Semi-static Hedging of Variable Annuities

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Abstract

This paper focuses on hedging financial risks in variable annuities with guarantees. We first review standard hedging strategies to identify pros and cons of each of them. Using standard dynamic hedging techniques, we show that best hedging performance is obtained when the hedging strategy is rebalanced at the dates when the variable annuity fees are collected. Thus, our results suggest that the insurer should incorporate the specificity of the periodic payment of variable annuities fees to construct the hedging portfolio and focus on hedging the net liability. Since standard dynamic hedging is costly in practice because of the large number of rebalancing dates, we propose a new hedging strategy based on a semi-static hedging technique and thus with fewer rebalancing dates. We confirm that this new strategy outperforms standard dynamic hedging as well as traditional semi-static hedging strategies that do not consider the specificity of the payments of fees in their optimization. We also find that short-selling or using put options as hedging instruments gives better hedging performance.

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\textit{Keywords}: hedging variable annuities; semi-static hedging

1. Introduction

Life expectancies are steadily increasing and the population faces the need to guarantee a sustainable income after retirement. There has been a growing demand for variable annuities, which offer guaranteed income for post-retirement life: U.S. and Japan are the two first providers (Hardy (2003)). EIOPA (2011) illustrate a sharp increase in VA volume...
in Europe around 2011. In China, the first VAs were marketed in 2011, it is still a very small volume. A recent state-of-the-art on the Variable Annuities market can be found in Haefeli (2013) with figures and charts illustrating the Variable Annuities market in the U.S., Canada, Europe and Japan between 1990 and 2011.

A typical variable annuity contract consists of an accumulation phase, during which the policyholder invests an initial premium and possibly subsequent premiums into a basket of invested funds. At the end of the accumulation phase, the policyholder may receive a lump sum or may annuitize it to provide retirement income. Variable annuities contain both investment and insurance features. They improve upon traditional fixed annuity contracts that offer a stream of fixed retirement income, as policyholders of variable annuities could expect higher returns in a bull market. Variable annuities also provide some protection against downside risk with several kinds of optional benefit riders. Traditional annuities could simply be hedged by an appropriate portfolio of bonds. But the presence of protection benefits require more sophisticated hedging strategies. In a variable annuity, the policyholder invests her premiums in a separate account, which is protected from downside risks of investing in the financial market through additional options that are sold with the variable annuity and also called benefit riders. The simplest benefit is the Guaranteed Minimum Accumulation Benefit (GMAB). It guarantees a lump sum on a specific future date or anniversary. But it is not as popular as the following more complex benefits, GMIBs and GMWBs. Guaranteed Minimum Income Benefits (GMIB) guarantee a stream of lifetime annuity income after policyholder’s annuitization decision is made. Guaranteed Minimum Withdrawal Benefits (GMWB) guarantee the ability to withdraw a specified percentage of the benefit base during a specified number of years or it could be a lifetime benefit (Kling, Ruez, and Russ (2011)). A Guaranteed Minimum Death Benefit (GMDB) guarantees a specified lump sum benefit at the time of death of the policyholder. More details on each of these guarantees can be found in Hardy (2003).

There are lots of studies on the pricing of variable annuities and how to find the fair fee: Milevsky and Posner (2001) and Bacinello (2003) investigate the valuation of GMDB in variable annuities using risk-neutral pricing. Lin, Tan, and Yang (2009) price simple guarantees in VAs in a regime switching model. Marshall, Hardy, and Saunders (2010) study the value of a GMIB in a complete market, and the sensitivity of the value of GMIB to the financial variables is examined. They suggest that the fee rate charged by insurance companies for GMIB may be too low. GMWBs are studied intensively by several authors including Milevsky and Salisbury (2006), Chen, Vetzal, and Forsyth (2008), Dai, Kwok, and Zong (2008), Kolkiewicz and Liu (2012) and Liu (2010). Milevsky and Salisbury (2006) show that GMWB fees charged in the market are too low and not sustainable and argue that the fees have to increase or the product design should change to avoid arbitrage. Chen, Vetzal, and Forsyth (2008) also conclude that normally charged GMWB fees are not enough to cover the cost of hedging. Bacinello, Millossovich, Olivieri, and Pitacco (2011) and Bauer, Kling, and Russ (2008) propose general valuation framework for variable annuities.
There are fewer papers on hedging, although hedging embedded guarantees of variable annuities is a challenging and crucial problem for insurers. There are three main sources of risks, mortality risks, policyholders’ behavioral risks and financial risks. Let us discuss here these three sources, we will then focus on financial risks in the rest of the paper. When mortality risk is fully diversifiable, it is straightforward to hedge mortality risks by selling independent policies to a group of policyholders with similar risks of death. However, mortality risks cannot always be diversified and VAs are exposed to longevity risk (i.e. a systematic change in mortality risk affecting simultaneously all the population). It is a topic of research by itself and a thorough analysis of longevity risk in VAs can be found in Ngai and Sherris (2011), Hanewald, Piggott and Sherris (2012), Gatzert and Wesker (2012), Fung, Ignatieva, and Sherris (2013) for example.

Behavioral risks in variable annuities come from the uncertainty faced by the insurer about the policyholders’ decisions (choice of surrender, partial withdrawal, annuitization, reallocation, additional contributions, and so on). In general, they are hard to hedge. Under the assumption that investors do not act optimally and base their decisions on non-financial variables, behavioral risk can be diversified similarly as mortality risk by using a deterministic decision making process using historical statistics, which state for instance that $x\%$ of the policyholders follow a given behaviour in a specific situation. However, there is empirical evidence that policyholders may act optimally, or at least that their decision is correlated with some market factors and depends on the moneyness of the guarantee, so that all behavioral risks may not be diversified away (see for instance the empirical study of Knoller, Kraut, and Schoenmaekers (2015)). Kling, Ruez, and Russ (2014) study the impact of behavioral risks on the pricing as well as on the effectiveness of the hedging of VAs. They consider various assumptions on behaviors (from deterministic to optimal decision making). Interestingly, the impact of model misspecification on policyholders’ behaviors depends highly on the design of the policy. For instance, the effect of the surrender decision is more important in VAs without ratchets. MacKay, Augustyniak, Bernard, and Hardy (2015) design VAs that are never optimal to surrender. See also Augustyniak (2013) for a study of the effect of lapsations on the hedging effectiveness of the Guaranteed Minimum Maturity Benefit (GMMB).

In this paper, we ignore mortality and longevity risks, policyholder behavioral risks and focus on hedging financial risks. It can be done via delta hedging, semi-static hedging or static hedging. Guarantees in VAs are similar to options (financial derivatives) on the fund value and the insurer plays the role of option writer. However, they are also very different from standard derivatives. A crucial difference is that the costs of these options are not paid upfront like initial option premiums. By contrast, fees are paid periodically as a percentage of the fund value throughout the life of the contract. The fees collected should then be invested to hedge the provided guarantees. After finding the suitable level of fees to cover the guarantees (fair pricing of VAs), hedging the guarantees consists of investing these collected fees in an appropriate way so that at the time the guarantees must be paid the investment matches the guarantees. The main difficulty is that there
are no certain premiums paid upfront in a VA so that uncertainty and risk are inherent in the payoff and also in the premiums. Therefore, both components should be hedged. In addition, there is an unavoidable mismatch between the (random) value of fees collected from the policyholder’s account and the hedging cost, because the value of collected fees and the cost of hedging move in opposite directions. When the policyholder’s account value is high, the value of the fee is also high while the value of the embedded option in the guarantee is low. If the account value is low, the value of the fee is low but the insurer needs more money for the guaranteed benefits because the value of embedded option is high. This mismatch represents a real challenge for hedging guarantees in VAs. It is also exactly the reason why hedging guarantees in VAs differ from hedging standard options (for which fees are paid at inception only).

The most common approach to hedge financial risks is to perform a dynamic delta-hedging approach (Kling, Ruez, and Russ (2014) and Kling, Ruez, and Russ (2011)) to replicate the embedded guarantees. When hedging a guarantee that depends on a tradable fund \( F_t \) (or at least a replicable fund \( F_t \)), the hedger makes sure that he holds at any time a number of shares of fund equal to the delta of the guarantee (i.e. the sensitivity of the value of the guarantee to a change in the underlying price). Assume that the market is complete, with no transaction costs and the hedge is continuously rebalanced over time in a self-financing manner. Then, a dynamic delta hedging strategy achieves a perfect hedge of the guarantees at the time they must be paid out. An alternative popular dynamic hedging is based on a mean variance criteria and is sometimes referred as “optimal dynamic hedging” (Papageorgiou, Rémillard, and Hocquard (2008), Hocquard, Papageorgiou, and Remillard (2015, 2012)). This latter technique can be applied in incomplete markets.

Instead of dynamic hedging, Hardy (2003, 2000) and Marshall, Hardy, and Saunders (2010, 2012) investigate static hedging and suggest replicating maturity guarantees with a static position in put options. Static hedging consists of taking positions at inception in a portfolio of financial instruments that are traded in the market (at least over-the-counter) so that the future cash-flows of the VA match the future cashflows of the hedge as well as possible. Static hedging strategies have no intermediary costs between the inception and the maturity of the benefits, and tend to be highly robust to model risk because no rebalancing is involved. There are several issues with this approach, in particular the non-liquidity (and non-availability) of the long-term options needed to match the long-term guarantees as well as their exposure to counterparty risk. Also, most of guarantee benefits are path-dependent and therefore are hard to hedge with static hedging using the European path-independent options available in the market. Finally, static hedging of VAs tends to forget about the specificity of the options embedded in VAs. Their premiums are paid in a periodic way affecting the fund value. Thus a more natural hedging strategy should account for the periodicity of these premiums. In a static strategy, the insurer must borrow a large amount of money at the inception of the contract to purchase the hedge. This borrowed money will then potentially be offset by the future fees collected as a percentage of the fund. But the insurer is subject to the risk that the fees collected
in the future do not match the amount borrowed at the beginning to purchase the hedge. Typically, if the contract is fairly priced, only the expected value of the discounted future fees will indeed match the initial cost needed to hedge the guarantees.

Several authors including Coleman, Li, and Patron (2006), Coleman, Kim, Li, and Patron (2007), and Kolkiewicz and Liu (2012) have studied semi-static hedging for variable annuities using options as hedging instruments. In semi-static hedging, the hedging portfolio is rebalanced periodically by following an optimal hedging strategy for some optimality criterion. The hedging portfolio is not altered until the next rebalancing date. Coleman, Li, and Patron (2006) and Coleman, Kim, Li, and Patron (2007) investigate hedging of embedded options in GMDB with ratchet features. By assuming that mortality risk can be diversified away, they reduce the problem of hedging of variable annuities to the hedging of lookback options with fixed maturity. They show that semi-static hedging with local risk minimization is significantly better than delta hedging. Kolkiewicz and Liu (2012) reach similar conclusions for GMWBs. Semi-static hedging outperforms delta hedge, especially when there are random jumps in the underlying price. In this paper, we provide additional evidence of the superiority of semi-static hedging strategies over dynamic delta hedging in the context of hedging guarantees in VAs. We propose a new version of semi-static hedging which utilizes the collected fees right away to construct the hedging portfolio.

One focus in this paper is to explain how to use the periodic fees paid for the guarantees to develop an efficient hedge. Kolkiewicz and Liu (2012) and Augustyniak (2013)\footnote{We refer to Chapter 5 of Augustyniak (2013) on “Measuring the effectiveness of dynamic hedges for long-term investment guarantees”.} point out the importance of hedging the net liability taking into account the periodic fees that are paid as a percentage of the underlying fund. We propose here a new semi-static hedging strategy, which rebalances the hedging portfolio at the exact dates when the fees are paid. We assume that put options written on the underlying market index related to policyholder’s separate investment account as well as the underlying market index itself can be used as hedging instruments for semi-static hedging. In practice, there are several variable annuity products which allow policyholders to decide their own investment decision. In this case, the insurer faces basis risk arising from the fact that the fund in which the premiums are invested is not directly traded. It is thus not directly possible to take long or short positions in the fund and neither to buy options on the fund. For the ease of exposition, we assume that the premium paid by policyholder is invested in a market index, which is actively traded in the market, e.g. S&P 500, or highly correlated to the market index. This assumption gives us a liquid underlying index market and option market for our hedging strategies.\footnote{A rigorous account for basis risk is beyond the scope of this paper. Refer for instance to Adam-Müller and Nolte (2011) for a comparison of standard approaches to hedging basis risk. In the context of variable annuities, it is possible to mitigate basis risk by managing the fund such as to maintain a high correlation to traded indices. Note that we design a strategy that rebalances at each fee payment date. Between two fee payments, the dynamic of the fund is not affected by the fees and there is thus little basis risk if the fund is sufficiently diversified.}
We provide evidence of the following properties of the hedge of variable annuities. First, the effectiveness of the hedge is improved by focusing on the liability net of collected fees (see Section 3 for an illustration). Second, the insurer should generally start to borrow money at time 0 to hedge long term guarantees and anticipate that collected fees will cover this borrowed money in the future. Third, the insurer should be able to short-sell the underlying to perform an efficient hedge of long term guarantees. Fourth, hedging is improved significantly by including put options as hedging instruments in a semi-static hedge which is rebalanced at the dates fees are collected. We find that semi-static hedging is well adapted to VAs due to the periodicity of fee payments. Semi-static hedging is able to utilize short term options that are more liquid and traded at a wider range of strikes. Finally, semi-static hedging of VAs, when accounting for future collected fees, allows to decrease the initial borrowing significantly. We study the semi-static hedging strategies in three environments, first without any constraints when all positions in stocks, bonds and options can be taken, then with short-selling constraints, and finally when put options are not available. Note that the Solvency II regulation does not prevent insurers to short-sell or to trade options. However, variable annuity writers should use derivatives or short-selling for risk management purposes only, and not to speculate (Solvency II recommendations).

Our improved semi-static strategy outperforms the traditional semi-static strategies with local risk minimization. Especially, if short-selling is allowed, our strategy gives much better hedging performance than the other strategies with and without put options as hedging instruments. With our new strategy, the fee collected at time 0 is enough to construct the optimal hedging portfolio at time 0 and almost no borrowing is necessary. We also find that short-selling is necessary to hedge the options embedded in guarantees because the hedging targets are decreasing functions of the underlying index. In the presence of short-selling constraints, put options become more important hedging instruments. The larger the set of strike prices of the put options used for hedging, the better the hedging performance.

In Section 2, we describe a simple GMMB as the toy example to illustrate and compare the different hedging strategies. We then use a standard delta-hedging strategy to show the importance of hedging the net liability in Section 3. In Section 4, we show how to improve delta-hedging strategies by developing a new semi-static hedging strategy. Section 4.3 compares the hedging effectiveness of the optimal semi-static strategy to other hedging strategies. Section 5 concludes.

2. Description of a Variable Annuity Guarantee

To illustrate the hedging of variable annuities, we consider a single premium variable annuity contract sold at time 0. Let $F_0$ be the single premium paid by the policyholder at investment is invested in an index for which options are available.
time 0 and $F_T$ be the account value of the policyholder’s fund at time $T > 0$. We further assume that the variable annuity contract guarantees $K$ at time $T$, that is, the policyholder is guaranteed the payoff $G_T := \max\{F_T, K\}$ at time $T$. Our model can be used in the following situations:

- A GMAB, which guarantees a lump sum benefit $K$ at time $T$.
- A variable annuity contract with GMIB rider, which provides flat life annuity payment $b = \eta \max\{F_T, K\}$ with annuitization rate $\eta > 0$ (see Bacinello, Millossovich, Olivieri, and Pitacco (2011)).
- A variable annuity contract with GMDB rider which guarantees $K$ with assumption that the mortality risk can be diversified away (see Coleman, Li, and Patron (2006) and Coleman, Kim, Li, and Patron (2007)).

Notice that the payoff $G_T$ of the variable annuity contract can be written as

$$G_T = \max\{F_T, K\} = F_T + (K - F_T)^+,$$

with $x^+ := \max\{x, 0\}$. Therefore, the payoff $G_T$ corresponds to the policyholder’s account value at time $T$ plus an embedded option with payoff $V_T := (K - F_T)^+$ at time $T$, where $T$ can be interpreted as the maturity of a put option written on the fund.

### 2.1. Guarantee put option

The embedded option payoff $V_T$ must be funded by the fees collected from the policyholder’s account. Let us consider the time step size $\delta t := T/N$ corresponding to a number of periods $N > 0$ and time steps $t_k := k\delta t$ for $k = 0, 1, \ldots, N$, that is,

$$0 = t_0, t_1, \ldots, t_N = N\delta t = T.$$

Then we assume that the fees $\varepsilon F_{t_k}$ are withdrawn periodically at each time $t_k$, for $k = 0, 1, \ldots, N - 1$, from the policyholder’s account, where $F_{t_k}$ is the account value of the policyholder at time $t_k$ right before the withdrawal of the fee, and $\varepsilon$ is the fee rate. At time $t_{k-1}$, $k = 1, \ldots, N - 1$, the remaining account value $(1 - \varepsilon)F_{t_{k-1}}$ after the fee withdrawal stays invested in the fund whose value at time $t \geq 0$ is $S_t$. Therefore, at time $t_k$, the fund value satisfies the following relationship:

$$F_{t_k} = (1 - \varepsilon)F_{t_{k-1}} \frac{S_{t_k}}{S_{t_{k-1}}}, \quad k = 1, \ldots, N - 1.$$

At time $T = t_N$, $F_T$ is given by

$$F_T = F_{t_N} = (1 - \varepsilon)F_{t_{N-1}} \frac{S_{t_N}}{S_{t_{N-1}}}.$$
Consequently, $F_{tk}$ for $k = 0, 1, \ldots, N$ is given by

$$F_{tk} = (1 - \varepsilon)^k F_0 \frac{S_{tk}}{S_0}.$$ 

As already discussed in the introduction, to simplify the exposition, we neglect basis risk, which arises from the mismatch between the policyholder’s investment account and the available hedging instruments and assume that the money is fully invested in a traded index $S_t$ and such that put options written on $S_t$ can also be used as hedging instruments. For instance we can assume that $S_t$ is a market index, e.g. S&P 500 with a liquid market of options written on this index.\(^3\)

### 2.2. Fair valuation of the VA

Assume a constant risk-free interest rate $r$ and let $Z_T$ be the value at time $T$ of all accumulated fees taken from inception to time $T$, then $Z_T$ is given by

$$Z_T = \sum_{i=0}^{N-1} \varepsilon F_{ti} e^{r(T-t_i)} = \varepsilon F_0 e^{rN\delta t} + \varepsilon F_{t1} e^{r(N-1)\delta t} + \cdots + \varepsilon F_{TN-1} e^{r\delta t}. \quad (1)$$

The value of the payoff of the embedded option at time $T$ is

$$V_T = (K - F_T)^+. \quad (2)$$

Since the collected fees are used to hedge the embedded option, the fair value of the fee rate is determined by solving the following equation

$$\mathbb{E}^\mathcal{Q}[Z_T] = \mathbb{E}^\mathcal{Q}[V_T], \quad (2)$$

where $\mathcal{Q}$ is a risk-neutral measure. Here, using the risk-neutral measure is essential because we assume that financial risks can be hedged using the financial market and that a risk-neutral probability gives rise to prices that are consistent with an arbitrage-free financial market.

### 2.3. Liability and net liability

The VA terminal payoff with GMMB is equal to

$$G_T = F_T + (K - F_T)^+. \quad (3)$$

Since $F_T$ is policyholder’s fund value at maturity time $T$, it will be liquidated and transferred to the policyholder at $T$. Therefore, it does not need to be hedged per se and the remaining liability is the option payoff

$$V_T = (K - F_T)^+. \quad (3)$$

\(^3\)In practice, liquid index futures are used for hedging, however, we use the index directly for simplicity.
Using the accumulated fees $Z_T$ defined in (1), the liability at time $T$ net of collected fees or net liability is

\[(K - F_T)^+ - Z_T,\]  

and it should be distinguished from the liability in (3). The insurer should provide the guarantee $(K - F_T)^+$ but receives a random amount of fees at discrete dates. We will show that considering the net liability instead of the sole guarantee plays a crucial role in the efficiency of the hedge.

### 3. Delta Hedging

We consider the VA contract described in the previous section. This section is dedicated to delta hedging and to show how a delta hedging strategy of the net liability (4) outperforms a naive delta hedging strategy of the liability (3) very significantly. We illustrate this point with the Black-Scholes setting but the result is robust to more general assumptions on the market. The gap between the naive hedge and the hedge of the net liability is important and due to the hedging methodology and not due to the distribution of the underlying. In fact, our experiments have convinced us that adding stochastic volatility, stochastic interest rates or jumps will not change the conclusion that the hedge of the net liability accounting for the periodicity of the payment significantly outperforms the hedge of the liability. Thus, in what follows, the index follows a geometric Brownian motion

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dW_t
\]

with constant $\mu > 0$ and $\sigma > 0$ and standard Brownian motion $W_t$ under the real-world probability measure $\mathbb{P}$. Without loss of generality, we may assume that $S_0 = 1$. Recall that in the absence of market frictions such as transaction costs, and when the underlying asset is tradable in a complete market, the option can be hedged perfectly by rebalancing the hedging portfolio continuously with a dynamic delta hedging strategy. In practice, we cannot rebalance the hedging portfolio continuously and frequent rebalancing causes high transaction costs. Therefore, perfect hedging is not feasible and the hedging portfolio can be rebalanced only at a discrete set of times. In the case of a VA, it is natural to assume that the portfolio is rebalanced at the dates when the fees are collected: $t_0 = 0, t_1, t_2, ..., t_{N-1}$ and potentially at intermediary dates.

### 3.1. Profit and loss of the hedge

Recall that the only part that needs to be hedged in the VA is the maturity guarantee $V_T = (K - F_T)^+$. The collected fees taken periodically from the fund are utilized to construct the insurer’s hedging portfolio whose value at time $T$ is denoted by $X_T$.

To compare the performance of various hedging strategies, we consider the profit and loss of the hedge at time $t_N = T$ (surplus if it is positive and hedging error if it is negative).
We denote it by $\Pi_{t,N}$. It is equal to the difference between the hedging portfolio of the insurer at $T$, $X_T$, and the guarantee, $V_T$, that needs to be paid to the policyholder at time $T$

$$\Pi_{t,N} = X_T - V_T.$$ 

3.2. No hedge case

As a benchmark, we consider the case when there is no hedge at all from the insurance company and that all collected fees are directly invested in the bank account. Then, the profit and loss at time $T$ is given by

$$\Pi_{t,N} = Z_T - (K - F_T)^+, \quad Z_T = \sum_{i=0}^{N-1} \varepsilon F_t e^{r(T-t_i)} \text{(given in (1))}.$$

The expected value (under the real probability $\mathbb{P}$) of the profit and loss can be computed explicitly as follows

$$\mathbb{E}^{\mathbb{P}}[\Pi_{t,N}] = \mathbb{E}^{\mathbb{P}}\left[ \sum_{i=0}^{N-1} \varepsilon F_t e^{r(T-t_i)} - (K - F_T)^+ \right] = \sum_{i=0}^{N-1} \varepsilon e^{r(T-t_i)} \mathbb{E}^{\mathbb{P}}[F_t] - \mathbb{E}^{\mathbb{P}}[(K - F_T)^+]$$

$$= \sum_{i=0}^{N-1} \varepsilon e^{r(T-t_i)} (1 - \varepsilon) F_0 \mathbb{E}^{\mathbb{P}}[S_{t_i}] - e^{\mu T} \mathbb{E}^{\mathbb{P}}[e^{-\mu T}(K - F_T)^+]$$

$$= \sum_{i=0}^{N-1} \varepsilon e^{r(T-t_i)} (1 - \varepsilon) F_0 e^{\mu t_i} - \Phi(-\tilde{d}_2)K + e^{\mu T} \Phi(-\tilde{d}_1)(1 - \varepsilon) N F_0.$$

which is easily derived from the Black-Scholes formula (using $\mu$ instead of $r$) with

$$\tilde{d}_1 = \frac{\ln\left(\frac{(1 - \varepsilon) N F_0}{K}\right) + T(\mu + \frac{\sigma^2}{2})}{\sigma \sqrt{T}}, \quad \tilde{d}_2 = \tilde{d}_1 - \sigma \sqrt{T}.$$ 

This formula was checked by Monte Carlo simulations.

3.3. Delta Hedge of the Liability

A delta-hedge ($\Delta$-hedge) is a self-financing portfolio consisting of investment in the underlying index, $S$, and in a bond to replicate a target payoff. Our goal is to implement a $\Delta$-hedge with rebalancing at each dates $t_k$ for $k = 0, 1, \ldots, N - 1$. We start from $X_0 = \varepsilon F_0$ at time 0, and at each $t_k$, we construct a hedging portfolio consisting of $\pi_{t_k}$ units of underlying index and $X_{t_k} - \pi_{t_k} S_{t_k}$ risk-free bonds. Then, for time $t_k$, we can represent our self-financing replication relation as follows

$$X_{t_{k+1}} = (X_{t_k} - \pi_{t_k} S_{t_k}) e^{r_{t_k}} + \pi_{t_k} S_{t_{k+1}} + \varepsilon F_{t_{k+1}}, \quad k = 0, 1, \ldots, N - 2$$ 

(6)
with $X_0 = \varepsilon F_0$, and define the profit and loss at time $T$ as

$$\Pi_{t_n} = X_{t_n} - (K - F_T)^+$$

to investigate the hedging performance.

To replicate the payoff $(K - F_T)^+$ at each $t_k$ using $\Delta$-hedge, we compute $V_{t_k}$ the no-arbitrage value (risk-neutral expectation of discounted payoff conditional on $\mathcal{F}_{t_k}$) of $V_T$ at time $t_k$. We have

$$V_{t_k} = \mathbb{E}^Q \left[ e^{-r(T-t_k)}(K - F_T)^+ \mid \mathcal{F}_{t_k} \right] = \mathbb{E}^Q \left[ e^{-r(T-t_k)} \left( K - (1 - \varepsilon)^N F_0^S T S_0 \right)^+ \mid \mathcal{F}_{t_k} \right].$$

Let $\alpha = (1 - \varepsilon)^N F_0^S S_0$. Then,

$$V_{t_k} = \alpha \mathbb{E}^Q \left[ e^{-r(T-t_k)} \left( \frac{K}{\alpha} - S_T \right)^+ \mid \mathcal{F}_{t_k} \right] = \Phi(-d_2(k))Ke^{-r(T-t_k)} - \alpha \Phi(-d_1(k))S_{t_k}$$

which is also derived from the Black-Scholes formula with

$$d_1(k) = \frac{\ln(\frac{\alpha S_{t_k}}{K}) + (T - t_k)(r + \frac{\sigma^2}{2})}{\sigma \sqrt{T - t_k}}, \quad d_2(k) = d_1(k) - \sigma \sqrt{T - t_k}$$

and thus, it is then known that the number of shares to invest in shares to hedge the liability is

$$\pi_{t_k} = -\alpha \Phi(-d_1(k)). \quad (7)$$

### 3.4. Delta-Hedge of the Net Liability

Recall that the no-arbitrage value of $V_T$ at $t_k$ is $V_{t_k}$. Then, the no-arbitrage value of the net liability (4) at time $t_k$ is

$$V_{t_k} - \sum_{i=0}^{N-1} \varepsilon(1 - \varepsilon)^i F_0^S S_0 \mathbb{E}^Q \left[ S_{t_i} \mid \mathcal{F}_{t_k} \right] e^{-r(t_i - t_k)}$$

$$= V_{t_k} - \sum_{i=0}^{k-1} \varepsilon(1 - \varepsilon)^i F_0^S S_{t_i} e^{r(t_k - t_i)} - \sum_{i=k}^{N-1} \varepsilon(1 - \varepsilon)^i F_0^S S_{t_k}.$$

By differentiating with respect to $S_{t_k}$, the new number of shares is

$$\bar{\pi}_{t_k} := \pi_{t_k} - \frac{F_0^S}{S_0} (1 - \varepsilon)^k \left( (1 - (1 - \varepsilon)^{N-k}) \right) \quad (8)$$
where $\pi_{tk}$ indicates the number of shares (7) that is needed to hedge the terminal liability ignoring the intermediary payments of fees. The self-financing replication relation that corresponds to $\bar{\pi}_{tk}$ can be obtained by replacing $\pi_{tk}$ by $\bar{\pi}_{tk}$ (defined in (8)) in the expression of the self-financing condition (6).

Note that it is obvious from (7) and (8) that both $\pi_{tk}$ and $\bar{\pi}_{tk}$ are negative for all $k = 0, 1, \ldots, N - 1$. In other words, we always need a short position in the underlying index when we delta-hedge guarantees in VAs. Consequently, if there is any limitation on short-selling or the cost of short-selling is high, delta-hedging may not perform well.

Moreover, it can be seen from (8) that

$$\bar{\pi}_{tk} < \pi_{tk} < 0,$$

for all $k = 0, 1, \ldots, N - 1$.

This implies that the delta-hedge of the net liability requires more short-selling than the delta-hedge of the liability.

3.5. Numerical Illustration

We now illustrate and compare the hedging performance of the two proposed hedging strategies with numerical examples in the Black-Scholes framework as in (5). We generate sample paths of the underlying index from time 0 to $T$ at the rebalancing dates, i.e. $S_{tk}$ for $k = 0, 1, \ldots, N$. Since we assume (5), we have

$$S_{tk+1} = S_{tk}e^{(\mu - \frac{\sigma^2}{2})\delta t + \sigma(W_{tk+\delta t} - W_{tk})} \sim S_{tk}e^{(\mu - \frac{\sigma^2}{2})\delta t + \sigma \sqrt{\delta t}z},$$

where $z \sim N(0, 1)$. We generate 10,000 simulated paths. The benchmark parameters for the financial market and the variable annuity contract are given by

$$T = 10 \text{ (years)}, \quad N = 20, \quad \delta t = 0.5 \text{ (years)}, \quad F_0 = 100, \quad K = 120$$

$$r = 2\% \text{ (per year)}, \quad \sigma = 20\% \text{ (per year)}, \quad \mu = 6\% \text{ (per year)}.$$

Under these parameters, the fair fee rate is $\varepsilon = 0.0415$ (solving (2) where the dynamics of the fund under $Q$ is similar to (5) in which $\mu$ becomes $r$). This fee is slightly higher than usual fee rates charged in the industry. However, if we consider a longer maturity $T$, a smaller volatility $\sigma$ or a lower guaranteed value $K$, the fee rate becomes much lower than this example. Moreover, our findings do not hinge on the level of fee rate so we use this set of parameters for our numerical experiments.

Hedging effectiveness is assessed from a risk management perspective and not using standard performance measures as hedging is not meant to be directly profitable. Haefeli (2013) note that dynamic hedging programs “are designed and applied according to strict risk management rules to mitigate exposures to various market movements stemming from the guarantees provided to policyholders”. In Table 1 below, the expected return can be lower when hedging. However, profit of a hedging strategy should be seen for instance from
reduction in economic capital that is directly linked to the risk exposure of the company. We thus compare the hedging strategies based on risk measures and follow the methodology of Cotter and Hanly (2006) by comparing their five hedging effectiveness (HE) metrics.

1. HE Metric 1 based on the variance

\[
HE_1 = 1 - \left[ \frac{\text{Variance}_{\text{Hedged Portfolio}}}{\text{Variance}_{\text{Unhedged Portfolio}}} \right],
\]

where

\[
\text{Variance} = \mathbb{E} \left[ (\Pi_{t_N} - \mathbb{E}[\Pi_{t_N}])^2 \right].
\]

\(HE_1\) is the percentage reduction in the variance of \(\Pi_{t_N}\) with a hedging strategy as compared with the variance of unhedged case. If \(HE_1 = 1\), it indicates that the risk is completely eliminated by the hedging strategy while \(HE_1 = 0\) means that the hedging strategy does not reduce the variance of \(\Pi_{t_N}\).

2. HE Metric 2 based on the semivariance

\[
HE_2 = 1 - \left[ \frac{\text{SemiVariance}_{\text{Hedged Portfolio}}}{\text{SemiVariance}_{\text{Unhedged Portfolio}}} \right],
\]

where

\[
\text{SemiVariance} = \mathbb{E} \left[ (\max(0, -\Pi_{t_N}))^2 \right].
\]

The semivariance is a kind of downside risk measure, and generally defined by \(\mathbb{E}[(\max(0, \tau - R))^2]\), where \(R\) is the return on the hedged portfolio and \(\tau\) is the target return. Our case is obtained by setting \(R = \Pi_{t_N}\) and \(\tau = 0\).

3. HE Metric 3 based on the lower partial moment (LPM)

\[
HE_3 = 1 - \left[ \frac{\text{LPM}_3 \text{Hedged Portfolio}}{\text{LPM}_3 \text{Unhedged Portfolio}} \right],
\]

where the lower partial moment of order \(n\) is defined as

\[
\text{LPM}_n = \mathbb{E} \left[ (\max(0, -\Pi_{t_N}))^n \right].
\]

The LPM is a more general concept than the semivariance, and the semivariance is a special case of LPM with \(n = 2\). For \(HE_3\), we choose \(n = 3\) which corresponds to a risk-averse investor as argued by Cotter and Hanly (2006).

4. HE Metric 4 based on the Value-at-Risk

\[
HE_4 = 1 - \left[ \frac{\text{VaR}_{95\% \text{Hedged Portfolio}}}{\text{VaR}_{95\% \text{Unhedged Portfolio}}} \right].
\]

5. HE Metric 5 - Conditional Value at Risk

\[
HE_5 = 1 - \left[ \frac{\text{CVaR}_{95\% \text{Hedged Portfolio}}}{\text{CVaR}_{95\% \text{Unhedged Portfolio}}} \right].
\]
Table 1: Comparison of Hedging Performance by Monte Carlo simulations with $10^5$ simulations. The column corresponding to “No hedge” contains explicit computations (when available) or is obtained with 1,000,000 simulations so that all digits are significant. $\pi_0 < 0$ represents short-selling is necessary at time 0.

<table>
<thead>
<tr>
<th>Characteristics of $\Pi_{t_N}$</th>
<th>No hedge</th>
<th>$\Delta$-hedge with $N$ rebalancing dates</th>
<th>$\Delta$-hedge with $10N$ rebalancing dates</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\Delta$-hedge of Liability</td>
<td>$\Delta$-hedge of Net Liability</td>
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<tr>
<td>Mean</td>
<td>31.9007</td>
<td>12.49</td>
<td>-1.88</td>
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<tr>
<td>Median</td>
<td>22.1358</td>
<td>7.27</td>
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<td>Std</td>
<td>57.8022</td>
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<td>36.5202</td>
<td>17.68</td>
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<td>CVaR$_{90%}$</td>
<td>51.2175</td>
<td>23.95</td>
<td>16.23</td>
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<td>HE$_1$</td>
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<tr>
<td>$\pi_0$</td>
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<td>-36.01</td>
<td>-98.09</td>
</tr>
</tbody>
</table>

We illustrate the performance of a delta-hedging strategy using the computation of the shares as in (7) and (8) and report our results in Table 1.

From Table 1 and the two panels of Figure 1, we find the following properties of $\Delta$-hedging:

- **Hedging is necessary.** In Table 1, the column “no hedge” means that the insurer does not use the collected fees for trading the underlying index or put options. Instead he invests all fees in the risk-free asset as if the guarantees of the VA were fully diversifiable. All digits for the performance of “no hedge” are exact. Note that the no hedge performance is very poor: $\Pi_{t_N}$ has a very high standard deviation and the Value-at-Risk VaR$_{95\%}$ and VaR$_{90\%}$ are also very high. This implies that the insurer may encounter significant loss in unfavorable market conditions in the absence of hedging. Thus a good hedging strategy is necessary for the insurer to avoid catastrophic losses.

- **Short-selling is necessary.** The $\Delta$-hedging strategy does not require borrowing at time 0 but significant short-selling of the underlying index is needed.
• Hedging effectiveness is highest when hedging the net liability. First, we observe a significant reduction in the standard deviation and hedge improvement when $\Delta$-hedging the net liability instead of the liability. We find that the hedging effectiveness metrics for hedging the net liability are significantly higher than the effectiveness of the hedging of the liability.

• Rebalancing at more dates is only useful for $\Delta$-hedging the net liability. It is expected that when rebalancing more often, the performance of $\Delta$-hedging must improve. However, we observe that the uncertainty of the collected fees that is not taken into account in the $\Delta$-hedge of the liability prevents any significant improvement in performance of the $\Delta$-hedge. On the other hand, we observe a significant improvement in the case of $\Delta$-hedging the net liability, when the number of rebalancing dates is multiplied by 10.

Figure 1: Comparison of Probability Densities of $\Pi_t^N$

(a) Rebalancing 2 times a year  
(b) Rebalancing 20 times a year

4. Semi-Static Hedging

In this section, we develop a semi-static hedging technique that minimizes the variance of the hedge at a set of discrete dates by investing in stocks, risk-free bonds and some put options. For the ease of exposition, we assume that the hedging strategy utilizes the collected fees at the time they are paid to construct the hedging portfolio. Thus we assume that the rebalancing dates at which the portfolio is hedged coincide exactly with the dates $t_k$ for $k = 0, 1, \ldots, N - 1$ (time steps at which fees are taken from the policyholder’s account). It is possible to include other time steps, but this assumption simplifies the
model and does not affect our conclusions.\footnote{We have conducted analyses with one intermediary date and it does not change the results significantly.}

4.1. Two semi-static strategies

At time $t_k$, $k = 0, 1, \ldots, N - 1$, we construct a hedging portfolio $X_{tk}$ consisting of $\beta_{tk}$ risk-free bonds (in other words invested in a money market account), $\pi_{tk}$ units of underlying index (whose value at time $t_k$ is $S_{tk}$), and $\alpha_{tk}$ put options with strike price $K_{tk}$ and maturity $t_{k+1}$. Then the value of the hedging portfolio at time $t_k$ becomes

\[ X_{tk} := \pi_{tk} S_{tk} + \beta_{tk} + \alpha_{tk} P(S_{tk}, K_{tk}, \delta t), \quad k = 0, 1, \ldots, N - 1, \]

where $P(S_t, K, \tau)$ denotes the price of a put option at time $t$ with current underlying asset value $S_t$, strike price $K$, and time to maturity $\tau$. It is possible to add more put options but the exposition would be more complicated. This simplest situation with put options with only one given strike is already insightful.

At time $t_{k+1}$ before any rebalancing, the change in the value of hedging portfolio over $\delta t$ due to the movement of the underlying index is

\[ \pi_{tk}(S_{tk+1} - S_{tk}) + \beta_{tk}(e^{r \delta t} - 1) + \alpha_{tk}((K_{tk} - S_{tk+1})^+ - P(S_{tk}, K_{tk}, \delta t)). \]

Then we define the accumulated gain (possibly negative when it is a loss) at time $t_k$ as

\[ Y_{tk} := \sum_{n=0}^{k-1} \pi_{tn}(S_{tn+1} - S_{tn}) + \beta_{tn}(e^{r \delta t} - 1) + \alpha_{tn}((K_{tn} - S_{tn+1})^+ - P(S_{tn}, K_{tn}, \delta t)) \]

for $k = 1, 2, \ldots, N$ with $Y_0 = 0$, and the cumulative cost at time $t_k$ as

\[ C_{tk} := X_{tk} - Y_{tk}, \quad k = 0, 1, \ldots, N, \]

with $X_{TN} := V_T = (K - F_T)^+$. \par

**Remark 1.** The cost increment $C_{tk+1} - C_{tk}$ can be interpreted as the variation in the value of the hedging portfolio between $t_k$ and $t_{k+1}$ adjusted by potential gains and losses. It can be computed as

\[ C_{tk+1} - C_{tk} = X_{tk+1} - (X_{tk} + Y_{tk+1} - Y_{tk}) \]

with

\[ X_{tk+1} = \begin{cases} \pi_{k+1} S_{tk+1} + \beta_{tk+1} + \alpha_{tk+1} P(S_{tk+1}, K_{tk+1}, \delta t) & \text{if } k = 0, 1, \ldots, N - 2, \\ (K - F_T)^+ = (K - (1 - \epsilon)^N F_0 S_{0}/S_0)^+ & \text{if } k = N - 1, \end{cases} \]

\[ X_{tk} + Y_{tk+1} - Y_{tk} = \pi_{tk} S_{tk+1} + \beta_{tk} e^{r \delta t} + \alpha_{tk} (K_{tk} - S_{tk+1})^+, \quad k = 0, 1, \ldots, N - 1. \]
Ultimately, we are interested in the profit and loss of the hedging portfolio at the date $T$ when the payoff is paid. To do so, we need to take into account the collected fees and define the profit and loss $\Pi_t$ at any time $t$. At time 0, the insurer receives the fee $\varepsilon F_0$ and constructs a hedging portfolio whose value is $C_0 = X_0$. Therefore, the profit and loss at time 0 is

$$\Pi_0 = \varepsilon F_0 - C_0.$$  

At time $t_1$, the initial profit and loss (computed at time 0) becomes $\Pi_0 e^{r \delta t}$ and the insurer receives the fee $\varepsilon F_{t_1}$. However, additional cost $C_{t_1} - C_0$ is needed to construct the hedging portfolio at time $t_1$. As a result, the profit and loss at time $t_1$ is given by

$$\Pi_{t_1} = \Pi_0 e^{r \delta t} + \varepsilon F_{t_1} - (C_{t_1} - C_0) = (\varepsilon F_0 - C_0) e^{r \delta t} + \varepsilon F_{t_1} - (C_{t_1} - C_0).$$

A similar argument gives the following equation for the profit and loss at time $T$.

$$\Pi_T = \Pi_{t_N} = \sum_{k=0}^{N-1} \varepsilon F_{t_k} e^{r (N-k) \delta t} - C_0 e^{r k \delta t} - \sum_{k=1}^{N} (C_{t_k} - C_{t_{k-1}}) e^{r (N-k) \delta t}.$$

If $\Pi_{t_N}$ is positive, the insurer has positive surplus at $T$ after paying $(K - F_T)^+$ to the policyholder. On the other hand, if $\Pi_{t_N}$ is negative, the insurer is short of money to pay the guaranteed value at time $T$ to the policyholder. Thus $\Pi_{t_N}$ can be considered as the profit and loss and the hedging performance of strategies can be investigated using $\Pi_{t_N}$.

**Liability Hedge: Traditional semi-static hedging**

Consider a traditional semi-static hedging strategy (referred as liability hedge). Since perfect hedging is not feasible with semi-static hedging, some optimality criterion is necessary at each rebalancing date. In semi-static hedging of liability, the hedging strategy at each rebalancing date is determined to minimize local risk, which is defined as the conditional second moment of the cost increment during each hedging period. At each time step $t_k$ for $k = N - 1, N - 2, \ldots, 0$, starting from the last period, one solves the following optimization recursively

$$\min_{(\pi_{t_k}, \beta_{t_k}, \alpha_{t_k}, K_{t_k}) \in \mathcal{A}_{t_k}} L_k^\otimes(\pi_{t_k}, \beta_{t_k}, \alpha_{t_k}, K_{t_k}; s),$$

with

$$L_k^\otimes(\pi_{t_k}, \beta_{t_k}, \alpha_{t_k}, K_{t_k}; s) := \mathbb{E}_{t_k} \left[ \left( C_{t_{k+1}} - C_{t_k} \right)^2 \mid S_{t_k} = s \right] = \mathbb{E}_{t_k} \left[ \left\{ X_{t_{k+1}} - \left( X_{t_k} + Y_{t_{k+1}} - Y_{t_k} \right) \right\}^2 \mid S_{t_k} = s \right],$$

where $\mathcal{A}_{t_k}$ is some admissible set of controls at time $t_k$ and $\mathbb{E}_{t_k} [\cdot] := \mathbb{E}_{t_k} [\cdot | \mathcal{F}_{t_k}]$ is the conditional expectation on the information at time $t_k$. $\mathcal{A}_{t_k}$ may be defined differently.
depending on the constraints we consider. Equation (12) implies that, if we hedge the liability at time \( t_k \), \( X_{t_k+1} \) is the hedging target and our goal is to minimize the conditional second moment of increment of cost.

This optimality criterion was applied to semi-static hedging of variable annuities by Coleman, Li, and Patron (2006), Coleman, Kim, Li, and Patron (2007), and Kolkiewicz and Liu (2012). In these papers, the authors use several standard options with different strike prices in the hedging strategy, but as described above, we only consider a single put option and the strike price is determined by the optimization. Our model gives insights about the moneyness of the options needed for hedging variable annuities.

In (12), note that \( (X_{t_k} + Y_{t_k+1} - Y_{t_k}) \) is the amount available at time \( t_{k+1} \) from the hedging portfolio \( X_{t_k} \). However, the insurer receives the fee \( \varepsilon F_{t_k} \) at time \( t_k \) for \( k = 0, 1, \ldots, N - 1 \). If we hedge the liability, consistently with the previous literature, we only use \( (X_{t_k} + Y_{t_k+1} - Y_{t_k}) \) to construct the hedging portfolio at time \( t_{k+1} \) and the fee is not considered in the hedging strategy. The strategy below shows how to take this fee into account in the design of the semi-static hedging strategy.

**Net Liability Hedge: Improved semi-static hedging strategy**

For \( k = N - 1, N - 2, \ldots, 0 \), solve the following optimization recursively

\[
\min_{(\pi_{t_k}, \beta_{t_k}, \alpha_{t_k}, K_{t_k}) \in A_{t_k}} L^\bigodot_k(\pi_{t_k}, \beta_{t_k}, \alpha_{t_k}, K_{t_k}; s),
\]

with \( L^\bigodot_k(\pi_{t_k}, \beta_{t_k}, \alpha_{t_k}, K_{t_k}; s) := \mathbb{E}_{t_k} \left\{ \left( C_{t_k+1} - (C_{t_k} + \varepsilon F_{t_{k+1}} 1_{\{0 \leq k \leq N-2\}}) \right)^2 \middle| S_{t_k} = s \right\} \)

\[= \mathbb{E}_{t_k} \left\{ \left( X_{t_k+1} - \left( X_{t_k} + Y_{t_k+1} - Y_{t_k} + \varepsilon F_{t_k} 1_{\{0 \leq k \leq N-2\}} \right) \right)^2 \middle| S_{t_k} = s \right\}.
\]

The optimality criterion above enables us to utilize the collected fees right away in the hedging strategy. Notice that \( \Pi_{t_N} \) in (11) can be written as

\[
\Pi_{t_N} = (\varepsilon F_0 - C_0)e^{rN\delta t} + \sum_{k=1}^{N-1} (\varepsilon F_{t_k} + C_{t_{k-1}} - C_{t_k})e^{r(N-k)\delta t} + (C_{t_{N-1}} - C_{t_N}),
\]

and the conditional second moments of every term of \( \Pi_{t_N} \) except the first term in (13) are minimized when we hedge the net liability whereas the collected fees are not utilized when we hedge just the liability. Our numerical experiments show evidence that the net liability hedge gives a better hedge performance than the liability hedge.

We now illustrate the hedging performance of these two strategies (liability hedge and net liability hedge) under various market environments. The effect of fees on hedging performance under different assumptions on the market environment is also discussed.
4.2. Description

We describe the semi-static hedging procedure based on hedging the net liability. But similar explanations hold for the liability hedge because the only difference between the net liability hedge and the liability hedge is that the fees do not appear in the optimality criterion of the liability hedge. Therefore, we can obtain the solution for the liability hedge by removing the terms with the fees \( \varepsilon (1 - \varepsilon)^{k+1} F_0 S_{t_k+1} I_{0 \leq k \leq N-2} \) wherever they appear in the net liability hedge.

One of the variables included in the optimization is the optimal level of the strike price of the put options used in the hedging procedure. Since the strike prices of put options that are actively traded in the market are limited and related to the underlying index value at \( t_k \), we define a set \( K_{t_k}(S_{t_k}) \) of strike prices available in the market at time \( t_k \) with underlying index value \( S_{t_k} \). Specifically, we describe the available strikes as a percentage of the underlying value, so that a percentage equal to 100% represents an at-the-money option, a percentage higher (respectively lower) than 100% corresponds to an out-of-the-money put option (respectively in-the-money). Mathematically, it amounts to restricting the set of possible strikes to the following set \( K_{t_k}(S_{t_k}) \):

\[
K_{t_k}(S_{t_k}) = \{ \xi S_{t_k} | \xi \in \{ \xi_1, \xi_2, \ldots, \xi_M \} \}
\]

where \( \xi_1 < \xi_2 < \cdots < \xi_M \). Then we may examine the effect of the minimum available strike \( \xi_1 S_{t_k} \) at time \( t_k \) and the maximum available strike \( \xi_M S_{t_k} \) at time \( t_k \) on the hedging performance. It is also straightforward to extend our approach with additional constraints imposed on the other control variables \( \pi_{t_k}, \beta_{t_k}, \) and \( \alpha_{t_k} \).

We examine separately the effectiveness of the semi-static hedging strategy in three environments that we call “UC”, “SC” and “w/o Put”.

- “UC” refers to the situation where hedging is done without any constraints. Short selling of the underlying index is allowed and the index and put options on this index are used as hedging instruments (see details in Appendix A.1).

- “SC” refers to the situation where hedging is done under short-selling constraints on the underlying index. Put options are also used as hedging instruments (see details in Appendix A.2).

- “w/o Put” refers to the situation where hedging is done without any constraints on the underlying index but put options are not used as hedging instruments: Only the money market account and the underlying index can be used for hedging (see details in Appendix A.3).

4.3. Hedging Performance

For the ease of comparison with delta-hedging, we use the same parameter set as the numerical illustration in Section 3.5. The theoretical results needed for the numerical
implementation are given in Appendix A.

In Figures 2 and 3, and in Table 2, we use $\xi_1 = 0.7$ and $\xi_M = 1.1$. This choice is consistent with market data of available strike prices of put options on the S&P 500 index that are actively traded in the market.

Figure 2: Comparison of Probability Densities of $\Pi_{tN}$
We compare the probability distribution functions of $\Pi_{tN}$ for the two strategies described in Section 4 when there are no constraints on the semi-static hedging strategy and when put options are not available.

(a) Unconstrained, $\xi_1 = 0.7$, $\xi_M = 1.1$
(b) Without Put Options, $\xi_1 = 0.7$, $\xi_M = 1.1$

Figure 3: Hedging Targets with Short-selling Constraint $D_k = 0$, $\xi_1 = 0.7$, $\xi_M = 1.1$

From Figures 2 and 3, and Table 2, we make the following observations.

- The improved semi-static hedging strategy, the net liability hedge, outperforms the liability hedge. Figure 2a represents the probability distribution
of $\Pi_{t,N}$ of two strategies, the liability hedge and the net liability hedge, without any constraint. It can be directly compared with Figure 2b where the probability distributions of $\Pi_{t,N}$ of the liability hedge and net liability hedge are displayed when no put options are used in the hedging strategies. In both cases, we observe that the net liability hedge gives significantly better hedging performance. This can also be observed from the HE metrics in Table 2. In fact, this result is intuitive as better hedging performance with the net liability hedge is expected from the equation (13) because the net liability hedge minimizes each term in $\Pi_{t,N}$ while the liability hedge only focuses on the additional cost part of $\Pi_{t,N}$.

- **Less borrowing is needed at time 0 with the net liability hedge** Another advantage of the net liability hedge instead of the liability hedge is that the initial profit and loss $\Pi_0 = \varepsilon F_0 - X_0$ with the net liability hedge is close to zero and greater than that with the liability hedge as we can see in Table 2. Negative $\Pi_0$ means that the fee collected at time 0 is not enough to construct the optimal hedging portfolio $X_0$ at time 0 so that the insurer needs to borrow to construct the optimal hedging portfolio at time 0. Since the fee rate charged in this example is $\varepsilon = 0.0415$ and the single premium paid by the policyholder at time 0 is $F_0 = 100$, the fee collected at time 0 is $\varepsilon F_0 = 4.15$. The insurer needs to borrow more than 50 which is more than 1000% of $\varepsilon F_0$ to construct the optimal hedging portfolio for the liability hedge without constraints. On the contrary, the insurer only needs to borrow 0.02 which is less than 0.3% of $\varepsilon F_0$ to construct the optimal hedging portfolio for the net liability hedge. This is mainly due to the fact that we account for collected fees to construct the hedging portfolio at each rebalancing date when hedging the net liability.

- **Including put options as hedging instruments gives better hedging performance.** In Table 2, the HE metrics of the hedging strategy without put options (w/o Put) are lower than the HE metrics of the hedging strategy involving put options (UC). This means that better hedging performance can be achieved by including put options as hedging instruments. This result is consistent with the findings of Coleman, Li, and Patron (2006), Coleman, Kim, Li, and Patron (2007), and Kolkiewicz and Liu (2012).

- **Short-selling is of utmost importance for hedging VA guarantees.** From Table 2, the hedging performance under short-selling constraint $D_k = 0$ (SC) appears significantly worse than that without short-selling constraint (UC) for each strategy under study. Moreover, the hedging performance under short-selling constraints is even worse than that without put options as hedging instruments (w/o Put). This finding can be understood from the shape of the hedging target as a function of the underlying index. Figure 3 depicts the hedging target at time $t_5$ as an example.

Since the hedging targets of the two strategies are decreasing convex functions of the underlying index (Figure 3), a short position of the underlying index is essential to construct an efficient hedging portfolio and the combination of a short position in
the underlying index and put options allows to match the hedging target well and thus for a good hedging performance. Therefore, if short-selling is not allowed, we get much worse hedging performance.

Table 2: Comparison of Hedging Performances. $\xi_1 = 0.7, \xi_M = 1.1$

<table>
<thead>
<tr>
<th>Characteristics of $\Pi_{t_N}$</th>
<th>No hedge</th>
<th>Liability Hedge</th>
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<tr>
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<td>27.01</td>
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<td>-51.63</td>
<td>-52.94</td>
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</table>

UC: Unconstrained, SC: Short-selling constraint $D_k = 0$, w/o Put: Without put option

The next observations relate to Figure 4 and 5 and Table 3.

- **A larger set of strike prices gives a better hedge.** In Figure 4 and Table 3, we examine the hedging performance with $\xi_1 = 0.5$ and $\xi_M = 1.5$. This change gives the insurer a wider range of strike prices of put options that can be used to construct the hedging portfolio. It is clear that the hedging performance of strategies that use put options as hedging instruments, is improved when a wider range of strike prices is available. If we compare Table 2 and Table 3, the hedging performance without short-selling constraint is slightly improved. Noticeably, the hedging performance is significantly improved in the presence of the short-selling constraint (SC). For example, the standard deviation of $\Pi_{t_N}$ with the net liability hedge under short-selling constraint is 21.09 if $\xi_1 = 0.7$ and $\xi_M = 1.1$. By contrast, if $\xi_1 = 0.5$ and $\xi_M = 1.5$, the standard deviation is reduced dramatically and becomes 2.87. This significant improvement of hedging performance under short-selling constraint can also be observed in Figure 4. This is because the put option is the only hedging
instrument available to match the shape of the hedging target in the presence of short-selling constraints. Thus, put options become more useful hedging instruments when they are offered with a wider range of strike prices (which implies that extreme strikes are needed either deep out-of-the-money options or deep in-the-money options).

Figure 5 illustrates why the hedging performance under short-selling constraint is improved significantly when a larger set of strike prices is available. In Figure 5b, if $\xi_1 = 0.5$ and $\xi_M = 1.5$, the optimal strike prices without any constraints (UC) are far below the upper bound of available strike prices. However, the optimal strike prices under short-selling constraint (SC) are on the upper bound of available strike prices in Figure 5b. Therefore, in Figure 5a with $\xi_M = 1.1$, the upper bound of available strike prices is much lower and the optimal strike prices under short-selling constraint are far below the optimal levels attained in Figure 5b with $\xi_M = 1.5$. As a result, the hedging performance under short-selling constraint is affected significantly by the range of strike prices.

Figure 4: Comparison of Probability Densities of $\Pi_{t,N}$

We compare the probability distribution functions of $\Pi_{t,N}$ for the two strategies described in Section 4 when there are short-selling constraints on the semi-static hedging strategy and when put options are not available for a wider range of strikes.

(a) Short-selling constraint $D_k = 0, \xi_1 = 0.5, \xi_M = 1.5$  
(b) Without Put Option, $\xi_1 = 0.5, \xi_M = 1.5$
Figure 5: Comparison of Optimal Strike Prices at Time $t_1$
We compare the optimal strike prices for the net liability hedge at time $t_1$ for different set of strike prices.

(a) $\xi_1 = 0.7$, $\xi_M = 1.1$
(b) $\xi_1 = 0.5$, $\xi_M = 1.5$

Table 3: Comparison of Hedging Performances.  $\xi_1 = 0.5$,  $\xi_M = 1.5$

<table>
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<tr>
<th>Characteristics of $\Pi_{t_N}$</th>
<th>Liability Hedge</th>
<th>Net Liability Hedge</th>
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<td>$\text{VaR}_{90%}$</td>
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<td>$\Pi_0$</td>
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<td>-52.99</td>
</tr>
</tbody>
</table>

UC: Unconstrained,  SC: Short-selling constraint $D_k = 0$,  w/o Put: Without put option

5. Conclusions

This paper focuses on the hedging of financial guarantees in variable annuities. We review existing methods and introduce a new semi-static hedging strategy that outperforms
delta hedging and existing static and semi-static strategies presented in the literature. We show that it is crucial to take into account the fees collected periodically in the design of the hedging strategy. Another advantage of our new strategy is that almost no borrowing is needed at the beginning and the cost to construct the initial hedging portfolio can be matched to the fee collected initially.

We also show that short-selling and adding put options as additional hedging instruments gives better hedging performance of variable annuities. The larger the set of put options available in the market (wider range of strikes), the better the hedging performance with put options.

Finally, we want to point out that our study has been performed in the Black-Scholes setting. It already allows us to get good intuition. Semi-static strategies with local risk minimization are known to be more robust to model risk than delta-hedging strategies. Therefore, we expect that our qualitative results hold under more general market conditions.
Appendix A. Optimal Semi-Static Hedging Strategies

We here provide full details of the implementation of the semi-static strategy. We first consider the unconstrained optimization problem “UC” as a benchmark (Appendix A.1), and then we consider constrained problems “SC” (Appendix A.2) and “w/o Put” (Appendix A.3) to examine the effect of short-selling constraints and the absence of options in the hedging portfolio.

Appendix A.1. Unconstrained Semi-Static Hedging Strategy: “UC”

At each time step \( t_k, k = 1, 2, \ldots, N \), consider a discretization \( \{ S^i_{tk} \}_{i=1}^I \) of the underlying index. Then we can find the optimal hedging strategies recursively in backward from Appendix A.1. Unconstrained Semi-Static Hedging Strategy: “UC” in the hedging portfolio.

For each strike price \( K^i_{tk} \in \mathcal{K}_{tk}(S^i_{tk}) \), \( j = 1, 2, \ldots, M \), let us define \((\pi^i_{tk}, \beta^i_{tk}, \alpha^i_{tk})\) as

\[
(\pi^i_{tk}, \beta^i_{tk}, \alpha^i_{tk}) := \arg\min_{(\pi^i_{tk}, \beta^i_{tk}, \alpha^i_{tk}) \in \mathbb{R}^3} L^\bigcirc_k(\pi^i_{tk}, \beta^i_{tk}, \alpha^i_{tk}, K^i_{tk}, S^i_{tk})
\]

and find \((\pi^i_{tk}, \beta^i_{tk}, \alpha^i_{tk})\) for given \( K^i_{tk} \). Notice that \((\pi^i_{tk}, \beta^i_{tk}, \alpha^i_{tk})\) is the optimal choice of rebalancing decision for a given value of underlying index \( S^i_{tk} \) and a given strike price \( K^i_{tk} \) of the put option. If we choose \( K^i_{tk} \in \mathcal{K}_{tk}(S^i_{tk}) \) such that it gives the minimum value of \( L^\bigcirc_k(\pi^i_{tk}, \beta^i_{tk}, \alpha^i_{tk}, K^i_{tk}, S^i_{tk}) \), it must be the optimal strike price.

3. Let us define \( \tilde{K}^i_{tk} \) as

\[
\tilde{K}^i_{tk} := K^i_{tk} := \arg\min_{K^i_{tk} \in \mathcal{K}_{tk}(S^i_{tk})} L^\bigcirc_k(\pi^i_{tk}, \beta^i_{tk}, \alpha^i_{tk}, K^i_{tk}, S^i_{tk})
\]

then \( \tilde{K}^i_{tk} \) is the optimal strike price of the put option for given time \( t_k \) and underlying index \( S^i_{tk} \). The corresponding optimal rebalancing decisions are given by

\[
\tilde{\pi}^i_{tk} := \pi^i_{tk}, \tilde{\beta}^i_{tk} := \beta^i_{tk}, \tilde{\alpha}^i_{tk} :=\alpha^i_{tk}.
\]

4. Repeat the above procedures 1, 2, and 3 from \( k = N - 1 \) to \( k = 0 \). Then we obtain the optimal rebalancing decisions for the net liability hedge without any constraint.

Proposition 1. From (9) and (10),

\[
L^\bigcirc_k(\pi_{tk}, \beta_{tk}, \alpha_{tk}, K^i_{tk}, S^i_{tk}) = \mathbb{E}_k \left[ \left( X_{tk+1} - \pi_{tk} S_{tk+1} - \beta_{tk} e^{\rho t} - \alpha_{tk} (K^i_{tk} - S_{tk+1})^+ - \frac{\varepsilon(1 - \varepsilon)k+1 F_0 S_{tk+1} \mathbb{1}_{\{0 \leq k \leq N-2\}}}{S_0} \right)^2 S_{tk} = S^i_{tk} \right].
\]
Thus, the optimization problem in (A.1) is a least squares problem so that we can derive $(\bar{\pi}_{i,j}^{i,j}, \bar{\beta}_{i,j}^{i,j}, \bar{\alpha}_{i,j}^{i,j})$ as follows.

\[
\bar{\pi}_{i,j}^{i,j} = -\frac{1}{2} \frac{4b_k c_k g_k - 2b_k e_k i_k - 2c_k d_k h_k + d_k f_k i_k + e_k f_k h_k - f_k^2 g_k}{4a_k b_k c_k - a_k f_k^2 - b_k e_k^2 - c_k d_k^2 + d_k e_k f_k},
\]

\[
\bar{\beta}_{i,j}^{i,j} = -\frac{1}{2} \frac{4a_k c_k h_k - 2a_k f_k i_k - 2c_k d_k g_k + d_k e_k i_k + e_k f_k g_k - e_k^2 h_k}{4a_k b_k c_k - a_k f_k^2 - b_k e_k^2 - c_k d_k^2 + d_k e_k f_k},
\]

\[
\bar{\alpha}_{i,j}^{i,j} = -\frac{1}{2} \frac{4a_k b_k i_k - 2a_k f_k h_k - 2b_k e_k g_k + d_k e_k h_k + d_k f_k g_k - d_k^2 i_k}{4a_k b_k c_k - a_k f_k^2 - b_k e_k^2 - c_k d_k^2 + d_k e_k f_k},
\]

where

\[
a_k := \mathbb{E}_{t_k} \left[ S_{t_{k+1}}^2 \mid S_{t_k} = S_{t_k}^i \right]
\]

\[
b_k := e^{2r \delta t}
\]

\[
c_k := \mathbb{E}_{t_k} \left[ \left( \left( K_{t_k}^{i,j} - S_{t_{k+1}} \right)^+ \right)^2 \mid S_{t_k} = S_{t_k}^i \right]
\]

\[
d_k := 2\mathbb{E}_{t_k} \left[ S_{t_{k+1}} e^{r \delta t} \mid S_{t_k} = S_{t_k}^i \right]
\]

\[
e_k := \mathbb{E}_{t_k} \left[ S_{t_{k+1}} \left( K_{t_k}^{i,j} - S_{t_{k+1}} \right)^+ \mid S_{t_k} = S_{t_k}^i \right]
\]

\[
f_k := \mathbb{E}_{t_k} \left[ e^{r \delta t} \left( K_{t_k}^{i,j} - S_{t_{k+1}} \right)^+ \mid S_{t_k} = S_{t_k}^i \right]
\]

\[
g_k := \mathbb{E}_{t_k} \left[ S_{t_{k+1}} \left( \varepsilon(1 - \varepsilon)^{k+1} F_0 S_{t_{k+1}}^{i+1} N_{t_{k+1}}^{i} - X_{t_{k+1}} \right) \mid S_{t_k} = S_{t_k}^i \right]
\]

\[
h_k := \mathbb{E}_{t_k} \left[ e^{r \delta t} \left( \varepsilon(1 - \varepsilon)^{k+1} F_0 S_{t_{k+1}}^{i+1} N_{t_{k+1}}^{i} - X_{t_{k+1}} \right) \mid S_{t_k} = S_{t_k}^i \right]
\]

\[
i_k := \mathbb{E}_{t_k} \left[ \left( K_{t_k}^{i,j} - S_{t_{k+1}} \right)^+ \left( \varepsilon(1 - \varepsilon)^{k+1} F_0 S_{t_{k+1}}^{i+1} N_{t_{k+1}}^{i} - X_{t_{k+1}} \right) \mid S_{t_k} = S_{t_k}^i \right]
\]

\[
j_k := \mathbb{E}_{t_k} \left[ \left( \varepsilon(1 - \varepsilon)^{k+1} F_0 S_{t_{k+1}}^{i+1} N_{t_{k+1}}^{i} - X_{t_{k+1}} \right)^2 \mid S_{t_k} = S_{t_k}^i \right]
\]

To calculate the above conditional expectations, we need the hedging target $X_{t_{k+1}}$ as a function of underlying index $S_{t_{k+1}}$. However, we only have the values of $X_{t_{k+1}}$ which correspond to the discretized values $\{S_{t_{k+1}}^{i} \}_{i=1}^I$. Therefore, the hedging target $X_{t_{k+1}}$ corresponding to $S_{t_{k+1}}$ other than $\{S_{t_{k+1}}^{i} \}_{i=1}^I$ must be obtained by interpolation or extrapolation.

**Appendix A.2. With a Short-selling Constraint: “SC”**

Consider the following short-selling constraint

\[
\pi_{t_k} \geq -D_k, \quad D_k \geq 0, \quad k = 0, 1, \ldots, N - 1.
\]

Then the optimal hedging strategies under short-selling constraint can be obtained by following a similar procedure as we used for the unconstrained problem.
1. At time $t_k$, suppose that the hedging target $X_{t_{k+1}} = X_{t_{k+1}}(S_{t_{k+1}})$ is known as a function of $S_{t_{k+1}}$ from the previous optimization at time $t_k$. For each $S_{t_k}$, there is a corresponding set of strike prices $K_{t_k}(S_{t_k})$ of put options. Do the following steps 2 and 3 for all $i = 1, 2, \ldots, I$.

2. For each $K_{t_k}^i \in K_{t_k}(S_{t_k}^i)$, $j = 1, 2, \ldots, M$, let us define $(\hat{\pi}_{t_k}^{i,j}, \hat{\beta}_{t_k}^{i,j}, \hat{\alpha}_{t_k}^{i,j})$ as
\[
(\hat{\pi}_{t_k}^{i,j}, \hat{\beta}_{t_k}^{i,j}, \hat{\alpha}_{t_k}^{i,j}) := \arg\min_{(\pi_{t_k}, \beta_{t_k}, \alpha_{t_k}) \in \mathbb{R}^3} L_k^\otimes(\pi_{t_k}, \beta_{t_k}, \alpha_{t_k}, K_{t_k}^i, S_{t_k}^i)
\]
subject to $\pi_{t_k} \geq -D_k$.

(A.3)

3. Let us define $\hat{K}_{t_k}^i$ as
\[
\hat{K}_{t_k}^i := \hat{K}_{t_k}^{i,j} = \arg\min_{K_{t_k}^i \in K_{t_k}(S_{t_k}^i)} L_k^\otimes(\pi_{t_k}^{i,j}, \beta_{t_k}^{i,j}, \alpha_{t_k}^{i,j}, K_{t_k}^i, S_{t_k}^i),
\]
then $\hat{K}_{t_k}^i$ is the optimal strike price of put option, and corresponding optimal rebalancing decisions are given by
\[
\hat{\pi}_{t_k}^i := \hat{\pi}_{t_k}^{i,j}, \quad \hat{\beta}_{t_k}^i := \hat{\beta}_{t_k}^{i,j}, \quad \hat{\alpha}_{t_k}^i := \hat{\alpha}_{t_k}^{i,j}.
\]

4. Repeat the above steps 1, 2, and 3 from $k = N - 1$ to $k = 0$ to obtain the optimal rebalancing decisions for the net liability hedge under short-selling constraints (A.2) on $\pi$.

**Proposition 2.** Let $(\bar{\pi}_{t_k}^{i,j}, \bar{\beta}_{t_k}^{i,j}, \bar{\alpha}_{t_k}^{i,j})$ be the solution to the unconstrained optimization (A.1). Then the solution $(\hat{\pi}_{t_k}^{i,j}, \hat{\beta}_{t_k}^{i,j}, \hat{\alpha}_{t_k}^{i,j})$ of constrained optimization (A.3) can be determined as follows

If $\bar{\pi}_{t_k}^{i,j} \geq -D_k$,
\[
\hat{\pi}_{t_k}^{i,j} = \bar{\pi}_{t_k}^{i,j}, \quad \hat{\beta}_{t_k}^{i,j} = \bar{\beta}_{t_k}^{i,j}, \quad \hat{\alpha}_{t_k}^{i,j} = \bar{\alpha}_{t_k}^{i,j}.
\]

If $\bar{\pi}_{t_k}^{i,j} < -D_k$,
\[
\hat{\pi}_{t_k}^{i,j} = -D_k, \quad \hat{\beta}_{t_k}^{i,j} = \frac{(-2c_k h_k + f_k i_k) + (2c_k d_k - e_k f_k)D_k}{4b_k c_k - f_k^2}, \quad \hat{\alpha}_{t_k}^{i,j} = \frac{(-2b_k i_k + f_k h_k) + (2b_k e_k - d_k f_k)D_k}{4b_k c_k - f_k^2},
\]
where $b_k, c_k, d_k, e_k, f_k, h_k, i_k$ are obtained in the unconstrained problem “UC”.

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In the last market environment considered in this document, we assume that put options are not available and not used in the hedging procedure. Then the optimization problem for given $S^i_{tk}$ at time $t_k$ becomes the following simpler optimization problem.

$$(\tilde{\pi}^{i,j}_{tk}, \tilde{\beta}^{i,j}_{tk}) := \arg\min_{(\pi_{tk}, \beta_{tk}) \in \mathbb{R}^2} L^\otimes_k(\pi_{tk}, \beta_{tk}, 0, 0; S^i_{tk}),$$  \hspace{1cm} (A.4)

where

$$L^\otimes_k(\pi_{tk}, \beta_{tk}, 0, 0; S^i_{tk}) \hspace{1cm} (A.5)$$

$$= \mathbb{E}_k \left[ \left\{ X_{tk+1} - \pi_{tk}S_{tk+1} - \beta_{tk}e^{\delta t} - \varepsilon(1 - \varepsilon)^{k+1}F_0 \frac{S_{tk+1}}{S_0} \mathbf{1}_{\{0 < k < N - 2\}} \right\}^2 \bigg| S_{tk} = S^i_{tk} \right].$$

It can be verified that the optimal rebalancing decision without put option defined by (A.4) is

$$\tilde{\pi}^{i,j}_{tk} = \frac{-2b_k g_k + d_k h_k}{4a_k b_k - d_k^2},$$
$$\tilde{\beta}^{i,j}_{tk} = \frac{-2a_k h_k + d_k g_k}{4a_k b_k - d_k^2},$$

where $a_k, b_k, d_k, g_k,$ and $h_k$ are obtained in the unconstrained problem.

Note that the third argument of $L^\otimes_k$ in (A.5) is zero, i.e. $\alpha_{tk} = 0$, because we do not use put options in this case. The fourth argument of $L^\otimes_k$ in (A.5) corresponds to the strike price of the put option, it does not need to be zero because $L^\otimes_k$ is no longer affected by the strike price.

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