Optimal dynamic reinsurance policies under Mean-CVaR - a generalized Denneberg’s absolute deviation principle

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Abstract

This paper studies the optimal dynamic reinsurance policy for an insurance company whose surplus is modeled by the diffusion approximation of the classical Cramér-Lundberg model. We assume the reinsurance premium is calculated according to a proposed Mean-CVaR premium principle which generalizes Denneberg’s absolute deviation principle and expected value principle. Moreover, we require that both ceded loss and retention functions are non-decreasing to rule out moral hazard. Under the objective of minimizing the ruin probability, we obtain the optimal reinsurance policy explicitly and we denote the resulting treaty as the dual excess-of-loss reinsurance. This form of optimal treaty is new to the literature. It also demonstrates that reinsurance treaties such as the proportional and the standard excess-of-loss, which are typically found to be optimal in the dynamic reinsurance model, need not be optimal when we consider a more general optimization model.

Keywords: Dynamical reinsurance, Mean-CVaR premium principle, Denneberg’s absolute deviation principle, ruin probability

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1 Introduction

This paper studies the optimal reinsurance strategy by minimizing the ruin probability (or equivalently maximizing the survival probability) of an insurance company in a dynamic setting. This is one of the classical problems in actuarial and insurance risk analysis. While this problem has been studied extensively in the past few decades, the existing results are surprisingly quite restrictive. Their limitations stem from at least two aspects. The first aspect is that most of the optimal reinsurance strategies are obtained under the assumption of relatively simple premium principles such as the expected value premium principle and the variance premium principle. The second aspect is that the optimal reinsurance strategy is typically analyzed under a specified type of reinsurance treaty, such as the proportional reinsurance treaty, the excess-of-loss reinsurance, or their combinations. See for example Schmidli (2001, 2002), Taksar and Markussen (2003); Promislow and Young (2005); Luo et al. (2008); Zhang et al. (2016) that study optimal proportional reinsurance; Hipp and Vogt (2003); Dickson and Waters (2006); Zhang et al. (2016) that analyze optimal excess-of-loss reinsurance; and Zhang et al. (2007) that concludes the optimal combinational reinsurance strategy must be of a pure excess-of-loss type. All the studies mentioned above analyze the optimality of reinsurance scheme by minimizing an insurer’s probability of ruin and that the reinsurance premium is calculated based on the expected value principle. Other slightly more general results are established by Meng and Zhang (2010) and Hipp and Taksar (2010) who show that the excess-of-loss reinsurance treaty is optimal among the class of plausible reinsurance treaties but still under the expected value premium principle. Hipp and Taksar (2010) also demonstrate that proportional reinsurance treaty is optimal under the variance premium principle. See Albrecher et al. (2017) for a more complete and excellent exposition of the topic related to reinsurance.

By assuming the surplus of the insurer can be modeled by a diffusion approximation of the classical Cramér-Lundberg model,¹ this paper similarly investigates the

¹This assumption is commonly assumed in dynamic reinsurance models. Its justification can be found in Grandell (2012) and Taksar and Markussen (2003). Analyzing optimal reinsurance strat-
ruin probability minimization optimal reinsurance model but with the following two
distinctive features:

1. The losses assumed by both insurer and reinsurer increase with losses;

2. the reinsurance premium is determined by a newly proposed premium principle which we denote as the mean-CVaR premium principle (CVaR is short for conditional tail value at risk).

The first feature of our proposed dynamic reinsurance model allows us to investigate the optimal reinsurance strategy under a much wider class of reinsurance treaty. Recall that the existing optimal reinsurance strategies are typically obtained by confining to a particular type of reinsurance treaty such as the proportional reinsurance and/or the excess-of-loss-reinsurance. By considering the optimality under a class of reinsurance treaties that is increasing, it encompasses many of the existing results since both proportional reinsurance and excess-of-loss reinsurance are special cases of increasing reinsurance treaty. Furthermore, it is of significance interest to investigate if the excess-of-loss reinsurance is still optimal under a much wider family of reinsurance treaty since some existing results seem to suggest that the excess-of-loss reinsurance is robust to optimality (see Meng and Zhang, 2010; Hipp and Taksar, 2010).

There is another important reason requiring the losses assumed by both insurer and reinsurer to increase with losses. This condition helps alleviating the insurer’s ex post moral hazard. The ex post moral hazard is concerned with the effect of incentives on claiming losses. To demonstrate how a reinsurance treaty may trigger ex post moral hazard, it is instructive to consider the following example. Let $x$ be the insurer’s loss that is reported to the reinsurer and $(x - \gamma_1)_+\mathbb{I}_{\{x \leq \gamma_2\}}$ be the loss that is ceded to the reinsurer. Both $\gamma_1$ and $\gamma_2$ are pre-determined constants such that $\gamma_2 > \gamma_1 > 0$, $(x)_+ = \max(x, 0)$, and $\mathbb{I}_{\{D\}}$ denotes the indicator function of an event $D$. The ceded strategy under the classical Cramér-Lundberg model tends to be intractable, typically requires numerical solution as can be seen from Hipp and Vogt (2003) and Dickson and Waters (2006). On the other hand, under the diffusion approximation the resulting dynamic reinsurance model becomes much more tractable and hence explains its popularity. See for example Højgaard and Taksar (1998a,b), Zhang et al. (2016), Liang and Yuen (2016), Meng and Siu (2011), Zhou and Yuen (2012), Asmussen et al. (2000), Bai and Guo (2008, 2010), Cao and Wan (2009), Gu et al. (2012), Meng and Zhang (2010), Schmidli (2002), Zhang et al. (2007).
loss function \((x - \gamma_1) + \mathbb{1}_{x \leq \gamma_2}\) is commonly known as the truncated stop-loss function and it has been shown to be an optimal reinsurance treaty under a number of (static) reinsurance models. See for example Gajek and Zagrodny (2004), Kaluszka (2005), Kaluszka and Okolewski (2008), Bernard and Tian (2009), and Chi and Tan (2011).

Despite the optimality of the truncated stop loss function, it is not a practical nor a desirable function as it leads to ex post moral hazard. To see this, let us first note that under the truncated stop loss reinsurance treaty, the reinsurer is only responsible for losses that are in the range \((\gamma_1, \gamma_2)\). In particular, for losses that exceed \(\gamma_2\), the indemnity from the reinsurer to the insurer drops from the maximum amount of \(\gamma_2 - \gamma_1\) to zero. Hence the amount ceded to the reinsurer is not an increasing function of \(x\) though the losses retained by the insurer is non-decreasing in \(x\). The decline in indemnity could incentivize insurer to underreport its actual loss, especially for losses that are above \(\gamma_2\); thus gives rise to ex post moral hazard. If both the ceded loss function and the retained loss function are increasing function of \(x\), the ex post moral hazard can be alleviated and hence signifies the importance of imposing the “increasing” condition. See also Cai et al. (2008), Cheung (2010), Chi and Tan (2013), Cui et al. (2013), Lu et al. (2013), and Weng and Zhuang (2016) for some other single-period optimal reinsurance models that emphasize the importance of increasing ceded loss functions.

The intertwine issue between moral hazard and optimal insurance is a widely studied topic among economists and actuaries. For a historical account of the term “moral hazard”, see Rowell and Connelly (2012). See Shavell (1979), Huberman et al. (1983), Müller and Brammertz (1986), Picard (2000), Winter (2013), and Drèze and Schokkaert (2013) for some related studies as well as Dionne and St-Michel (1991), Cummins and Tennyson (1996), and Butler et al. (1996) on empirical evidences of moral hazard. Despite the importance of moral hazard in contract theory, to the best of our knowledge this may be the first time the non-decreasing constraint is imposed explicitly in the dynamic reinsurance model to address the undesirable feature of moral hazard.

The second feature of our optimal reinsurance model is prompted by the existing
results that are either based on expected value premium principle or variance premium principle. By relaxing this assumption and considering a more general premium principle provides an additional insight to the optimal choice of reinsurance strategy. As will be explained in Section 2, the proposed mean-CVaR premium principle (see Definition 2.2) can be considered as a generalization of the absolute deviation premium principle proposed by Denneberg (1990). Therefore the proposed premium principle encompasses both Denneberg’s absolute deviation principle and the expected premium principle as special cases. The proposed mean-CVaR premium principle also satisfies most of the desirable properties of premium principles given in Young (2004). Finally, the proposed mean-CVaR premium principle can be interpreted as a weighted average of the risk’s mean and CVaR. Furthermore, the premium principle is completely specified by three parameters $\alpha$, $\theta$ and $\beta$. The parameter $\alpha$ determines the confidence level of the CVaR and measures the degree to which the reinsurer penalizes tail risk. The tail risk loading $\beta$ measures the importance of the tail risk relative to the mean. The parameter $\theta$ can be interpreted as the premium loading and is analogous to the safety loading in the classical expected value premium principle. Each combination of these parameters reflects a specific reinsurer’s preference in pricing risk.

Incorporating the above two features to the dynamic reinsurance model substantially increases the difficulty of deriving an optimal reinsurance strategy. To resolve this, we rely on dynamic programming and obtain the optimal contract and the value function explicitly. We demonstrate, under the mean-CVaR premium principle, the optimal reinsurance policies are special cases of dual excess-of-loss reinsurance policy, depending on the relative magnitude of premium loading $\theta$ and the tail risk loading $\beta$. The dual excess-of-loss reinsurance is a newly defined treaty that is a combination of both capped excess-of-loss reinsurance and a standard excess-of-loss reinsurance. This is a novel finding as it contradicts to the existing results that standard reinsurance treaties such as the excess-of-loss reinsurance and proportional reinsurance need not always be optimal in the dynamic setting. By relaxing the feasible reinsurance treaty to a class of increasing functions, the more complicated dual excess-of-loss reinsur-
ance with multiple layers can be optimal in the dynamic reinsurance model.

Both optimal reinsurance policies have some similarities and differences. A key similarity is that the insurer in both cases is concerned with insolvency triggered by large losses. Hence insurer reinsurance large losses in order to minimize the desirable ruin probability. Another similarity found in the two optimal reinsurance policies is that the indemnity is capped at a fixed amount for medium losses. A main difference between the two reinsurance policies is on reinsuring small losses. If \( \theta \leq \beta \), the reinsurer imposes greater penalty on large losses relative to small losses. In this case it is desirable for the insurer to fully insure small losses. On the other hand, if \( \theta > \beta \), i.e., it is relatively more costly to insure against small losses, then it is optimal for the insurer to retain small losses.

The main contributions of our paper are as follows. First, we propose the Mean-CVaR premium principle which generalizes the expected value principle and Denneberg’s absolute deviation principle. The new premium principle is flexible enough to reflect the reinsurer’s risk attitude. Second, we study dynamic reinsurance design beyond the standard expected value principle and variance principle. Moreover, we require the insurer and reinsurer are obligated to pay more for a larger loss to rule out moral hazard. These new features can provide new directions in the future dynamic reinsurance research. Third, we derive two types of the optimal contracts explicitly which are new in the dynamic reinsurance literature. The forms of these contracts are interesting in their own rights.

The rest of the paper is organized as follows. Section 2 introduces the Denneberg’s absolute deviation principle and Mean-CVaR premium principle. Section 3 formulates the dynamic reinsurance model. In Section 4, we derive the Hamilton-Jacobi-Bellman equation and characterize the optimal reinsurance strategy explicitly. Section 5 concludes the paper and Appendix A contains the proof to Lemma A.1.
2 A proposed mean-CVaR premium principle

We begin the section by defining $Z$ as a non-negative loss random variable with distribution function $F_Z$. We further assume that a (re)insurance policy written on a risk $Z$ is priced according to a selected premium principle. Mathematically, a premium principle $\pi(Z)$ is a mapping from $Z$ to the nonnegative real line. There exists a wide array of premium principles; see, for example, Young (2004) who provides a comprehensive survey on premium principles, together with their desirable properties. Among these premium principles, one particular premium principle that is of interest to this paper is attributed to Denneberg (1990). This principle is known as the Denneberg’s absolute deviation principle and is formally defined as follows:

**Definition 2.1.** The Denneberg’s absolute deviation principle is given by

$$\pi(Z) = \mathbb{E}(Z) + \rho \tau(Z).$$

where $0 \leq \rho < 1$ and $\tau(Z)$ is the average absolute deviation from the median.

As shown in Denneberg (1990), this premium principle recovers the expected value premium principle

$$\pi(Z) = (1 + \rho)\mathbb{E}(Z)$$

for some special distributions satisfying $F_Z(0) \geq 0.5$. Moreover, the Denneberg’s absolute deviation principle is related to two popular risk measures known as the Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR). To see this, let us denote the generalized inverse $F_Z^{-1}$ as the quantile function of $F_Z$. Formally, the quantile function of $Z$ is defined as

$$F_Z^{-1}(t) = \inf\{z \in \mathbb{R} : F_Z(z) \geq t\}, 0 < t \leq 1.$$  

Then VaR and CVaR are defined as belows:

**Definition 2.2.** The VaR and CVaR of a random variable $Z$ at a confidence level $\alpha$
(with \(0 < \alpha < 1\)) are defined, respectively, as

\[
VaR_\alpha(Z) \triangleq \inf \{ z \in \mathbb{R} : F_Z(z) \geq \alpha \} = F_Z^{-1}(\alpha),
\]

(2)

and

\[
CVaR_\alpha(Z) \triangleq \frac{1}{1 - \alpha} \int_\alpha^1 VaR_\alpha(Z)ds = \frac{1}{1 - \alpha} \int_\alpha^1 F_Z^{-1}(s)ds,
\]

(3)

provided that the integral exists.

The relation between the Denneberg’s absolute deviation principle and VaR is obvious by recognizing that when \(\alpha = 0.5\), \(VaR_{0.5}(Z)\) becomes the median of \(Z\). Its connection to CVaR can be shown as follow:

\[
\pi(Z) = \mathbb{E}(Z) + \rho \tau(Z)
\]

\[
= \int_0^1 F_Z^{-1}(t)dt + \rho \int_0^1 |F_Z^{-1}(t) - F_Z^{-1}(0.5)|dt
\]

\[
= \int_0^{1/2} F_Z^{-1}(t)(1 - \rho)dt + \int_{1/2}^1 F_Z^{-1}(t)(1 + \rho)dt
\]

\[
= (1 - \rho) \int_0^1 F_Z^{-1}(t)dt + 2\rho \int_{1/2}^1 F_Z^{-1}(t)dt
\]

\[
= (1 - \rho) \mathbb{E}(Z) + \rho CVaR_{1/2}(Z).
\]

The above representation provides a useful interpretation of the Denneberg’s absolute deviation principle. It demonstrates that under the Denneberg’s absolute deviation principle, the pricing of a risk is a weighted average between the mean and the \(CVaR\) at the 50% confidence level.

Inspired by the above insight, we propose the following premium principle, which we denote as the Mean-CVaR premium principle:

**Definition 2.3.** The Mean-CVaR premium principles is defined as

\[
\pi(Z) = \frac{1 + \theta}{1 + \beta} \left[ \mathbb{E}(Z) + \beta CVaR_\alpha(Z) \right],
\]

(4)

where \(\theta, \beta \geq 0\) and \(\alpha \in [0, 1]\).
Note that for a given insurance risk $Z$, the proposed Mean-CVaR premium principle can similarly be interpreted as some kind of weighted average in term of its mean and its $CVaR$ at the confidence level $\alpha$. Furthermore, the proposed premium principle is completely specified by three parameters $\alpha, \theta, \beta$. Each of these parameters plays an intricate role in pricing risk. More specifically, $\alpha$ measures the risk aversion of the reinsurer for underwriting tail risk. A higher $\alpha$ implies that the reinsurer is more concerned with the underwritten tail risk and thus imposes a greater reinsurance premium (since CVaR increases with $\alpha$). The magnitude of the penalty for accepting the tail risk is exacerbated by the parameter $\beta$, which is similar to the weighting parameter $\rho$ in the Denneberg’s absolute deviation principle. A larger $\beta$ leads to a greater penalty on the tail risk and hence a larger premium. For this reason, $\beta$ can be interpreted as the tail risk loading. Finally, the parameter $\theta$ is analogous to the safety loading in the classical expected value premium principle. For this reason, we refer $\theta$ as the premium loading.

The flexibility provided by parameters $\alpha, \theta, \beta$ suggests that some important premium principles can be viewed as special cases of the proposed premium principle. In particular, by setting $\beta = 0$ or $\alpha = 0$, the Mean-CVaR premium principle reduces to the expected value premium principle. When $\theta = 0$, $\beta = \frac{\rho}{1-\rho}$, and $\alpha = 0.5$, the Mean-CVaR premium principle reduces to the Denneberg’s absolute deviation principle. In other words, when $\theta = 0$ and $\beta = \frac{\rho}{1-\rho}$, the Mean-CVaR premium principle can be regarded as a generalization of the Denneberg’s absolute deviation principle for $\alpha \in [0, 1]$.

To conclude this section, we point out that the proposed Mean-CVaR premium principle satisfy many of the desirable properties of premium principles, such as independence, risk loading, maximal loss, translation invariance, scale invariance, sub-additivity, comonotonic additivity, monotonicity, preservation of the first stochastic dominance ordering, preservation of the stop-loss ordering and continuity, as stipulated in Young (2004).
3 Formulation of the dynamic reinsurance model

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with filtration \(\{\mathcal{F}_t\}\). We assume that an insurer’s surplus process can be described by the following classical Cramér-Lundberg (CL) model:

\[
R_t = R_0 + pt - \sum_{i=1}^{N_t} Z_i, \tag{5}
\]

where \(p > 0\) is the premium rate on the risk that the insurer is insuring, \(R_0\) is the initial reserve (or surplus), \(N_t\) is the Poisson process of the arising claims and \(Z_i\) is a sequence of i.i.d. random variables that represents the sizes of successive losses (i.e. the payout on the insurance that the insurer is providing). Without loss of generality, the intensity of the process \(N_t\) is assumed to be 1. The loss random variable \(Z_i\) is also assumed to have cumulative distribution function \(F_Z(\cdot)\), quantile function \(F_Z^{-1}(\cdot)\) and that \(p > E[Z_i]\).

Consider now the insurer is using reinsurance to risk managing its risk exposure. Without reinsurance, the insurer is fully responsible for all arising losses \(Z_i, i = 1, 2, \ldots\). In the presence of reinsurance, a portion of each arising loss \(Z_i\) will be ceded to the reinsurer. The precise coverage is dictated by the reinsurance policy \(I(Z_i)\), which is a function of \(Z_i\). For this reason, \(I(Z_i)\) is known as the ceded loss function and the residual loss that is retained by the insurer, i.e. \(H(Z_i) = Z_i - I(Z_i)\) is known as the retained loss function.

The presence of reinsurance modifies the risk exposure of the insurer by distorting the cash inflow and the cash outflow of the surplus model (5). By denoting \(p(I)\) as the reinsurance premium rate that corresponds to a given reinsurance policy \(I\), then the insurer’s surplus process in the presence of reinsurance, \(R^I_t\), becomes

\[
R^I_t = R_0 + (p - p(I))t - \sum_{i=1}^{N_t} (Z_i - I(Z_i)). \tag{6}
\]

Note that with the reinsurance strategy, the insurer’s risk exposure of each arising claim is reduced from \(Z_i\) to \(Z_i - I(Z_i)\) but at the expense of a smaller net premium income rate which declines from \(p\) to \(p - p(I)\). In practice reinsurance strategy exists
in various forms and shapes, and hence so is the associated cost. Thus the quest to optimal reinsurance boils down to an optimal tradeoff between risk transfer and risk retention.

Corresponding to the surplus process (6), some measures that are of interest to the insurer are the ruin time and the probability of ruin. Formally the ruin time, $\tau^I$, is defined as

$$\tau^I = \inf\{t > 0 | R^I_t \leq 0\},$$

which gives the first time the insurer becomes insolvent, or equivalently the first time the insurer’s surplus falls below zero, given its initial surplus level $R_0$ and reinsurance policy $I$. By denoting $V^I(x)$ as the insurer’s probability of ruin given a reinsurance policy $I$, then it follows from the definition of ruin time that

$$V^I(x) = P(\tau^I < \infty \,| \, R^I_0 = x).$$

(7)

Since the surplus process (6) depends on the reinsurance policy $I$ and its reinsurance premium $p(I)$, so is the ruin probability $V^I$.

In practice the pricing of the reinsurance policy is exogeneous to the insurer but the choice of reinsurance policy is not. Accordingly an optimal choice of reinsurance policy can be selected depending on the chosen objective. As pointed out in the introduction, determining an optimal reinsurance strategy by minimizing the ruin probability $V^I(x)$ is one of the commonly employed criterion (see Schmidli, 2001, 2002; Taksar and Markussen, 2003; Promislow and Young, 2005; Luo et al., 2008; Zhang et al., 2016; Hipp and Vogt, 2003; Dickson and Waters, 2006; Zhang et al., 2016). In this paper we similarly adopt the same criterion.

We now describe the additional assumptions of our optimal reinsurance model. Let us first discuss the reinsurance treaty $I$. Recall that $Z_t = I(Z_t) + H(Z_t)$. Here both $I$ and $H$ are measurable functions. As pointed out in the introduction, the reinsurance treaty $I(Z_t)$ is typically assumed to be a proportional reinsurance or an excess-of-loss reinsurance. In our formulation, the ceded loss function $I$ is restricted to the following
set $\mathcal{I}$:

$$
\mathcal{I} := \left\{ I : [0, \infty] \to [0, \infty] \mid 0 \leq I(y) - I(x) \leq y - x, \ \forall \ x \leq y \right\}.
$$

(8)

The increasing assumption on the ceded loss function $I$ and the retained loss function $H$ is vital in that it reduces the insurer’s moral hazard. While the increasing and Lipschitz continuous condition is often imposed in static optimal reinsurance models (see for example Chi and Tan, 2011; Chi and Lin, 2014; Xu et al., 2015; Boonen et al., 2016; Zhuang et al., 2016), to the best of our knowledge such condition has not been explicitly imposed in the optimal dynamic reinsurance model.

The second assumption relates to the pricing of reinsurance treaty. We assume that $p(I)$ is determined by

$$
p(I) = \pi(I(Z_i)),
$$

where $\pi(\cdot)$ is our proposed Mean-CVaR premium principle defined in (4). By using the change-of-variable technique, we obtain

$$
\pi(I(Z_i)) = \frac{1 + \theta}{1 + \beta} \left[ \mathbb{E}[I(Z_i)] + \beta CVaR_\alpha(I(Z_i)) \right] \\
= \frac{1 + \theta}{1 + \beta} \left[ \int_0^1 F_{I(Z_i)}^{-1}(t)dt + \frac{\beta}{1 - \alpha} \int_0^1 I(Z_i)^{-1}(t)dt \right] \\
= \frac{1 + \theta}{1 + \beta} \left[ \int_0^1 I(F_Z^{-1}(t))dt + \frac{\beta}{1 - \alpha} \int_\alpha^1 I(F_Z^{-1}(t))dt \right] \\
= \frac{1 + \theta}{1 + \beta} \left[ \int_0^\infty I(z)dF_Z(z) + \frac{\beta}{1 - \alpha} \int_{F_Z^{-1}(\alpha)}^\infty I(z)dF_Z(z) \right] \\
= \int_0^{+\infty} I(z)g(z)dF_Z(z)
$$

where

$$
g(z) = \begin{cases} 
1 + k_2, & 0 \leq z \leq F_Z^{-1}(\alpha), \\
1 + k_1, & F_Z^{-1}(\alpha) \leq z.
\end{cases}
$$

with

$$
k_1 = \frac{(1 + \theta)(1 - \alpha + \beta)}{(1 - \alpha)(1 + \beta)} - 1
$$

(9)
This result implies that $\pi(I(Z))$ can equivalently be represented in term of $g(\cdot)$.

In addition to the above assumed premium principle, we further impose the following condition:

**Assumption 3.1.** $E[Z_i] < p < \pi(Z_i) = \int_0^{+\infty} zg(z)dF_Z(z)$.

This condition $E[Z_i] < p$ is fundamental to the insurance pricing as it stipulates that the insurer has a positive safety loading on the underwritten risk; otherwise ruin is certain. Moreover, the condition $p < \pi(Z_i)$ ensures that reinsurance is noncheap. If reinsurance is cheap, then the insurer can simply eliminate all its risk exposure by ceding all arising claims to the reinsurer, reaping the certain profit of $p - \pi(Z_i)$ with zero ruin probability.

Finally, instead of analyzing the surplus process (6) directly, we focus on a diffusion process of the following form

$$dR_t = (p - p(I) - E[Z_i - I(Z_i)])dt + \sqrt{E[(Z_i - I(Z_i))^2]}dB_t,$$

where $B_t$ is a standard Brownian motion. The above diffusion process is an approximation to the surplus process (6), as justified in Grandell (2012).

To continue, it is useful to define the following notation:

- $c := p - E[Z_i]$,
- $\mu(I) := -p(I) + E[I(Z_i)] = -\int_0^{+\infty} I(z)dF_Z(z)$,
- $\sigma^2(I) := E[(Z_i - I(Z_i))^2] = \int_0^{+\infty} (z - I(z))^2dF_Z(z)$,

where

$$d(z) := g(z) - 1 = \begin{cases} k_2, & 0 \leq z < F_Z^{-1}(\alpha), \\ k_1, & F_Z^{-1}(\alpha) \leq z. \end{cases}$$

Suppose the insurer chooses the reinsurance policy $I_t$ at time $t$. An admissible reinsurance policy $I$ is described by a $(\mathcal{F}_t)_{t\geq0}$-adapted stochastic process $I_t(Z)$, where
$I_t \in \mathcal{I}, \forall t \geq 0$. Also, by denoting $R_t^I$ as the reserve process corresponding to an admissible policy $I$, the dynamics of the reserve process can be written as

$$dR_t^I = (c + \mu(I_t))dt + \sigma(I_t)dB_t.$$  \hspace{1cm} (11)

The objective of this paper therefore boils down to the determination of an optimal reinsurance treaty $I_t$ that minimizes the insurer’s ruin probability $V^I(R_0)$. Formally the optimal reinsurance model can be formulated as follows:

$$\inf_{I \in \mathcal{I}} V^I(R_0).$$  \hspace{1cm} (12)

As shown in the next section, the above optimization problem can be solved via the dynamic programming approach.

## 4 Solution

In this section, we demonstrate that the dynamic programming approach can be used to solve the optimal reinsurance model (12). This entails solving the Hamilton-Jacobi-Bellman (HJB) equation as shown in the following subsection.

### 4.1 HJB equation

Let $V(x)$ be the minimal ruin probability, i.e., $V(x) = \inf_{I \in \mathcal{I}} V^I(x)$. Following the standard argument in the literature (e.g. Yong and Zhou, 1999; Taksar and Markussen, 2003; Fleming and Soner, 2006), $V$ solves the following Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \min_{I \in \mathcal{I}} \left[ (c + \mu(I))V'(x) + \frac{1}{2}\sigma^2(I)V''(x) \right].$$  \hspace{1cm} (13)

In what follows we seek a twice continuously differentiable solution to (13) subject
to the following boundary conditions

\[ V(0) = 1 \quad \text{and} \quad V(\infty) = 0. \]

These boundary conditions reflect the fact that the probability of ruin approaches 1 when the surplus declines to 0 and the probability of ruin approaches 0 when the surplus tends to infinity.

We make the ansatz \( V(x) = e^{-ax} \) for some positive constant \( a \). The inner optimization in (13) then becomes

\[
h(a) := \min_{I \in \mathcal{I}} J(I; a), \tag{14}
\]

where

\[
J(I; a) := -\mu(I) + \frac{a}{2} \sigma^2(I) = \int_0^{+\infty} \left\{ I(z)d(z) + \frac{a}{2}(z - I(z))^2 \right\} dF_Z(z). \tag{15}
\]

Note that the optimization problem (14) is a function of \( a \). If such optimization problem can be solved with a fixed \( a > 0 \), then the solution to the HJB equation (13) can be determined by solving \( h(a) = c \).

### 4.2 Relaxed Solutions

We just pointed out the reason for considering the optimization problem (14). Unfortunately it is difficult to solve (14) directly due to the requirement \( I \in \mathcal{I} \). To circumvent this difficulty, we relax (14) by modifying the feasible set of \( I \) from \( \mathcal{I} \) to \( \{0 \leq I(z) \leq z\} \) and consider the following relaxed problem:

\[
\min_{0 \leq I(z) \leq z} J(I; a) = \min_{0 \leq I(z) \leq z} \int_0^{+\infty} \left\{ I(z)d(z) + \frac{a}{2}(z - I(z))^2 \right\} dF_Z(z). \tag{16}
\]

The above optimization problem is readily solved by recognizing that \( I(z)d(z) + \frac{a}{2}(z - I(z))^2 \) is strictly convex in \( I(z) \). Thus, by the pointwise optimization the optimal
solution of the relaxed problem (16) can be represented as

$$\tilde{I}(z; a) = \min \left\{ z, \max \left[ 0, z - \frac{d(z)}{a} \right] \right\}. \quad (17)$$

To further analyze the above optimal solution, it is useful to note that $k_1 = k_2 + \frac{(1+\theta)\beta}{(1+\beta)(1-\alpha)}$, $k_2 \leq k_1$ and $k_1 > 0$. The first property follows immediately from the definitions of $k_1$ and $k_2$ given in (9) and (10). The second property $k_2 \leq k_1$ follows from the conditions $\theta, \beta \geq 0$ and $\alpha \in [0, 1]$. The third property can be justified via contradiction. Suppose $k_1 \leq 0$. Then the second property implies that $k_2 \leq 0$. Furthermore, we also have

$$\int_0^{+\infty} zg(z)dF_Z(z) - \int_0^{+\infty} zdF_Z(z) = \int_0^{+\infty} zd(z)dF_Z(z) \leq 0,$$

which violates Assumption 3.1. Hence $k_1$ must be strictly positive.

We now demonstrate that by exploiting the properties of $k_1$ and $k_2$ enable us to re-express the optimal solution $\tilde{I}(z; a)$ into the following two representations, depending on the relative magnitude of $\theta$ and $\beta$:

**Case 4.1.** $k_1 > 0$ and $\theta \leq \beta$.

We have $k_2 \leq 0$ so that $\tilde{I}(z; a)$ can equivalently be represented as

$$\tilde{I}(z; a) = \begin{cases} z, & 0 \leq z < F_Z^{-1}(\alpha), \\ \max\{0, z - \frac{k_1}{a}\}, & F_Z^{-1}(\alpha) \leq z. \end{cases} \quad (18)$$

**Case 4.2.** $k_1 > 0$ and $\theta > \beta$.

We have $k_2 > 0$ so that $\tilde{I}(z; a)$ can be represented as

$$\tilde{I}(z; a) = \begin{cases} \max\{0, z - \frac{k_2}{a}\}, & 0 \leq z < F_Z^{-1}(\alpha), \\ \max\{0, z - \frac{k_1}{a}\}, & F_Z^{-1}(\alpha) \leq z. \end{cases} \quad (19)$$

It is important to point out that both (18) and (19) do not satisfy the requirement that $\tilde{I}(z; a) \in \mathcal{I}$ as they are not monotone at $F_Z^{-1}(\alpha)$. To highlight this, the ceded
loss functions given by (18) and (19) are depicted in Figures 1 and 2, respectively. For (18) there are two possible shapes as shown in either blue dashed lines or red solid lines in Figure 1, depending on the relative magnitude of $F^{-1}_Z(\alpha)$ and $\frac{k_2}{a}$. For (19) we only plot the ceded loss functions for cases $\frac{k_2}{a} < F^{-1}_Z(\alpha) \leq \frac{k_1}{a}$ and $\frac{k_2}{a} < \frac{k_1}{a} < F^{-1}_Z(\alpha)$, as shown in blue solid lines and red dashed lines, respectively, in Figure 2. Note that we do not illustrate the case $F^{-1}_Z(\alpha) \leq \frac{k_2}{a} < \frac{k_1}{a}$ as the optimal contract reduces to the standard excess-of-loss treaty. Clearly, all of the ceded loss functions $\tilde{I}(z; a)$ depicted in both Figures 1 and 2 do not satisfy the condition that $\tilde{I}(z; a) \in \mathcal{I}$. As pointed out in the introduction that this phenomenon could trigger ex post moral hazard and therefore signifies the importance of imposing condition (8) for determining the optimal dynamic reinsurance policies.

Figure 1: The illustrative ceded loss functions (18). The blue dashed lines correspond to the case $F^{-1}_Z(\alpha) > \frac{k_1}{a}$ and the red solid lines correspond to the case $F^{-1}_Z(\alpha) \leq \frac{k_1}{a}$. 
Figure 2: The illustrative ceded loss functions (19). The blue solid lines correspond to the case $\frac{k_1}{a} < F_Z^{-1}(\alpha) \leq \frac{k_1}{a}$ and the red dashed lines correspond to the case $\frac{k_1}{a} < F_Z^{-1}(\alpha)$.

4.3 Modified Solutions

In this subsection, we modify (18) and (19) so that they can satisfy the requirement that $I \in \mathcal{I}$.

Lemma 4.1. Consider Case 4.1 with $k_1 > 0$ and $\theta \leq \beta$. For any given feasible reinsurance treaty $I \in \mathcal{I}$, there exists a reinsurance treaty of the form

$$I^*(z) = I(z; a, m) = \begin{cases} 
    z, & 0 \leq z \leq m, \\
    m, & m < z \leq m + \frac{k_1}{a}, \\
    z - \frac{k_1}{a}, & m + \frac{k_1}{a} < z,
\end{cases} \quad (20)$$

for some $m \leq F_Z^{-1}(\alpha)$, such that $J(I^*; a) \leq J(I; a)$.

Proof. For any given feasible contract $I \in \mathcal{I}$, let us construct a reinsurance treaty of the form (20) based on

$$m := \max \left[ I(F^{-1}(\alpha)), F^{-1}(\alpha) - \frac{k_1}{a} \right].$$
Then, using (15), we have

\[
J(I^*; a) - J(I; a) \leq \int_m^{k_1 + m} \left\{ I^*(z)d(z) + \frac{a}{2}(z - I^*(z))^2 \right\} dF_Z(z) \\
- \int_m^{k_1 + m} \left\{ I(z)d(z) + \frac{a}{2}(z - I(z))^2 \right\} dF_Z(z) \\
\leq 0,
\]

where the last inequality follows from the fact that \( yd(z) + \frac{a}{2}(z - y)^2 \) is strictly convex in \( y \) and that

\[
\tilde{I}(z; a) \geq I^*(z) \geq I(z), \quad m \leq z \leq F_Z^{-1}(\alpha), \\
\tilde{I}(z; a) \leq I^*(z) \leq I(z), \quad F_Z^{-1}(\alpha) < z \leq \frac{k_1}{a} + m.
\]

Analogously we have a similar lemma based on Case 4.2. The proof of this lemma is also very similar to the proof of Lemma 4.1, as we will see shortly.

**Lemma 4.2.** Consider Case 4.2 with \( k_1 > 0 \) and \( \theta > \beta \). For any given feasible reinsurance treaty \( I \in \mathcal{I} \), there exists a reinsurance treaty of the form

\[
I^*(z) = I(z; a, m) = \begin{cases} 
0, & 0 \leq z \leq \frac{k_2}{a}, \\
\quad z - \frac{k_2}{a}, & \frac{k_2}{a} < z \leq m + \frac{k_2}{a}, \\
\quad m, & m + \frac{k_2}{a} < z \leq m + \frac{k_1}{a}, \\
\quad z - \frac{k_1}{a}, & m + \frac{k_1}{a} < z,
\end{cases}
\]

(21)

for some \( m \leq \max\{0, F_Z^{-1}(\alpha) - \frac{k_2}{a}\} \), such that \( J(I^*; a) \leq J(I; a) \).

**Proof.** For any given feasible reinsurance treaty \( I \in \mathcal{I} \), let us construct a reinsurance treaty of the form (21) with \( m \) determined from

\[
m := \min \left\{ \max \left[ 0, F_Z^{-1}(\alpha) - \frac{k_2}{a} \right], \max \left[ I(F_Z^{-1}(\alpha)), F_Z^{-1}(\alpha) - \frac{k_1}{a} \right] \right\}
\]

Suppose that \( m > 0 \). Using (15), together with the fact that \( yd(z) + \frac{a}{2}(z - y)^2 \) is
strictly convex in $y$ and that
\[
\tilde{I}(z; a) \geq I^*(z) \geq I(z), \quad \frac{k_2}{a} + m \leq z \leq F^{-1}_Z(\alpha),
\]
\[
\tilde{I}(z; a) \leq I^*(z) \leq I(z), \quad F^{-1}_Z(\alpha) < z \leq \frac{k_1}{a} + m,
\]
we similarly establish the required result that
\[
J(I^*; a) - J(I; a) \leq \int_{\frac{k_1}{a} + m}^{k_2 + m} \left\{ I^*(z) d(z) + \frac{a}{2} (z - I^*(z))^2 \right\} dF_Z(z)
- \int_{\frac{k_2}{a} + m}^{k_1 + m} \left\{ I(z) d(z) + \frac{a}{2} (z - I(z))^2 \right\} dF_Z(z)
\leq 0.
\]

Otherwise, $m = 0$ and $I^*(z) = \max\{0, z - \frac{k_1}{a}\} = \tilde{I}(z; a)$ which is optimal. \hfill \square

Armed with the above two lemmas, let us now consider the following optimization problem:
\[
h(a) = \min_{m \leq F^{-1}_Z(\alpha)} J(I(z; a, m); a)
\]  
(22)

where $J(I; a)$ is given by (15) and $I(z; a, m)$ has the form (20) if $\theta \leq \beta$ or (21) if $\theta > \beta$. The significance of the above formulation of the optimization problem is twofold. First, both Lemma 4.1 and Lemma 4.2 imply that (22) is equivalent to (14). Second, and more importantly, in comparison to (14), (22) is a relatively easier problem in that it is a one-dimensional optimization problem. These features are used explicitly in the following lemma.

**Lemma 4.3.** For any $a > 0$, there exists a unique $I^*(z; a) \in \mathcal{I}$ such that $I^*(z; a)$ solves (14), i.e. $\min_{I \in \mathcal{I}} J(I; a)$. Moreover, $I^*(z; a) = I(z; a, m^*)$ where $m^*$ is the unique minimizer of (22).

**Proof.** From Lemma 4.1 and Lemma 4.2, (14) is equivalent to (22). The existence of the optimal $m^*$ can be proved by the Weierstrass extreme value theorem.

Suppose that both $m_1$ and $m_2$ are optimal to (22), then both $I^1 := I(z; a, m_1)$ and
\(I^2 := I(z; a, m_2)\) are optimal solutions of (14). By defining
\[
\bar{I} := \frac{1}{2} I^1 + \frac{1}{2} I^2,
\]
it is easy to see that \(\bar{I}\) is still a feasible solution. Because \(J(I; a)\) is strictly convex in \(I\), we have
\[
J(\bar{I}; a) < \frac{1}{2} J(I^1; a) + \frac{1}{2} J(I^2; a) = J(I^1; a) = J(I^2; a),
\]
which contradicts the optimality of \(I^1\) and \(I^2\). Thus \(m^*\) is the unique minimizer of (22) and this completes the proof.

Recall that ultimately we are interested in solving (13). We have also pointed out that solving (13) is facilitated by seeking \(a^*\) that is the solution to \(h(a^*) = c\). The lemma below establishes the uniqueness of the positive solution \(a^*\).

**Lemma 4.4.** There is a unique positive \(a^*\) such that \(h(a^*) = c\).

**Proof.** In view of the envelope theorem, \(h(a)\) is continuous and strictly increasing. Because \(I(z) \equiv 0\) is a feasible solution, we have
\[
h(a) \leq \int_0^{+\infty} \frac{a}{2} z^2 dF_Z(z).
\]
Thus,
\[
\lim_{a \downarrow 0} h(a) \leq \lim_{a \downarrow 0} \int_0^{+\infty} \frac{a}{2} z^2 dF_Z(z) = 0.
\]
Next,
\[
\lim_{a \uparrow +\infty} h(a) \geq \lim_{a \uparrow +\infty} \int_0^{+\infty} \left\{ \bar{I}(z; a) d(z) + \frac{a}{2} (z - \bar{I}(z; a))^2 \right\} dF_Z(z)
\]
\[
= \lim_{a \uparrow +\infty} \int_0^{+\infty} \left\{ zd(z) 1_{d(z) \leq 0} + |zd(z)| - \left( \frac{d(z)}{2a} \right)^2 [1_{0 < \frac{d(z)}{a} < z} + \frac{a}{2} z^2 1_{\frac{d(z)}{a} \geq z}] \right\} dF_Z(z)
\]
\[
\geq \lim_{a \uparrow +\infty} \int_0^{+\infty} \left\{ zd(z) 1_{d(z) \leq 0} + |zd(z)| - \left( \frac{d(z)}{2a} \right)^2 [1_{0 < \frac{d(z)}{a} < z}] \right\} dF_Z(z)
\]
\[
= \int_0^{+\infty} zd(z) dF_Z(z),
\]
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where the convergence follows from the monotone convergence theorem. From Assumption 3.1, it follows that
\[
\lim_{a \uparrow \infty} h(a) > c
\]
and hence there is a unique positive \(a^*\) such that \(h(a^*) = c\).

Collecting all the above results, the procedure for finding the optimal solution \(I^*\) of (13) involves the following three steps:

**Step 1.** Fix \(a\) and determine the minimizer \(m^*(a)\) of (22). Note that \(m^*(a)\) is a function of \(a\).

**Step 2.** Find \(a^*\) such that \(h(a^*) = c\).

**Step 3.** Find the optimal contract \(I^*\).

If \(\theta \leq \beta\), then
\[
I^*(z) = \begin{cases} 
  z, & 0 \leq z \leq m^*(a^*), \\
  m^*(a^*), & m^*(a^*) < z \leq m^*(a^*) + \frac{k_1}{a^*}, \\
  z - \frac{k_2}{a^*}, & m^*(a^*) + \frac{k_1}{a^*} < z.
\end{cases}
\]

Else if \(\theta > \beta\), then
\[
I^*(z) = \begin{cases} 
  0, & 0 \leq z \leq \frac{k_2}{a^*}, \\
  z - \frac{k_2}{a^*}, & \frac{k_2}{a^*} < z \leq m^*(a^*) + \frac{k_2}{a^*}, \\
  m^*(a^*), & m^*(a^*) + \frac{k_2}{a^*} < z \leq m^*(a^*) + \frac{k_1}{a^*}, \\
  z - \frac{k_1}{a^*}, & m^*(a^*) + \frac{k_1}{a^*} < z.
\end{cases}
\]

### 4.4 Verification Theorem

The objective of this subsection is proving the verification theorem.

**Theorem 4.1** (Verification theorem). The optimal reinsurance treaty is either of the form (23) if \(\theta \leq \beta\) or (24) if \(\theta > \beta\), and the minimal ruin probability is
\[
V(x) = e^{-a^*x},
\]
where \( a^* \) solves \( h(a^*) = c \).

**Proof.** It is not difficult to see that \( V(x) \) given by (25) is

\[
P(\tau^I < \infty | R_0^I = x),
\]

where \( I^* \) corresponds to either (23) or (24). It suffices to show that \( I^* \) is optimal.

For any admissible policy \( I \) such that the stochastic differentiate equation (11) admits a unique solution for \( R_0 = x, \forall x \geq 0 \), by Dynkin’s formula we have that

\[
\mathbb{E}[V(R_t^I | R_0^I = x)] = V(x) + \mathbb{E} \int_0^{\tau^I \wedge t} (c + \mu(I_s)V'(R_s^I) + \frac{1}{2} \sigma^2(I_s)V''(R_s^I)) ds | R_0 = x
\]

Then the HJB equation (13) yields

\[
\mathbb{E}[V(R_t^I | R_0^I = x)] \geq V(x).
\]

Since \( V \) is continuous and bounded by 1, we then have

\[
\lim_{t \to \infty} \mathbb{E}[V(R_t^I | R_0^I = x)] = \lim_{t \to \infty} \mathbb{E}[V(R_t^I)I_{\tau^I = \infty} | R_0^I = x] + \mathbb{E}[V(R_t^I)I_{\tau^I < \infty} | R_0^I = x] = \mathbb{E}[\lim_{t \to \infty} e^{-a^* R_t^I} I_{\tau^I = \infty} | R_0^I = x] + \mathbb{E}[e^{-a^* R_t^I} I_{\tau^I < \infty} | R_0^I = x].
\]

It follows from Lemma A.1 that either \( \tau < \infty \) or \( \lim_{t \to \infty} R_t^I = \infty \). Consequently,

\[
P(\tau^I < \infty | R_0^I = x) = \lim_{t \to \infty} \mathbb{E} V(R_t^I | R_0^I = x) \geq V(x).
\]

Theorem 4.1 highlights the key contribution of the paper. It attests that in the dynamic setting the optimal reinsurance does not need to be the standard excess-of-loss reinsurance. In particular the optimal reinsurance treaties stipulated by both (23) and (24) are considerably more involved, depending on the relationship between the premium loading \( \theta \) and the risk loading \( \beta \). Both (23) and (24) resemble a reinsurance treaty with multiple layers. More specifically both of these optimal reinsurance are
special cases of the dual excess-of-loss reinsurance.

**Definition 4.1.** The dual excess-of-loss reinsurance treaty with attachment points \((l_1, l_2, l_3)\) is defined as

\[
I_{l_1,l_2,l_3}(z) = \min \left\{ \max\{z - l_1, 0\}, l_2 - l_1 \right\} + \max\{z - l_3, 0\}.
\]

where \(l_1 \leq l_2 \leq l_3\).

The dual excess-of-loss reinsurance with attachment points \((l_1, l_2, l_3)\) is a newly defined reinsurance treaty such that the first attachment point \(l_1\) gives the upper limit of the first layer of loss that is retained by the insurer. For losses that fall in the second layer of loss, i.e. losses between the first and second attachment points \(z \in (l_1, l_2)\), the reinsurer is responsible for the amount of losses in excess of \(l_1\) up to the limit \(l_2\). For losses that lie between second and third attachment points; i.e. third layer of loss with \(z \in (l_2, l_3)\), the reinsurer indemnifies a constant cap amount of \(l_2\). Finally for the spill-over layer of loss \(z > l_3\), the reinsurer is responsible for any additional loss in excess of \(l_3\).

Graphically the dual excess-of-loss reinsurance treaty is depicted in Figure 3. Visually, the dual excess-of-loss reinsurance treaty can be interpreted as a combination of a capped excess-of-loss reinsurance (with retention \(l_1\) and capped \(l_2\)) and a standard excess-of-loss reinsurance (with retention \(l_3\)). This explains using the term “dual excess-of-loss reinsurance” (with appropriate attachment points) to denote the combined ceded loss functions.

Adopting the above definition, it is easy to see that the optimal ceded loss function (23) is a special case of the dual excess-of-loss reinsurance with attachment points

\[
l_1 = 0, \quad l_2 = m^*(a^*) + \frac{k^*_1}{a^*}, \quad l_3 = m^*(a^*) + \frac{k^*_1}{a^*}.
\]

Similarly, the optimal ceded loss function (24) is also a special case of the dual excess-
Figure 3: Dual excess-of-loss reinsurance with attachment points \((l_1, l_2, l_3)\).
of-loss reinsurance with attachment points

\[ l_1 = \frac{k_2}{a^*} , \quad l_2 = m^*(a^*) + \frac{k_2}{a^*} , \quad l_3 = m^*(a^*) + \frac{k_1}{a^*} . \]

Figures 4 and 5 produce plots of these optimal ceded loss functions (23) and (24), respectively. Both optimal policies have some common characteristics but also have a distinctive difference. In both cases, it is optimal for the insurer to reinsure losses beyond a certain large losses in order to minimize its ruin probability, as desired by its objective. For medium losses, it is optimal for the insurer to reinsure a fixed losses. The key distinction between the two optimal reinsurance policies lies on how the initial small losses are reinsured. For (23) we have \( \theta \leq \beta \); i.e. the tail risk loading \( \beta \) is greater than or equal to the premium loading \( \theta \). This implies that the reinsurer imposes a greater penalty on the tail risk by charging (relatively) higher premium on the tail losses than the smaller losses. Hence this encourages insurer to fully reinsure its losses that are smaller than \( m^*(a^*) \). If \( \theta > \beta \) with the optimal reinsurance policy (24), it is relatively more costly to reinsure small losses. As a result, it is optimal for the insurer to retain losses up \( \frac{k_2}{a^*} \). For losses greater than \( \frac{k_2}{a^*} \), the optimal reinsurance is then similar to (23).

To conclude this subsection, we emphasize this maybe the first paper demonstrating that the optimal reinsurance in the dynamic setting can admit more complicated structures (i.e. (23) and (24)). This is in contrast to the various studies that have documented the optimality of the standard excess-of-loss reinsurance. In particular, Meng and Zhang (2010) and Hipp and Taksar (2010) have concluded that the excess-of-loss reinsurance is optimal when the premium is calculated according to the expected value premium principle. Theorem 4.1, however, can be shown to be consistent with some existing results. More specifically, the remark below provides two special cases for which Theorem 4.1 recovers the standard result that the excess-of-loss reinsurance is optimal.
Figure 4: Dual excess-of-loss reinsurance but with attachment points $(0, m^*(a^*), m^*(a^*) + \frac{k_1}{a^*})$; i.e. optimal reinsurance policy (23) with $\theta \geq \beta$.

Figure 5: Dual excess-of-loss reinsurance with attachment points $(\frac{k_2}{a^*}, m^*(a^*) + \frac{k_2}{a^*}, m^*(a^*) + \frac{k_3}{a^*})$; i.e. optimal reinsurance policy (24) with $\theta > \beta$. 

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Remark 4.1. If $\beta = 0$, then the Mean-CVaR premium principle reduces to the expected value principle. In this case, we have
\[ k_1 = k_2 = \theta, \]
and
\[ I^*(z) = \max \left[ 0, z - \frac{\theta}{a^*} \right], \]
where $a^*$ satisfies $h(a^*) = c$.

If $\alpha = 0$, then the optimal $m^*$ in (20) and (21) is equal to 0. Then the optimal reinsurance strategy in both cases reduce to the excess-of-loss treaty, which is consistent with the results in Meng and Zhang (2010) and Hipp and Taksar (2010).

Remark 4.2. When $\theta = 0$, $\beta = \frac{\rho}{1-\rho}$, and $\alpha = 0.5$, i.e., the Mean-CVaR premium principle recovers the Denneberg’s absolute deviation principle, we have $\theta < \beta$ so that the optimal reinsurance contract has the form of (23).

5 Conclusion

In this paper we derive explicitly the optimal reinsurance contract under the Mean-CVaR premium principle, which is a generalization of Denneberg’s absolute deviation principle and classical expected value principle. We impose a monotonicity constraint on the reinsurance contract to eliminate the moral hazard risk, and overcome the difficulties arising from this constraint. The method in deriving the optimal contract is interesting in its own right. We find that there are two types of the optimal reinsurance contract, depending on the premium loading $\theta$ and tail risk loading $\beta$. Moreover, the optimal contracts involve multiple reinsurance layers and are special cases of dual excess-of-loss reinsurance. These findings are in sharp contrast to the existing results in that the optimal reinsurance policies can be more complicated than the standard excess-of-excess reinsurance.

The models and problems considered in this paper can be explored further in different ways. For example, it would be interesting to generalize our results to a more
general class of premium principles, such as the Wang’s premium principle (Wang, 1995, 1996; Wang et al., 1997). In addition, we can consider alternative objectives such as the expected discounted dividends objective as in Taksar and Zhou (1998), Jgaard and Taksar (1999), Asmussen et al. (2000), Choulli et al. (2003), He and Liang (2009). We leave these problems for our future research.

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References


### A Appendix

**Lemma A.1.** For any policy $I$ and any $N > 0$,

$$
\tau^I_N < \infty
$$

almost surely, where $\tau^I_N = \inf\{t \geq 0 : R^I_t \notin (0, N)\}$ with $R^I_t$ given by (11).
Proof. Define
\[ \psi(t) = \int_0^t \sigma^2(I_s) ds, \]
\[ \gamma(t) = \int_0^t (c + \mu(I_s)) ds, \]
\[ \Delta = \int_0^{+\infty} zg(z) dF_Z(z) - p. \]

Assumption 3.1 implies that \( \Delta > 0 \). Define
\[ \eta_0 = 0, \eta_{k+1} = \inf\{ t > \eta_k : \psi(t) - \psi(\eta_k) = \frac{\Delta}{2} \}. \]  

(27)

We next give an upper bound of \( \gamma(t) \) on \( \{ \psi(\eta_{k+1}) - \psi(\eta_k) \leq \frac{\Delta}{2} \} \).

\[ \gamma(t) = \int_{\eta_k}^{\eta_{k+1}} (p - \int_0^\infty I_s(z) g(z) dF_Z(z) + \int_0^\infty (I_s(z) - z) dF_Z(z)) \, ds \]
\[ = \int_{\eta_k}^{\eta_{k+1}} (p - \int_0^\infty zg(z) dF_Z(z) + \int_0^\infty (I_s(z) - z) g(z) dF_Z(z) \]
\[ + \int_0^\infty (I_s(z) - z) dF_Z(z)) \, ds \]
\[ < \int_{\eta_k}^{\eta_{k+1}} (\int_0^\infty (z - I_s(z)) g(z) dF_Z(z) + \int_0^\infty (I_s(z) - z) dF_Z(z)) \, ds \]
\[ \leq \int_{\eta_k}^{\eta_{k+1}} (\sqrt{\int_0^\infty (z - I_s(z))^2 dF_Z(z)} \sqrt{\int_0^\infty g(z)^2 dF_Z(z)} \]
\[ + \sqrt{\int_0^\infty (I_s(z) - z)^2 dF_Z(z)) \, ds} \]
\[ \leq \frac{\Delta}{2} \left( 1 + \sqrt{\int_0^\infty g(z)^2 dF_Z(z)} \right). \]

Suppose that \( \mathbb{P}(\eta_k < \infty) = 1, \forall k \). Following the arguments in Taksar and Markussen (2003), we have that

\[ \mathbb{P}(\tau_N > \eta_n) \leq (1 - \delta_0)^n, \]

with
\[ \delta_0 = \Phi \left( -N + \frac{\Delta}{2} (1 + \sqrt{\int_0^\infty g(z)^2 dF_Z(z)}) \right). \]

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Here $\Phi$ is the cumulative distribution function of a standard normal random variable. Letting $n \to \infty$, we then have that

$$P(\tau^I_N < \infty) = 1.$$ 

Next we suppose that $P(\eta_{m+1} = \infty) = \delta$ for some $m \geq 0$. Then similar to the above derivation we have that

$$\gamma(t) \leq \int_t^\infty -\Delta ds + \frac{\Delta}{2} \left( 1 + \sqrt{\int_0^\infty g(z)^2 dF_Z(z)} \right) = -\infty$$

Define

$$\xi_0 = \eta_m,$$

$$\eta'_{k+1} = \inf \{ s > \xi_k : \psi(s) - \psi(\eta_k) = 1 \},$$

$$\eta''_{k+1} = \inf \{ s : s > \xi_k, \gamma(s) - \gamma(\xi_k) < -K \},$$

$$\xi_{k+1} = \min \{ \eta'_{k+1}, \eta''_{k+1} \},$$

where $K$ is chosen in a way such that

$$\frac{1}{(N - K)^2} \leq \frac{\delta}{2}.$$ 

Then the arguments in Taksar and Markussen (2003) yield that

$$P(\tau^I_N > \xi_n) < (1 - \frac{\delta}{2})^n.$$ 

This completes the proof. \hfill \blacksquare