CVaR-based Optimal Partial Hedging

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Abstract

In this paper, we consider the problem of optimal partial hedging for a contingent claim subject to a preset hedging budget constraint. Under some technical assumptions on the hedged loss function and the market pricing functional, the optimal partial hedging strategy, which minimizes the conditional Value-at-Risk (CVaR) of the hedger’s total risk exposure, is derived explicitly. Some in-depth analysis is conducted for a utility based indifference pricing functional. Ample numerical examples are presented to highlight the comparative advantages of the proposed CVaR-based hedging strategy relative to other hedging strategies including the expected shortfall hedging, the VaR-based hedging strategies and the CVaR hedging strategy of Melnikov and Smirnov (2012). Among these hedging strategies, the numerical examples demonstrate that our proposed CVaR-based hedging is more robust and more effective in terms of managing the tail risk of the hedger’s risk exposure.

Key words Partial hedging, conditional value-at-risk (CVaR), bull call spread hedging, utility based indifference pricing, stop-loss.

1 Introduction

The quest for optimal hedging strategies has remained an interesting and important problem in the field of financial engineering. Various hedging strategies have been proposed in the literature. For example, it is well-known that under the market completeness assumption we can construct a perfect hedging strategy that perfectly replicates the payout of any contingent claim via a self-financing portfolio. On the other hand, when the market is incomplete, the perfect hedging is typically not possible and the superhedging strategy has been suggested. The principle underlying the superhedging is to use the smallest amount to construct a self-financing portfolio whose payout is no smaller than the contingent claim in all possible scenarios. By this strategy, the hedger is ensured to have sufficient fund to cover all his future obligations. The downside is that the strategy is too costly to be of practical interest.

Motivated by the above shortcomings, a compromised strategy, known as the partial hedging, has been advocated. Rather than requiring all future obligations to be hedged perfectly or “overly”, the partial hedging relaxes this stringent condition by only partially hedging the future obligation. The cost of the partial strategy (i.e. the hedging budget) is typically much smaller than the corresponding perfect hedging and superhedging strategies and it also leads to a strategy of making a profit. An example of a partial hedging strategy is the quantile hedging which is proposed by Föllmer and Leukert (1999). The quantile hedging ensures that, for a given budget constraint, the probability of meeting the future obligation is optimally maximized.

Other variants of optimal partial hedging strategies have also been proposed in the literature. For example, Föllmer and Leukert (2000) consider the optimal partial hedging problem by taking into account the magnitude of the hedging shortfall and using a loss function to
describe the hedger’s attitude towards the shortfall. Nakano (2004) considers the problem in the context of minimizing some coherent risk measures, which essentially can be represented as the expected value of the shortfall under a particular probability measure. Rudloff (2007) generalizes these results to some convex risk measures. More recently, Melnikov and Smirnov (2012) adopt a criterion of minimizing the conditional Value-at-Risk (CVaR) of the hedger’s portfolio in a complete market, and they derive a semi-explicit solution to the optimal partial hedging problem by applying some results from Föllmer and Leukert (2000) and Rockafellar and Uryasev (2002). By minimizing the Value-at-Risk (VaR) of the hedger’s total risk exposure, Cong, et al. (2013) derive analytically that the optimal partial hedging strategy is either a knock-out call strategy or a bull call spread strategy, depending on the admissible classes of hedging strategies. Other related results on optimal partial hedging can be found in Cvitanić (2000), Nakano (2007), Sekine (2004) and references therein. Also, the quantile hedging has been successfully applied to many specific financial and insurance contracts; see, for example, Sekine (2000), Melnikov and Skornyakova (2005), Wang (2009), Klusik and Palmowski (2011).

This paper is motivated by Cong, et al. (2013) and Melnikov and Smirnov (2012) in that by using a framework similar to the former reference, it derives analytically the optimal partial hedging strategy under the CVaR, a risk measure which is used in the latter reference. By restricting to an admissible set of the hedged loss functions (see Section 2 for the precise definition and justifications), we explicitly obtain the optimal partial hedging strategies with desirable properties. Our results indicate that the optimal hedging strategy based on the CVaR minimization is the so-called bull call spread hedging, which amounts to constructing a bull call spread written on the contingent claim itself.

We now describe in greater details the similarities and the differences between the present paper and the works of Cong, et al. (2013) and Melnikov and Smirnov (2012) in order to highlight the contributions of the present paper. As argued in Cong, et al. (2013), the approach that is used to derive the partial hedging problem differs from those existing literature (including Melnikov and Smirnov, 2012) in at least the following three aspects. First, while most existing literatures typically formulate the problem as one of identifying the most powerful test for a corresponding hypothesis testing problem, we achieve the objective by first investigating an optimal partition between the hedged loss and retained loss (by the hedger) and then analyzing the specific hedging strategy for the hedged loss. Second, while the structure of optimal solutions obtained in the existing literature usually depends on the specific dynamic of the underlying asset, the structure of our proposed CVaR-based partial hedging is usually independent of it. For example, the optimal quantile hedging strategies under a Black-Scholes model turn out to be highly dependent on the specific values of the model parameters (expected return and volatility). Furthermore, if the assumed model is sufficiently complex, it is difficult to obtain analytic solution to a problem of optimal quantile hedging. In contrast, the optimal partial hedging strategy can always be obtained explicitly for our CVaR-based model, and the optimal solution is always a bull call spread written on the claim itself. Third, in many cases our optimal solutions involve hedging an instrument which has the same structure (with different parameter values though) as the claim we aim to
partially hedge. If such an instrument exists in the market, then we can achieve our objective
of partial hedging by a simple static hedging strategy; otherwise, the hedging budget can
be used to construct a portfolio that dynamically replicates the payout of such a bull call
spread.

Because CVaR is a more desirable risk measure than VaR, such as more adequate in
capturing the magnitude of the risk on the tail (see Artzner, et al., 1999), the optimal
partial hedging derived under this risk measure will be of more relevant and more important.
While the same basic framework is used, it is important to emphasize that the construction
technique employed in Cong, et al. (2013) cannot be directly applied in the present paper.
Instead, we need to impose some different assumptions on the feasible hedging strategy and
utilize a different construction method in order to derive the analytically the optimal partial
hedging strategy. Furthermore, the optimal solution is analyzed under the stop-loss order
reserving pricing and the marginal utility-based pricing.

It is also worth noting that the approach of partitioning a risk into two parts (i.e. hedged
risk and retained risk) is commonly used in the context of optimal reinsurance; see, for
and references therein. In particular, Chi and Tan (2013) is the one which has the most
similar formulation as the present paper. Nevertheless, there is an important difference
between the feasible set considered in their model and the one proposed in the present
paper. In their paper, both hedged risk and retained risk are nondecreasing as functions of
the original risk. However, in the present paper we only assume that the risk retained by
the hedger is nondecreasing.

We now contrast our proposed optimal partial hedging model to that in Melnikov and
Smirnov (2012). On one hand, both models are similar in that both utilize the same CVaR
risk measure to determine the optimal hedging strategy. On the other hand, our proposed
CVaR-based model is less general in the sense that we impose two additional assumptions
on the hedged loss function, namely not globally over-hedged assumption and not locally
over-hedged assumption, in addition to the usual nonnegativity assumption. As justified in
Subsection 2.2, these additional assumptions are reasonable. There are several important
advantages to analyze our proposed optimal partial hedging model, albeit it is more restric-
tive. First, the analysis of the optimal partial hedging is considerable simpler and more
transparent by resorting to a two-step procedure as described earlier. In contrast, Melnikov
and Smirnov (2012) follow the same procedure as Föllmer and Leukert (1999) and study
the optimal partial hedging strategies by searching for the most powerful test for a corre-
sponding hypothesis testing problem. Second, in the general case the solution to the optimal
partial hedging strategy of Melnikov and Smirnov (2012) is only semi-explicit. Even if we
were to confine to one of the simplest models such as the Black-Scholes model, the procedure
for obtaining the optimal hedging strategy can be quite involved as we will demonstrate in
Example 4.3. In other words, it in general, can be challenging to numerically solve for the
partial hedging strategy of Melnikov and Smirnov (2012). On the other hand, the optimal
partial hedging strategy of our proposed model is explicit and is relatively easy to deduce,
not just for the Black-Scholes model but also for other more sophisticated and more complex
market models. Third, the optimal partial hedging strategy of our proposed model has the same functional form, irrespective of the specification of the market model assumptions. The functional form of the optimal partial hedging strategy of Melnikov and Smirnov (2012), on the other hand, could be sensitive to the specification of the market models as well as the corresponding parameter values. For instance, for the Black-Scholes model considered in Example 4.3, the optimal partial hedging strategy of Melnikov and Smirnov (2012) for hedging a European call option could either be a knock-in call or a knock-out call, depending on the relative magnitude of interest rate and the drift coefficient. This implies that the optimal partial hedging strategy needs to be evaluated case by case depending on the values of these parameters. In contrast, our proposed optimal partial hedging strategy is consistently a bull call spread. Because of this property, the optimal partial hedging of our proposed model is said to be more robust than that of Melnikov and Smirnov (2012).

The rest of the paper is organized as follows. Section 2 describes our proposed CVaR-based partial hedging model as well as provide justifications to our proposed model. The optimal solutions to our CVaR-based model are derived explicitly in Section 3. In Section 4, some numerical examples are presented to highlight some comparative advantages of our CVaR-based partial hedging strategies relative to some other existing partial hedging strategies. Finally, Section 5 concludes the paper, and the appendix collects some of the proofs.

2 CVaR based partial hedging model

2.1 Model setup

We begin by assuming that a hedger is exposed to a future obligation $X$ at time $T$ for which the hedger intends to hedge with a pre-specified budget $\pi_0$. We emphasize that $X$ is not necessarily the payout of a European option. Instead, it can be any functional of a specific stock price process, i.e., $X = H(S_t, 0 \leq t \leq T)$, where $S_t$ denotes the time-$t$ price of a specific stock and $H$ is a functional. Without any loss of generality, we assume that $X$ is a non-negative random variable with cumulative distribution function $F_X(x) = \mathbb{P}(X \leq x)$ and $\mathbb{E}^\mathbb{P}(X) < \infty$, where $\mathbb{E}^\mathbb{P}$ denotes the expectation under the physical probability measure $\mathbb{P}$. The exact expression of $F_X(x)$ may be unknown.

We resolve the problem of optimal hedging by dividing it into two subproblems and then solving them in two independent steps. In the first step, we study the optimal partitioning of $X$ into $f(X)$ and $R_f(X)$ such that $X = f(X) + R_f(X)$, where $f(X)$ denotes the part of the payout which should be hedged with the initial capital budget, while $R_f(X)$ represents the part of the payout the hedger retains. We will refer $f(X)$ and $R_f(X)$ as hedged loss and retained loss, respectively. As functions of $x$, we call $f(x)$ and $R_f(x)$ as hedged loss function and retained loss function respectively. In the second step, we investigate the possibility of replicating the time-$T$ payout $f(X)$ in the market. It is a quite interesting prospective to study the optimal partial hedging problem by considering the optimal hedged loss function with respect to the exposed risk $X$. In this way, we do not have to emphasize the specific
form of the contract under consideration until we have established the optimal hedged loss \( f^*(X) \) and consider how to hedge \( f^*(X) \).

To proceed, it is convenient to introduce some additional notation. Let \( \Pi \) denote the risk pricing functional so that \( \Pi(X) \) is the time-0 market price of the contingent claim with payout \( X \) at time \( T \). Similarly, \( \Pi(f(X)) \) is the time-0 market price of \( f(X) \) which also corresponds to the cost of hedging \( f(X) \). In this paper, we do not need to specify a pricing functional \( \Pi(\cdot) \) for main results, but it will be assumed to preserve the so-called stop-loss ordering; see Definition 2.2 and Proposition 2.1 for details. The sum

\[
T_f(X) := R_f(X) + e^{rT}\Pi(f(X))
\]  

(2.1)
can be interpreted as the total time-\( T \) risk exposure of the hedger for implementing the partial hedge strategy since \( R_f(X) \) captures the part of the risk that is retained by the hedger while \( e^{rT}\Pi(f(X)) \) is the time-\( T \) value of the cost for hedging \( f(X) \), where \( r \) is the risk-free interest rate. Given the initial hedging budget \( \pi_0 \), an amount up to which the hedger is willing to spend on hedging, the first immediate task of the hedger is to address the optimal partitioning of \( X \) into \( f(X) \) and \( R_f(X) \) while meeting the cost constraint \( \pi_0 \). This can be accomplished by solving the following optimization problem:

\[
\left\{ \begin{array}{l}
\min_{f \in \Omega} \rho(T_f(X)) \\
\text{s.t. } \Pi(f(X)) \leq \pi_0.
\end{array} \right. 
\]  

(2.2)
where \( \rho(\cdot) \) is an appropriate chosen risk measure for quantifying the total risk exposure \( T_f(X) \) and \( \Omega \) denotes the admissible set of hedged loss functions.

We emphasize that the above risk measure based partial hedging model is quite general in that it permits any arbitrary risk measure as long as it reflects and quantifies the hedger’s attitude towards risk. Risk measures such as the Value-at-Risk (VaR), the conditional value-at-risk (CVaR), expected shortfall, among others could be reasonable choices. In this paper, we focus on CVaR, since it is one of the coherent risk measures and it is very popular among financial institutions and insurance companies. The precise definition of CVaR is defined as follows:

**Definition 2.1.** The CVaR of a risk \( X \) at the confidence level \( (1 - \alpha) \), where \( 0 < \alpha < 1 \), is defined as

\[
CVaR_\alpha(X) = \frac{1}{\alpha} \int_0^{\alpha} VaR_s(X) ds
\]

where

\[
VaR_\alpha(X) = \inf\{x \geq 0 : \mathbb{P}(X > x) \leq \alpha\}.
\]

With the CVaR employed as the optimality criterion, problem (2.2) becomes

\[
\left\{ \begin{array}{l}
\min_{f \in \Omega} CVaR_\alpha(T_f(X)) \\
\text{s.t. } \Pi(f(X)) \leq \pi_0.
\end{array} \right. 
\]  

(2.3)
Remark 2.1. It is interesting to compare the above risk measure based partial hedging framework to the quantile hedging and the expected shortfall hedging. Recall that the quantile hedging, which is proposed by Föllmer and Leukert (1999), targets to maximize the probability of meeting the future obligation subject to a preset budget constraint. Mathematically, this boils down to solving the following optimization problem:

\[
\begin{align*}
\max_{f \in \Omega} & \quad \mathbb{P}(f(X) \geq X) \\
\text{s.t.} & \quad \Pi(f(X)) \leq \pi_0,
\end{align*}
\]  

(2.4)

where \( \mathbb{P} \) is the physical probability measure for the financial market. The expected shortfall hedging, which is proposed in Föllmer and Leukert (2000), aims to minimize the expected shortfall of the hedger, and it can be formulated as the following optimization problem:

\[
\begin{align*}
\min_{f \in \Omega} & \quad \mathbb{E}^\mathbb{P} [(X - f(X))_+] \\
\text{s.t.} & \quad \Pi(f(X)) \leq \pi_0,
\end{align*}
\]  

(2.5)

where \( \mathbb{E}^\mathbb{P} \) denotes the expectation under the physical probability measure \( \mathbb{P} \). Some examples will be provided in Section 4 to demonstrate the optimal solutions under both partial hedging strategies.

Remark 2.2. If we were to interpret \( X \) as the risk exposure faced by an insurer so that \( T_f(X) \) becomes the insurer’s total risk exposure, \( \Pi(\cdot) \) as the premium premium charged by the reinsurer, and \( \pi_0 \) as the budget the insurer is willing to spend on reinsurance purchase, then the optimization model (2.2) corresponds to an optimal reinsurance model. The objective in this case is to determine the optimal reinsurance policy \( f(X) \) that minimizes the insurer’s total risk exposure; see, for example, Cai, et al. (2008), Chi and Tan (2013), Tan, et al. (2011), and Tan and Weng (2012).

To conclude this subsection, we recall the definition of stop-loss ordering between two risks as below.

Definition 2.2. Suppose \( X \) and \( Y \) are two random variables with finite means under a probability measure \( \mathbb{P} \). If

\[
\mathbb{E}^\mathbb{P} [(X - d)_+] \leq \mathbb{E}^\mathbb{P} [(Y - d)_+], \quad \forall \, d \in \mathbb{R},
\]

then we say that \( X \) is smaller than \( Y \) in stop-loss order under \( \mathbb{P} \) and is denoted by \( X \preceq_{sl} Y \).

We remark that the stop-loss ordering may be defined in some other equivalent ways (see Hurlimann (1998)).

2.2 Admissible sets of hedged loss functions

In this section, we define the admissible set \( \Omega \) for our CVaR-based partial hedging model (2.3). The admissible set \( \Omega \) cannot be the universal set as this leads to a model that is ill-posed in that a position with an infinite number of certain assets (long or short) in the market
could be optimal. Indeed, the issue of ill-posedness arises in the context of partial hedging for almost every optimality criterion used in the literature, and a standard technique to resolve this issue is to impose some additional conditions or constraints on the optimization formulation. For example, Föllmer and Leukert (1999, 2000) respectively solve the optimal quantile hedging and expected shortfall hedging by confining the hedge loss function to an admissible set of nonnegative functions; Cong et al. (2013) analyze the optimal VaR-based partial hedging by imposing a set of conditions on the hedged loss function; Alexander, et al. (2004) consider the optimal hedging problem by minimizing the CVaR of the hedging error, and to prevent the problem from being ill-imposed, they introduce an additional term (which reflects the cost of holding an instrument) to the objective function. It is worth emphasizing that, while Alexander, et al. (2004) also employ CVaR as the risk measure in their optimal hedging strategy construction, their analysis is from a computational perspective, and moreover, there is no hedging cost budget incorporated in their formulation.

We impose the following assumptions on the hedged loss functions:

**A1.** Not globally over-hedged: \( f(x) \leq x \) for all \( x \geq 0 \);

**A2.** Not locally over-hedged: \( f(x_2) - f(x_1) \leq x_2 - x_1 \) for all \( 0 \leq x_1 \leq x_2 \);

**A3.** Nonnegativity of the hedged loss: \( f(x) \geq 0 \) for all \( x \geq 0 \).

Note that assumption **A2** is equivalent to the following;

**A2’.** Monotonicity of the retained loss function: \( R_f(x_2) \geq R_f(x_1) \) \( \forall \ 0 \leq x_1 \leq x_2 \).

For our CVaR-based optimization problem, we will analyze the optimal partial hedging strategy among all the hedged loss functions satisfying assumptions **A1-A3** in the above. Furthermore, without loss of too much generality we assume that the retained loss function \( R_f(x) \) is a left continuous function with respect to \( x \). By combining this assumption with the above conditions **A1-A3**, the admissible set \( \Omega \) in the formulation (2.3) is formally defined as follows:

\[
\Omega = \{ 0 \leq f(x) \leq x : R_f(x) \equiv x - f(x) \text{ is a nondecreasing and left continuous function} \}.
\] (2.6)

We now provide some justifications to the assumptions **A1-A3** we imposed on the hedged loss function. First, assumption **A1** is reasonable as it ensures that the hedged loss should be uniformly bounded from above by the original risk to be hedged. Second, assumption **A2** indicates that the increment of the hedged part should not exceed the increment of the risk itself. If the hedger feels comfortable to have a nondecreasing retained loss function, then **A2** will be necessary. While imposing assumption **A2** makes the admissible set of the hedging functions more restrictive, it is reassuring from the numerical examples to be presented in Section 4 that the expected shortfall of the optimal partial hedging strategy derived based on minimizing CVaR with this condition is still significantly smaller than the expected shortfall of the quantile hedging strategy. In the numerical examples, we will also
see that the expected shortfall of our proposed CVaR-based partial hedging strategy is quite close to that of the expected shortfall hedging strategy, which achieves the minimal expected shortfall. Moreover, with assumption \( A2 \) imposed on the hedged loss function, the resulting optimal partial hedging strategy will be model independent, i.e., the structure of the optimal hedging strategy does not depend on the underlying market model; otherwise, the solutions may vary with the specification on the dynamics of the underlying asset price. Nevertheless it is possible to relax assumption \( A2 \) to the following relatively weaker condition:

\[
R_f(x_2) \geq R_f(VaR_\alpha(X)) \geq R_f(x_1) \quad \forall \ 0 \leq x_1 \leq VaR_\alpha(X) \leq x_2
\]

where \( \alpha \) reflects the confidence level adopted by the hedger as in Definition 2.1. This can be accomplished by a simple modification of the proof to our main results to be presented in Theorem 3.1. Third, assumption \( A3 \) is not only commonly imposed in the quantile hedging, its importance is further highlighted in an example in Cong, et al. (2013), which shows that the partial hedging problem (2.2) is still ill-posed if we only impose assumptions \( A1 \) and \( A2 \) but without \( A3 \).

### 2.3 Utility based indifference pricing methods

As mentioned earlier, we assume that the pricing functional \( \Pi \) in our CVaR based formulation (2.3) preserves the stop-loss ordering. One typical example in finance satisfying this property is the utility based indifference pricing (UBIP). In this subsection, we first summarize some fundamental facts about UBIP and make some relevant remarks. We then introduce one special but important UBIP called the marginal utility-based pricing (MUBP).

In a complete market, the price of a contingent claim is uniquely determined as an amount of the cost needed to replicate the claim. In reality, however, the financial market is far from being complete, particularly when we realize the existence of those factors such as transactions costs, non-traded securities, portfolio constraints and so forth. When the market is incomplete, the arbitrage free price is no longer unique, and there are quite a few prevalent pricing approaches, among which is the UBIP method. One of the pioneering works is attributed to Hodges and Neuberger (1989) where the authors discussed how to apply the UBIP method for options in the presence of transaction cost. Some other interesting literature include Henderson and Hobson (2004), Musiela and Zariphopoulou (2004), Mania and Schweizer (2005), Klöppel and Schweizer (2007), Monoyios (2008) and the references therein.

While the UBIP method can further be categorized into many different groups depending on the choices of utility functions and some other relevant factors, the method is based on the same basic idea, where the attitude of every investor in the market towards risks are assumed to be fully described by a utility function and all the investors are assumed to maximize their expected utility of wealth. The utility indifference price of a given contract is then defined as an amount which makes no difference to the investor’s expected utility whether an investor chooses to add the contract into or exclude it from the portfolio. As such, the utility indifference price does not rely on the completeness of the market and does not distinguish between the presence or the absence of the market frictions.
Below are some fundamental properties of the UBIP.

1. Recovery of complete market price.
   If the contract can be replicated in the market, then the utility indifference price coincides with the cost of replicating that contract. A brief justification of this property can be found in Henderson and Hobson (2004). It is a very desirable property of the utility indifference pricing, since it is compatible with the complete market price.

2. Monotonicity.
   If the payoff of contract A is larger than or equal to that of contract B, then the utility indifference price of contract A is also larger than or equal to that of contract B. This property guarantees that the utility indifference price of a contingent claim lies between the super-replicating price and the sub-replicating price.

3. Concavity.
   The utility indifference price of the convex combination of contract A and contract B is larger than or equal to the convex combination of the utility indifference price of contract A and contract B. This is also due to the concavity of the utility function.

4. Preserving stop-loss order.
   As will be shown in Proposition 2.1 below, as long as the preference of the investor can be described by an increasing concave utility function and the investor is aiming at maximizing the expected utility of wealth, the market price must preserve the stop-loss order, ie, \( \Pi(X) \leq \Pi(Y) \) if \( X \leq_{\text{sl}} Y \).

**Proposition 2.1.** If the preference of the investors can be fully described by an increasing concave utility function and the investors are aiming at maximizing his expected utility of wealth, then the market price must preserve the stop-loss order, ie, \( \Pi(X) \leq \Pi(Y) \) whenever \( X \leq_{\text{sl}} Y \).

**Proof.** See Appendix. \( \square \)

As discussed above, in contrast to the complete market price and many alternative pricing methods, the UBIP is generally non-linear (concave), a property inherited from the concavity assumption on the utility functions of the investors. At the first glance, it seems that the non-linear property may cause the UBIP an inappropriate pricing method as it contradicts to the principle of no arbitrage. Nevertheless, on the one hand, the utility indifference pricing still can eliminate the arbitrage opportunity in the market if we take into account those practical factors such as the existence of market friction. On the other hand, a marginal version of the UBIP developed by Davis (1997) is indeed a linear pricing functional. This marginal utility-based pricing (MUBP) gives the utility indifference price for an infinitesimal position in claims, which is unique, lies in the no-arbitrage interval and also lies in the bid and ask utility-based price for a finite position in claims. Consequently, the MUBP functional is endowed with those desirable properties such as recovery of complete market price, monotonicity,
stop-loss order preserving. More interestingly, the MUBP pricing functional has a very elegant representation as the risk neutral pricing. By using the Markov process theory, Davis (1997) proved that the marginal utility-based pricing (MUBP) can be expressed as a discounted expectation under a unique probability measure and a unique discount factor. For the rigorous definition of the MUBP, we refer to Davis (1997); see also Monoyios (2008) for a different but equivalent definition.

3 CVaR optimization

3.1 Under stop-loss order reserving pricing functional

Recall that our proposed optimal partial hedging model corresponds to the optimization problem (2.2). By using CVaR as the relevant risk measure $\rho$ for a given confidence level $1 - \alpha \in (0,1)$, the objective of this section is to identify the solution to the optimization problem (2.3) under the admissible set of $\Omega$ in (2.6). The technical method we will use is similar to that used in Cong, et al. (2013). In Cong, et al. (2013), the VaR-based partial hedging problem is investigated under the no-arbitrage pricing functional. For the CVaR optimization problem, we need some stronger assumptions on the market pricing functional $\Pi(\cdot)$. We assume it preserves the stop-loss order so that all the UBIP functionals apply. For presentation convenience, we assume that the risk free interest rate is constant hereafter, though our results can easily be extended to the case with a stochastic interest rate.

Remark 3.1. As commented in Remark 2.2, due to the connection between our proposed optimal partial hedging model and the optimal reinsurance model, the approach we use to derive the optimal hedged loss function is similar to that in Chi and Tan (2013). Yet, it is worth noting that both the retained loss function and the ceded loss function are assumed to be nondecreasing in Chi and Tan (2013) while we only require the nondecreasing assumption on the retained loss function in this paper. This implies that the admissible set we consider in this paper strictly contains the one in Chi and Tan (2013).

To proceed, it is useful to present the following two lemmas. Their proofs are relegated to the appendix.

Lemma 3.1. For a given random variable $X$ and any function $f \in \Omega$, let

$$g_f(x) = \min \{(x + f(VaR_\alpha(X)) - VaR_\alpha(X))_+, \bar{u}\}, \quad (3.1)$$

where $\bar{u}$ is determined by $CVaR_\alpha(R_f(X)) = CVaR_\alpha(R_{g_f}(X))$, and $R_f(X) := X - f(X)$ so that $R_{g_f}(X) = X - g_f(X)$. Then, $g_f$ is well defined and $g_f \in \Omega$ for any $g \in \Omega$.

Lemma 3.2. For a given payout $X$ and any function $f \in \Omega$, the function $g_f \in \Omega$ as constructed in Lemma 3.1 is smaller than $f(X)$ in stop-loss order under the physical probability measure $P$:

$$g_f(X) \leq_{sl}^{P} f(X). \quad (3.2)$$
Hence, if the market pricing functional $\Pi$ preserves the stop-loss ordering, we have

$$\Pi(g_f(X)) \leq \Pi(f(X)).$$

Lemma 3.2 indicates that, for any given payout $X$, the cost of the hedging strategy $g_f(X)$ will not be higher than that of the hedging strategy $f(X)$, provided that the market pricing functional $\Pi(\cdot)$ preserves the stop-loss order.

Remark 3.2. Let $v = \text{VaR}_\alpha(X)$, $d = \text{VaR}_\alpha(X) - f(\text{VaR}_\alpha(X))$ (which is obviously nonnegative) and $u = \bar{u} + d$. Then, the function $g_f$ given in (3.1) can be rewritten as $g_f(x) = \min\{(x - d)_+, u - d\}$, where $0 \leq d \leq v$. Since $\bar{u}$ is determined by the equation $\text{CVaR}_\alpha(R_f(X)) = \text{CVaR}_\alpha(R_{g_f}(X))$, $\bar{u} \geq 0$ and hence $u \geq 0$; consequently, $g_f$ can be further rewritten as follows

$$g_f(x) = (x - d)_+ - (x - u)_+$$

(3.3)

where $0 \leq d \leq v$, $u \geq d$. This implies that $g_f(X)$ is the payout of a bull call spread written on $X$.

The following theorem shows that the optimal hedged loss function has a form as given in (3.1) (or equivalently (3.3)).

Theorem 3.1. Assume that the market pricing functional preserves the stop-loss order. For any hedging function $f \in \Omega$, the function $g_f$ as constructed in (3.1) belongs to $\Omega$ and satisfies

$$\text{CVaR}_\alpha(T_{g_f}(X)) \leq \text{CVaR}_\alpha(T_f(X))$$

(3.4)

Moreover, $\Pi(f(X)) \leq \pi_0$ implies $\Pi(g_f(X)) \leq \pi_0$.

Proof: For any function $f \in \Omega$, it follows from Lemma 3.1 that $g_f \in \Omega$, and from Lemma 3.2 that $\Pi(g_f(X)) \leq \Pi(f(X))$; thus we must have $\Pi(g_f(X)) \leq \pi_0$ as long as $\Pi(f(X)) \leq \pi_0$. It remains to verify inequality (3.4). This can be justified as follows:

$$\text{CVaR}_\alpha(T_{g_f}(X)) = \text{CVaR}_\alpha(R_{g_f}(X)) + e^T \Pi(g_f(X))$$

$$= \text{CVaR}_\alpha(R_f(X)) + e^T \Pi(f(X))$$

$$\leq \text{CVaR}_\alpha(R_f(X)) + e^T \Pi(f(X))$$

$$= \text{CVaR}_\alpha(T_f(X)),$$

where the first and the last equalities are due to the translation invariance property of the risk measure CVaR, the second equality is because of the construction of $g_f$, and the inequality results from the fact that $\Pi(g_f(X)) \leq \Pi(f(X))$. □

For brevity, it is convenient to introduce the following function:

$$G(x; d, u) = (x - d)_+ - (x - u)_+$$

for $u \geq d$, and $x, d, u \in \mathbb{R}$. (3.5)

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Remark 3.3. (a) With an aid from Theorem 3.1, for the solution to the CVaR-optimization problem, it is sufficient for us to concentrate on the set of hedged loss functions of the form (3.1) or equivalently (3.3). This means that it suffices to focus on the following 2-dimensional optimization problem with decision variables $d$ and $u$:

$$\begin{align*}
\min_{0 \leq d \leq v, u \geq d} & \quad CVaR_\alpha \left\{ X - (X - d)_+ + (X - u)_+ + e^{rT} \cdot \Pi [G(X; d, u)] \right\} \\
\text{s.t.} & \quad \Pi [G(X; d, u)] \equiv \Pi [(X - d)_+ - (X - u)_+] \leq \pi_0
\end{align*}$$

which, obviously can be simplified, by using the translation invariance property of CVaR risk measure, to as follows

$$\begin{align*}
\min_{0 \leq d \leq v, u \geq d} & \quad d + CVaR_\alpha [(X - u)_+] + e^{rT} \cdot \Pi [G(X; d, u)] \\
\text{s.t.} & \quad \Pi [G(X; d, u)] \equiv \Pi [(X - d)_+ - (X - u)_+] \leq \pi_0.
\end{align*}$$

Once we obtain the optimal solution $(d^*, u^*)$ to the above 2-dimensional optimization problem (3.7), the optimal hedged loss function to our CVaR-minimization problem is

$$f^*(x) = (x - d^*)_+ - (x - u^*)_+.$$

(b) The above theorem shows that the optimal hedging strategy, if possible, is to buy a call option on the payout $X$ with strike price $d^*$, while selling a call option on the payout $X$ with strike price $u^*$, where $d^*$ and $u^*$ are the optimizers of the optimization problem (3.7).

(c) As long as the call options on the payout $X$ exist in the market, our proposed partial hedging strategy, which is the bull call spread hedging strategy, can be replicated by the portfolio of options without rebalancing. Therefore, in many cases, our proposed partial hedging strategy is a static hedging strategy.

(d) Since we did not specify the dynamics of the underlying assets, the result in Theorem 3.1 is model independent.

### 3.2 Under the marginal utility-based pricing

According to the previous subsection (see Theorem 3.1 and Remark 3.3), for the specific optimal partial hedging strategy, we need to solve the optimization problem (3.7), which is not trivial in the general case. In this subsection, we will analyze the optimal partial hedging strategies under the marginal utility-based pricing (MUBP) method. As introduced in Subsection 2.3, the MUBP functional can be expressed as a discounted expectation under a uniquely determined probability measure $Q$, and hence it is linear. Without losing generality, hereafter we assume that it admits the following representation

$$\Pi(Z) = e^{-rt} \mathbb{E}^Q[Z] \text{ for any time-}t \text{ contingent claim } Z \quad (3.8)$$

where $\mathbb{E}^Q$ denotes the expectation under $Q$, $e^{-rt}$ is the discounted factor. If the risk free rate is constant in the market, then the discount rate is the same as the risk free rate (see Davis
(1997)). For the details on how to obtain the probability measure $Q$ and the discount rate $r$ in more general cases, we refer to Davis (1997).

Under the MUBP functional with the presentation (3.8), the optimization problem (3.7) can be simplified as shown in the following proposition.

**Proposition 3.1.** Assume the pricing functional $\Pi$ is such that $\Pi(Z) = e^{-rt}E_{Q}[Z]$ for any time-$t$ contingent payout $Z$, where $E_{Q}$ denotes the expectation under a fixed probability measure $Q$, and $r$ is the discount rate. The solutions to the following optimization problem (3.9) solve problem (3.7) and they share the same optimal value.

$$\begin{aligned}
\min_{0\leq d\leq v, u\geq v} & \{ d + \frac{1}{\alpha} E_{P}(X - u)_{+} + e^{rt} \cdot \Pi(G; d, u) \} \\
\text{s.t.} & \Pi[(X - d)_{+}] - \Pi[(X - u)_{+}] \leq \pi_{0}.
\end{aligned}$$

(3.9)

**Proof:** See Appendix A. \(\square\)

**Lemma 3.3.** Assume that the conditions in Proposition 3.1 are satisfied with a hedging budget $\pi_{0} \leq e^{-rt}E_{Q}[(X - \tilde{d})_{+} - (X - v)_{+}]$, where

$$v = \text{VaR}_{\alpha}(X) \text{ and } \tilde{d} = \sup\{d \in \mathbb{R} : Q(X \leq d) = 0\}.\quad (3.10)$$

Then the budget constraint in problem (3.9) is binding under the optimal hedging strategy.

**Proof:** See Appendix A. \(\square\)

**Remark 3.4.** (a) When the hedging budget $\pi_{0} \leq e^{-rt}E_{Q}[(X - \tilde{d})_{+} - (X - v)_{+}]$, by using Lemma 3.3, we can rewrite the optimization problem (3.9) as follows

$$\begin{aligned}
\min_{0\leq d\leq v, u\geq v} & \left\{ d + \frac{1}{\alpha} E_{P}(X - u)_{+} \equiv d + \frac{1}{\alpha} \int_{u}^{\infty} P(X > x)dx \right\} \\
\text{s.t.} & \Pi[(X - d)_{+}] - \Pi[(X - u)_{+}] = \pi_{0}.
\end{aligned}$$

(3.11)

(b) In many interesting situations, $\tilde{d}$ defined in (3.10) is obviously 0, and hence for a sufficient small $\alpha$, $e^{-rt}E_{Q}[(X - d)_{+} - (X - v)_{+}]$ is very close to the market price of the full payout $X$. Therefore, an investigation in such a case is of interest.

Combining Lemma 3.3 and Remark 3.4 provides an explicit way of identifying the optimal hedged loss function as stated in the following theorem:

**Theorem 3.2.** Assume that the conditions in Lemma 3.3 are satisfied. Then, the optimal hedged loss function $g_{f}^{*}$ is given by

$$g_{f}^{*}(x) = (x - d_{f}^{*})_{+} - (x - u_{f}^{*})_{+},$$

where $(d_{f}^{*}, u_{f}^{*})$ satisfies the following equations

$$\begin{aligned}
e^{-rt} \int_{d_{f}^{*}}^{u_{f}^{*}} Q(X > x)dx = \pi_{0} \\
\mathbb{P}(X > u_{f}^{*}) = \alpha \cdot \frac{Q(X > u_{f}^{*})}{Q(X > d_{f}^{*})}.
\end{aligned}$$

(3.12)
Proof: Consider the following Lagrangian function of problem (3.11):

\[ L(d,u,\lambda) = d + \frac{1}{\alpha} \int_u^{+\infty} \mathbb{P}(X > x)dx + \lambda \left( e^{-rT} \cdot \int_d^u \mathbb{Q}(X > x)dx - \pi_0 \right). \]

By letting

\[ \frac{\partial L}{\partial d} = \frac{\partial L}{\partial u} = \frac{\partial L}{\partial \lambda} = 0, \]

we immediately obtain the desired results. \qed

The following corollary provides one of the possible candidates of the optimal hedged loss functions.

Corollary 3.1. Assume the conditions in Lemma 3.3 are satisfied, then one of the possible optimal hedging functions is given by

\[ g^*_f(x) = (x - d^*_f)_+, \]

where \( d^* \) is the solution to the following equation

\[ e^{-rT} \int_{d^*}^{+\infty} \mathbb{Q}(X > x)dx = \pi_0. \]

This means that constructing a call option on the contract \( X \) is one of the possible optimal hedging strategies.

Proof: The result follows trivially from Theorem 3.2 by taking \( u^* = \infty \). \qed

4 Comparison with other partial hedging strategies

In this section, we will present two numerical examples to compare and contrast our proposed CVaR-based hedging to other strategies that have appeared in the literatures, namely the well-known quantile hedging and expected shortfall hedging respectively developed by Föllmer and Leukert (1999, 2000), and the two hedging strategies proposed by Cong, et al. (2013) from a perspective of minimizing VaR of the hedger’s resulting risk exposure. We use the same numerical setting as in Föllmer and Leukert (1999). In Example 4.1, we analyze and evaluate all four hedging strategies in term of the shape of optimal hedged loss functions, the expected shortfall of the hedging strategies and CVaR of the hedger’s total risk exposure. By confining to the proposed CVaR-based hedging strategy and the expected shortfall hedging strategy, Example 4.2 is then used to highlight the relative effectiveness of these strategies. Finally, Example 4.3 is used highlight the advantages of our proposed CVaR-based partial hedging to the work of Melnikov and Smirnov (2012).
Example 4.1. We consider the Black-Scholes model with the dynamics of the stock price $S_t$ at time $t$ given by

$$dS_t = S_t(\sigma dW_t + mdt)$$

where $W$ is a Wiener process under the physical probability measure $\mathbb{P}$, $\sigma$ and $m$ are, respectively, the constant volatility and the return rate. The objective is to hedge a European call option with payoff $X_T = (S_T - K)_+$, where $T$ is the expiration date and $K$ is the strike price. We further assume

$$T = 0.25, \quad K = 110, \quad r = 0, \quad S_0 = 100, \quad m = 0.08,$$

where $r$ is the risk free rate, $S_0$ is the initial stock price, and the following three distinct sets of values for the volatility $\sigma$ and the hedging budget $\pi_0$:

(i) $\sigma = 0.3$, $\pi_0 = 1.5$,

(ii) $\sigma = 0.3$, $\pi_0 = 0.5$,

(iii) $\sigma = 0.2$, $\pi_0 = 0.5$.

Using the Black-Scholes formula, the prices of the corresponding call options are

$$P_C = \begin{cases} 2.50 & \text{for } \sigma = 0.3 \\ 0.95 & \text{for } \sigma = 0.2. \end{cases}$$

Since the hedging budget for $\sigma = 0.3$ is either 0.5 or 1.5 and for $\sigma = 0.2$ is 0.5, this implies that we do not have sufficient fund to perfectly hedge the option. If the hedger is still interested in some sort of hedging subject to the limited budget, partial hedging is one alternative. We now investigate the following four optimal partial hedging strategies:

(a) **Quantile hedging strategy.** Using the results derived in Föllmer and Leukert (1999), the optimal quantile hedging strategies for the three cases are

(i): $(S_T - 110)_+ - (S_T - 129.47)_+ - 19.47 \cdot I(S_T > 129.47)$

(ii): $(S_T - 110)_+ - (S_T - 118.69)_+ - 8.69 \cdot I(S_T > 118.69)$

(iii): $(S_T - 110)_+ - (S_T - 119.98)_+ - 9.98 \cdot I(S_T > 119.98) + (S_T - 1323)_+ + 1213 \cdot I(S_T > 1323)$.

(b) **VaR-based hedging strategy.** As shown in Cong, et al. (2013) that when VaR is used as the criterion, the knock-out call hedging strategy and the bull call spread hedging strategy are respectively the optimal hedging strategies for the two different admissible sets. At 95% confidence level, the optimal knock-out call hedging and the optimal bull call spread hedging are

(b.1) **Knock-out call hedging strategy:**

...
(i): \((S_T - 110)_+ - (S_T - 129.11)_+ - 19.11\mathbb{I}_{(S_T \geq 129.11)}\)

(ii): \((S_T - 116.67)_+ - (S_T - 129.11)_+ - 12.44\mathbb{I}_{(S_T \geq 129.11)}\)

(iii): \((S_T - 110)_+ - (S_T - 119.66)_+ - 9.66\mathbb{I}_{(S_T \geq 119.66)}\).

(b.2) Bull call spread hedging strategy:

(i): \((S_T - 113.30)_+ - (S_T - 129.11)_+\)

(ii): \((S_T - 120.88)_+ - (S_T - 129.11)_+\)

(iii): \((S_T - 112.18)_+ - (S_T - 119.66)_+\).

(c) Expected shortfall hedging strategy. Using the results from Föllmer and Leukert (2000), the optimal hedging strategies under the expected shortfall hedging are

(i): \((S_T - 110)_+ \cdot \mathbb{I}_{(S_T \geq 123.85)}\)

(ii): \((S_T - 110)_+ \cdot \mathbb{I}_{(S_T \geq 137.31)}\)

(iii): \((S_T - 110)_+ \cdot \mathbb{I}_{(S_T \geq 119.19)}\).

(d) CVaR-based partial hedging. As shown earlier that under our proposed CVaR-based hedging strategy, the optimal solution is to follow the bull call spread hedging strategy of the form

\[
[S_T - (K + d^*)]_+ - [S_T - (K + u^*)]_+,
\]

where \(d^*\) and \(u^*\) can be determined numerically from (3.12) of Theorem 3.2. The corresponding values for all three cases at 95% confidence level are

(i): \(d = 5.13\) and \(u = \infty\);

(ii): \(d = 15.08\) and \(u = \infty\);

(iii): \(d = 3.72\) and \(u = \infty\).

This leads to the following optimal hedged loss functions:

(i): \((S_T - 115.13)_+\);

(ii): \((S_T - 125.08)_+\);

(iii): \((S_T - 113.72)_+\).

Figures 1-3 provide a graphical comparison of all the optimal hedged loss functions for all three cases of parameter values and all the five aforementioned hedging strategies. While all these strategies are optimal depending on their adopted objectives, there are some notable differences among the optimal loss functions, as depicted in Figures 1-3. In particular, one key distinction among them is that strategies such as the quantile hedging and the expected shortfall hedging are of the type “all-or-nothing” while the strategy such as the CVaR-based hedging is not. By “all-or-nothing”, we mean that the hedger is perfectly hedged for some
Figure 1: Optimal hedging strategies under scenario (i)
Figure 2: Optimal hedging strategies under scenario (ii)
Figure 3: Optimal hedging strategies under scenario (iii)
part of the losses but unhedge for other part of losses. To elaborate this point, let us consider quantile hedging the risk exposure in case (i). In this case, as long as \( S_T \leq 129.47 \) the hedge is perfect. The perfect hedging, however, is accomplished at the expense of having a naked exposure for \( S_T > 129.47 \); ie, no protection is provided whenever \( S_T > 129.47 \) and as a result the hedger is exposed to an unlimited loss due to the infinite payout of the European call option. The optimal hedged function for hedging the call option for the expected shortfall hedging is also another “all-or-nothing” strategy. The main difference here is that the expected shortfall hedging perfectly hedges large losses while leaving the small losses unhedged. For instance in case (i) of our example the expected shortfall hedging yields a strategy with perfect hedge for \( S_T > 123.85 \) but for \( 110 < S_T < 123.85 \), the strategy does not provide any protection. Hence the maximum risk exposure is capped at 13.85, which is an improvement over the quantile hedging with an infinite risk exposure. For this reason, the expected shortfall hedging is considered more desirable than the quantile hedging.

In contrast, the CVaR-based hedging is not a “all-or-nothing” strategy. In fact it is truly a partial hedging strategy in the sense that whenever there is a loss, the hedging position is never perfect unless the hedging budget is large enough for implementing a perfect hedge. In other words, the hedger typically needs to absorb a certain level of loss whenever there is a payout from the option. The potential loss, however, is normally kept to a manageable level so that the tail risk is managed effectively. For instance, under the CVaR-based hedging the maximum loss exposure of the hedger is never more than 5.13 in case (i). This compares favorably to the quantile hedging with an infinite loss and the expected shortfall hedging with a maximum loss of 13.85.

The differences among the optimal loss functions also have important implication on the effectiveness of the hedge. By construction, each strategy is optimal to the designated criterion. It is therefore of great interest to evaluate the performance of these “optimal” strategies if other criteria have been used instead. Table 1 provides some insights. The values in the table are the resulting expected shortfall of the hedging strategy and the CVaR of the hedger’s total risk exposure for each of the five “optimal” hedging strategies. It is not surprising that the expected shortfall hedging and the CVaR-based hedging, respectively, lead to the smallest expected shortfall of the hedging strategy and the smallest CVaR of the hedger’s total risk exposure. It is, however, important to note that while the expected shortfall hedging has the smallest expected shortfall of the hedging strategy (by design), other hedging strategies also have expected shortfall that are close to the optimal value. This is desirable for other hedging strategies even though they are not specifically design to achieve this. In contrast, if the CVaR of the hedger’s total risk exposure were used to assess these strategies, the situation is quite different. In this case, other hedging strategies have a much larger CVaR than the optimal CVaR-based hedging strategy. Let us exemplify this by considering case (i). The CVaR of the hedger’s total risk exposure for other strategies ranges from 1.9 times to 4.4 times relative to the CVaR-based hedging strategy. This cautions the hedger that while other strategies may be optimal in their designated criteria, but when the optimal strategy is used to measure the CVaR of the total risk exposure, the resulting CVaR can be unreasonably large and hence expose the hedger to unexpectedly large loss exposure. This
also suggests the sensitivity of these hedging strategies and their ineffectiveness in managing the tail risk as measured by the CVaR. This phenomenon is even more pronounced when we consider the relative effectiveness of the CVaR-based hedging and the expected shortfall hedging on hedging a put option. We will demonstrate this in the next example.

Table 1: The resulting total risk exposure of hedging a call option

<table>
<thead>
<tr>
<th>Case</th>
<th>Expected shortfall of the hedging strategy</th>
<th>CVaR of hedger’s total risk exposure</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(i) (ii) (iii)</td>
<td>(i) (ii) (iii)</td>
</tr>
<tr>
<td>CVaR hedging</td>
<td>1.20 2.45 0.62</td>
<td>6.63 15.58 4.22</td>
</tr>
<tr>
<td>Quantile hedging</td>
<td>1.35 2.56 0.72</td>
<td>28.46 28.17 14.83</td>
</tr>
<tr>
<td>Knock-out call hedging</td>
<td>1.36 2.52 0.72</td>
<td>29.15 28.17 15.34</td>
</tr>
<tr>
<td>Bull call spread hedging</td>
<td>1.25 2.48 0.66</td>
<td>13.36 19.94 7.87</td>
</tr>
<tr>
<td>Expected shortfall hedging</td>
<td>1.16 2.43 0.60</td>
<td>12.76 21.07 7.48</td>
</tr>
</tbody>
</table>

Example 4.2. The set up of this example is similar to Case (i) of Example 4.1 (ie, $\sigma = 0.03$ and $\pi_0 = 1.5$) except that we are interested in hedging a European put option with payoff $X_T = (95 - S_T)^+$. Based on the assumed parameter values, the Black-Scholes put option price is $P_P = 3.6659$ and thus is is not possible to perfectly hedge the put option from the given budget $\pi_0 = 1.5$. The purpose of this example is to compare our proposed CVaR-based hedging to the expected shortfall hedging. For this example, the optimal hedged loss function becomes

$$ (95 - S_T)^+ \cdot 1_{(S_T \geq 83.0297)} $$ (4.1) 

for the expected shortfall hedging and

$$ (87.8215 - S_T)^+ $$ (4.2) 

for the CVaR-based hedging.

Note that with the put option, the hedger is concerned with declining stock prices. Hence to control the risk of large loss exposure, the hedger should pay special attention when the stock prices have depreciated substantially. We saw in the last example that the expected shortfall hedging strategy was effective at managing the tail risk of the call option by perfectly hedging large losses. What is striking (and counter-intuitive) about the expected shortfall hedging strategy in hedging the put option is that the optimal hedging function (4.1) suggests that it is optimal not to hedge large losses. More specifically, the optimal strategy dictated by the expected shortfall hedging is to only perfectly hedge the put option for $83.0297 \leq S_T \leq 95$ and is unhedged for $S_T < 83.0297$. Clearly this strategy the hedger exposed to undesirable potentially large losses. On the other hand, the optimal hedging strategy from the CVaR-based hedging is consistent with what we observed earlier. Whenever there is a payout from
the put option, hedger needs to incur some losses. However, the loss is never more than 7.1785, regardless of the level of the stock prices. In this aspect, the CVaR-based hedging can be considered as more effective in managing tail risk than the expected shortfall hedging.

The above phenomenon is further highlighted by comparing the resulting CVaR of the hedger’s risk exposure and the expected shortfall of the hedged position under both hedging strategies. The results, which provide additional support in favor of the CVaR-based hedging, are shown in Table 2. While the CVaR hedging is only designed to optimally minimize the CVaR of the hedger’s risk exposure, its expected shortfall of the hedging strategy is just slightly larger than the optimal expected shortfall hedging. In contrast, even though the expected shortfall hedging optimally minimizes the expected shortfall of the hedged position, the resulting optimal strategy gives rise to the CVaR of the hedger’s total risk exposure that is a few times larger than the corresponding CVaR-based hedging. This calls for a concern with the expected shortfall hedging strategy in that while the expected shortfall of the hedging strategy is optimally minimized, the resulting CVaR of the hedger’s risk exposure can be unexpectedly large. In fact, the CVaR of the hedger’s total risk exposure resulting from the expected shortfall hedging strategy is even larger than that without any hedging. The reason is that the expected shortfall hedging strategy does not hedge the risk of the option beyond its VaR level and thus the additional hedging budget leads to an increase in CVaR.

Table 2: The effectiveness of hedging a put option using the CVaR-based hedging and the expected shortfall hedging

<table>
<thead>
<tr>
<th>Hedging Strategy</th>
<th>CVaR</th>
<th>Expected Shortfall</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected shortfall of the hedging strategy</td>
<td>1.8479</td>
<td>1.6984</td>
</tr>
<tr>
<td>CVaR of hedger’s total risk exposure</td>
<td>8.6785</td>
<td>22.3548</td>
</tr>
</tbody>
</table>

The following example is designed to compare and contrast our proposed CVaR-based optimal partial hedging to the CVaR hedging strategy of Melnikov and Smirnov (2012). As pointed out earlier that it is a non-trivial exercise to obtain the optimal hedging strategy of Melnikov and Smirnov (2012) in the general case. For this reason, we only focus on the Black-Scholes model below where the optimal partial hedging is derived in Melnikov and Smirnov (2012).

**Example 4.3.** Similar to Example 4.1 and Example 4.2, we assume that the dynamics of the stock price $S_t$ at time $t$ are governed by

$$dS_t = S_t(\sigma dW_t + m dt)$$

and that we are interested in partial hedging a European call option with parameter values

$T = 0.25, \ K = 110, \ r = 0.05, \ S_0 = 100, \ \sigma = 0.3, \ \pi_0 = 1$

and the following two scenarios of drift coefficient $m$:
Under the Black-Scholes model, the price of the call option is independent of the drift of the stock price. Hence the price of the European call option is calculated to be \( P_C = 2.8444 \), irrespective of the drift coefficients.

Let us now investigate the optimal partial hedging of Melnikov and Smirnov (2012). The optimal form of their partial hedging depends on the relative magnitude of the risk-free rate \( r \) and the drift coefficient \( m \). This entails analyzing the optimal hedging strategy of Melnikov and Smirnov (2012) separately depending on the hypothetical scenarios, as shown below:

(i) For \( m = 0.06 \) so that this corresponds to the case \( r < m \). In this particular case, Melnikov and Smirnov (2012) demonstrates that the optimal hedging strategy is given by

\[
(H - \hat{z}_1)_+ \cdot \mathbb{I}(S_T > e^{rT} \cdot \tilde{b}_1(\hat{z}_1))
\]  

where \( \hat{z}_1 \) is the minimizer of function \( c_1(z) \) defined by

\[
c_1(z) = z + \frac{1}{1-\alpha} \cdot (S_0 \cdot e^{(m-r)T} \cdot \tilde{A}_+(K(z), \tilde{b}_1(z)) - K(z) \cdot \tilde{A}_-(K(z), \tilde{b}_1(z))),
\]

and \( \tilde{b}_1(z) \) is the solution for the following system

\[
\begin{align*}
S_0 \Phi_+(b) - K(z) \Phi_-(b) &= \pi_0 \\
b &\geq K(z)
\end{align*}
\]

In the above,

\[
K(z) = K \cdot e^{-rT} + z,
\]

\[
\tilde{A}_+(x,y) = \Phi_+(x \cdot e^{-(m-r)T}) - \Phi_+(y \cdot e^{-(m-r)T}),
\]

\[
\tilde{A}_-(x,y) = \Phi_-(x \cdot e^{-(m-r)T}) - \Phi_-(y \cdot e^{-(m-r)T}),
\]

\[
\Phi_+(x) = \Phi \left( \frac{\ln S_0 - \ln K}{\sigma \sqrt{T}} + 0.5 \sigma \sqrt{T} \right),
\]

\[
\Phi_-(x) = \Phi \left( \frac{\ln S_0 - \ln K}{\sigma \sqrt{T}} - 0.5 \sigma \sqrt{T} \right),
\]

and \( \Phi(\cdot) \) is the standard normal distribution. Solving the above system numerically, the optimal partial hedging strategy is in the form of \((S_T - 120.4472)_+\).

(ii) For \( m = 0.04 \) so that this corresponds to the case \( r > m \). In this scenario, the optimal partial hedging is given by

\[
(H - \hat{z}_2)_+ \cdot \mathbb{I}(S_T < e^{rT} \cdot \tilde{b}_2(\hat{z}_2))
\]  

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where \( \hat{z}_2 \) is the minimizer of \( c_2(z) \) defined by

\[
c_2(z) = z + \frac{1}{1 - \alpha} \cdot (S_0 \cdot e^{(m-r)T} \cdot \Phi_+(\tilde{b}_2(z) \cdot e^{-(m-r)T}) - K(z) \cdot \Phi_-(\tilde{b}_2(z) \cdot e^{-(m-r)T}),
\]

and \( \tilde{b}_2(z) \) is the solution for the following system

\[
\begin{align*}
S_0 \Lambda_+(K(z), b) - K(z) \Lambda_-(K(z), b) &= \pi_0 \\
b &\geq K(z).
\end{align*}
\]

In the above,

\[
\Lambda_+(x, y) = \Phi_+(x) - \Phi_+(y),
\]

\[
\Lambda_-(x, y) = \Phi_-(x) - \Phi_-(y),
\]

where \( K(z), \Phi_+(\cdot), \Phi_-(\cdot) \) and \( \Phi(\cdot) \) are defined in the previous scenario.

By solving the above system numerically, the optimal partial hedging strategy is found to be \((S_T - 120.4472)_+\).

It is interesting to note that in both cases, the optimal partial hedging strategies are identical though this is not true in general. More importantly, the above optimal form of the partial hedging satisfies all the assumptions of our proposed optimal partial hedging. This implies that our model should similarly deduce the same optimal partial hedging strategy. Indeed, it follows easily from Theorem 3.2 that for both cases \( m = 0.06 \) and \( m = 0.04 \), our proposed CVaR-based partial hedging is the same with the optimal hedged loss function given by \((S_T - K - d^*_r)_+ - (S_T - K - u^*_r)_+\), where \( d^*_r \) and \( u^*_r \) are determined by (3.12). Numerically solving (3.12) yields \( d^*_r = 10.4472 \) and \( u^*_r = \infty \) and the optimal hedged loss function is identical to that of Melnikov and Smirnov (2012).

We now draw the following remarks based on the above Example 4.3.

**Remark 4.1.** (a) The above example clearly demonstrates that the optimal partial hedging strategy of our proposed model is numerically much easier to obtain as it boils down to solving a two dimensional optimization problem given by (3.12). While we have assumed the Black-Scholes model in the above example, it should be emphasized if we were to consider a more sophisticated (and complex) model, the basic procedure of obtaining the optimal partial hedging strategy applies with the same level of complexity. In contrast, it can be very challenging to numerically obtain the optimal partial hedging strategy of Melnikov and Smirnov (2012), particularly if the model assumption deviates from the Black-Scholes model.

(b) The above example again ascertains that the functional form of our proposed CVaR-based partial hedging is robust with respect to the specification of the underlying asset price process. It is always a bull call spread hedging on the risk itself. On the contrary, the general functional form of the hedging strategy of Melnikov and Smirnov (2012) depends on the specific model for the underlying asset price process. Furthermore, as highlighted in (4.3) and (4.4) that the optimal hedging strategy of Melnikov and Smirnov (2012) is either a knock-in call or a knock-out call, depending on the relative magnitude of the interest rate \( r \) and the drift coefficient \( m \).
(c) It is easy to show that for the Black-Scholes model with \( m < r \), the optimal partial hedging strategy of Melnikov and Smirnov (2012) satisfies all the assumptions we imposed on the hedged loss function of our proposed CVaR-based hedging problem. Consequently, the optimal hedging strategy from both models are exactly the same, as confirmed in the numerical example above. Our approach, as opposed to Melnikov and Smirnov (2012), is easier and more flexible.

5 Conclusion

In this paper, we discuss how to optimally hedge a contingent claim that minimizes the CVaR of the hedger’s total risk exposure subject to a hedging cost constraint. Our results show that a bull call spread on the claim itself is optimal provided that the pricing functional preserves the stop-loss order. The optimal partial hedging problem consequently boils down to solving a two-dimensional optimization problem and thus is very tractable. We discuss the optimal partial hedging strategies with more in-depth under the utility based indifference pricing methods. Many numerical examples are provided to demonstrate how to partially hedge the European call and put options using our proposed CVaR-based hedging. The effectiveness of our proposed partial hedging strategy is compared with the other hedging strategies in the literature such as quantile hedging, VaR-based hedging, expected shortfall hedging and the CVaR-based partial hedging strategies proposed by Melnikov and Smirnov (2012). The results indicate that our proposed CVaR-based hedging has some competitive advantages in the sense of managing the tail risk of the hedger when it is compared to the quantile hedging, VaR-based hedging and expected shortfall hedging strategies. Relative to the CVaR-based hedging strategies by Melnikov and Smirnov (2012), our proposed CVaR-based partial hedging has the advantage of explicitness, tractability and transparency.

Appendix A

Proof of Proposition 2.1: We begin by assuming that an investor, with an initial cash endowment \( w \), has an increasing concave utility function \( U(\cdot) \). When the investor’s trading strategy is \( \xi \in \Xi \), where \( \Xi \) denotes the admissible set of trading strategies, the cash value of his dynamic portfolio at time \( t \) is denoted as \( W_w^\xi(t) \). The objective of the investor is to maximize his expected utility of wealth at the terminal time \( T \) as given below:

\[
V(w) = \sup_{\xi \in \Xi} \mathbb{E}[U(W_w^\xi(T))].
\]

Next, we assume that there exists two random variables \( X \) and \( Y \) satisfying \( \Pi(X) > \Pi(Y) \) and \( X \leq_{st} Y \) simultaneously. We will then complete the proof by contradiction. Indeed, if we let \( \xi^* \) be the optimal strategy of the investor, we have the following contradiction: for
small enough $\delta > 0,$
\begin{align*}
V(w) &= \mathbb{E} \left[ U(W_w^\xi^*(T)) \right] \\
&\geq \mathbb{E} \left[ U \left( W_{w+\delta \frac{\Pi(X)}{\Pi(Y)}}^\xi^*(T) - \frac{\delta}{\Pi(Y)} X \right) \right] \\
&\geq \mathbb{E} \left[ U \left( W_{w+\delta \frac{\Pi(X)}{\Pi(Y)}}^\xi^*(T) - \frac{\delta}{\Pi(Y)} Y \right) \right] \\
&\geq \mathbb{E} \left[ U \left( W_w^\xi^*(T) + \delta \left( \frac{\Pi(X)}{\Pi(Y)} - 1 \right) \right) \right] \\
&> \mathbb{E} \left[ U(W_w^\xi^*(T)) \right] \\
&= V(w).
\end{align*}

The first and the third inequalities are due to the optimality of $\xi^*$ while the second inequality follows from
\begin{equation}
-\mathbb{E} \left[ U \left( W_{w+\delta \frac{\Pi(X)}{\Pi(Y)}}^\xi^*(T) - \frac{\delta}{\Pi(Y)} X \right) \right] \leq -\mathbb{E} \left[ U \left( W_{w+\delta \frac{\Pi(X)}{\Pi(Y)}}^\xi^*(T) - \frac{\delta}{\Pi(Y)} Y \right) \right],
\end{equation}
which in turn follows from the increasing property and convexity of
\begin{equation}
- \left[ U \left( W_{w+\delta \frac{\Pi(X)}{\Pi(Y)}}^\xi^*(T) - \frac{\delta}{\Pi(Y)} X \right) \right],
\end{equation}
as a function of $X$ and the assumption $X \leqslant_{st} Y.$

**Proof of Lemma 3.1:** To show that function $g_f$ is well-defined, it is sufficient to verify the equation $\text{CVaR}_\alpha(R_f(X)) = \text{CVaR}_\alpha(R_{g_f}(X)).$ To this end, we first note that
\begin{equation}
R_{g_f}(x) = x - g_f(x) = x - \min \{(x-d)_+, \bar{u}\} \tag{A.1}
\end{equation}
where $d := \text{VaR}(X) - f(\text{VaR}(X)) \geq 0.$ From (A.1), $\text{CVaR}_\alpha(R_{g_f}(X))$ is continuous as a function of $\bar{u}.$ Moreover, when $\bar{u} = 0,$ $R_{g_f}(x) = x \geq R_f(x),$ and hence $\text{CVaR}_\alpha(R_{g_f}(X)) \geq \text{CVaR}_\alpha(R_f(X)).$ Thus, to show that $g_f$ is well defined, it is sufficient to establish
\begin{equation}
\lim_{\bar{u} \to \infty} \text{CVaR}_\alpha(R_{g_f}(X)) \leq \text{CVaR}_\alpha(R_f(X)). \tag{A.2}
\end{equation}
Indeed, since $R_f(x) = x - f(x)$ is nondecreasing and left continuous. Thus, using Theorem 1 in Dhaene et al. (2002), we have $R_f(\text{VaR}_\alpha(X)) = \text{VaR}_\alpha(R_f(X))$ and hence
\begin{align*}
\text{CVaR}_\alpha(R_f(X)) &\geq \text{VaR}_\alpha(R_fX) = R_f(\text{VaR}_\alpha(X)) \\
&= \text{VaR}_\alpha(X) - f(\text{VaR}_\alpha(X)) \\
&= d. \tag{A.3}
\end{align*}
Moreover,
\[
\lim_{u \to \infty} \text{CVaR}_\alpha(R_{g_f}(X)) = \text{CVaR}_\alpha \left( \lim_{u \to \infty} R_{g_f}(X) \right) = \text{CVaR}_\alpha(X - (X - d)_+) = \text{CVaR}_\alpha(\min\{X, d\}) \leq d.
\] (A.4)

Combining (A.3) and (A.4) yields (A.2).

To show \( g_f \in \Omega \), we need to demonstrate that \( R_{g_f}(x) \) is nondecreasing and left continuous as a function of \( x \). It is clear if we notice the expression (A.1). Thus, the proof is complete. □

Proof of Lemma 3.2: We only need to establish (3.2). Let \( u_\alpha \) denote a random variable uniformly distributed on \([0, \alpha]\), and assume it is independent of all other random variables involved in the paper. If

\[
g_f(VaR_{u_\alpha}(X)) \leq P_{sl} f(VaR_{u_\alpha}(X)) \tag{A.5}
\]

holds, then, for any \( d \in \mathbb{R} \), we have

\[
\mathbb{E}^P [(g_f(X) - d)_+] = \int_0^1 (g_f(VaR_s(X)) - d)_+ ds \\
= \int_0^\alpha (g_f(VaR_s(X)) - d)_+ ds + \int_\alpha^1 (g_f(VaR_s(X)) - d)_+ ds \\
= \alpha \mathbb{E}^P [(g_f(VaR_{u_\alpha}(X)) - d)_+] + \int_\alpha^1 (f(VaR_s(X)) - d)_+ ds \\
\leq \alpha \mathbb{E}^P [(f(VaR_{u_\alpha}(X)) - d)_+] + \int_\alpha^1 (f(VaR_s(X)) - d)_+ ds \\
= \int_0^1 (f(VaR_s(X)) - d)_+ ds \\
= \mathbb{E}^P [(f(X) - d)_+],
\]

which leads to the desired result (3.2). Thus, it remains to show (A.5).

To demonstrate (A.5), we shall use a well known sufficient condition for the stop-loss order (see, for example, Rolski et al. (1999)). For two random variables \( Z_1 \) and \( Z_2 \) with finite means, a sufficient condition for \( Z_1 \leq_{sl} Z_2 \) is as follows:

(i) \( \mathbb{E}^P[Z_1] \leq \mathbb{E}^P[Z_2] \), and

(ii) There exists \( t_0 \in \mathbb{R} \) such that \( \mathbb{P}(Z_1 \leq t) \leq \mathbb{P}(Z_2 \leq t) \) for \( t < t_0 \) while \( \mathbb{P}(Z_1 \leq t) \geq \mathbb{P}(Z_2 \leq t) \) for \( t > t_0 \).

Consequently, we only need to verify the above two conditions with \( Z_1 = g_f(VaR_{u_\alpha}(X)) \) and \( Z_2 = f(VaR_{u_\alpha}(X)) \). In fact, using the nonincreasing and left continuous property of
\( R_f(x) = x - f(x) \) as a function of \( x \) and Theorem 1 in Dhaene et al. (2002), we have

\[
\text{CVaR}_\alpha(R_f(X)) = \frac{1}{\alpha} \int_0^\alpha \text{VaR}_s(R_f(X)) ds
\]
\[
= \frac{1}{\alpha} \int_0^\alpha R_f(\text{VaR}_s(X)) ds
\]
\[
= \frac{1}{\alpha} \int_0^\alpha (\text{VaR}_s(X) - f(\text{VaR}_s(X))) ds
\]
\[
= \text{CVaR}_\alpha(X) - \mathbb{E}^P \left[ f(\text{VaR}_{u_\alpha}(X)) \right]
\]

and similarly

\[
\text{CVaR}_\alpha(R_{g_f}(X)) = \text{CVaR}_\alpha(X) - \mathbb{E}^P \left[ g_f(\text{VaR}_{u_\alpha}(X)) \right].
\]

The above results, combining with the fact that \( g_f \) is constructed such that \( \text{CVaR}_\alpha(R_f(X)) = \text{CVaR}_\alpha(R_{g_f}(X)) \), imply

\[
\mathbb{E}^P \left[ f(\text{VaR}_{u_\alpha}(X)) \right] = \mathbb{E}^P \left[ g_f(\text{VaR}_{u_\alpha}(X)) \right].
\]

This means that condition (i) in the above is met by \( Z_1 = g_f(\text{VaR}_{u_\alpha}(X)) \) and \( Z_2 = f(\text{VaR}_{u_\alpha}(X)) \).

To verify condition (ii), we first note that \( \text{VaR}_{u_\alpha}(X) \geq \text{VaR}_\alpha(X) \) and \( \text{VaR}_{u_\alpha}(X) - f(\text{VaR}_{u_\alpha}(X)) \geq \text{VaR}_\alpha(X) - f(\text{VaR}_\alpha(X)) \) due to the nondecreasing property of \( R_f(x) \), and consequently, for any \( t < \bar{u} \),

\[
\mathbb{P}(g_f(\text{VaR}_{u_\alpha}(X)) \leq t) = \mathbb{P}(\text{VaR}_{u_\alpha}(X) - \text{VaR}_\alpha(X) + f(\text{VaR}_\alpha(X)) \leq t)
\]
\[
\leq \mathbb{P}(f(\text{VaR}_{u_\alpha}(X)) \leq t), \quad (A.6)
\]

where the equality is due to the construction (3.1) for \( g_f \). Moreover, for any \( t > \bar{u} \), the construction (3.1) of \( g_f \) implies that

\[
\mathbb{P}(g_f(\text{VaR}_{u_\alpha}(X)) \leq t) = 1 \geq \mathbb{P}(f(\text{VaR}_{u_\alpha}(X)) \leq t).
\]

By (A.6) and (A.7), condition (ii) is also satisfied by \( Z_1 = g_f(\text{VaR}_{u_\alpha}(X)) \) and \( Z_2 = f(\text{VaR}_{u_\alpha}(X)) \), and thus the proof is complete. \( \square \)

**Proof of Proposition 3.1:** We first note that \( \text{CVaR}_\alpha([X - u]_+) = \frac{1}{\alpha} \mathbb{E} \left[ (X - u)_+ \right] \) for \( u \geq v \). Thus, problem (3.7) reduces to (3.9) if we confine to \( u \geq v \). Consequently, it suffices to show that that the optimal value of problem (3.7) for \( u < v \) cannot be smaller than that for \( u = v \).

We now consider a generic point \((d_1, u_1)\) from the feasible set of problem (3.7) with \( u_1 < v \). Let \( d_2 \) be a number satisfying the following equation

\[
\Pi(G(X; d_2, v)) = \Pi(G(X; d_1, u_1)),
\]

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where $G$ is defined in (3.5). The last equation is obviously equivalent to
\[
\int_{d_1}^{u_1} Q(X > x)dx = \int_{d_2}^{v} Q(X > x)dx,
\]
which, along with the fact that $Q(X > x)$ is decreasing as function of $x$, further implies $v - d_2 \geq u_1 - d_1$. Consequently,
\[
d_2 + CVaR_{\alpha} [(X - v)_+] = d_2 + CVaR_{\alpha}(X) - v \\
\leq d_1 + CVaR_{\alpha}(X) - u_1 \\
= d_1 + CVaR_{\alpha} [(X - u_1)_+].
\]
This means that the optimal value of the problem (3.7) for $u < v$ cannot be smaller than that for $u = v$. Thus, the proof is complete. \hfill \Box

**Proof of Lemma 3.3:** Let $(d^*, u^*)$ be one optimal solution to problem (3.9). We will complete the proof by contradiction. We first note that the objective function in problem (3.7) is nondecreasing as a function of $d$ as indicated below:
\[
\frac{\partial}{\partial d} \left( d + \frac{1}{\alpha} E^P(X - u)_+ + \int_{d}^{u} Q(X > x)dx \right) = 1 - Q(X > d) \geq 0.
\]
Thus, if the constraint in problem (3.9) is loose at $(d^*, u^*)$, we would have $d^* \in [0, \tilde{d})$, and consequently
\[
e^{-rT} E^Q [(X - d^*)_+ - (X - u^*)_+] < \pi_0 \leq e^{-rT} E^Q \left[ (X - \tilde{d})_+ - (X - v)_+ \right].
\]
It follows from the last inequality that
\[
E^Q [(X - u^*)_+] - E^Q [(X - v)_+] > E^Q [(X - d^*)_+] - E^Q \left[ (X - \tilde{d})_+ \right] \geq 0
\]
and hence $u^* < v$, which contradicts the assumption that $(d^*, u^*)$ is an optimal solution to problem (3.9). Hence the proof is complete. \hfill \Box

**References**


