted terminal value

\[
V_{T^+}^* = \frac{V_{T^+}}{F_{T}} = \frac{C_{T^+} + F_{T}}{F_{T}} = C_{T^+}^* + 1.
\]

Consequently, it is prudent to consider the following Omega measure (at a constant threshold \( L \)) for the CPPI portfolio:

\[
\Omega(V_{T^+}^* | H_0) = \frac{\mathbb{E} \left[ (V_{T^+}^* - L)^+ | H_0 \right]}{\mathbb{E} \left[ (L - V_{T^+}^*)^+ | H_0 \right]} = \frac{\mathbb{E} \left[ (C_{T^+}^* - (L - 1))^+ | H_0 \right]}{\mathbb{E} \left[ ((L - 1) - C_{T^+}^*)^+ | H_0 \right]},
\]

(38)

As a performance measure for portfolios with a guarantee, the Omega measure is typically set with a threshold no less than the guarantee level, and thus, \( L \) in (40) should be chosen no less than 1. When \( L = 1 \), the Omega measure reduces to

\[
\Omega(V_{T^+}^* | H_0) = \frac{\mathbb{E} \left[ (C_{T^+}^*)^+ | H_0 \right]}{\mathbb{E} \left[ (-C_{T^+}^*)^+ | H_0 \right]} = \frac{\text{UEG}_{H_0}}{\text{UES}_{H_0}},
\]

which is computable by using Propositions 4.2 and 4.3.

Next, consider the case with \( L > 1 \) and let \( d = (L - 1)/C_{t^*_0}^* \). Then, the Omega measure can be equivalently rewritten as follows

\[
\Omega(V_{T^+}^*, d | H_0) = \frac{\mathbb{E} \left[ (C_{T^+}^*/C_{t^*_0}^* - d)^+ | H_0 \right]}{\mathbb{E} \left[ (C_{T^+}^*/C_{t^*_0}^* - d)^+ | H_0 \right] + d - \mathbb{E} \left[ C_{T^+}^*/C_{t^*_0}^* | H_0 \right]},
\]

(39)

where \( d \) in \( \Omega(V_{T^+}^*, d | H_0) \) is used to signify the dependence of the Omega measure on the quantity \( d \). Note that \( \mathbb{E} \left[ (C_{T^+}^*/C_{t^*_0}^* | H_0) = (\text{UEG}_{H_0} - \text{UES}_{H_0}) / C_{t^*_0}^* \right] \) is computable by using Propositions 4.2 and 4.3. Thus, by the expression (39), it suffices to calculate \( \mathbb{E} \left[ (C_{T^+}^* - d)^+ | H_0 \right] \) in order to obtain the value of the Omega measure. However, directly computing \( \mathbb{E} \left[ (C_{T^+}^* - d)^+ | H_0 \right] \) seems technically impossible, and thus we resort to a Laplace transform methodology. To proceed, write \( d = e^{-u} \) so that

\[
\mathbb{E} \left[ (C_{T^+}^*/C_{t^*_0}^* - d)^+ | H_0 \right] = A(u | H_0),
\]

where

\[
A(u | H_0) : = \mathbb{E} \left[ (e^Z - e^{-u})^+ I_{(C_{T^+}^* > 0)} | H_0 \right] \\
= \mathbb{E} \left[ (e^Z - e^{-u})^+ | C_{T^+}^* > 0 | H_0 \right] \cdot \Pr (C_{T^+}^* > 0 | H_0), \quad u \in \mathbb{R},
\]
and $Z = \ln \left( C_{T+}^\ast / C_{t_0^*}^\ast \right)$. In terms of $A(u)$, the Omega measure can be equivalently rewritten as

$$
\Omega(V_{T+}^* ; d|H_0) = \frac{A(u|H_0)}{A(u|H_0) + e^{-u} - (\text{UEG}_{H_0} - \text{UES}_{H_0})/C_{t_0^*}^\ast},
$$

with $u = -\ln d$. (40)

To proceed, let $i$ denote the complex unity so that $(-i)^2 = -1$ and define

$$
C_+ (u|H_0) := \mathbb{E} \left[ e^{iuz} \mathbb{I}[(C_{T+}^\ast > 0)] H_0 \right],
$$

for which an explicit formula will be developed in Proposition 4.4 below. For $\sigma > 0$, the double-sided Laplace transform of $A(u)$ can be computed in terms of $C_+$ as follows:

$$
\mathcal{L}_A(\sigma + i\omega|H_0) = \int_{-\infty}^{\infty} e^{-i(\sigma + \omega)u} A(u) du
$$

$$
= \int_{-\infty}^{\infty} e^{-i(\sigma + \omega)u} \mathbb{E} \left[ (e^Z - e^{-u}) \right| C_{T+}^\ast > 0] \Pr (C_{T+}^\ast > 0) du
$$

$$
= \int_{-\infty}^{\infty} e^{-i(\sigma + \omega)u} \int_{-u}^{\infty} (e^Z - e^{-u}) f_{Z_+}(z) dz du \cdot \Pr (C_{T+}^\ast > 0)
$$

$$
= \int_{-\infty}^{\infty} \left[ f_{Z_+}(z) \cdot \Pr (C_{T+}^\ast > 0) \right] \int_{-\infty}^{\infty} e^{-i(\sigma + \omega)u} (e^Z - e^{-u}) du dz
$$

$$
= \left( \frac{1}{\sigma + i\omega} - \frac{1}{1 + \sigma + i\omega} \right) \int_{-\infty}^{\infty} e^{iz\left[i\omega - i(\sigma + \omega)\right]} f_{Z_+}(z) \cdot \Pr (C_{T+}^\ast > 0) dz
$$

$$
= \frac{1}{[\sigma + \sigma^2 - \omega^2] + i\omega [2\sigma + 1]} \mathcal{C}_+ (\omega - i (1 + \sigma) |H_0) \mathrm{(42)}
$$

where $f_{Z_+}(z)$ denotes the density function of $Z$ conditional on $C_{T+}^\ast > 0$.

For an explicit expression of the double-sided Laplace transform $\mathcal{L}_A(\cdot)$, it remains to analyze $C_+$, which is defined in (43). To proceed, denote

$$
\psi^+(u) = (\psi_1^+(u), \ldots, \psi_r^+(u))' \text{ and } \phi^+(u) = (\phi_1^+(u), \ldots, \phi_r^+(u))',
$$

where $\psi_j^+(u) = \ln \mathbb{E} \left[ e^{u \ln \mathbb{I}(\varepsilon_{k^*}^j > 0)} \right]$, $\phi_j^+(u) = \ln \mathbb{E} \left[ e^{u \ln \Lambda_{k^*}^j \mathbb{I}(\Lambda_{k^*}^j > 0)} \right]$, $j = 1, \ldots, r$, and by convention $\ln x = \ln |x| + \arg(x)$ for a complex number $x$.

**Proposition 4.4.** Denote $D_{\phi^+}(u) = \text{diag} \left( e^{\phi^+_1(u)}, \ldots, e^{\phi^+_r(u)} \right)$ and $e^{\psi^+(u)} = \left( e^{\psi_1^+(u)}, \ldots, e^{\psi_r^+(u)} \right)'$.

The functions $C_+ (u|H_0)$ defined in (43) can be computed explicitly as follows:

$$
C_+ (u|H_0) = \left< e^{\psi^+(u)}, P \left( D_{\phi^+}(u) P \right)^{n-1} H_0 \right>. \mathrm{(43)}
$$
Proof. Recall the expression for $C_T^*$ in (21), and note that $\{C_T^* > 0\} = \{\varrho > n\}$. Also note the fact that $\{(\Lambda_t, \Xi_t), t = 1, \ldots, k\}$ are independent and identically distributed random vectors given $\mathcal{F}_n^H$. Thus,

\[
\mathbb{E} \left[ e^{iux} \mathbb{1}_{\{C_T^* > 0\}} \Big| \mathcal{F}_n^H \right] = \mathbb{E} \left[ e^{iu \ln \Xi_n} \mathbb{1}_{\{\Xi_n > 0\}} \cdot \prod_{j=1}^{n-1} \left( e^{iu \ln \Lambda_j} \mathbb{1}_{\{\Lambda_j > 0\}} \right) \Big| \mathcal{F}_n^H \right] = \mathbb{E} \left[ e^{iu \ln \Xi_n} \mathbb{1}_{\{\Xi_n \leq 0\}} \cdot \prod_{j=1}^{n-1} \mathbb{E} \left[ e^{iu \ln \Lambda_j} \mathbb{1}_{\{\Lambda_j > 0\}} \Big| H_j \right] \right],
\]

where $E_j = i$ if $H_j = h_i$ for $i = 1, \ldots, r$ and $j = 1, \ldots, n$. Consequently, the desired result follows from part (a) of Lemma as follows

\[
C_+ (u|H_0) = \mathbb{E} \left[ \exp \left\{ \psi_{E_n}^+ (u) + \sum_{j=1}^{n-1} \phi_{E_j}^+ (u) \right\} \Big| H_0 \right] = \mathbb{E} \left[ e^{\psi^+ (u)} \cdot \mathbb{E} \left[ H_n \cdot \exp \left\{ \sum_{j=1}^{r} \left( \Gamma_j (n-1) \cdot \phi_j^+ (u) \right) \right\} \Big| H_0 \right] \right] = \left\langle e^{\psi^+ (u)}, P \left( D \phi^+ (u) \right)^{n-1} H_0 \right\rangle.
\]

\[\square\]

The expression for $\mathcal{L}_A (\cdot)$ in (42) and equation (43) for $C_+ (\cdot|H_0)$ in Proposition (4.4) allow us to adopt certain Laplace inversion algorithm to compute the Omega measure defined in (38). In the numerical analysis in next section, a double-sided Laplace inversion algorithm recently developed by Cai et al. (2013) will be applied for this purpose.

5. Numerical analysis

5.1. Numerical setting

Throughout the section, a CPPI portfolio will be studied with weekly rebalancing between a risky asset and a bond for $T = 2$ (years). Assume that there are 52 weeks in one year so that there are $n = 104$ weeks in total over the investment horizon. Further assume that the portfolio starts from an initial value of $V_0 = 1,000$ and targets to guarantee a value of $F_T = 1,000$ in the terminal. Compared with the stocks, a bond is much less risky and therefore, the main uncertainty of the terminal portfolio value stems from the fluctuation of the stock index. To simplify our analysis, we assume the yield of the bond is constant at an annual rate of 5.2% so that its weekly yield rate is 0.1%. Such an assumption implies
a starting cushion of $C_0 = V_0 - F_T \times e^{-T \times 5.2\%} = 98.7747$ and a floor $F_t = 901.2253e^{0.001t}$, $0 \leq t \leq 104$.

In the subsequent numerical analysis, the S&P 500 index is chosen as the risky asset and the financial market is assumed to be governed by a Markov-modulated regime switching model with three distinct regimes, which respectively interpreted as bullish, normal and bearish states. Under each regime, the log-return of the index is subject to a normal distribution with mean $\mu_i$ and standard deviation $\sigma_i$ given as follows:

$$
\mu_1 = -0.0159, \quad \mu_2 = 0.0013, \quad \mu_3 = 0.0027, \quad \sigma_1 = 0.0609, \quad \sigma_2 = 0.0254, \quad \sigma_3 = 0.0141.
$$

The transition probability matrix for the regime process is as follows

$$
P = (p_{i,j})_{i,j=1,2,3} = \begin{pmatrix}
0.8241, & 0.0154, & 0.0000 \\
0.1759, & 0.9743, & 0.0069 \\
0.0000, & 0.0103, & 0.9931
\end{pmatrix},
$$

where $p_{i,j} = \Pr(\xi_t = i | \xi_{t-1} = j)$, $i, j = 1, 2, 3$. The above parameters for the risky asset are calibrated from 1,248 weekly closing data points of the S&P 500 index from January 01, 1990 throughout November 10, 2013. With the above estimates, the first regime might be interpreted as the bear market state, as it has a negative expected return accompanied with a large volatility. The third regime is the bull state in that it has a positive expected return and a small volatility. Compared with third regime, the second one has a smaller expected return and a relatively larger volatility; thus the second state can be viewed as the normal state.

### 5.2. Numerical results

With the numerical settings specified in the preceding subsection, shortfall probabilities, unconditional expected shortfall and unconditional expected gain can be computed by their formulas given in Propositions 4.1, 4.2 and 4.3, respectively. The results for the shortfall probabilities are demonstrated in Figure 1 for various combinations of a different starting market state, transaction cost and multiplier. The results for $1,000 \times \text{UES}_{H_0}$ are reported in Figure 2. Note that $1,000 \times \text{UES}_{H_0} = \mathbb{E}[C_{T+} | \varrho \leq n]$ and it is the expected dollar amount of the risk on the guarantor at the terminal transaction time $T$. Similarly, the results reported in Figure 3 are for $1000 \times \text{UEG}_{H_0} = \mathbb{E}[C_{T+} | \varrho > n]$, which is the real expected value of the terminal payoff for the investors.

Moreover, formula (40) is used to compute the Omega measures at a series of different threshold $d$, with results demonstrated in Figure 4. In the specific implementation, the values of $\text{UES}_{H_0}$ and $\text{UEG}_{H_0}$ are directly computed by their formulas given in Propositions (4.2) and (4.3), and the values of $A(u|H_0)$ are obtained based on the Laplace transform (42) by the double-side Laplace inversion algorithm from Cai et al. (2013) with algorithm parameters $\sigma = 0.3$, $C = 10$ and $N = 350$.

For presentation convenience, the CPPI portfolios with initial market states $H_0 = c(1, 0, 0)$, $H_0 = c(0, 1, 0)$ and $H_0 = c(0, 0, 1)$ will be referred to as “bearish-start portfolio”, “normal-start portfolio” and “bullish-start portfolio”, respectively. The results in Figures 1-4 show that the regime switching feature has a significant effect on the performance of the CPPI
portfolio. A different market state at the inception of the investment will lead to a quite different risk-and-reward profile for the CPPI portfolio, with specific comments as below.

1. First, at any of the four levels of transaction cost, the shortfall probability is substantially larger for a bearish-start portfolio than a portfolio starting from either of the other two market states; see Figure 4.1. For example, for \( m = 8 \) and \( \theta = 0 \), there is a probability of more than 20\% to realize a shortfall within the investment horizon if the CPPI portfolio starts in a bearish market, while the probability is only about 4\% (13\%) if it starts in a bullish (normal) market.

2. Second, the unconditional expected shortfall is also obviously larger for a bearish-start portfolio than a portfolio starting in either of the other two states, and it is true at each of the four levels of transaction cost; see Figure 2. This implies that the risk on the guarantor substantially differs from one initial market state to another. There is more risk on the CPPI fund guarantor if the investors enter the fund in a bear market than bull market, which in turn hints that more premium should be charged on the investors if they choose to enter the fund in a bear market.

3. Third, the risk-and-reward profile for the investors is also quite different if they start their investment in a different initial market state; see Figure 3. For all of the four transaction cost levels, the investors should expect to receive a much high unconditional expected gain if they start their portfolio in a bullish market than a normal or a bearish market.

4. Fourth, the Omega measures demonstrated in Figure 4 for different transaction cost all consistently confirm an order among the three portfolios starting from different market states with respect to their performance. The bullish-start one is the best, and the bearish one is the worst.

Moreover, Figures 1-4 also illustrate what an effect the transaction cost can exert on each risk-and-reward profile measure of the CPPI portfolio, with more comments as below.

1. First, Figure 1 clearly shows that the higher the transaction cost, the higher the shortfall probability, which is in accordance to our intuition that the transaction cost reduces the portfolio value and therefore increases the chances for the portfolio value to drop below the floor.

2. Second, as demonstrated by the plots in Figure 2, the existence of the transaction cost indeed lessen the risk on the fund guarantor, as it decreases the unconditional expected shortfall, though very slightly. At the first thought, it seems a contradiction to our intuition. Nevertheless, it indeed accords with the mechanism of the CPPI strategy well. The existence of the transaction cost reduces the level of cushion over each portfolio revision, and therefore the exposure in the risky asset is also reduced accordingly. This, in turn, alleviates the shortfall level once a gap risk occurs.
Third, the effect of the transaction cost on the unconditional expected gain is the most intricate; see Figure 3. The higher the transaction cost, the lower the investor’s unconditional expected gain. This confirms the unsurprising fact that the transaction cost hampers the investor’s returns. More interestingly, the transaction cost changes the effect of the multiplier on the unconditional expected gain. It has been widely recognized that the expected gain of a CPPI portfolio is increasing in the multiplier $m$ when no transaction cost is taken into account. This is confirmed again by plot a in Figure (3). Nevertheless, when the transaction is not cost free, the expected gain is not necessarily increasing all the way with the multiplier $m$, as confirmed by the other three plots in Figure (3). In particular, when the transaction cost proportional factor $\theta$ is as large as 1%, the expected gain for both the bearish-start portfolio and the normal-start portfolio is decreasing in $m$ over the whole interval [3, 11], and it is increasing with $m$ up to 6 and then decreasing all the way after 6.

Fourth, the Omega measures demonstrated in Figure 4 consistently show a negative effect the transaction cost exerts on the performance of the CPPI portfolio. For all the three starting market states, an increase in transaction cost substantially reduces the Omega measure.

6. Conclusion

The regime switching framework for modeling econometric series offers a transparent and intuitive way to capture market behavior through different economic conditions. It has been widely used in econometrics since the pioneering work of Hamilton (1989). The present paper developed a framework for studying the performance of discrete-time CPPI portfolio in the presence of proportional transaction cost and regime switching of the financial market. Analytically tractable expressions for the shortfall probability, expected shortfall, and expected gain of the “gap risk” are derived under a general Markov-modulated regime switching model. Moreover, a double-sided Laplace inversion method is developed for computing the Omega measure of the discrete-time CPPI portfolio. These results can not only help the investors to develop a comprehensive understanding on their risk-and-reward profiles, but also offer an effective framework for CPPI fund guarantors to conduct stress tests by adjusting input parameters. Our results show that the behavior of the CPPI portfolio can differ significantly with a different market state at the inception of the investment.

References


Figure 1: Shortfall probability as a function of the multiplier $m$ for CPPI portfolios starting from three different market states and at four distinct levels of transaction cost.
Figure 2: Unconditional expected shortfall as a function of the multiplier $m$ for CPPI portfolios starting from three different market states and at four distinct levels of transaction cost.
Figure 3: Unconditional expected gain as a function of the multiplier $m$ for CPPI portfolios starting from three different market states and at four distinct levels of transaction cost.
Figure 4: Omega measure of the terminal portfolio value as a function of the threshold $d$ for CPPI portfolios starting from three different market states and at four distinct levels of transaction cost: $m = 6$ and $T = 2$. 

a) $\theta = 0.0\%$

b) $\theta = 0.3\%$

c) $\theta = 0.6\%$

d) $\theta = 1.0\%$


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