VaR-based Optimal Partial Hedging

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Abstract. Hedging is one of the most important topics in finance. When a financial market is complete, every contingent claim can be hedged perfectly to eliminate any potential future obligations. When the financial market is incomplete, the investor may eliminate his risk exposure by superhedging. In practice, both hedging strategies are not satisfactory due to their high implementation costs which erode the chance of making any profit. A more practical and desirable strategy is to resort to the partial hedging which hedges the future obligation only partially. The quantile hedging of Föllmer and Leukert (1999), which maximizes the probability of a successful hedge for a given budget constraint, is an example of the partial hedging. Inspired by the principle underlying the partial hedging, this paper proposes a general partial hedging model by minimizing any desirable risk measure of the total risk exposure of an investor. By confining to the Value-at-Risk (VaR) measure, analytic optimal partial hedging strategies are derived. The optimal partial hedging strategy is either a knock-out call strategy or a bull call spread strategy, depending on the admissible classes of hedging strategies. Our proposed VaR-based partial hedging model has the advantage of its simplicity and robustness. The optimal hedging strategy is easy to determine. Furthermore, the structure of the optimal hedging strategy is independent of the assumed market model. This is in contrast to the quantile hedging which is sensitive to the assumed model as well as the parameter values. Extensive numerical examples are provided to compare and contrast our proposed partial hedging to the quantile hedging.
Key words. Quantile hedging, partial hedging, optimal strategy, Value-at-Risk (VaR), bull call spread, knock-out call

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1 Introduction

Under the classical option pricing theory, when the market is complete the payout of any contingent claim can be duplicated perfectly by a self-financing portfolio and this gives rise to the so-called perfect hedging strategy. When the market is incomplete, the perfect hedging is typically not possible and the superhedging strategy has been proposed as an alternative. The superhedging strategy involves seeking a cheapest self-financing portfolio with payout no smaller than that of the contingent claim in all scenarios. While superhedging ensures that the hedger always has sufficient fund to cover his future obligation arising from the sale of the contingent claim, the strategy, however, is too costly to be of practical interest. A more desirable strategy is to resort to the partial hedging which hedges the future obligation only partially. By relaxing the objective of perfectly hedging or superhedging the future obligation, the initial cost of the strategy (i.e. the hedging budget) is typically smaller than the proceeds from the sale of the contingent claim. This implies that such a strategy creates an opportunity of making a profit. The pioneering work of optimal partial hedging is attributed to Föllmer and Leukert (1999) who propose a hedging strategy that maximizes the probability of meeting the future obligation under a given budget constraint. This strategy is commonly known as the quantile hedging.

The classical quantile hedging has been generalized in a number of interesting directions. One extension is to study the quantile hedging under more sophisticated market structures. For example, Spivak and Cvitanić (1999) study the problem of quantile hedging and re-derive the complete market solution by using a duality method. They also demonstrate how to modify their approach to deal with the problem in a market with partial information. They define a market with partial information as a market where the hedger only knows a prior distribution of the vector of returns of the risky assets. Krutchenko and Melnikov (2001) study the quantile hedging strategy under a special case of jump-diffusion market. They obtain the hedging strategy by deducing the corresponding stochastic differential equation. Bratyk and Mishura (2008) consider an incomplete market with finite number of independent and fractional Brownian motions. In particular, they estimate the successful probability for quantile hedging when the price process is modeled by two Wiener processes and two fractional Brownian motions.

Another extension is to investigate the partial hedging strategies using some other optimization criteria, as opposed to maximizing the probability of a successful hedge as in the quantile hedging. The optimal partial hedging in Föllmer and Leukert (2000), for example, takes into account the magnitude of the shortfall, instead of the probability of its occurrence. They use a loss function to describe the hedger's attitude towards the shortfall and derive the optimal hedging strategy. Nakano (2004) attempts to minimize some coherent risk measures of the shortfall under a similar model setup as in Föllmer and Leukert (1999). Nakano (2007) represents the risk measure as the expected value of the loss under a certain probability measure, and then addresses the optimization problem by constructing the most powerful test in a way similar to Föllmer and Leukert (2000). Rudloff (2007) considers similar hedging problem in an incomplete market by using convex risk measures. More recently, Melnikov and Smirnov (2012) study the optimal hedging strategies by minimizing the conditional value-at-risk of the portfolio in a complete market. By exploiting the results from Föllmer and Leukert (2000) and Rockafellar and Uryasev (2002), they derive some semi-explicit solutions. Many other generalizations can be seen in Cvitanić (2000), Nakano (2007), Sekine (2004) and references therein. The quantile hedging has also been successfully applied to a variety of specific financial and insurance contracts; see, for example, Sekine (2000), Melnikov and Skornyakova (2005), Wang (2009) and Klusik and Palmowski (2011).

In this paper, a general risk measure based optimal partial hedging model is first proposed. Then by confining to a special case which involves minimizing the Value-at-Risk (VaR) of the total exposed risk of a hedger for a given hedging budget constraint, analytic solutions under two admissible sets of hedging strategies (see Section 2 for their definitions and justifications) are derived. Our results indicate that the optimal hedging strategy is either a knock-out call hedging or a bull call spread hedging, depending on the prescribed admissible set. Our approach in solving the partial hedging problem differs from the existing literature in at least the following aspects. First, while most existing literature typically formulate the problem as one of identifying the most powerful test, we achieve the objective by first investigating an optimal partition between the hedged loss and the retained loss, and then analyzing the specific hedging strategy. This approach is commonly employed in the context of optimal reinsurance; see, for example, Cai et al. (2008), Tan and Weng (2011), Tan et al. (2011) and Chi and Tan (2011, 2013). Second, while the structure of optimal solutions obtained in the existing literature usually depends on the specific form of the market model, the structure of optimal strategies derived under our framework is independent of the market model. For example, the optimal quantile hedging strategy highly depends on the parameter values of the assumed market model. It is usually challenging to obtain an analytic form of the optimal quantile hedging for a market model which is more complicated than the Black-Scholes model. In contrast, our proposed model is tractable and it is relatively easy to derive its optimal partial hedging strategy. Moreover, the optimal partial hedging strategy under our framework is robust in the sense that it is either a bull call spread or a knock-out call strategy, and is independent of the market model. Third, in many cases our optimal solutions involve hedging an instrument which has the same structure (with different parameter values though) as the risk we are aiming to hedge partially. If such an instrument is available in the market, then we are able to achieve our objective by a simple static hedging strategy; otherwise, the hedging budget can be used to construct a portfolio to dynamically replicate the payout of such an instrument.

The rest of the paper is organized as follows. Section 2 introduces our proposed risk measure based partial hedging model and provides some justifications on the admissible sets of hedging strategies that we are analyzing. Section 3 derives the optimal solutions to the proposed VaR-based partial hedging model. Section 4 consists of two subsections. The first subsection gives some numerical examples to illustrate our proposed partial hedging strategies. The second subsection compares and contrasts our proposed partial hedging to the quantile hedging. Section 5 concludes the paper.

2 Risk Measure Based Partial Hedging Model

2.1 Model Description

We suppose that a hedger is exposed to a future obligation X at time T and that his objective is to hedge X. We emphasize that X is not necessarily the payout of a European option. It can be any functional of a specific stock price process, i.e. $X = H(S_t, 0 \leqslant t \leqslant T)$, where S_t denotes the time-t price of a stock and H is a functional. Without any loss of generality, we assume that X is a non-negative random variable with cumulative distribution function (c.d.f.) $F_X(x) = \mathbb{P}(X \leqslant x)$ and $\mathbb{E}(X) < \infty$ under the physical probability measure $\mathbb{P}$. The exact expression of the c.d.f. $F_X(x)$ may be unknown.

Our approach of addressing the optimal partial hedging problem is conducted in two steps. In the first step, we study the optimal partitioning of X into $f(X)$ and $R_f(X)$; i.e. $X = f(X) + R_f(X)$. Here $f(X)$ denotes the part of the payout to be hedged with an initial capital budget, and $R_f(X)$ represents the part of the payout to be retained. We use π_0 to denote the initial hedging budget. As functions of x, we call $f(x)$ and $R_f(x)$ the hedged loss function and the retained loss function respectively. In the second step, we investigate the possibility of replicating the time-T payout $f(X)$ in the market.

Let Π denote the risk pricing functional so that $\Pi(X)$ is the time-0 market price of the contingent claim with payout X at time T. Similarly, $\Pi(f(X))$ is the time-0 market price of $f(X)$ and this also corresponds to the time-0 cost of hedging $f(X)$. In this paper, we do not need to specify the pricing functional $\Pi(\cdot)$, but we assume that it admits no arbitrage opportunity in the market.

Assuming the initial cost of hedging $f(X)$ accumulates with interest at a risk-free rate r, then $T_f(X)$, which is defined as

$$T_f(X) = R_f(X) + e^{rT}\Pi(f(X)), \tag{2.1}$$

can be interpreted as the hedger's total time-T risk exposure from implementing the partial hedge strategy since $R_f(X)$ denotes the time-T retained risk exposure. Note that $T_f(X)$ also succinctly captures the risk and reward tradeoff of the partial hedging strategy. On the one hand, if the hedger is more conservative in that he is willing to spend more on hedging, then a greater portion of the initial risk will be hedged so that the retained risk $R_f(X)$ will be smaller. On the other hand, if the hedger is more aggressive in that he is willing to spend less on hedging, then this can be achieved at the expense of a higher retained risk exposure $R_f(X)$. Consequently, the problem of partial hedging boils down to the optimal partitioning of X into $f(X)$ and $R_f(X)$ for a given hedging budget constraint π_0, and one possible formulation of the optimal partial hedging problem can be described as follows:

$$\begin{cases} \min\limits_{f \in \mathcal{L}} & \rho(T_f(X)) \\ \text{s.t.} & \Pi(f(X)) \leqslant \pi_0, \end{cases} \tag{2.2}$$

where $\rho(\cdot)$ is an appropriately chosen risk measure for quantifying the total risk exposure $T_f(X)$ and $\mathcal{L}$ denotes an admissible set of hedged loss functions.

We emphasize that the risk measure based partial hedging model (2.2) is quite general in that it permits an arbitrary risk measure as long as it reflects and quantifies the hedger's attitude towards risk. Risk measures such as the Value-at-Risk (VaR), the conditional value-at-risk (CVaR), expected shortfall, among many others, are reasonable choices. In this paper, we focus on the optimal partial hedging by considering VaR. This is motivated by its popularity among financial institutions, insurance companies and regulatory authorities for quantifying risk (Jorion 2006). Formally, VaR is defined as follows:

Definition 2.1. *The VaR of a non-negative variable X at the confidence level $(1-\alpha)$ with $0<\alpha<1$ is defined as*

$$VaR_\alpha(X)=\inf\{x\geqslant 0:\mathbb{P}(X>x)\leqslant\alpha\}.$$

Remark 2.1. *It is instructive to compare the risk measure based partial hedging framework* (2.2) *to the quantile hedging. Recall that the quantile hedging maximizes the probability of meeting the future obligation for a given budget constraint, with a formal mathematical formulation as follows:*

$$\begin{cases}\max\limits_{f\in\mathcal{L}} & \mathbb{P}(f(X)\geqslant X)\\ s.t. & \Pi(f(X))\leqslant\pi_0,\end{cases}\tag{2.3}$$

where $\mathbb{P}$ is the physical probability measure for the financial market. A comprehensive example will be provided in Subsection 4.2 to further highlight the difference between these two partial hedging frameworks.

Remark 2.2. *If we were to interpret X as the risk exposure faced by an insurer so that $T_f(X)$ becomes the insurer's total risk exposure, $\Pi(\cdot)$ as the premium principle adopted by the reinsurer, and π_0 as the budget the insurer is willing to spend on transferring part of his risk to the reinsurer, then the optimization model* (2.2) *corresponds to an optimal reinsurance model. The objective in this case is to determine an optimal reinsurance policy $f(X)$ that minimizes the insurer's total risk exposure; see, for example, Cai et al. (2008), Tan et al. (2011), Tan and Weng (2012), and Chi and Tan (2011, 2013).*

2.2 Admissible Sets Of Hedged Loss Functions

In addition to specifying the risk measure ρ in model (2.2), we also need to define the admissible set $\mathcal{L}$; otherwise, the formulation is ill-posed in that a position with an infinite number of certain assets (long or short) in the market is optimal. Similar issue has been observed in the quantile hedging and the CVaR hedging, and a standard technique of alleviating this issue is to impose some additional conditions or constraints on the optimization problem. For example, the hedged loss functions in both the quantile hedging of Föllmer and Leukert (1999) and the CVaR dynamic hedging of Melnikov and Smirnov (2012) are restricted to be nonnegative. Alexander, et al. (2004), on the other hand, introduce an additional term (which reflects the cost of holding an instrument) to the objective function in a CVaR-based hedging problem.

Before specifying the admissible sets of the hedged loss function, we now consider the following properties:

P1. Not globally over-hedged: $f(x) \leqslant x$ for all $x \geqslant 0$.

P2. Not locally over-hedged: $f(x_2) - f(x_1) \leqslant x_2 - x_1$ for all $0 \leqslant x_1 \leqslant x_2$.

P3. Nonnegativity of the hedged loss: $f(x) \geqslant 0$ for all $x \geqslant 0$.

P4. Monotonicity of the hedged loss function: $f(x_2) \geqslant f(x_1)\ \forall\ 0 \leqslant x_1 \leqslant x_2$.

Note that property **P2** is equivalent to the following

P2'. Monotonicity of the retained loss function: $R_f(x_2) \geqslant R_f(x_1)\ \forall\ 0 \leqslant x_1 \leqslant x_2$.

In this paper, we analyze the optimal partial hedging strategy under two overlapping admissible sets of hedged loss functions. The first set assumes that the hedged loss functions satisfies properties **P1-P3** while the second set imposes property **P4** in addition to **P1-P3**. Without loss of too much generality we assume that the retained loss function $R_f(x)$ is left continuous with respect to x. These two admissible sets, with formal definitions given below, are labeled as $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively:

$$\mathcal{L}_1 = \{0 \leqslant f(x) \leqslant x : R_f(x) \equiv x - f(x) \text{ is a nondecreasing and left continuous function}\}, \tag{2.4}$$

$$\mathcal{L}_2 = \{0 \leqslant f(x) \leqslant x : \text{both } R_f(x) \text{ and } f(x) \text{ are nondecreasing functions, } R_f(x) \text{ is left continuous}\}. \tag{2.5}$$

Note that $\mathcal{L}_2 \subset \mathcal{L}_1$.

We now provide some justifications on the above properties for the hedged loss functions. Property **P1** is reasonable as it ensures that the hedged loss should be uniformly bounded from above by the original risk to be hedged. Property **P2** indicates that the increment of the hedged part should not exceed the increment of the risk itself. Imposing **P2** implies that the retained loss function is nondecreasing. While imposing **P2** makes the admissible set of the hedging functions more restrictive, it is reassuring from the numerical examples to be presented in Subsection 4.2 that the expected shortfall of the optimal partial hedging strategy is still significantly smaller than that under the quantile hedging strategy. Moreover, it will become clear shortly that with property **P2**, the resulting optimal partial hedging strategy will be model independent. This means that the structure of the optimal hedging strategy remains unchanged irrespective of the assumptions on the dynamics of the underlying asset price. Part (b) of Remarks 3.1 and 3.2 in Section 3 will further elaborate this point.

We note that it is possible to relax property **P2** to a relatively weaker condition of the form

$$R_f(x_2) \geqslant R_f(VaR_\alpha(X)) \geqslant R_f(x_1)\ \forall 0 \leqslant x_1 \leqslant VaR_\alpha(X) \leqslant x_2$$

where $1 - \alpha$ is the confidence level adopted by the hedger. This can be accomplished by a simple modification on the proof of our main results in Theorem 3.1 and Theorem 3.3.

Property **P3** is not only commonly imposed in the quantile hedging, its importance is further highlighted in the following example which shows that the partial hedging problem (2.2) is still ill-posed if we only impose properties **P1** and **P2**.

Example 2.1. *Suppose we wish to partially hedge a payout X, which is nondecreasing as a function of the stock price S so that S is nondecreasing in X as well. Take a constant K_0 large enough such that $K_0 > VaR_\alpha(S)$, and consider the hedged loss $f_n(X) = -n(S - K_0)_+$ indexed by positive integers n. Clearly, in this case both properties* **P1** *and* **P2** *are satisfied by the hedged loss $f_n(X)$. Since $K_0 > VaR_\alpha(S)$ and X is nondecreasing in S, $VaR_\alpha(X) = VaR_\alpha(X - f_n(X))$ for any $n > 0$, which implies that, if we do not consider the premium received by the hedger, the payout of selling the call option with strike price K_0 will not affect the VaR of the hedger. Therefore, by selling one unit of the call option with strike price K_0, the hedger can decrease his VaR by the premium he receives, which is the price of the call option. It follows that the more units the hedger sells the call options, the smaller the VaR of his total exposed risk. In this case, the optimal hedging strategy is to sell an infinite units of the call options with strike price K_0. With this hedging strategy, the VaR of the hedger's total exposed risk is negative infinity. However, such a hedging strategy is not a desirable hedging strategy as it is obviously is a kind of gamble.* □

Remark 2.3. *(a) In Example 2.1, selling the call option on the stock S with strike price K_0 is not the only choice to decrease the VaR of the hedger's total exposed risk. In fact, selling any contract whose payout is zero with probability larger than $1 - \alpha$ is able to decrease the VaR of the hedger's total exposed risk.*

(b) Example 2.1 indicates that if we only impose properties **P1** *and* **P2***, the optimal hedging strategy is to sell as many "lotteries" as possible. Here, the term "lottery" refers to a financial contract whose payout is zero with very high probability (larger than $1 - \alpha$ in the above example).*

(c) The situation illustrated above is not unique to the VaR-based partial hedging model. It also occurs in the context of quantile hedging; see equation (2.3) *of Föllmer and Leukert (1999).*

We assume that the hedger's primary objective is to hedge the payout X rather than gambling. Consequently selling "lottery" is not an acceptable partial hedging strategy. This situation can be avoided by imposing some additional constraints on the admissible set $\mathcal{L}$, in addition to properties **P1** and **P2**. This leads to property **P3**; the same condition is also imposed in Föllmer and Leukert (1999) to eliminate the ill-posedness of the quantile hedging problem.

Apart from analyzing the optimal hedging strategy under properties **P1-P3**, we are also interested in the optimal solution of the partial hedging problem by imposing the monotonicity condition on the hedged loss function (i.e. property **P4**). By doing so, the admissible set $\mathcal{L}_2$ is even more restrictive than the admissible set $\mathcal{L}_1$. However, the monotonicity condition of the hedged loss function is crucial, especially if the hedger has a greater concern with the tail risk. Property **P4** ensures that the protection level will not decline as the risk exposure X gets larger. Without such a condition, it is possible for the hedger to have some or full protection for small losses and yet no protection against the extreme losses. This phenomenon seems counter-intuitive, particularly from the risk management point of view. We will further highlight this situation in the numerical examples in Subsection 4.2.

3 VaR-based Optimal Partial Hedging

Recall that our proposed optimal partial hedging model corresponds to the optimization problem (2.2). By using VaR as the relevant risk measure ρ for a given confidence level $1-\alpha \in (0,1)$, the objective of this section is to identify the solution to the optimization problem (2.2) under either the admissible set $\mathcal{L}_1$ as defined in (2.4) or $\mathcal{L}_2$ as defined in (2.5). These two cases are discussed in details in Subsections 3.1 and 3.2 respectively. As indicated in Remark 2.2, the connection between our proposed optimal partial hedging model and the optimal reinsurance model enables us to employ a similar technical approach as in Chi and Tan (2011, 2013) to derive the optimal hedged loss function. Nevertheless, it is worth noting that Chi and Tan (2011, 2013) obtain the optimal reinsurance policy, respectively, for the expectation premium principle and over a specified class of premium principles that preserve stop-loss order, while the optimal hedged loss function in this paper is determined for an arbitrary pricing functional Π which it admits no arbitrage opportunity in the market.

3.1 Optimality Of The Knock-out Call Hedging

This subsection focuses on the VaR-based optimal partial hedging problem under the admissible set $\mathcal{L}_1$ as defined in (2.4). We will show that the so-called knock-out call hedging is optimal among all the hedging strategies in $\mathcal{L}_1$. We achieve this objective by demonstrating that given any partial hedging strategy f from the admissible set $\mathcal{L}_1$, the knock-out call hedging strategy g_f constructed from f leads to a smaller VaR of the total risk exposure of the hedger. More precisely, suppose g_f is constructed from $f \in \mathcal{L}_1$ as follows:

$$g_f(x) = \begin{cases} (x+f(v)-v)_+ \quad , & \text{if} \quad 0 \leqslant x \leqslant v, \\ \\ 0, & \text{if} \quad x > v, \end{cases} \tag{3.6}$$

where $v = \mathrm{VaR}_\alpha(X)$ and $(x)_+$ equals to x if $x > 0$ and zero otherwise. We first note that for any $f \in \mathcal{L}_1$, the function g_f constructed according to (3.6) is an element in $\mathcal{L}_1$. Second, for an arbitrary choice of f, $g_f(X)$ is the knock-out call option written on X with retention level $v - f(v)$ and knock-out barrier v. For any given hedged loss function $f \in \mathcal{L}_1$, (3.6) provides a corresponding hedged loss function $g_f \in \mathcal{L}_1$ in the form of a knock-out call hedging strategy. If we can demonstrate that the hedged loss function g_f outperforms the hedged loss function f in the sense that former function results in a smaller VaR of the hedger's risk exposure, then we can conclude that the knock-out call hedging g_f is optimal among all the admissible strategies in $\mathcal{L}_1$. The following Theorem 3.1 confirms our assertion.

Theorem 3.1. *Assume that the pricing functional* Π *admits no arbitrage opportunity in the market. Then, the knock-out call hedged function* g_f *of the form* (3.6) *satisfies the following properties: for any* $f \in \mathcal{L}_1$,

(a) $\Pi(f(X)) \leqslant \pi_0$ *implies* $\Pi(g_f(X)) \leqslant \pi_0$*, and*

(b) $VaR_\alpha(T_{g_f}(X)) \leqslant VaR_\alpha(T_f(X))$.

Proof: (a) It follows from properties **P1-P3** that, for any $f \in \mathcal{L}_1$,

$$f(x) \geqslant (x + f(v) - v)_+ = g_f(x), \forall\ 0 \leqslant x \leqslant v$$

and

$$g_f(x) = 0 \leqslant f(x), \forall\ x > v.$$

Thus, $g_f(x) \leqslant f(x), \forall\ x \geqslant 0$ and the no arbitrage assumption implies $\Pi(g_f(X)) \leqslant \Pi(f(X))$, which in turn leads to the required result.

(b) The translation invariance property of the VaR risk measure leads to

$$\begin{aligned}
\mathrm{VaR}_\alpha(T_f(X)) &= \mathrm{VaR}_\alpha(R_f(X)) + \Pi(f(X)) \\
&= R_f(\mathrm{VaR}_\alpha(X)) + \Pi(f(X)) \\
&= \mathrm{VaR}_\alpha(X) - f(\mathrm{VaR}_\alpha(X)) + \Pi(f(X)) \\
&\geqslant \mathrm{VaR}_\alpha(X) - h_f(\mathrm{VaR}_\alpha(X)) + \Pi(h_f(X)) \\
&= \mathrm{VaR}_\alpha(T_{h_f}(X)),
\end{aligned}$$

where the second equality is due to the left continuity and nondecreasing properties of $R_f(x)$ and Theorem 1 in Dhaene et al. (2002). □

Remark 3.1. *(a) Theorem 3.1 indicates that the knock-out call hedging strategy is optimal among all the strategies in* $\mathcal{L}_1$*. We note that the optimal knock-out call hedging is the one written on the risk* X *itself, instead of a barrier option written on the asset that underlies the risk* X*.*

(b) The optimality of the knock-out call hedging is model independent. Even though the market is incomplete, as long as the market admits no arbitrage opportunity, the knock-out call hedging strategy given in Theorem 3.1 is always optimal.

(c) If the knock-out call option on the risk X *is available from the financial market, then the optimal partial hedging strategy can easily be implemented via a simple static hedging strategy without the need of rebalancing. The numerical examples in Subsection 4.1 will exemplify this point.*

If we denote $d = \mathrm{VaR}_\alpha(X) - f(\mathrm{VaR}_\alpha(X)) = v - f(v)$, then the knock-out call function g_f defined in (3.6) can be succinctly represented as

$$g_f(x) = (x - d)_+ \cdot \mathbb{I}(x \leqslant v),$$

where $\mathbb{I}(\cdot)$ is the indicator function. Furthermore, it follows from Theorem 3.1 that the VaR-based partial hedging problem (2.2) can equivalently be rewritten as

$$\begin{cases} \min\limits_{0 \leqslant d \leqslant v} & \mathrm{VaR}_\alpha \left(X - (X-d)_+ \cdot \mathbb{I}(X \leqslant v) + e^{rT} \cdot \Pi \left[g_f(X) \right] \right) \\ \\ \text{s.t.} & \Pi[g_f(X)] \equiv \Pi \left[(X-d)_+ \cdot \mathbb{I}(X \leqslant v) \right] \leqslant \pi_0. \end{cases} \tag{3.7}$$

This is simply an optimization problem of only one variable and technically it is easily solved as demonstrated in the following theorem.

Theorem 3.2. *Assume that the pricing functional* Π *admits no arbitrage opportunity in the market.*

(a) If the hedging budget $\pi_0 \geqslant \Pi \left[X \cdot \mathbb{I}(X \leqslant v) \right]$*, then the optimizer to problem* (3.7) *is* $d^* = 0$ *and the corresponding minimal VaR of the hedger's total risk exposure at time* T *is* $e^{rT} \cdot \Pi \left[X \cdot \mathbb{I}(X \leqslant v) \right]$.

(b) If the hedging budget $\pi_0 < \Pi \left[X \cdot \mathbb{I}(X \leqslant v) \right]$*, then the optimizer to problem* (3.7) *is given by the solution* d^* *to the following equation*

$$\Pi \left[(X-d)_+ \cdot \mathbb{I}(X \leqslant v) \right] = \pi_0, \tag{3.8}$$

and the corresponding minimal VaR of the hedger's total risk exposure at time T *is* $d^* + e^{rT} \cdot \pi_0$.

Proof: First note that the pricing formula $e^{-rT}\mathbb{E}^{\mathbb{Q}} \left[(X-d)_+ \cdot \mathbb{I}(X \leqslant v) \right]$ in the constraint is clearly nonincreasing in d. Thus, it is sufficient to show that the objective in problem (3.7) is nondecreasing in d as well.

Let $B(x) = x - (x-d)_+ \cdot \mathbb{I}(x \leqslant v)$. Then, the objective in problem (3.7) can be expressed as $\mathrm{VaR}_\alpha(B(X) + e^{rT} \cdot \Pi(g_f(X)))$. Moreover, the function B is obviously left continuous and nondecreasing, and hence a direct application of Theorem 1 in Dhaene et at. (2002) implies that $\mathrm{VaR}_\alpha(B(X)) = B(\mathrm{VaR}_\alpha(X))$, which, together with the fact that $d < \mathrm{VaR}_\alpha(X) \equiv v$, leads to

$$\begin{aligned} \mathrm{VaR}_\alpha(B(X)) &= \mathrm{VaR}_\alpha \left(X - (X-d)_+ \cdot \mathbb{I}(X \leqslant v) \right) \\ &= \mathrm{VaR}_\alpha(X) - [\mathrm{VaR}_\alpha(X) - d]_+ \\ &= d. \end{aligned}$$

Consequently, the translation invariance property of VaR implies equivalence between the objective function in (3.7) and the following expression

$$\mathrm{VaR}_\alpha(B(X) + e^{rT} \cdot \Pi(g_f(X))) = d + e^{rT} \cdot \Pi \left[(X-d)_+ \cdot \mathbb{I}(X \leqslant v) \right]. \tag{3.9}$$

Hence, it remains to show that the right-hand-side of (3.9) is indeed nondecreasing in d. We verify this by contradiction.

Assume that (3.9) is strictly decreasing in d. Then, there must exist two constants d_1 and d_2 satisfying $d_1 < d_2$ and

$$d_1 + e^{rT} \cdot \Pi\left[(X-d_1)_+ \cdot \mathbb{I}(X \leqslant v)\right] > d_2 + e^{rT} \cdot \Pi\left[(X-d_2)_+ \cdot \mathbb{I}(X \leqslant v)\right]. \tag{3.10}$$

Indeed, this condition implies an arbitrage opportunity which can be exploited by constructing the following portfolio:

(i) selling the contract $(X-d_1)_+ \cdot \mathbb{I}(X \leqslant v)$,

(ii) buying the contract $(X-d_2)_+ \cdot \mathbb{I}(X \leqslant v)$,

(iii) putting the net premium $\Delta := \Pi\left[(X-d_1)_+ \cdot \mathbb{I}(X \leqslant v)\right] - \Pi\left[(X-d_2)_+ \cdot \mathbb{I}(X \leqslant v)\right]$ in the bank account to earn interest at a constant rate r.

Since Π is assumed to admit no arbitrage opportunity, we must have $\Delta \geqslant 0$, which means that there is no initial cost to create the above portfolio. Nevertheless, its payoff at the expiration date T is positive almost surely as shown below:

$$\begin{aligned}
&e^{rT} \cdot \Pi\left[(X-d_1)_+ \cdot \mathbb{I}(X \leqslant v)\right] - e^{rT} \cdot \Pi\left[(X-d_2)_+ \cdot \mathbb{I}(X \leqslant v)\right] \\
&+(X-d_2)_+ \cdot \mathbb{I}(X \leqslant v) - (X-d_1)_+ \cdot \mathbb{I}(X \leqslant v) \\
\geqslant& e^{rT} \cdot \Pi\left[(X-d_1)_+ \cdot \mathbb{I}(X \leqslant v)\right] - e^{rT} \cdot \Pi\left[(X-d_2)_+ \cdot \mathbb{I}(X \leqslant v)\right] + d_1 - d_2 \\
>&0,
\end{aligned}$$

where the first step is due to the fact that

$$(X-d_1)_+ \cdot \mathbb{I}(X \leqslant v) - (X-d_2)_+ \cdot \mathbb{I}(X \leqslant v) \leqslant d_2 - d_1,$$

and the second step is because of (3.10). The existence of an arbitrage opportunity violates our assumption on the pricing functional Π and thus this completes the proof. □

Remark 3.2. *By Theorem 3.2, the optimal partial hedged loss is given by $f(X) = X \cdot \mathbb{I}(X \leqslant v)$ for sufficiently large hedging budget (no less than $\Pi\left[X \cdot \mathbb{I}(X \leqslant v)\right]$). This implies that the optimal strategy is to hedge the entire risk up to the threshold level v. If the risk X is so large that it exceeds v, then the optimal hedging strategy is not to hedge at all. On the other hand when the hedging budget is limited, it is then optimal to hedge $f(X) = (X-d^*)_+ \cdot \mathbb{I}(X \leqslant v)$ and exhaust the entire hedging budget by determining the positive retention d^* which satisfies* (3.8)*. Note that in either scenario, it is optimal not to hedge at all when the risk is so extreme that it exceeds v. Even though such a hedged loss function seems counterintuitive, it can still be optimal, since the hedged loss function needs not be nondecreasing under the admissible set $\mathcal{L}_1$.*

3.2 Optimality Of The Bull Call Spread Hedging

In this subsection, we investigate the optimal solution of the VaR-based partial hedging problem under the admissible set $\mathcal{L}_2$ as defined in (2.5). Recall that comparing to the admissible set $\mathcal{L}_1$ analyzed in the preceding subsection, the admissible set $\mathcal{L}_2$ is more restrictive in that it imposes the additional monotonicity condition on the hedged loss functions. As a result, the undesirable characteristic of the optimal hedging solution observed in the last subsection (see Remark 3.2) is excluded.

We will see shortly that the same technique can be used to derive the optimal hedging strategy under the more restrictive admissible set $\mathcal{L}_2$, and the so-called bull call spread hedging is an optimal hedging strategy to the VaR-based partial hedging problem. To proceed, for any hedged loss function $f \in \mathcal{L}_2$, we construct h_f as follows:

$$\begin{aligned} h_f(x) &= \min\left\{(x + f(\mathrm{VaR}_\alpha(X)) - \mathrm{VaR}_\alpha(X))_+, f(\mathrm{VaR}_\alpha(X))\right\}, \\ &= (x-d)_+ - (x-v)_+. \end{aligned} \tag{3.11}$$

Recall that $v = \mathrm{VaR}_\alpha(X)$ and $d = \mathrm{VaR}_\alpha(X) - f(\mathrm{VaR}_\alpha(X))$. Clearly, for any $f \in \mathcal{L}_2$, h_f constructed according to (3.11) is also an element in $\mathcal{L}_2$. The function $h_f(X)$ is commonly known as the bull call spread written on X; i.e. it consists of a long and a short call option written on the same underlying risk X with respective strike prices d and v such that $0 \leqslant d \leqslant v$. The following Theorem 3.3 states that the bull call spread on the underlying risk X is an optimal partial hedging strategy among $\mathcal{L}_2$.

Theorem 3.3. *Assume that the pricing functional* Π *admits no arbitrage opportunity in the market. Then, the bull call spread hedged function* h_f *defined in* (3.11) *satisfies the following properties: for any* $f \in \mathcal{L}_2$*,*

(a) $\Pi(f(X)) \leqslant \pi_0$ *implies* $\Pi(h_f(X)) \leqslant \pi_0$*, and*

(b) $VaR_\alpha(T_{h_f}(X)) \leqslant VaR_\alpha(T_f(X))$*.*

Proof. The proof is similar to that of Theorem 3.1. (a). Due to the no-arbitrage assumption on Π, the result $\Pi(h_f(X)) \leqslant \Pi(f(X))$ follows if we can show that $h_f(x) \leqslant f(x)$ for all $x \geqslant 0$. Indeed, properties **P1-P3** imply $f(x) \geqslant [x + f(v) - v]_+ = h_f(x)$ for $0 \leqslant x \leqslant v$, and property **P4** implies $h_f(x) = f(v) \leqslant f(x), \forall\ x \geqslant v$. (b). The proof is in parallel with that of part (b) of Theorem 3.1 and hence is omitted. □

Remark 3.3. *The comments we made in Remark 3.1 for the solutions among* $\mathcal{L}_1$ *are similarly applicable to the solutions among* $\mathcal{L}_2$ *established in Theorem 3.3. In particular, we draw the following conclusions.*

(a) Theorem 3.3 indicates that the bull call spread hedging strategy is optimal among all the strategies in $\mathcal{L}_2$*. We note again that the optimal bull call spread hedging strategy is the one that is written on the risk* X *and not on the asset that underlies* X*.*

(b) The optimality of bull call spread hedging is model independent. Even though the market is incomplete, as long as no arbitrage is admitted in the market, the bull call spread hedging strategy given in Theorem 3.3 remains optimal.

(c) If the bull call spread written on the risk X is available from the financial market, then the optimal partial hedging can be achieved via a simple static hedging strategy.

Based on the results from Theorem 3.3, it is easy to see that the VaR-based partial hedging problem (2.2) can be equivalently cast as

$$\begin{cases} \min\limits_{0 \leqslant d \leqslant v} & \mathrm{VaR}_\alpha \left\{ X - (X-d)_+ + (X-v)_+ + e^{rT} \cdot \Pi\left[h_f(X)\right] \right\} \\ \text{s.t.} & \Pi\left[h_f(X)\right] = \Pi\left[(X-d)_+ - (X-v)_+\right] \leqslant \pi_0. \end{cases} \tag{3.12}$$

The optimal partial hedging problem is similarly reduced to an optimization problem of a single variable. Consequently, we have the following Theorem 3.4 as a counterpart of Theorem 3.2 that we have established in the previous subsection.

Theorem 3.4. *Assume that the pricing functional Π admits no arbitrage opportunity.*

(a) If the hedging budget $\pi_0 \geqslant \Pi\left[X - (X-v)_+\right]$, then the optimizer of the problem (3.12) is $d^* = 0$ and the corresponding minimal VaR of the hedger's total risk exposure at the expiration date T is $\Pi\left[X - (X-v)_+\right]$.

(b) If the hedging budget $\pi_0 < \Pi\left[X - (X-v)_+\right]$, then the optimizer of the problem (3.12) is given by the solution d^* to the following equation

$$\Pi\left[(X-d^*)_+ - (X-v)_+\right] = \pi_0, \tag{3.13}$$

and the corresponding minimal VaR of the hedger's total risk exposure at the expiration date T is $d^* + e^{rT} \cdot \pi_0$.

Proof: The proof is similar to that of Theorem 3.2 and hence is omitted. □

Remark 3.4. *Theorem 3.4 provides a very simple way of identifying the parameter values of the optimal hedged loss function. If the hedging budget is sufficiently large (i.e. greater than or equal to $\Pi\left[X - (X-v)_+\right]$), then the optimal strategy is to hedge all the risk except in the tail. In this case the optimal hedged loss is given by $f(X) = \min(X, v)$. However, if the hedging budget is limited, then the optimal strategy is to implement the bull call spread hedging and exhaust the entire hedging budget by determining the positive retention d^* with equation* (3.13)*.*

4 Partial Hedging Examples: VaR vs. Quantile

In the previous section, we have analyzed the optimal hedged loss function among the admissible sets $\mathcal{L}_1$ (see (2.4)) and $\mathcal{L}_2$ (see (2.5)) respectively. The optimal solution among $\mathcal{L}_1$ is the knock-out call hedging as formally established in Subsection 3.1 while the optimal solution among $\mathcal{L}_2$ is the bull call spread hedging as shown in Subsection 3.2. In Remarks 3.1 and 3.3, we respectively commented that the knock-out call hedging and the bull call spread hedging can usually be achieved by a static strategy in many situations. Subsection 4.1 provides some examples to further illustrate such a statement. The contingent claim X we will consider include forward, European put option, Asian option and barrier option. Subsection 4.2 gives some interesting comparison between our proposed partially hedge strategy and the quantile hedging strategy.

4.1 Partial Hedging Examples

Example 4.1. *Suppose that a hedger has written a forward contract on a stock price S_T and that he intends to partially hedge the time-T payout $X = S_T$ of the forward contract. The optimal VaR-based partial hedging strategies under the respective admissible sets $\mathcal{L}_1$ and $\mathcal{L}_2$ are as follows:*
(a) Under admissible set $\mathcal{L}_1$: According to Theorem 3.1, the optimal hedging] strategy among $\mathcal{L}_1$ is to hedge the part of loss given by

$$(X - d^*)_+ \cdot \mathbb{I}(X \leqslant v) = (S_T - d^*)_+ - (S_T - v)_+ - (v - d^*) \cdot \mathbb{I}(S_T \leqslant v),$$

where d^* is determined by the hedging budget π_0 as specified in Theorem 3.2. Therefore, the optimal hedging strategy is to long a knock-out call option on the underlying stock with barrier v and with strike price as low as possible.
(b) Under admissible set $\mathcal{L}_2$: It follows from Theorem 3.3 that the optimal hedging strategy among $\mathcal{L}_2$ is to hedge the part of risk given by

$$(X - d^*)_+ - (X - v)_+ = (S_T - d^*)_+ - (S_T - v)_+,$$

where d^* depends on the hedging budget π_0 as specified in Theorem 3.4. We assume that $v \equiv VaR_\alpha(X) = VaR_\alpha(S_T)$ is known. Given that the call option $(S_T - v)_+$ is available from the market, then from Theorem 3.4 the optimal partial hedging strategy can be constructed as follows, depending on the relative magnitude of π_0:

- *(i) If the hedging budget π_0 is large enough such that $\pi_0 \geqslant S_0 - \Pi[(S_T - v)_+]$, then the optimal hedging strategy is to long the stock and short a call option on the stock with strike price v. Under this strategy, the hedger only retains the risk in the tail.*
- *(ii) If the hedging budget π_0 is of small amount satisfying $\pi_0 < S_0 - \Pi[(S_T - v)_+]$, then the optimal hedging strategy is to first short a call option on the underlying stock with strike price v. The proceeds received from the short position, i.e. $\Pi[(S_T - v)_+]$, together with the initial hedging budget π_0, is used to invest in a call option on the same underlying stock with a strike price as low as possible so as to exhaust the entire amount of $\Pi[(S_T - v)_+] + \pi_0$. Consequently, the budget constraint is binding and the hedging strategy mimics a bull call spread on the underlying stock.*

For brevity, the remaining examples only discuss the optimal partial hedging strategies among $\mathcal{L}_2$. The optimal partial hedging strategies among $\mathcal{L}_1$ can be constructed in a similar fashion.

Example 4.2. *This example is concerned with partial hedging a European put option with its time-T payout given by $X = (K - S_T)_+$. Using Theorem 3.3, the optimal hedging strategy among $\mathcal{L}_2$ is to hedge the part of risk given by*

$$\begin{aligned}(X - d^*)_+ - (X - v)_+ &= ((K - S_T)_+ - d^*)_+ - ((K - S_T)_+ - v)_+ \\ &= (K - d^* - S_T)_+ - (K - v - S_T)_+,\end{aligned}$$

where d^* is determined by the hedging budget π_0 as specified in Theorem 3.4.

As in Example 4.1, we assume that $v \equiv VaR_\alpha(X)$ is known and the market price of the European put option with strike price $(K - v)_+$ (i.e. $\Pi[(K - v - S_T)_+]$) is observable from the market. Then, the optimal hedging strategy can be constructed according to Theorem 3.4 as follows:

(i) If the hedging budget π_0 is large enough such that $\pi_0 \geqslant \Pi[(K - S_T)_+] - \Pi[(K - v - S_T)_+]$, then the optimal hedging strategy is to long a European put option on the stock with strike price K and at the same time short a European put option on the same underlying stock with strike price $(K - v)_+$.

(ii) If the hedging budget is of small amount satisfying $\pi_0 < \Pi[(K - S_T)_+] - \Pi[(K - v - S_T)_+]$, then the optimal hedging strategy consists of a short position in a European put option on the stock with strike price $(K - v)_+$ and a long position in a European put option on the same stock with a strike price as high as possible to exhaust the entire budget.

Remark 4.1. *(a) In the previous example, if the options used to construct the hedging portfolio are not available in the market, we may directly replicate the payout of these options by a continuously rebalancing strategy on the stock.*

(b) The European call option can be partially hedged in a way similar to the European put option. We omit the specific procedure for brevity.

Example 4.3. *Suppose a hedger is to partially hedge an Asian call option with a time-T payout given by $X = \left(\frac{1}{T}\int_0^T S_t dt - K\right)_+$. According to Theorem 3.3, the optimal hedging strategy among $\mathcal{L}_2$ is to hedge the part of risk given by*

$$\begin{aligned}(X - d^*)_+ - (X - v)_+ &= \left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)_+ - d^*\right]_+ - \left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)_+ - v\right]_+ \\ &= \left(\frac{1}{T}\int_0^T S_t dt - K - d^*\right)_+ - \left(\frac{1}{T}\int_0^T S_t dt - K - v\right)_+,\end{aligned}$$

where d^* is determined by the hedging budget π_0 according to Theorem 3.4.

Again, we assume that $v \equiv VaR_\alpha(X)$ *is known and the option price* $\Pi\left[\left(\frac{1}{T}\int_0^T S_t dt - K - v\right)_+\right]$ *can be observed from the market. Then, Theorem 3.4 implies the following optimal hedging strategies*

(i) *If the hedging budget is large enough such that*

$$\pi_0 \geqslant \Pi\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)_+\right] - \Pi\left[\left(\frac{1}{T}\int_0^T S_t dt - K - v\right)_+\right],$$

then the optimal hedging strategy is to long an Asian call option on the stock with strike price K *and at the same time short an Asian option on the same stock with strike price* $(K+v)$*.*

(ii) *If, however, the hedging budget is relatively small with*

$$\pi_0 < \Pi\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)_+\right] - \Pi\left[\left(\frac{1}{T}\int_0^T S_t dt - K - v\right)_+\right],$$

then the optimal hedging strategy is to first short an Asian call option on the stock with strike price $(K+v)$*, and then use the proceeds, together with the hedging budget*

$$\Pi\left[\left(\frac{1}{T}\int_0^T S_t dt - K - v\right)_+\right] + \pi_0,$$

to invest in an Asian call option on the same stock with a strike price as low as possible so as to exhaust the entire amount.

Remark 4.2. *While Example 4.3 illustrates how to apply Theorem 3.4 to construct the partial hedging strategy for the Asian call option, we can similarly construct the optimal hedging strategy for the Asian put option. However, when the strike price is floating, instead of being fixed, Theorem 3.4 cannot be applied directly for an effective hedging strategy, as in this case, the optimal hedged loss obtained from Theorem 3.4 may not be attainable in the market.*

Example 4.4. *Suppose a hedger is to partially hedge an up-and-in call option with a time-*T *payout* $X = (S_T - K)_+ \cdot \mathbb{I}\left(\max\limits_{0\leqslant t\leqslant T} S_t \geqslant H\right)$*, where* S_t *is the time-*t *price of the stock. According to Theorem 3.3, the optimal hedging strategy among* $\mathcal{L}_2$ *is to hedge the part of risk given by*

$$\begin{aligned}
& (X - d^*)_+ - (X - v)_+ \\
= & \left[(S_T - K)_+ \cdot \mathbb{I}\left(\max_{0\leqslant t\leqslant T} S_t \geqslant H\right) - d^*\right]_+ - \left[(S_T - K)_+ \cdot \mathbb{I}\left(\max_{0\leqslant t\leqslant T} S_t \geqslant H\right) - v\right]_+ \\
= & (S_T - K - d^*)_+ \cdot \mathbb{I}\left(\max_{0\leqslant t\leqslant T} S_t \geqslant H\right) - (S_T - K - v)_+ \cdot \mathbb{I}\left(\max_{0\leqslant t\leqslant T} S_t \geqslant H\right),
\end{aligned}$$

where d^* is determined by the hedging budget π_0 according to Theorem 3.4.

As in the previous examples, we assume we can accurately determine the value of $v \equiv VaR_\alpha(X)$ and can observe from the market the corresponding price

$$\Pi\left[(S_T - K - v)_+ \cdot \mathbb{I}\left(\max_{0\leqslant t\leqslant T} S_t \geqslant H\right)\right]$$

of the barrier option. As a result, Theorem 3.4 implies the following optimal partial hedging strategies.

(i) If the hedging budget is large enough such that

$$\pi_0 + \Pi\left[(S_T - K - v)_+ \cdot \mathbb{I}\left(\max_{0\leqslant t\leqslant T} S_t \geqslant H\right)\right] \geqslant \Pi\left[(S_T - K)_+ \cdot \mathbb{I}\left(\max_{0\leqslant t\leqslant T} S_t \geqslant H\right)\right],$$

then the optimal hedging strategy is to long an up-and-in call option on the stock with strike price K and barrier H and at the same time short an up-and-in call option on the same stock with strike price $(K+v)$ and barrier H.

(ii) If, however, the hedging budget is relatively small such that

$$\pi_0 + \Pi\left[(S_T - K - v)_+ \cdot \mathbb{I}\left(\max_{0\leqslant t\leqslant T} S_t \geqslant H\right)\right] < \Pi\left[(S_T - K)_+ \cdot \mathbb{I}\left(\max_{0\leqslant t\leqslant T} S_t \geqslant H\right)\right],$$

then the optimal hedging strategy is to first short an up-and-in call option on the stock with strike price $(K+v)$ and barrier H, and then the proceeds together with the hedging budget, i.e. $\Pi\left[(S_T - K - v)_+ \cdot \mathbb{I}\left(\max_{0\leqslant t\leqslant T} S_t \geqslant H\right)\right] + \pi_0$, are used to invest in an up-and-in call option on the same stock with barrier H and a strike price as low as possible to exhaust the entire amount.

4.2 A Comparison Between Partial Hedge And Quantile Hedge

In this subsection, we will conduct an example to highlight the difference between our proposed VaR-based partial hedging strategy and the well-known quantile hedging strategy proposed by Föllmer and Leukert (1999). We assume that the standard Black-Scholes market applies so that the dynamic of the stock price process is governed by the following stochastic differential equation:

$$dS_t = S_t m dt + S_t \sigma dW_t, \;\; t \geqslant 0,$$

where W is a Wiener process under the physical measure $\mathbb{P}$, and σ and m are respectively the constant volatility and return rate of the underlying stock. The contingent claim that we are interested in hedging is a European call option with payout $X_T = (S_T - K)_+$. We use the same set of parameter values as in Föllmer and Leukert (1999):

$$S_0 = 100, \;\; r = 0, \;\; m = 0.08, \;\; T = 0.25 \text{ and } K = 110.$$

To establish the optimal partial hedging strategy, we need to further specify the values of the volatility σ and the hedging budget π_0. We consider the following three scenarios:

(i) $\sigma = 0.3$, $\pi_0 = 1.5$,

(ii) $\sigma = 0.3$, $\pi_0 = 0.5$,

(iii) $\sigma = 0.2$, $\pi_0 = 0.5$.

Using the Black-Scholes formula, the prices of the corresponding European call options are

$$P_C = \begin{cases} 2.50, & \text{for } \sigma = 0.3; \\ 0.95, & \text{for } \sigma = 0.2. \end{cases}$$

By comparing the budget of the respective hypothetical volatility scenario to the above option prices, it is clear that the European call options cannot be hedged perfectly. Given the limited hedging budget, it is therefore instructive and useful to develop alternate partial hedging strategies involving the quantile hedging, the knock-out call hedging, and the bull call spread hedging. These three hedging strategies are discussed in details below.

(a) Quantile hedging strategy:

For our assumed Black-Scholes model, Föllmer and Leukert (1999) show that the quantile hedging strategy admits different form depending on the relative magnitude of m and σ. In particular we need to consider the following two cases:

(1) When $m \leqslant \sigma^2$, the optimal quantile hedging strategy is $f(X_T) = X_T \mathbb{I}_{\{X_T < c\}}$ and in our European call option case, this becomes

$$(S_T - K)_+ - (S_T - c)_+ - (c - K) \cdot \mathbb{I}(S_T > c), \tag{4.14}$$

where the constant c is determined by the following two equations through an auxiliary variable b:

$$\begin{cases} c = S_0 \exp\left(\sigma b - \frac{1}{2}\sigma^2 T\right) \\ \pi_0 = P_C - S_0 \Phi\left(\dfrac{-b + \sigma T}{\sqrt{T}}\right) + K\Phi\left(\dfrac{-b}{\sqrt{T}}\right). \end{cases} \tag{4.15}$$

In the above, Φ denotes the standard normal cumulative distribution function.

(2) When $m > \sigma^2$, the optimal quantile hedging strategy is $f(X_T) = X_T \mathbb{I}_{\{X_T < c_1, \text{ or } X_T > c_2\}}$ and in our European call option case, this becomes

$$(S_T - K)_+ - (S_T - c_1)_+ - (c_1 - K) \cdot \mathbb{I}_{(S_T > c_1)} + (S_T - c_2)_+ + (c_2 - c_1)\mathbb{I}_{S_T > c_2}, \tag{4.16}$$

where c_1 and c_2 are two distinct constants satisfying the following system of equations with auxiliary variables b_1, b_2 and λ:

$$\begin{cases} c_1^{\frac{m}{\sigma^2}} = \lambda(c_1 - K)_+ \\ c_2^{\frac{m}{\sigma^2}} = \lambda(c_2 - K)_+ \\ c_1 = S_0 \exp\left(\sigma b_1 - \frac{1}{2}\sigma^2 T\right) \\ c_2 = S_0 \exp\left(\sigma b_2 - \frac{1}{2}\sigma^2 T\right) \\ \pi_0 = P_C - S_0\Phi\left(\frac{-b_1 + \sigma T}{\sqrt{T}}\right) + K\Phi\left(\frac{-b_1}{\sqrt{T}}\right) + S_0\Phi\left(\frac{-b_2 + \sigma T}{\sqrt{T}}\right) + K\Phi\left(\frac{-b_2}{\sqrt{T}}\right). \end{cases} \tag{4.17}$$

With the above setup, we are now ready to obtain the optimal quantile hedging strategies under each of the three scenarios (i)-(iii) specified above. For scenarios (i) and (ii), we have $m < \sigma^2$ so that the optimal quantile hedging strategy is of the form (4.14). The required constant c can be deduced by substituting the corresponding parameter values into the set of equations (4.15). To summarize, the optimal quantile hedging strategy is of the form

$$(S_T - 110)_+ - (S_T - 129.47)_+ - 19.47 \cdot \mathbb{I}(S_T > 129.47)$$

for scenario (i), and

$$(S_T - 110)_+ - (S_T - 118.69)_+ - 8.69 \cdot \mathbb{I}(S_T > 118.69)$$

for scenario (ii). For scenario (iii), we have $m > \sigma^2$ so that the optimal quantile hedging strategy is of the form (4.16), and the respective constants c_1 and c_2 can be obtained by solving the system of equations (4.17) using the assumed parameter values. The resulting optimal quantile hedging strategy becomes

$$(S_T - 110)_+ - (S_T - 119.98)_+ - 9.98 \cdot \mathbb{I}(S_T > 119.98) + (S_T - 1323)_+ + 1203.02 \cdot \mathbb{I}(S_T > 1323).$$

The optimal hedged loss functions for the three scenarios are demonstrated in Figure 1.

(b) Knock-out call hedging strategy:

We consider the optimal partial hedging strategies by minimizing $\mathrm{VaR}_{0.95}$ of the hedger's total risk exposure among the admissible set $\mathcal{L}_1$ defined in (2.4). Using Theorems 3.1 and 3.2, the optimal choice of the hedger is to adopt the following knock-out call hedging strategy

$$[S_T - (K + d^*)]_+ - [S_T - (K + v)]_+ - (v - d^*)\mathbb{I}_{(S_T \geqslant K+v)},$$

where $v = \mathrm{VaR}_{0.95}((S_T - K)_+)$ and d^* is again implied by the budget π_0 as asserted in Theorem 3.2. These values are readily determined and are summarized as follows for the three scenarios:

$$\begin{cases} v = 19.11, & d^* = 0, \quad \text{for } \sigma = 0.3 \text{ and } \pi_0 = 1.5, \\ v = 19.11, & d^* = 6.67, \quad \text{for } \sigma = 0.3 \text{ and } \pi_0 = 0.5, \\ v = 9.66, & d^* = 0, \quad \text{for } \sigma = 0.2 \text{ and } \pi_0 = 0.5. \end{cases}$$

Accordingly, the corresponding optimal knock-out call hedging strategies are

$$\begin{cases} (S_T - 110)_+ - (S_T - 129.11)_+ - 19.11\mathbb{I}_{(S_T \geqslant 129.11)} \\ (S_T - 116.67)_+ - (S_T - 129.11)_+ - 12.44\mathbb{I}_{(S_T \geqslant 129.11)} \\ (S_T - 110)_+ - (S_T - 119.66)_+ - 9.66\mathbb{I}_{(S_T \geqslant 119.66)} \end{cases}$$

The optimal hedged loss functions are illustrated in Figure 2.

(c) Bull call spread hedging strategy:

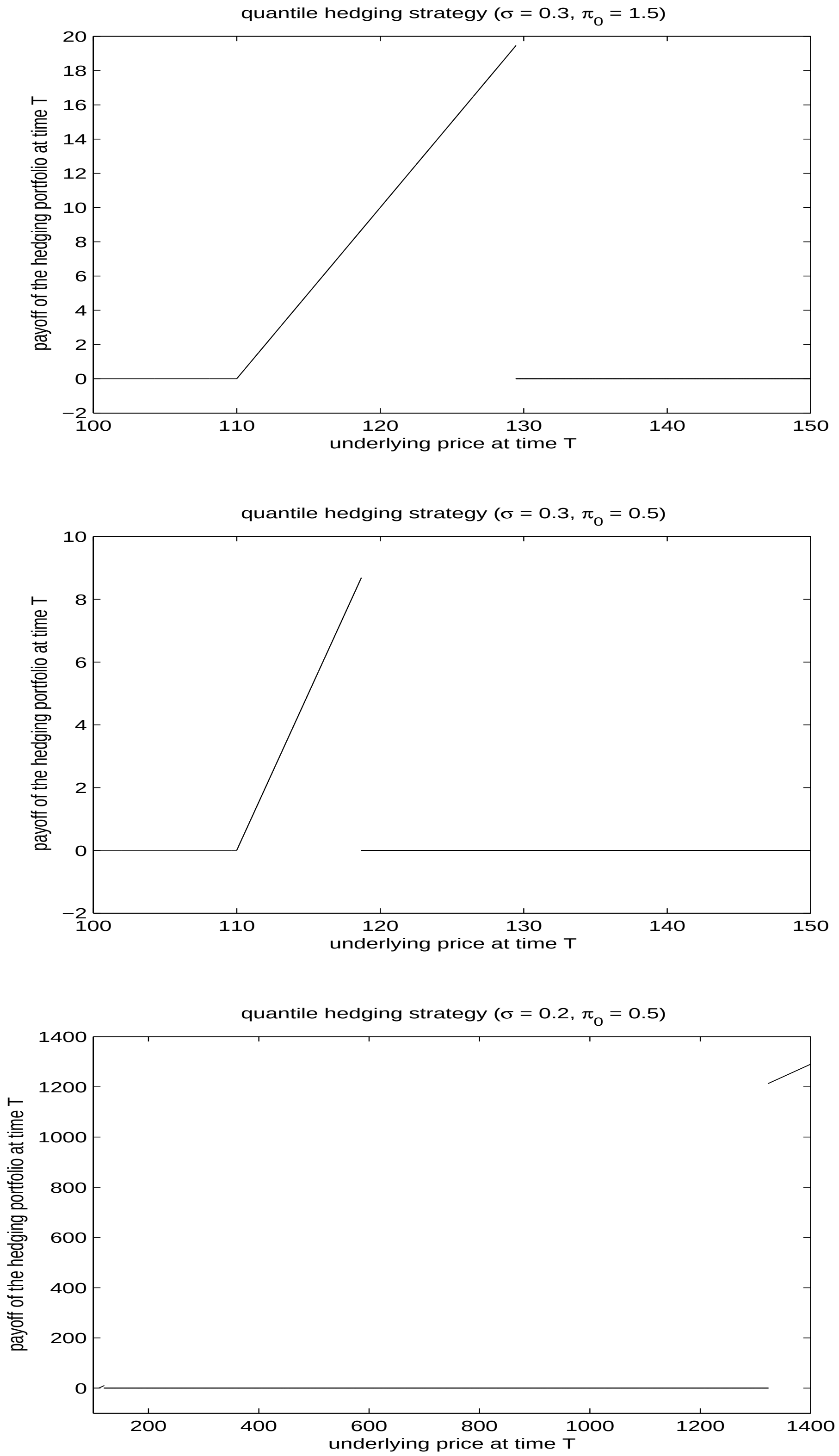

Figure 1: Optimal quantile hedging strategies under the three scenarios

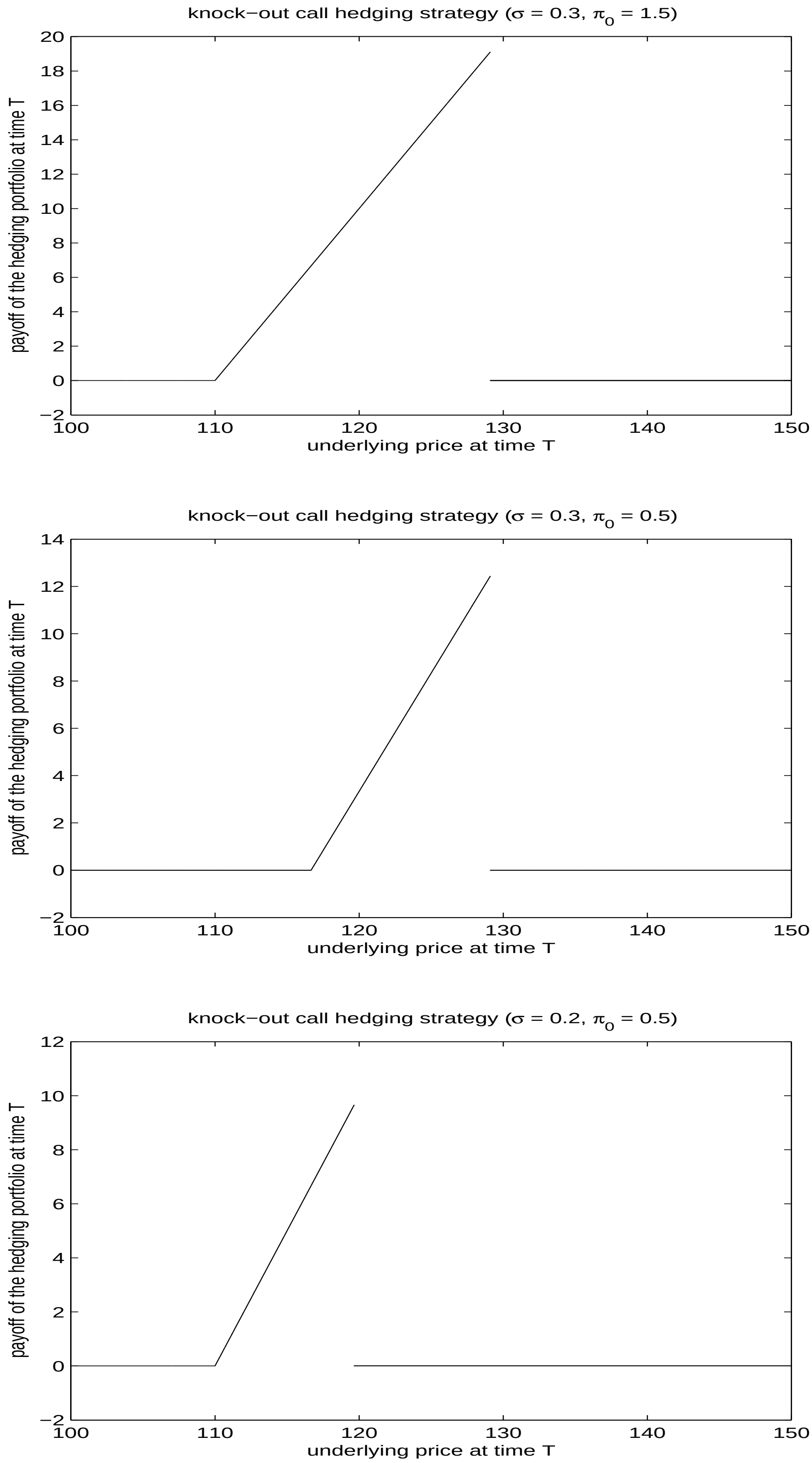

Figure 2: Optimal knock-out call hedging strategy under the three scenarios

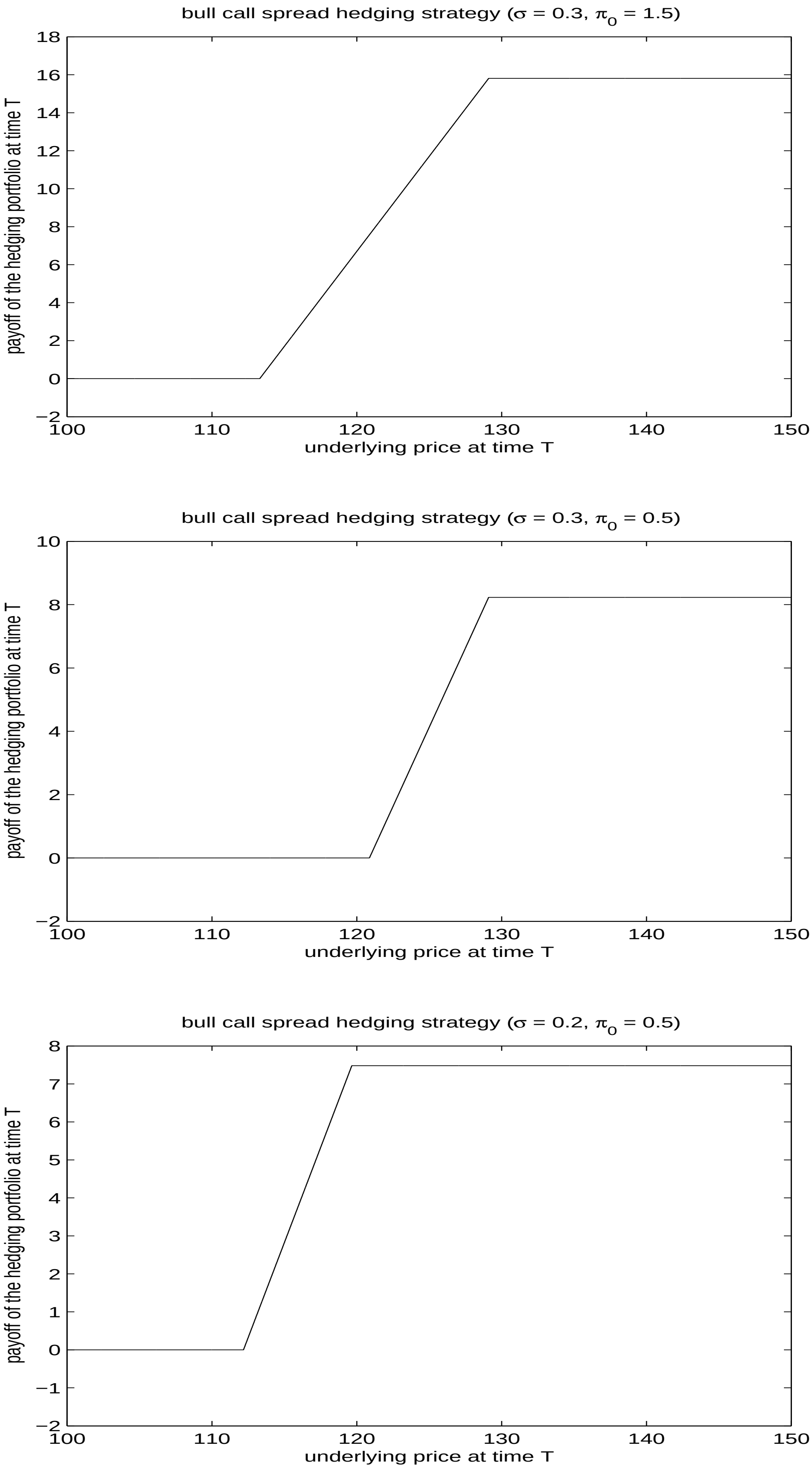

Figure 3: Optimal bull call spread hedging strategy under the three scenarios

We consider the optimal partial hedging strategies by minimizing $\mathrm{VaR}_{0.95}$ of the hedger's total risk exposure among the admissible set $\mathcal{L}_2$ defined in (2.5). It follows from Theorems 3.3 and 3.4 that the optimal bull call spread hedging strategy is of the form

$$[S_T - (K + d^*)]_+ - [S_T - (K + v)]_+,$$

where $v = \mathrm{VaR}_{0.95}((S_T - K)_+)$ and d^* is determined by the budget π_0 as asserted in Theorem 3.4. These values are easily determined and are summarized as follows for the three specified scenarios:

$$\begin{cases} v = 19.11, & d^* = 3.30, \quad \text{for } \sigma = 0.3 \text{ and } \pi_0 = 1.5, \\ v = 19.11, & d^* = 10.88, \quad \text{for } \sigma = 0.3 \text{ and } \pi_0 = 0.5, \\ v = 9.66, & d^* = 2.18, \quad \text{for } \sigma = 0.2 \text{ and } \pi_0 = 0.5. \end{cases}$$

Accordingly, the corresponding optimal bull call spread hedging strategies are

$$\begin{cases} (S_T - 113.30)_+ - (S_T - 129.11)_+ & \text{for } \sigma = 0.3 \text{ and } \pi_0 = 1.5, \\ (S_T - 120.88)_+ - (S_T - 129.11)_+ & \text{for } \sigma = 0.3 \text{ and } \pi_0 = 0.5, \\ (S_T - 112.18)_+ - (S_T - 119.66)_+ & \text{for } \sigma = 0.2 \text{ and } \pi_0 = 0.5. \end{cases}$$

The optimal hedged loss functions are demonstrated in Figure 3.

Based on these numerical results, we draw the following observations with respect to the optimal hedging strategies.

a). Let us first consider the quantile hedging. Recall that scenario (i) has a higher hedging budget than scenario (ii) and this is their only difference. As a result, the shapes of both optimal quantile hedging are the same for both scenarios; see the top and middle graphs in Figure 1. The European call option is fully hedged for $S_T \leqslant 118.69$. For $S_T > 118.69$, the optimal quantile hedging under scenario (ii) changes drastically from the fully hedged position to the naked position, as induced by the limited hedging budget, and the hedger is exposed to the entire potential obligation of $S_T - 110$. Moreover, because the first scenario has a higher budget, the option remains to be hedged until S_T increases to 129.47, beyond which the hedger is again exposed to the naked position, as in the second scenario. From the risk management viewpoint, the above optimal hedging strategy seems counterintuitive, since generally a hedger should be more concerned with larger losses. Yet the strategy dictated by the quantile hedging only produces perfect hedging for small losses and completely no hedging for lager losses. This phenomenon is attributed to the criterion stipulated by the quantile hedging that it only focuses on the likelihood of a successful hedge while ignores completely the tail risk.

   We now compare the optimal quantile hedging between scenario (ii) and scenario (iii); see the middle and bottom graphs in Figure 1. The only difference between these two scenarios is the volatility parameter σ. By merely decreasing σ from 30% to 20%, it is striking to learn that the shape of the hedging strategy changes quite substantially. In particular, the call option is perfectly hedged up to $S_T = 119.98$ and then completely unhedged, just like the first two scenarios. More interestingly, when S_T becomes really large such as exceeding 1323, the option is completely hedged again. The optimal hedged loss function displayed at the bottom most panel of Figure 1 seems to indicate that it is flat at zero for most of S_T. However, it should be pointed out that this is just an optical illusion due to the scale of the plot. Figure 4 magnifies the portion of the optimal hedged loss function for $100 \leqslant S_T \leqslant 150$ and confirms that for low values of S_T, the optimal hedged loss function from scenario (iii) resembles the first two scenarios.

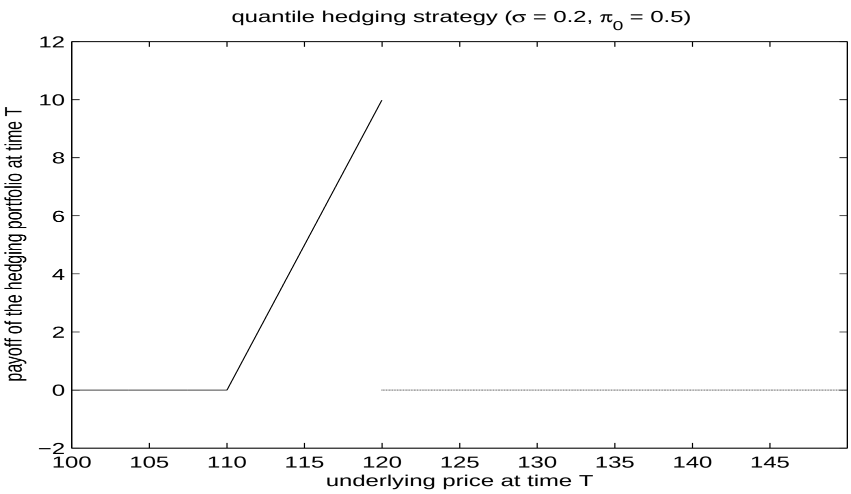

Figure 4: The optimal quantile hedging strategy for scenario (iii) over the range $100 \leqslant S_T \leqslant 150$

b). Unlike the quantile hedging, the optimal knock-out call partial hedging strategy has the same consistent shape in all three scenarios (see Figure 2). Moreover, their shapes resemble that of the quantile hedging in the first two scenarios. Just to elaborate, the knock-out call hedging strategy for scenario (i) provides a perfect hedge for S_T up to 129.11 and then switches to a naked position for $S_T > 129.11$. On the other hand, the optimal partial hedging under the lower hedging budget of scenario (ii) is accomplished at the expense of not perfectly hedging the call option. In particular, the hedger absorbs the loss of amount $S_T - 110$ for $110 < S_T < 116.67$ and up to a fixed amount of 6.67 for $S_T \in [116.67, 129.11]$. For $S_T > 129.11$ the hedger does not hedge anything at all as in the first scenario.

c). The optimal partial hedging under the bull call spread strategy generates a very different but more desirable solution (see Figure 3). First, we emphasize that the optimal shapes of the hedged loss functions are again consistently the same among the three scenarios; they are all bull call spread strategies. Second, the bull call spread hedging provides some partial hedging, even for large losses. This contradicts the preceding two methods (except the quantile hedging under scenario (iii)) which do not provide any protection on the right tail. This is a consequence of imposing the nondecreasing property **P4** on the hedged loss functions. Third, because of enforcing some partial hedging on large losses, the bull call spread strategies sacrifice the chance of perfect hedging for small losses. To see this, let us recall that for scenario (i) the knock-out call strategy perfectly hedges the call option for $S_T \in [110, 129.11]$. For the bull call spread hedging, the optimal strategy only begins partial hedging from $S_T = 113.30$ using an option that pays $S_T - 113.30$ for $S_T \in [113.30, 129.11]$. This implies that over the same range of stock prices, the hedger is exposed to a constant loss of 3.30 for the bull call spread hedging while zero loss for the knock-out call strategy. Fourth, while the bull call spread hedging provides some partial hedging on the tail, it is still not satisfactory in view that the amount being hedged remains constant after a threshold level. For instances, when $S_T > 129.11$ the optimal bull call spread hedging yields a constant hedged amount of 15.81 for scenario (i). This implies that the hedger is still subject to a potential loss of $S_T - 110 - 15.81 = S_T - 125.81$ for $S_T > 129.11$.

d). The plots of the optimal strategies in Figures 1-3 again highlight the sensitivity of the shape of the optimal hedged loss functions for the quantile hedging to the parameter values of the assumed model. The shape of the optimal hedged loss function changes depending on the ratio m/σ^2 being greater or smaller than 1. In contrast, Figure 2 and Figure 3 re-assure that the optimal hedging strategies are always the knock-out call strategy and the bull call spread strategy respectively. These results demonstrate the stability or the robustness of the VaR-based hedging strategy in that the optimal hedging strategy always admits the same structure and it is independent of the assumed market model.

e). Additional insight on these hedging strategies can be gained by comparing the expected shortfall of the hedger under each of these three strategies. The results, which are depicted in Table 1, indicate that the expected shortfall of the hedger's total risk under the bull call spread hedging strategy is always the smallest among the three hedging strategies and in all three scenarios. This is consistent with our intuition as the bull call hedging strategy is derived as an optimal solution under the additional assumption of **P4**, which reflects the hedger's concern on the right tail risk. In other words, bull call spread hedging provides some partial hedging on the tail risk.

	Quantile hedging	Knock-out call hedging	Bull call spread hedging
Scenario (i)	1.35	1.36	1.25
Scenario (ii)	2.56	2.52	2.48
Scenario (iii)	0.72	0.72	0.66

Table 1: Expected shortfall of the hedger under each of the optimal partial hedging strategies

5 Conclusion

In this paper, we propose a general framework for determining an optimal partial hedging strategy. The proposed model involves minimizing an arbitrary risk measure of a hedger's risk exposure. We derive the analytic solutions by specializing to the Value-at-Risk measure and under two admissible classes of hedging strategies. We analytically obtain the optimal hedging solution as either the knock-out call hedging strategy, which involves constructing a knock-out call on the payout, or the bull call spread hedging strategy, which involves constructing a bull call spread on the payout. Through many examples, we show that, in implementing our optimal hedging strategies, we often only need to hedge an instrument which has the same structure (with different parameter values though) as the risk we aim to partially hedge. Therefore, if such an instrument exists in the market, we are then able to achieve our objective by a static hedging strategy. Even if such an instrument is not available in the market, our results provide some important insights on which part of the risk should be hedged as an optimal partial hedging strategy.

In comparison to the well-known quantile hedging, our proposed VaR-based partial hedging has a number of advantages. Notably, the structure of the optimal hedging strategy is independent of the assumed market model, the optimal solution is relatively easy to determine and robust, and it is also better at capturing the tail risk when we impose the monotonicity on the hedging strategy.

Although the proposed VaR-based partial hedging model and the resulting optimal strategies have the above appealing features, it is also important to point out their potential limitations. In particular, VaR suffers from the typical criticisms that it is not a coherent risk measure and that it is a quantile risk measure. The latter property implies that as long as the probability of loss is within the prescribed tolerance of the hedger, the optimal VaR-based partial hedging strategy is to leave the risk unhedged. For example, if the probability of a loss on a particular risk exposure is less than 5%, then the optimal VaR-based partial hedging strategy under 95% confidence level is not to hedge any part of the risk.

While we have confined our analysis to VaR, it should be emphasized that our proposed partial hedging model is quite general in that it can be applied to other risk measures including the conditional value-at-risk (CVaR). It would be of great interest to investigate the optimal hedging strategy under CVaR since CVaR is known to have some desirable properties. These include the coherence property, spectral property, and the capability of capturing the tail risk. Melnikov and Smirnov (2012) investigate the problem of partial hedging by minimizing the CVaR of the portfolio in the complete market. Their solution exploits the properties of CVaR risk measure and also relies on the Neyman-Pearson lemma approach, a method which is used extensively in the quantile hedging. On the other hand, the proposed model and the approach used in this paper provide a possible different perspective on studying the optimal partial hedging problem involving CVaR. We report this in details in our companion paper Cong, et al. (2012).

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