Price of Reinsurance Bargaining with Monetary Utility Functions

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Abstract

Optimal contracts have widely been studied in the literature, yet the bargaining for optimal prices has remained relatively unexplored. Therefore the key objective of this paper is to analyze the price of reinsurance contracts. We use a novel way to model the bargaining powers of the insurer and reinsurer, which allows us to generalize the contracts according to the Nash bargaining solution, full competition, and the equilibrium contracts. We illustrate these pricing functions by means of inverse-S shaped distortion functions of the insurer and the Value-at-Risk for the reinsurer.

1 Introduction

This paper analyzes optimal reinsurance design and its pricing when firms are endowed with monetary utility functions. Broadly speaking there are two recent streams of literature that consider risk sharing with monetary utility functions. Both streams study roughly the same objective function in mathematical terms, but with different motivations. First, several authors study optimal risk sharing and Pareto equilibria (see, e.g., Filipović and Kupper, 2008; Jouini et al., 2008; Ludkovski and Young, 2009; Boonen, 2015). Second there is a stream in the literature that studies optimal (re)insurance contract design with a given premium principle analogue to a monetary utility function (see, e.g., Asimit et al., 2013; Cui et al., 2013; Chi and Meng, 2014; Assa, 2015; Boonen et al., 2015; Cheung and Lo, 2015). This paper combines both settings in the sense that we use a bargaining

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approach for optimal risk sharing contracts in the context of optimal reinsurance contract design. To the best of our knowledge, we are the first to explicitly combine both streams of literature.

Pricing of insurance and reinsurance contracts is typically done by assuming full competition. Then, the price is set such that the reinsurer or insurer is indifferent to selling the contract or not. In this way, one determines the zero-utility premium. Moreover, there is a stream in the literature that focuses on empirical data on insurance prices, and try to derive the implied pricing functions. The problem with such approach is that the number of transactions in reinsurance is typically limited. Our approach is different from both approaches. We determine the prices via a cooperative bargaining process.

Kihlstrom and Roth (1982), Schlesinger (1984), and Quiggin and Chambers (2009) all use the Nash bargaining solution for an insurance contract between a client and an insurer. Moreover, Aase (2009) uses the Nash bargaining problem to price reinsurance risk as well. Specifically for longevity risk, Boonen et al. (2012) and Zhou et al. (2012) use Nash bargaining solutions to price longevity-linked Over-The-Counter contracts. All these authors focus on firms that maximize Von Neumann-Morgenstern expected utility. In contrast, we use a cooperative bargaining approach to derive optimal reinsurance contracts and their corresponding prices via the Nash bargaining solutions. Moreover, we let firms maximize a monetary utility function. We provide a unique mechanism that allows us to generalize optimal contracts even if there is asymmetric bargaining power such as for the asymmetric Nash bargaining solution (Kalai, 1977). This mechanism allows us to include full competition as well, which leads to the extreme case that one firm is indifferent from trading, and the other firm gains maximally. This assumption is popular in the economic and actuarial literature, dating back from the concept of Bertrand equilibria (Bertrand, 1883).

This paper contributes to the literature in the following ways. We characterize the optimal hedge benefits (alternative interpreted as welfare gains) from bilateral bargaining for reinsurance. In the special case in which the preferences are given by a distortion risk measure, we derive a simple expression of the hedge benefits. Moreover, we derive bounds on the individual rational prices of a specific Pareto optimal contract, and provide to any price a corresponding bargaining power for the asymmetric Nash bargaining solution. To highlight our results, we illustrate the construction of the premium principle under the special case that the insurer is endowed with preferences given by an inverse-S shaped distortion risk measure, and the reinsurer optimizes a trade-off between the expected value and the Value-at-Risk (VaR). This leads to a discontinuous pricing function. Inverse-S shaped distortion risk measures are getting more popular to use as preferences since Quiggin (1982, 1991, 1992) and Tversky and Kahneman (1992).
This paper is set out as follows. Section 2 provides all general results on bargaining with monetary utility functions. Section 3 shows how these results translate to preferences given by a distortion risk measure. Section 4 provides important insight on the premium principle for the class of inverse-\(S\)-shaped distorted preferences and the well-known Value-at-Risk.

2 Model formulation

We consider a one-period model involving two firms, with one firm representing an insurer (I) and the other firm representing the reinsurer (R). Let \((\Omega, F, \mathbb{P})\) be a probability space, and \(L^\infty(\Omega, F, \mathbb{P})\) be the class of bounded random variables on it. When there is no confusion, we simply write \(L^\infty = L^\infty(\Omega, F, \mathbb{P})\). The total insurance liabilities that the insurer faces is given by the non-negative, bounded risk \(X \in L^\infty\). Here we assume that the insurer is interested in transferring a part of this risk to a reinsurer. Let us denote \(M = \text{esssup } X = \inf\{a \in \mathbb{R} : \mathbb{P}(X > a) = 0\}\).

The reinsurance contract is given by the tuple \((f, \pi)\), where \(f(X)\) is the indemnity paid by the reinsurer to the insurer and \(\pi \geq 0\) is the price (or premium) paid by the insurer to the reinsurer. It is natural to assume that \(f \in F\), where

\[
\mathcal{F} = \{f : \mathbb{R}_+ \to \mathbb{R}_+ \mid 0 \leq f(x) - f(y) \leq x - y, \forall x \geq y \geq 0, f(0) = 0\},
\]

i.e., we assume that the reinsurance contract \(f \in \mathcal{F}\) is 1-Lipschitz. 1-Lipschitz contracts account for moral hazard (Bernard and Tian, 2009), and is often used in the literature on reinsurance contract design (see, e.g., Young, 1999; Asimit et al., 2013; Chi and Meng, 2014; Assa, 2015; Xu et al., 2015).

By denoting \(W_k\) as the deterministic initial wealth for firm \(k\), where \(k \in \{I, R\}\), and \(\pi_I\) as the premium received by the insurer for accepting risk \(X\), then without the reinsurance the wealth at a pre-determined future time for the insurer and reinsurer are \(W_I + \pi_I - X\) and \(W_R\), respectively. If the insurer were to transfer part of its risk to a reinsurer using \(f(X)\) with corresponding price \(\pi\), then the wealth at a pre-determined future time for the insurer becomes

\[
W_I + \pi_I - X + f(X) - \pi. \tag{1}
\]

Similarly, the wealth for the reinsurer changes to

\[
W_R - f(X) + \pi. \tag{2}
\]

To assess if there should be a risk transfer between both firms, we need to make additional assumption on how firms evaluate such preference. In particular, we assume that firm \(k\) uses a monetary utility function \(V_k\). A preference relation \(V_k\) is monetary if it is monotone with respect to the order of \(L^\infty\), satisfies the
normalization condition $V_k(0) = 0$ and has the cash-invariance property $V_k(X + a) = V_k(X) + a$ for every $X \in L^\infty$ and $a \in \mathbb{R}$. Moreover, we assume that $V_k$ is comonotonic additive, i.e., if $X$ and $Y$ are comonotonic, then $V_k(X + Y) = V_k(X) + V_k(Y)$. This implies that the Choquet representation of Schmeidler (1986) can be used to formalize $V_k$. It is well-known that the initial wealth and $\pi_I$ are irrelevant for preferences given by monetary utility functions.

Pareto optimal reinsurance contracts are such that there does not exist another contract that is weakly better for both firms, and strictly better for at least one firm. Since we consider 1-Lipschitz reinsurance contracts, we have that $-X + f(X)$ and $-f(X)$ are comonotonic for all $f \in \mathcal{F}$. Furthermore, the comonotonic additivity of $V_k$ implies that the utility function $V_k$ is additive (and hence concave) on the domain $\mathcal{F}$.

In risk sharing, the problem of finding an $f \in \mathcal{F}$ and a price $\pi \in \mathbb{R}$ is analogous to the problem of finding comonotonic risk sharing contracts. If there is no constraints on $\pi$, it is shown by Jouini et al. (2008) that all Pareto optimal reinsurance contracts are obtained by

$$\max_{f \in \mathcal{F}} V_I(-X + f(X)) + V_R(-f(X)). \quad (3)$$

Note that $W_k, k \in \{I, R\}$, $\pi_I$ and $\pi$ do not appear in the above objective function due to the cash invariance property of $V_k, k \in \{I, R\}$; the $\pi$’s cancel out each other. The non-negativity constraint influences the Pareto optimal set only for contracts with $V_R(-f(X) + \pi) < V_R(0)$, which holds true for all $\pi < 0$ and $f \in \mathcal{F}$. The following asserts the existence of the Pareto optimal reinsurance contract.

**Proposition 2.1** Under the assumption that $f \in \mathcal{F}$ and $V_k < \infty, k \in \{I, R\}$ are monetary utility function, then there exists a Pareto optimal reinsurance contract $f^*$; i.e. $f^*$ is the optimal solution to (3).

**Proof** By defining the norm $d(f^1, f^2) = \max_{t \in [0, M]} | f^1(t) - f^2(t) |$, for any $f^1, f^2 \in \mathcal{F}$, then the set $\mathcal{F}$ is compact under this norm $d$. Moreover, we have $V_k(X) < \infty$ for $k \in \{I, R\}$. Hence, an optimal solution to (3) exists.

We now consider the benefits of risk sharing to both firms. Recall that without risk sharing, the utility of the insurer for insuring risk $X$ is $V_I(W_I + \pi_I - X)$ and the utility of the reinsurer is simply $V_R(W_R)$. If both firms agree to a risk sharing indemnity function $f(X)$ with corresponding price $\pi$, then the resulting utility of the insurer changes to $V_I(W_I + \pi_I - X + f(X) - \pi)$ so that the difference

$$V_I(W_I + \pi_I - X + f(X) - \pi) - V_I(W_I + \pi_I - X) = -V_I(-f(X)) - \pi \quad (4)$$
can be interpreted as the hedge benefit to the insurer using the risk sharing strategy $f \in \mathcal{F}$. The right hand side of the above equation follows from comonotonic additivity and cash invariance of $V_I$. Similarly, from the perspective of the reinsurer its hedge benefit can easily be shown to be
\[ V_R(W_R - f(X) + \pi) - V_R(W_R) = V_R(-f(X)) + \pi. \] (5)
Positive difference implies that there is an incentive with the risk sharing due to the gain in monetary utility. By denoting $HB(f)$ as the aggregate hedge benefit or the aggregate utility gains in the market for exercising the risk sharing strategy $f \in \mathcal{F}$, then we have
\[ HB(f) = V_R(-f(X)) - V_I(-f(X)). \]
Note that $HB(f)$ is simply the sum of the hedge benefit of both insurer and reinsurer and hence for brevity we refer to $HB(f)$ as the (aggregate) hedge benefit for a given risk sharing $f \in \mathcal{F}$. Note also that $HB(f)$ can be positive, negative, or zero, depending on $f(X)$ and the heterogeneous preferences of insurer and reinsurer. Since the utility functions are monetary, the hedge benefit $HB(f)$ is expressed in monetary terms as well.

If the risk sharing strategy corresponds to a Pareto optimal $f^*$, then the maximum achievable hedge benefit of the market is given by $HB^* \equiv HB(f^*)$; i.e.
\[ HB^* = HB(f^*) = V_R(-f^*(X)) - V_I(-f^*(X)) \geq 0. \] (6)
The inequality follows immediately from the fact that $f(X) = 0$ is a feasible strategy in $\mathcal{F}$. Also if $V_I = V_R$, then the comonotonic additivity of $V_k$ leads to $HB^* = 0$; i.e. there is no gain in welfare in the market regardless of the risk sharing $f \in \mathcal{F}$.

Depending on the market conditions, the hedge benefit $HB^*$ will be shared among both firms. Particularly, we require that the following two conditions are satisfied:

- Pareto optimality,

- individual rationality, or both firms are better off than when they do not trade: $V_I(-X + f(X) - \pi) \geq V_I(-X)$ and $V_R(-f(X) + \pi) \geq 0$.

Recall that the Pareto optimal $f^*$ does not depend on the price $\pi$, the following proposition establishes the lower and upper bounds of the individual rational price corresponding to $f^*$. The key to deriving these bounds is based on the minimum acceptable price that a reinsurer is willing to accept the risk from an insurer and the maximum price that an insurer is willing pay to transfer its risk to a reinsurer.
Proposition 2.2 The set of Pareto optimal and individual rational prices is given by the interval $[-V_R(-f^*(X)), -V_I(-f^*(X))]$, where $f^*$ is a solution of (3).

Proof For any $\pi \geq 0$, the solution $f^*$ is optimal (see Proposition 2.1). Note that due to cash-invariance of $V$, we get that $V_I(-X + f^*(X) - \pi)$ is strictly decreasing and continuous in $\pi$, and $V_R(-f^*(X) + \pi)$ is strictly increasing and continuous in $\pi$. Hence, the set of individual rational pricing is given by an interval, where the lower bound is such that $V_R(-f^*(X) + \pi) = V_R(0)$, and the upper bound such that $V_I(-X + f^*(X) - \pi) = V_I(-X)$. The lower bound follows directly from cash-invariance and $V_R(0) = 0$, and the upper bound follows directly from cash-invariance, comonotonic additivity, and the fact that $-X + f^*(X)$ and $-f^*(X)$ are comonotonic. Finally, $-V_R(-f^*(X)) \leq -V_I(-f^*(X))$ follows from (6). This concludes the proof.

Suppose now for a given indemnity function $f$, we define $\alpha \in [0,1]$ as the proportion of the hedge benefit that is allocated to the insurer; i.e. $\alpha HB(f)$ hedge benefit is assigned to the insurer and the remaining $(1-\alpha)HB(f)$ hedge benefit to the reinsurer. Corresponding to the indemnity function $f$ and the hedge benefit allocation $\alpha$, it is of interest to determine the resulting price of the reinsurance contract. To do this, it is useful to interpret the price $\pi$ as a function of both $f$ and $\alpha$ so that $\pi \equiv \pi(\alpha, f)$ represents the price of a reinsurance contract $f(X)$ with the insurer receives $\alpha HB(f)$ hedge benefit and reinsurer receives the remaining $(1-\alpha)HB(f)$ hedge benefit. Note that for a given allocation $\alpha$ the posterior utility of the insurer and the reinsurer are given by $V_I(W_I + \pi_I - X) + \alpha HB(f)$ and $V_R(W_R) + (1-\alpha)HB(f)$, respectively.

For a given $f \in \mathcal{F}$, the function $V_I(W_I + \pi_I - X + f(X) - \pi)$ is continuous and strictly decreasing in $\pi$, and the function $V_R(W_R - f(X) + \pi)$ is continuous and strictly increasing in $\pi$. Therefore, we get that the pricing function for a given $f \in \mathcal{F}$ and $\alpha$ can be defined as the solution to the following optimization problem:

$$
\pi(\alpha, f) = \arg \max_{\pi} V_I(W_I + \pi_I - X + f(X) - \pi)
$$

s.t. $V_R(W_R - f(X) + \pi) \geq V_R(W_R) + (1-\alpha)HB(f)$.

Due to the cash invariance property of $V_k$, $k \in \{I, R\}$, the above optimization problem is equivalent to

$$
\pi(\alpha, f) = \arg \max_{\pi} V_I(-X + f(X) - \pi)
$$

s.t. $V_R(-f(X) + \pi) \geq (1-\alpha)HB(f)$.

The following proposition explicitly provides an expression for determining the price $\pi(\alpha, f)$.
Proposition 2.3 For any \( f \in \mathcal{F} \) and \( \alpha \in [0, 1] \), we have \( \pi(\alpha, f) = -(1 - \alpha)V_I(-f(X)) - \alpha V_R(-f(X)) \).

Proof The optimal price \( \pi(\alpha, f) \) with \( \alpha \in [0, 1] \) is such that \( V_I(-X + f(X) - \pi(\alpha, f)) = V_I(-X + \alpha \cdot HB(f)) \). Since \( V_I \) is cash-invariant we get \( V_I(-X + f(X)) - \pi(\alpha, f) = V_I(-X) + \alpha \cdot HB(f) \) so that

\[
\pi(\alpha, f) = V_I(-X + f(X)) - V_I(-X) - \alpha HB(f) = V_I(-X) - V_I(-f(X)) - \alpha [V_I(-f(X)) + V_R(-f(X))] = -(1 - \alpha) V_I(-f(X)) - \alpha V_R(-f(X)).
\]

Here, (11) follows from comonotonic additivity of \( V_I \) and the fact that \(-X + f(X)\) and \(-f(X)\) are comonotonic since \( f \in \mathcal{F} \). This concludes the proof.

Note that \( HB(f) \) can be negative, but it is bounded from above by \( HB(f^*) \). Moreover, by using the property that \( V_I \) is monotone and \( V_R(0) = 0 \), we can see that \( \pi(\alpha, f) \) is always non-negative even when \( HB(f) \) is negative. Note that if \( HB(f) \) is negative, any contract \((f, \pi)\) is not individual rational.

A more interesting situation to analyze is the Pareto optimal case with \( f = f^* \), where \( f^* \) is the optimal solution to (3). Recall that the resulting \( HB^* \) gives the highest attainable hedge benefit among the insurer and the reinsurer and that \( \alpha \) captures the proportion of \( HB^* \) that is assigned to insurer. As \( \alpha \) increases from 0 to 1, the portion of the hedge benefit that is allocated to the insurer increases until \( \alpha = 1 \) with the insurer receives the entire hedge benefit. Consequently the parameter \( \alpha \) measures the bargaining power of the insurer; the higher the \( \alpha \), the greater the bargaining power of the insurer. The extreme cases \( \alpha = 0, 1 \) reflect cases of full competition: all hedge benefits in the market are shifted to one party. It is easy to show that the competitive equilibrium outcome, also called tâtonnement outcome (see, e.g., Zhou et al., 2015), corresponds to \( \alpha = 1 \).

Next, we provide a characterization of our mechanism which is well-studied in the classical economic literature. In particular, the asymmetric Nash bargaining solution (Kalai, 1977) with asymmetry parameter \( \alpha \) is given by:

\[
\arg \max_{(f, \pi) \in \mathcal{F} \times \mathbb{R}_+} \left\{ -V_I(W_I + \pi_I - X + f(X) - \pi) - V_I(W_I + \pi_I - X)^\alpha \times [V_R(W_R - f(X) + \pi) - V_R(W_R)]^{1-\alpha} \right\} \quad \text{(13)}
\]

s.t. \( V_I(W_I + \pi_I - X + f(X) - \pi) \geq V_I(W_I + \pi_I - X) \), \( V_R(W_R - f(X) + \pi) \geq V_R(W_R) \). \quad \text{(14)}
**Proposition 2.4** If $HB^* > 0$ and $f^* \in \mathcal{F}$ solves (3), the price $\pi(\alpha, f^*)$ with $\alpha \in (0, 1)$ coincides one-to-one with the asymmetric Nash bargaining solution with asymmetry parameter $\alpha$.

**Proof** It is well-known that the asymmetric Nash bargaining solution is Pareto optimal (Kalai, 1977).

For any Pareto optimal contract $f^* \in \mathcal{F}$ that solves (3), and together with (4) and (5), the objective in (13) can be reformulated as

$$\arg\max_{a \in \mathbb{R}} [aHB^*]^\alpha \cdot [(1 - a)HB^*]^{1-\alpha}.$$  \hspace{1cm} (16)

Moreover, (14) implies $a \geq 0$ and (15) implies $a \leq 1$. Obviously, the solution of (16) does not depend on $HB^* > 0$. Then, we derive straightforwardly that the solution $a$ is given by $a = \alpha$. Finally, $\pi = \pi(\alpha, f^*)$ follows by definition. This concludes the proof.

The case $\alpha = \frac{1}{2}$ corresponds to equal sharing of the hedge benefits, and corresponds to the Nash bargaining solution (Nash, 1950). The following proposition characterizes the pricing rule $\pi(\alpha, f^*)$ in a way that is commonly used in economic theory as well.

**Proposition 2.5** The optimal solution $(f^*, \pi(\alpha, f^*))$ is an element of

$$\arg\max_{(f, \pi) \in \mathcal{F} \times \mathbb{R}^+} \left\{ V_I(-X + f(X) - \pi) \right. \left. \text{s.t. } V_R(-f(X) + \pi) \geq (1 - \alpha) \cdot HB^* \right\}$$  \hspace{1cm} (17)

**Proof** First, we show that every solution to (17) is Pareto optimal. Let $(f, \pi) \in \mathcal{F} \times \mathbb{R}^+$ solve (17) and suppose that it is not Pareto optimal. Then, there exist $(\hat{f}, \hat{\pi})$ such that $V_I(-X + \hat{f}(X) - \hat{\pi}) \geq V_I(-X + f(X) - \pi)$ and $V_R(-\hat{f}(X) + \hat{\pi}) \geq V_R(-f(X) + \pi)$ with one strict inequality. Since $(f, \pi)$ solve (17), we get that $V_I(-X + \hat{f}(X) - \hat{\pi}) = V_I(-X + f(X) - \pi)$, and, hence, $V_R(-\hat{f}(X) + \hat{\pi}) > V_R(-f(X) + \pi)$. The function $V_I(-X + f(X) - \pi)$ is continuous and strictly decreasing in $\pi$, and the function $V_R(-f(X) + \pi)$ is continuous and strictly increasing in $\pi$. Hence, there exist $(\bar{f}, \bar{\pi})$, with $\bar{\pi} < \hat{\pi}$ such that $V_I(-X + \bar{f}(X) - \bar{\pi}) > V_I(-X + f(X) - \pi)$ and $V_R(-\bar{f}(X) + \bar{\pi}) > V_R(-f(X) + \pi)$. This is a contradiction. Hence, $(f, \pi)$ is Pareto optimal. So, there exists a Pareto optimal solution solving (17). Hence, $f = f^*$ is a solution to (17).

The result that $\pi = \pi(\alpha, f^*)$ follows from the definition in (7)-(8). This concludes the proof.
To conclude this section, we point out that if the reinsurance contracts $f$ are allowed to have any shape, i.e., not only in set $F$, and that the preferences are assumed to be concave and law-invariant, then Proposition 2.1 still holds. However, Proposition 2.4 only applies to Pareto optimal contracts that are in the set $F$, i.e., that are 1-Lipschitz. It is important to note that Landsberger and Meilijson (1994) show that there exists a Pareto optimal contract $f$ that is in $F$.

3 Distortion risk measures

In this section, we assume that insurer $I$ and reinsurer $R$ are endowed with a particular type of monetary utility function. More specifically, their preferences are given by a distortion risk measure (see Wang et al., 1997), i.e., we have

$$V_k(W) := -E^k[-W] = \int_{-\infty}^{0} [1 - g_k(S_W(z))] dz - \int_{0}^{\infty} g_k(S_W(z)) dz,$$

(18)

where $k \in \{I, R\}$, $S_W(z) = 1 - F_W(z)$ is the survival function of $-W$, and $g_k : [0, 1] \rightarrow [0, 1]$ is a non-decreasing function such that $g_k(0) = 0$ and $g_k(1) = 1$. A non-decreasing function $g_k : [0, 1] \rightarrow [0, 1]$ with $g(0) = 0$ and $g(1) = 1$ is called a distortion function. As a special case, when $Y \geq 0$, we have

$$E^k(Y) = \int_{0}^{\infty} g_k(S_Y(z)) dz.$$  

(19)

Moreover, distortion risk measures are convex if the distortion functions are concave. This particular class of risk measures is popular as it is related to the dual theory (Yaari, 1987) and the coherent risk measures (Artzner et al., 2001). Maximizing dual utility is equivalent to minimizing a distortion risk measure. Risk-aversion for distortion risk measures is equivalent to using a concave distortion function (Yaari, 1987). The Value-at-Risk (VaR) and all coherent risk measures satisfying law-invariance and comonotonic additivity are distortion risk measures (see Wang et al., 1997). Also, maximizers of a risk-reward trade-off $V_k(W) = (1 - \gamma)E[W] - \gamma \rho_k(W), \alpha \in [0, 1]$, are captured by this preference relation.

Pareto optimal reinsurance contracts $f^*$ for distortion risk measures follow from Cui et al. (2013) and Assa (2015), and are given by “layering” of $X$. In this case, every optimal solution $f^*$ can be shown to satisfy the following relationship:

$$f^*'(z) = \begin{cases} 
1 & \text{if } g_I(S_X(z)) > g_R(S_X(z)), \\
\beta(z) & \text{if } g_I(S_X(z)) = g_R(S_X(z)), \\
0 & \text{otherwise},
\end{cases}$$

(20)
for all \( z \geq 0 \) almost surely, where \( \beta_i(z) \in [0, 1] \). Note that the reinsurance contracts \( f \in \mathcal{F} \) are 1-Lipschitz, and therefore absolutely continuous. Hence the derivative of \( f \) exists almost everywhere.

An interesting consequence of using the distortion risk measure to capture the monetary utility function is that the hedge benefit \( HB^* \) can be determined without knowing the Pareto optimal contract \( f^* \). This is asserted in the following proposition.

**Proposition 3.1** If both firms use a distortion risk measure, we get

\[
HB^* = \int_0^\infty \Delta g_+(S_X(z)) \, dz, 
\]

where \( \Delta g_+ = (g_I - g_R)_+ \) and \( (y)_+ = \max\{y, 0\} \).

**Proof** It follows from (6) that

\[
HB^* = -V_I(-f^*(X)) + V_R(-f^*(X)) 
\]

\[
= E^{g_I}[f^*(X)] - E^{g_R}[f^*(X)] 
\]

\[
= \int_0^\infty g_I(S_{f^*(X)}(z)) \, dz - \int_0^\infty g_R(S_{f^*(X)}(z)) \, dz 
\]

\[
= \int_0^\infty g_I(S_X(z)) f^*(z) \, dz - \int_0^\infty g_R(S_X(z)) f^*(z) \, dz 
\]

\[
= \int_0^\infty (g_I - g_R)(S_X(z)) f^*(z) \, dz 
\]

\[
= \int_0^\infty (g_I - g_R)(S_X(z))1_{(g_I(S_X(z)) > g_R(S_X(z)))} \, dz 
\]

\[
= \int_0^\infty \Delta g_+(S_X(z)) \, dz 
\]

where (26) follows from (20). This concludes the proof.

**Corollary 3.2** Let \( X \) have a compact support \([0, M]\). It holds that \( HB^* = 0 \) if and only if the Lebesgue measure of the set \( \{z \in [0, M] : g_I(S_X(z)) - g_R(S_X(z)) > 0\} \) is zero. Furthermore, if \( X \) has a positive density on its support \([0, M]\), then it holds that \( HB^* = 0 \) if and only if the Lebesgue measure of the set \( \{z \in [0, M] : g_I(z) - g_R(z) > 0\} \) is zero.

Recall that \( HB^* = 0 \) signifies the situation that both firms are not able to strictly benefit from risk sharing. In the economic literature on risk sharing, this situation is also called no-trade (De Castro and Chateauneuf, 2011).

We next derive a pricing function associates with the distortion risk measures.
**Proposition 3.3** The price for $f^*$ that satisfies (20) is given by

$$\pi(\alpha, f^*) = E^{(1-\alpha)g_I + \alpha g_R}[f^*(X)].$$

**Proof** The fact that $f^*$ satisfies (20) implies $f^*(X) \geq 0$. If $V_k, k \in \{I, R\}$, are distortion risk measures, then by substituting (19) in (9)-(12), we obtain

$$\pi(\alpha, f^*) = (1-\alpha)\int_0^\infty g_I(S_{f^*(X)}(z)) \, dz + \alpha \int_0^\infty g_R(S_{f^*(X)}(z)) \, dz$$

$$= \int_0^\infty [(1-\alpha)g_I(S_{f^*(X)}(z)) + \alpha g_R(S_{f^*(X)}(z))] \, dz$$

$$= E^{(1-\alpha)g_I + \alpha g_R}[f^*(X)].$$

This concludes the proof.

Proposition 3.3 establishes that the pricing function is a distortion premium principle. The use of distortion premium principles to price risk has gained popularity in the actuarial literature. See for example De Waegenaere et al. (2003), Cui et al. (2013), and Assa (2015). Note that these authors all assume that the distortion premium principles are given, whereas we derive it from cooperative bargaining.

To conclude this section, we point out that suppose the insurer is a risk-averse distortion risk measure minimizer, and the reinsurer is risk-neutral, then it is optimal to reinsure all risk to the reinsurer, i.e., $f^*(X) = X$. However, the reinsurer might ask for a mark-up above the expected value premium if $\alpha < 1$. Therefore, the premium is always larger than the expected value of a risk.

### 4 An illustration: inverse-$S$ shaped distorted preferences and VaR

The objective of this section is to provide explicitly the pricing function under some additional assumptions on the preferences of insurer and reinsurer. We assume that the monetary utility function for the insurer is dictated by an inverse-$S$ shaped distorted function while the reinsurer relies on the value-at-risk (VaR). We consider inverse-$S$ shaped distortion risk measures because of their desirable properties in modeling human behavior and their popularity in recent years (Quiggin 1982, 1991, 1992; Tversky and Kahneman 1992; Tversky and Fox 1995; Wu and Gonzalez, 1999; Abdellaoui, 2000; Rieger and Wang, 2006; Jin and Zhou, 2008; He and Zhou, 2011; Xu and Zhou, 2013; Bernard et al., 2015). Similarly, we adopt VaR in our example because it is a prominent measure of risk among financial institutions.
and insurance companies. It is also a regulatory risk measure adopted by the Solvency II regulations for insurance companies in European Union.

We first focus on the insurer’s monetary utility function and then followed by the reinsurer’s. For the insurer, we additionally assume that the adopted distortion function $g_I$ is continuously differentiable so that for all $W \in L^\infty$, (18) can be written as

$$V_I(W) = \int_0^1 F_W^{-1}(s)g_I'(s)ds,$$

(28)

where $F_W^{-1}(s) = \inf\{z \in \mathbb{R} : F_W(z) \geq s\}, s \in [0, 1]$. The above representation demonstrates the role of the shape of the distortion function on evaluating wealth. If the function $g_I$ is strictly concave; i.e. $g_I'(0) > 1$ and $g_I'(1) < 1$, then (28) implies that the good outcomes receive higher weights and the bad ones get smaller weights. If the function $g_I$ is strictly convex; i.e. $g_I'(0) < 1$ and $g_I'(1) > 1$, then the good outcomes get smaller weights and the bad ones receive higher weights. On the other hand, inverse-S shaped preferences ($g_I'(0) > 1$ and $g_I'(1) > 1$) are such that both bad outcomes and good outcomes are heavily weighted. This is consistent with numerous psychological experiments conducted to study individual’s risk aversion (Tversky and Kahneman, 1992; Tversky and Fox, 1995). Therefore, we consider an inverse-S shaped function in our example.

We now formally provide the definition of an inverse-S shaped distortion function.

Definition 4.1 A distortion function $g$ is called inverse-S shaped if:

- it is continuously differentiable;
- there exists $b \in (0, 1)$ such that $g$ is strictly concave on the domain $(0, b)$ and strictly convex on the domain $(b, 1)$;
- it holds that $g'(0) = \lim_{s \downarrow 0} g'(s) > 1$ and $g'(1) = \lim_{s \uparrow 1} g'(s) > 1$.

The point $b$ in the above definition is the inflection point such that the $g$ changes from locally concave to locally convex. Many distortion functions used in the literature are examples of inverse-S shaped. For example, let us consider the function proposed by Tversky and Kahneman (1992), which is parameterized by:

$$g^\zeta(s) = \frac{s^\zeta}{(s^\zeta + (1-s)^\zeta)^{\frac{1}{\zeta}}} \quad \text{for all } s \in [0, 1],$$

(29)

where $\zeta > 0$. Figure 1 plots (29) using $\zeta = 0.5$. Rieger and Wang (2006) point out that (29) is increasing and inverse-S shaped for $\zeta \in (0.279, 1)$. 

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Recall that Proposition 3.3 formally establishes that the distortion function in premium principle is given by a convex combination of both functions $g_I$ and $g_R$. If the distortion functions $g_I$ and $g_R$ are inverse-$S$ shaped, the pricing function is a distortion premium principle with an increasing distortion function. Any shape can be generated by a choice of $\alpha, g_I$ and $g_R$.

Remark If the functions $g_I$ and $g_R$ are concave, then $\pi$ is a concave distortion premium principle. If the functions $g_I$ and $g_R$ are convex, then $\pi$ is a convex distortion premium principle. If the functions $g_I$ and $g_R$ are inverse-$S$ shaped and the inflection points of $g_I$ and $g_R$ are the same, then $\pi$ is an inverse-$S$ shaped distortion premium principle.

We now discuss the reinsurer’s monetary utility function. As pointed out earlier that the reinsurer’s monetary utility function is based on VaR. Formally, the VaR of the wealth $W$ at a confidence level $\beta$, $0 < \beta < 1$, is given by $VaR_\beta(W) = E^\beta(-W)$ with $g(s) = 1_{s > \beta}$. This risk measure is connected to the quantile function via $VaR_\beta(W) = -F_W^{-1}(\beta)$. We assume that the preferences of the reinsurer are given

\[A concave distortion premium principle follows from bargaining between two firms that both use a concave distortion risk measure. Concave distortion risk measures resemble risk-averse preferences (Yaari, 1987).\]
by
\[ V_R(W) = \gamma E[W] - (1 - \gamma) VaR_\beta(W), \tag{30} \]
for all \( W \in L^\infty \), where \( \gamma \in [0, 1], \beta \in (0, 1) \). So, firms optimize a trade-off between expected return and risk as measured by VaR. By construction, the price incorporates an expected value as well. In the literature, this is also called the risk-adjusted value of the liabilities (for more detailed information, see De Giorgi and Post, 2008; Chi, 2012; Chi and Weng, 2013; Cai and Weng, 2014; Cheung and Lo, 2015). The preference relation in (30) is a distortion risk measure, which corresponds to setting the distortion function \( g_R \) as a weighted average of \( g(s) = s \) and \( g(s) = 1 - s > \beta \):
\[
g_R(s) = \begin{cases} \gamma s & \text{if } s \leq \beta, \\ \gamma s + (1 - \gamma) & \text{if } s > \beta. \end{cases}
\]

Moreover, we assume that the insurer’s preferences are given by an inverse-S shaped distortion risk measure with the distortion function \( g_I \). For inverse-S shaped distortion functions, the next property of the function
\[
p(s) = \frac{1 - g_I(s)}{1 - s}, \text{ for all } s \in [0, 1),
\]
follows from Xu et al. (2015).

**Lemma 4.2** The function \( p \) is continuous. Moreover, there exists \( a \in (0, b) \) such that \( p \) is strictly decreasing on \([0, a]\) and strictly increasing on \([a, 1)\).

The point \( a \) is illustrated in Figure 1. If \( \gamma < 1 \), there are five cases to consider for the Pareto optimal insurance contracts given by (3). These five cases are illustrated in Figure 2.

**Case 4.1** In Figure 2, we have \((\beta, \gamma \beta) \in A\) and \((\beta, \gamma \beta + 1 - \gamma) \in 1\), i.e., \( g_I(\beta) \geq \gamma \beta + 1 - \gamma \). Then, there exists \( c \in [\beta, 1) \) such that \( g_R(s) < g_I(s) \) for \( s \in (0, c) \) and \( g_R(s) > g_I(s) \) for \( s \in (c, 1) \). The optimal solution in (20) is given by
\[
f^*(z) = \begin{cases} 0 & \text{if } 0 \leq z \leq VaR_c(-X), \\ z - VaR_c(-X) & \text{if } z > VaR_c(-X). \end{cases}
\]
or, equivalently, \( f^*(X) = (X - VaR_c(-X))_+ \).

**Case 4.2** In Figure 2, we have \((\beta, \gamma \beta) \in A\) and \((\beta, \gamma \beta + 1 - \gamma) \in 2\), i.e., \( \gamma \beta < g_I(\beta) < \gamma \beta + 1 - \gamma < \gamma > p(a) \) and \( \beta \geq a \). Then, we have \( g_R(s) < g_I(s) \) for \( s \in (0, \beta) \) and \( g_R(s) > g_I(s) \) for \( s \in (\beta, 1) \). The optimal solution in (20) is given by
\[
f^*(X) = (X - VaR_\beta(-X))_+.
\]
Figure 2: The solid line is an inverse-S shaped distortion function $g_I$. The five different cases are indicated via the areas $A, B, 1, 2, 3, 4$.

**Case 4.3** In Figure 2, we have $(\beta, \gamma \beta) \in A$ and $(\beta, \gamma \beta + 1 - \gamma) \in 3$, i.e., $\gamma \beta < g_I(\beta) < \gamma \beta + 1 - \gamma$ and $\gamma \leq p(a)$. The optimal solution coincides with the solution of Case 4.2.

**Case 4.4** In Figure 2, we have $(\beta, \gamma \beta) \in A$ and $(\beta, \gamma \beta + 1 - \gamma) \in 4$, i.e., $\gamma \beta < g_I(\beta) < \gamma \beta + 1 - \gamma$, $\gamma \leq p(a)$, and $\beta \geq a$. Then, there exist two points $c \in (0, a)$ and $d \in (a, 1)$ such that $g_R(s) < g_I(s)$ for $s \in (0, \beta)$, $g_R(s) > g_I(s)$ for $s \in (\beta, c)$, $g_R(s) < g_I(s)$ for $s \in (c, d)$ and $g_R(s) > g_I(s)$ for $s \in (d, 1)$. The optimal solution in (20) is given by

$$f^*(z) = \begin{cases} 
0 & \text{if } 0 \leq z \leq VaR_d(-X), \\
z - VaR_d(-X) & \text{if } VaR_d(-X) < z \leq VaR_c(-X), \\
VaR_c(-X) - VaR_d(-X) & \text{if } VaR_c(-X) < z \leq VaR_\beta(-X), \\
z - VaR_\beta(-X) + VaR_c(-X) - VaR_d(-X) & \text{if } z > VaR_\beta(-X),
\end{cases}$$

or, equivalently, $f^*(X) = \min\{(X - VaR_d(-X))_+, VaR_c(-X) - VaR_d(-X)\} + (X - VaR_\beta(-X))_+$.

**Case 4.5** In Figure 2, we have $(\beta, \gamma \beta) \in B$, i.e., $g_I(\beta) \leq \gamma \beta$. Then, there exists $e \in [\beta, 1)$ such that $g_R(s) < g_I(s)$ for $s \in (0, e)$ and $g_R(s) > g_I(s)$ for $s \in (e, 1)$. The optimal solution in (20) is given by $f^*(X) = (X - VaR_e(-X))_+$.
The case $\gamma = 1$ is analogue to Case 4.1, where, without loss of generality, we set $\beta = 0$.

In Figure 3, we graphically illustrate the pricing premium principle of Proposition 3.3 for Case 4.1. We observe that the pricing distortion function is discontinuous and piecewise concave.

![Figure 3: We graphically display the premium principle. The preferences of the insurer are inverse-$S$ shaped (the solid line) and the preferences of the reinsurer are given in (30), with $\gamma = 0.9$ and $\beta = 0.1$ (the dotted line). The optimal reinsurance contract is derived from Case 4.1, and is given by $f^*(X) = (X - \text{VaR}_c(-X))_+$ for $c \approx 0.45$ in this figure. The line of crosses is the distortion function that serves as a premium principle after bargaining with $\alpha = 0.5$.](image)

5 Conclusion

This paper studies bargaining for optimal reinsurance contracts with monetary preferences. In classical economics, one often assume that all profits from trading in economic markets are borne by one party. This leads to no profits for the other parties. Very few papers in the literature consider the fact in Over-The-Counter trades, benefits from sharing risk are shared between the two parties. Some exceptions are Kihlstrom and Roth (1982), Schlesinger (1984), Aase (2009), Boonen et al. (2012) and Zhou et al. (2012). All these authors consider the Nash bargaining solution.
If firms have monetary preferences and use the Nash bargaining solution, the profits are shared equally between the two parties. We generalize this concept by parameterizing the share of the hedge benefits that are assigned to the insurer. We derive an implicit premium principle, which is analogue to a monetary utility function.

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