How Robust is the Value-at-Risk of Credit Risk Portfolios?

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Abstract

In this paper, we assess the magnitude of model uncertainty of credit risk portfolio models, i.e., what is the maximum and minimum Value-at-Risk (VaR) of a portfolio of risky loans that can be justified given a certain amount of available information. Puccetti and Rüschendorf (2012b) and Embrechts et al. (2013) propose the rearrangement algorithm (RA) as a general method to approximate VaR bounds when the loss distributions of the different loans are known but not their interdependence. Their numerical results show that the gap between worst-case and best-case VaR is typically very high, a feature that can only be explained by lack of using dependence information.

In this paper, we propose a modification of the RA that makes it possible to approximate sharp VaR bounds when besides the marginal distributions also higher order moments of the aggregate portfolio such as variance and skewness are available as sources of dependence information.

A numerical study shows that the use of moment information makes it possible to significantly improve the (unconstrained) VaR bounds. However, VaR assessments of credit portfolios that are performed at high confidence levels (as it is the case in Solvency II and Basel III) remain subject to significant model uncertainty and are not robust. We provide some suggestions for strengthening future (capital) regulation.

Keywords Rearrangement algorithm, Moment bounds, Value-at-Risk, Credit risk portfolio. Minimum variance

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1 Introduction

The financial crisis that emerged in 2008 has shown that management of credit risk is of utmost importance for the stability of the worldwide financial system. Such stability is intimately connected to the amount of capital that is available as a cushion against adverse events, and financial institutions as well as regulatory authorities use models to determine these capital buffers. In this regard, many industry participants as well as Basel III and Solvency II regulatory frameworks rely on the so-called “Merton’s model of the firm” to estimate Value-at-Risk\(^1\) (VaR) of their credit risk portfolios and use this risk number as input to establish capital requirements.

In the industry, this model is also essentially known as the KMV model (see also Gordy (2000)) and we refer to this name without further ado. However, like any other credit risk portfolio model, the KMV model requires several ad-hoc assumptions that are hard to justify, and it is thus inherently subject to model uncertainty. The basic reason for this feature is that large losses of a credit portfolio occur when several loans default together, but lack of default data implies that these joint probabilities are very hard to specify\(^2\) (joint defaults are “rare events”).

To illustrate that model uncertainty is a real concern in the context of credit risk portfolio modeling, Chernih et al. (2010) describe a portfolio model that is statistically indistinguishable of the KMV model in the sense that it uses exactly the same basic parameters: These parameters are the probabilities of default (PDs), the exposures at default (EADs) and the loss given defaults (LGDs) of all individual loans as well as their default (asset) correlations used to describe the interactions among the loans. Yet, these authors show that under their model the VaR of a portfolio can be more than fifteen times larger than when using the KMV model. At first, it may seem surprising that the VaRs of two models can be so different. However, (asset) credit correlations that are used to specify the dependence, in reality only reveal information on the likelihood that exactly two loans default together, but they do not make it possible to determine the likelihood that three or more loans default together. The KMV model effectively deals with this issue by imposing that default of a loan occurs when the assets of the underlying debtor are insufficient to meet the liabilities and also assumes that the asset returns are multivariate normally distributed (with some asset correlation matrix).\(^3\) By contrast, Chernih et al. (2010) make another than Gaussian dependence assumption (while preserving the same correlations) and this choice significantly impacts the VaRs.

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\(^1\)It is well known that VaR does not capture tail risk and does not reward diversification. Its main challenger is Tail-Value-at-Risk (TVaR), also known as Expected Shortfall (ES), which in essence is the mean of the tail distribution and which addresses several shortcomings of VaR. However, TVaR has its own deficiencies. For example, Kou and Peng (2014a) show that it is more sensitive to model mis specification than VaR. As a better alternative for VaR, they propose the Median shortfall (MS) which is the median of tail loss distribution; see also the discussion in Kou and Peng (2014b). In this paper we focus on VaR, which is the preferred reference measure for risk quantification and (regulatory) solvency assessment for banks and insurers.

\(^2\)Note also that the independence assumption cannot be invoked here as the credit quality of different credit loans depend on common factors such as the state of the global economic environment, industry, geography, monetary policy and so on.

\(^3\)The use of multivariate normal models is often based on the (wrong) intuition that correlations are enough to model dependence. It is however clear that correlations only are not enough to model dependence, as a number (i.e., the correlation) can never be sufficient to describe the complex interaction between variables unless additional assumptions are made (see e.g. Embrechts et al. (2013)). This fallacy may then also partially explain why the KMV model has gained so much support in the industry.
Regulators are also increasingly concerned with model uncertainty and consistency of models. In a discussion paper, the Basel Committee 2010) explicitly states that a desired objective of a solvency framework concerns comparability: “Two banks with portfolios having identical risk profiles apply the framework’s rules and arrive at the same amount of risk-weighted assets and two banks with different risk profiles should produce risk numbers that are different proportionally to the differences in risk”. However, there are a number of other reasons that explain why model uncertainty is important for business. Indeed, an important task of an Enterprise Risk Management (ERM) framework concerns capital (risk) allocation, i.e., the allocation of total capital held by the insurer across its various constituents (subgroups) such as business lines, risk types, geographical areas, among others. Doing so makes it possible to redistribute the cost of holding capital across the various constituents so that it can be transferred back to the clients in the form of charges (premiums). Risk allocation makes it also possible to assess the performance of the different business lines by determining the return on allocated capital for each line. Finally, looking for risk allocation may help to identify areas of risk consumption within a given organization and thus to support the decision making process concerning business expansions and reductions.

In this paper, we aim at assessing the magnitude of model uncertainty of credit risk portfolio models, i.e., what is the maximum (or minimum) value for a certain risk measure (typically the VaR) that can be justified given a certain set of information? In the unconstrained case (i.e., when all PDs, LGDs and EADs are assumed to be known but not the dependence), some explicit bounds were found by Rüschendorf (1982) for the two-dimensional case and by Puccetti and Rüschendorf (2012b) for homogeneous portfolios in higher dimensions. However, the problem is fairly more complicated when the portfolio is heterogeneous. In this regard, Puccetti and Rüschendorf (2012a) and Embrechts et al. (2013) propose the Rearrangement Algorithm (RA) to approximate the unconstrained VaR bounds of an heterogeneous portfolio. While their numerical examples provide evidence that the RA is indeed able to approximate the sharp bounds accurately, they also show that the gap between the minimum and the maximum possible VaR is typically very high. In particular, the upper bound on VaR is always larger than the VaR when all risks are assumed to be maximally correlated (comonotonic), a situation that is hard to accept by practitioners.

Hence, sharpening the VaR bounds by considering the presence of dependence information (constrained case) is of great practical relevance but also difficult to do because, as pointed out before, knowledge of the joint default probabilities is not in reach practically. By contrast, the variance and perhaps also the skewness and the kurtosis of the aggregate portfolio can be estimated statistically and can potentially be used as a source of dependence information making it possible to obtain improvements of the VaR bounds. This idea is actually inherent in Bernard et al. (2013) who propose a version of the RA that incorporates a variance constraint and who show that such constraint can have significant impact on the VaR bounds.

Our paper is a further development of theirs. We focus on credit risk and study portfolios of risky loans. In this context, the marginal distributions correspond to (scaled) Bernoulli distributions that are characterized by a risk exposure (i.e., the effective loss in case of default) and a default probability. We provide several contributions. Our first main contribution is that we propose an efficient modification of the original RA to approximate sharp VaR bounds. When there are no dependence constraints our modified algorithm does not perform better than the original one. However, as compared to the original RA our
The algorithm is also directly applicable when there is dependence information available through
knowledge of some higher order moment constraints (variance, skewness, kurtosis, etc.). In
this regard, we point out that our approach is designed to incorporate the statistical uncer-
tainty on these additional moment constraints. Indeed, in practice the available information
on moments appears through statistical point estimates and moments are never known with
certainty. Hence, rather than imposing equality constraints for the moment information we
work with inequality constraints as a prudent approach to estimate VaR. In the paper we
also provide specialized results for VaR bounds in the case of credit portfolios that are ho-
mogeneous in exposures (but not in probability). We also derive the dependence that yields
the minimum variance of the credit portfolio. As a second major contribution we provide a
detailed numerical study showing that VaR assessments of credit risk portfolios that focus
on “deep in the tail events” are not stable and prone to significant model uncertainty that
we are able to quantify. In fact, while the use of moment information certainly makes it
possible to narrow the distance between the VaR upper bound and lower bound, the gap
may remain significant. This observation implies that other sources of dependence inform-
ation are needed to obtain further reductions. For example, knowledge on (the direction
of) the interrelations that exist among some of the credit risks in times of stress could be of
significant value in obtaining improvements. We conclude that credit risk capital require-
ments that are based on VaRs computed at a very high confidence level (e.g., 99.5%) using
some industry model may provide a false feeling of safety and lead to unfair competition
among financial institutions. We provide some policy suggestions about how regulation can
address these issues.

2 Problem description

We consider loan portfolios under the so-called default mode paradigm. Hence, a credit loss
occurs if the loan (i.e., the underlying obligor) defaults during the considered time horizon
and other value changes (e.g., due to a downgrade) are not recognized. Hence, let $I_i$ be the
indicator variable, which is equal to one if the $i$-th loan defaults and to zero otherwise. The
default probability is denoted by $p_i$:

$$p_i := P[I_i = 1].$$

Further, let $EAD_i$ denote the “Exposure-At-Default” and $LGD_i$ the “Loss-Given-Default”
of risk $i$. The “Exposure-At-Default” is the maximum amount of loss on the $i$-th loan, pro-
vided that there is a default. The “Loss-Given-Default” is the percentage of the maximum
amount that is effectively lost in the event of a default. We assume that all $EAD_i$ and
$LGD_i$ are deterministic and known. The portfolio loss $S$ during the reference period is then
given by

$$S = \sum_{i=1}^{n} X_i,$$

in which $X_i = v_i I_i$ and $v_i = EAD_i LGD_i$. Hence, the credit losses $X_i$ follow a scaled
Bernoulli distribution (with known scaling factor $v_i$), i.e., $X_i \sim v_i B(p_i)$. We denote its
distribution by $F_i$. Without loss of generality, we assume that $v_1 \geq v_2 \geq \ldots \geq v_n > 0$. We
aim at computing the worst-case outcome, i.e., the Value-at-risk (VaR), of the portfolio loss

\footnote{In presence of inequality constraints, the bounds on VaR will become wider as compared to a situation in which all moments are assumed to be known with certainty.}
$S$ at a given confidence level $q$ ($0 < q < 1$). Hence, we are interested in $\text{VaR}_q^+[S]$, which is defined as

$$\text{VaR}_q^+[S] = \sup \{ x \in \mathbb{R} \mid F_S(x) \leq q \},$$

in which $F_S(x)$ is the distribution function of $S$.

It is then clear that a precise computation of VaRs of the portfolio loss $S$ can only be obtained if and only if one knows the joint distribution of the default vector $(I_1, I_2, \ldots, I_n)$. However, this joint distribution is hard to obtain. In this regard, we point out that financial institutions typically use models that allow specification of default probabilities and default correlations. However, whilst default probabilities and correlations together reveal the level of all pairwise default probabilities (i.e., the specification of the distributions of the pairs $(I_i, I_j)$), the probability that three or more loans default together is not known. In fact, lack of sufficient default statistics (joint defaults are rarely observed) makes it hard, if not impossible, to specify that several loans default together so that the joint distribution of $(I_1, I_2, \ldots, I_n)$ cannot readily be specified. In other words, all models that assess VaRs of credit risk portfolios strictly require additional ad-hoc (hard to justify) assumptions to describe the full dependence and all provide different VaR numbers. In this paper we aim at quantifying this inherent uncertainty on VaR estimates.

We first assume that (besides the information on the net exposures $v_i$) the only information that is available concerns the probabilities of default $p_i$ of each loan, i.e., the distributions of the different default events $I_i$ ($i = 1, 2, \ldots, n$) are known (but not their joint distribution). In this context, we solve for the maximum and minimum VaR of the portfolio of loans (Section 3). The (unconstrained) bounds that we obtain are very wide confirming that using dependence information is crucial in improving the bounds.

However, it appears realistic to have a reasonable estimate for the variance and perhaps even the skewness of the portfolio loss $S$, providing indirect information on the dependence among credit loans. Hence, in this paper we are interested in the maximum possible VaR of a portfolio of loans in which the loss distributions $F_i$ ($i = 1, 2, \ldots, n$) of the constituent risky loans are known as well as some higher order moments of the portfolio loss (revealing information on dependence). Of course, these moments are typically not precisely known but have to estimated from available data. In order to capture the statistical uncertainty on these estimates, we propose a robust approach in the sense that we only assume a maximum value $c_k$ for each unknown higher-order moments of $S$ for ($k = 2, 3, \ldots, K$). Note that using inequality constraints is prudent in the sense that the VaR bounds will be wider as compared to a situation in which the moments are assumed to be known (and equal to $c_k$). Typically, $c_k$ is the point estimate of the $k$-th moment but it can also be a higher value. In summary, we consider the following problem,

$$M = \sup \text{VaR}_q^+[S] \text{ subject to } X_j \sim F_j \text{ and } \mathbb{E}(S^k) \leq c_k \ (k = 2, 3, \ldots, K). \quad (1)$$

As for the lower bound for VaR, we consider the problem

$$m = \inf \text{VaR}_q[S] \text{ subject to } X_j \sim F_j \text{ and } \mathbb{E}(S^k) \leq c_k \ (k = 2, 3, \ldots, K), \quad (2)$$

in which $\text{VaR}_q[S]$ is defined as

$$\text{VaR}_q[S] = F_S^{-1}(q) = \inf \{ x \in \mathbb{R} \mid F_S(x) \geq q \}.$$
In what follows, we always tacitly assume that the problems (1) and (2) are well posed in the sense that there exist portfolios that satisfy the constraints. In particular, by denoting $E(S) := \mu$ and observing that (since $S \geq 0$),

$$\mu^k \leq E[S^k],$$  \hspace{1cm} (3)

it follows that $c_k \geq \mu^k$ ($k = 2, 3, \ldots, K$) will hold.

In our analysis, we make extensively use of two other risk measures, namely Tail Value-at-Risk (TVaR) and Left Tail Value-at-Risk (LTVaR), denoted by $TVaR_q[S]$ and $LTVaR_q[S]$, and defined as

$$TVaR_q[S] = \frac{1}{1-q} \int_q^1 \text{VaR}_u^+[S]du$$

and

$$LTVaR_q[S] = \frac{1}{q} \int_0^q \text{VaR}_u^+[S]du,$$

respectively. Loosely speaking, TVaR$_q$ is the average of all upper VaRs and LTVaR$_q$ is the average of all lower VaRs.

3 VaR Bounds when only the default probabilities are known

In this section, we first recall VaR bounds that were established in prior literature in the case that no dependence information is used at all. In other words, we consider the problems (1) and (2) in which all $c_k = \infty$ ($k = 2, 3, \ldots, K$). In general, these bounds are not sharp. Therefore, we provide a rearrangement algorithm (RA) that makes it possible to determine (approximate) sharp VaR bounds when there is no information on the dependence available. The algorithm that we propose is inspired by the original RA of Puccetti and Rüschendorf (2012a) and Embrechts et al. (2013). It is as good as the original RA in the unconstrained case (only default probabilities are assumed to be known), but it is also directly applicable when the correlations among the risks are known or in the presence of higher order moment information on the portfolio (Sections 4 and 5). In the special case of credit risk portfolio with the same loss exposure ($\forall i, v_i = v$), we are able to derive explicit sharp bounds. We refer to this situation as a portfolio that is homogeneous in its structure of exposures.

3.1 Analytical VaR Bounds

In what follows, we represent (without any loss of generality) the $n$ credit risks $X_i$ ($i = 1, 2, \ldots, n$) as $X_i = f_i(U)$ for some random variable $U$ that has a uniform distribution over $(0, 1)$, i.e., $U \sim U(0, 1)$. Each outcome $u$ of $U$ can be effectively interpreted as “a scenario” and translates into a particular loss $f_i(u)$ (that is either 0 or $v_i$). It follows that $f_i(U)$ and $F_i^{-1}(U)$ have the same distribution, namely $F_i$, and we say that $f_i$ is a “rearrangement” of $F_i^{-1}$ on $[0, 1]$. In fact, the rearrangements make it possible to describe dependence among the risks and one can show that $(X_1, X_2, \ldots, X_n) =_d (f_1(U), f_2(U), \ldots, f_n(U))$ in which the

\footnote{See e.g., Acerbi (2001, Rockafellar and Uryasev (2000).}
(i.e., the rearrangements) are suitably chosen and “\(d\)” reflects equality in distribution; (see Rüschendorf (Lemma 1, 1983) and Puccetti and Rüschendorf (2012)).

Example 3.1 (perfectly dependent risks). Let \(n = 2\) and take \(f_1(U) = F_1^{-1}(U)\) and \(f_2(U) = F_2^{-1}(U)\), then the risks \(X_1\) and \(X_2\) are both increasing in the same variable \(U\). They are perfectly positively dependent (also called comonotonic). By contrast, taking \(f_1(U) = F_1^{-1}(U)\) and \(f_2(U) = F_2^{-1}(1-U)\) results in risks \(X_1\) and \(X_2\) that are perfectly negatively dependent (also called antimonotonic).

When the individual risks \(X_i\) are comonotonic (i.e., when \(f_i = F_i^{-1}\)) we denote them by \(X_i^c\) and in this instance the portfolio loss is denoted by \(S^c\). It seems intuitive that the highest possible VaR for the portfolio loss occurs when the risks are comonotonic. While this intuition turns out to be incorrect (as we show below), the comonotonic situation is still of great interest for finding VaR bounds. We explain this further as follows.

First, let us observe that for every dependence among the risks (thus also for the comonotonic dependence), the portfolio sum \(S = X_1 + X_2 + \cdots + X_n\) satisfies the following inequalities

\[
A := \text{LTVaR}_q[S^c] \leq \text{VaR}_q[S] \leq \text{VaR}_q^+[S] \leq B := \text{TVaR}_q[S^c].
\]  

A proof for these inequalities can for instance be found in Bernard et al. (2013). Note that \(A\) and \(B\) can be expressed as \(A = \sum_{i=1}^n \text{LTVaR}_q[X_i]\) and \(B = \sum_{i=1}^n \text{TVaR}_q[X_i]\), respectively. The inequalities (4) show that it is not possible to construct rearrangements \(f_i\) of \(F_i^{-1}\) with the property that \(\text{VaR}_q^+[S] > B\).

![Figure 1: Representation of VaR\(_q^+\) as a function of the level \(q \in (0, 1)\) for the comonotonic portfolio sum \(S^c\).](image)

In Figure 1, we depict the VaRs of the portfolio loss \(S^c\) constructed from comonotonic losses. For a given probability level \(q\), \(B\) is the average of the upper VaRs (from level \(q\) onwards) in the comonotonic case. It is an upper bound for \(\text{VaR}_q^+\) of the comonotonic sum. As the graph in Figure 1 indicates, in order to obtain the best case and worst case VaR one has to choose rearrangements \(f_i\) such that the quantile function of \(S\) assumes the value \(A\) on \([0, q]\) and the value \(B\) on \([q, 1]\). Thus, we look for rearrangements \(f_i^*\) such that the

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6The traditional way to describe dependence is to use copulas. Indeed, Sklar’s theorem states that for a multivariate vector \((X_1, X_2, \ldots, X_n)\) it holds that \((X_1, X_2, \ldots, X_n) \overset{d}{=} (F_{X_1}^{-1}(U_1), F_{X_2}^{-1}(U_2), \ldots, F_{X_n}^{-1}(U_n))\) for some suitable chosen vector \((U_1, U_2, \ldots, U_n)\) in which the \(U_i\) are uniformly distributed. The joint distribution of \((U_1, U_2, \ldots, U_n)\) is called a copula.
portfolio sum $S^* = \sum_{i=1}^{n} f_i^*(U)$ takes two values only. Specifically, we aim for

$$S^* = \sum_{i=1}^{n} f_i^*(U) = \begin{cases} A & \text{if } U \in [0, q[ \\ B & \text{if } U \in [q, 1] \end{cases}$$

(5)

In general, the lower bound $A$ and the upper bound $B$ are not sharp (attainable), as it is often not possible to change the dependence among the risks such that the quantile function of the portfolio sum $S$ becomes flat on $[0, q]$ and $[q, 1]$, respectively.

3.2 Algorithm to approximate sharp VaR bounds

As mentioned, an explicit dependence structure among the loss variables $X_1, \ldots, X_n$ that makes it possible to achieve the bounds $A$ and $B$ generally does not exist. We thus propose an algorithm that approximates sharp bounds by optimizing over all possible dependence among the $X_1, \ldots, X_n$. In this regard, it is useful to define the auxiliary (extra) variable $X_{n+1}$,

$$X_{n+1} = \begin{cases} -B & \text{with probability } 1 - q \\ -A & \text{with probability } q \end{cases}$$

(6)

and note that $X_{n+1}$ can be represented as $X_{n+1} = F_{X_{n+1}}^{-1}(U)$ in which $F_{X_{n+1}}$ is denoting a rearrangement of $F_{X_{n+1}}^{-1}$ and $U \sim U(0, 1)$ is a standard uniformly distributed random variable. Hence, the bounds $A$ and $B$ are sharp if and only if we find rearrangements $f_i^*$ ($i = 1, 2, \ldots, n + 1$) such that

$$\sum_{i=1}^{n+1} f_i^*(U) = 0$$

(7)

or equivalently,

$$\text{var} \left( \sum_{i=1}^{n+1} f_i^*(U) \right) = 0$$

(8)

Without much loss of (practical) generality, we assume that the confidence level $q$ and the default probabilities $p_j$ are rational numbers so that we we can choose integer numbers $d$, $d_j$ ($j = 1, 2, \ldots, n$) and $k$ such that

$$\forall j \in \{1, 2, \ldots, n\}, \quad p_j = \frac{d_j}{d}$$

(9)

and

$$1 - q = \frac{k}{d}$$

(10)

We sample each risk\(^7\) $X_j$ ($j = 1, 2, \ldots, n$) into $d$ equiprobable values. Hence, every $X_j$ takes $d$ values $x_{ij}$ ($i = 1, 2, \ldots, d$) all occurring with probability $1/d$. We use these values to create a $d \times n$ matrix $(x_{ij})$. Specifically, in the $j$-th column ($j = 1, 2, \ldots, n$) the last $d_j$ observations take the value $v_j$ and the first $d - d_j$ observations take the value 0. Hence, the $d \times n$ matrix $(x_{ij})$ can be seen as a representation of a comonotonic loss vector $(X_1, X_2, \ldots, X_n)$ and the $j$-th column is a representation of a variable $X_j$ with loss distribution $F_j$.

To construct an approximation of the sharp bounds for the VaR, we make use of the following trick that consists in adding an extra column to the matrix. This added column

\(^7\)Note that we use the notation $X_j$ instead of $X_i$. The reason is that we represent risks as columns of a matrix and it is more natural to express the corresponding matrix as $(x_{ij})$ rather than $(x_{ij})$.
reflects the possible outcomes of a variable \( X_{n+1} \) that can take the value \(-B\) with probability \(1 - q\) and the value \(-A\) with probability \(q\), where \(A\) and \(B\) have been defined in (4). Hence, we obtain the \(d \times (n + 1)\) matrix \(M\),

\[
M := \begin{bmatrix}
x_{1,1} & x_{1,2} & \cdots & x_{1,n} & x_{1,n+1} \\
x_{2,1} & x_{2,2} & \cdots & x_{2,n} & x_{2,n+1} \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
x_{k,1} & x_{k,2} & \cdots & x_{k,n} & x_{k,n+1} \\
x_{k+1,1} & x_{k+1,2} & \cdots & x_{k+1,n} & x_{k+1,n+1} \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
x_{d,1} & x_{d,2} & \cdots & x_{d,n} & x_{d,n+1}
\end{bmatrix}, \tag{11}
\]

in which

\[x_{1,n+1} = x_{2,n+1} = \cdots = x_{k,n+1} = -B \quad \text{and} \quad x_{k+1,n+1} = x_{k+2,n+1} = \cdots = x_{d,n+1} = -A.\]

Note that if there exists a dependence structure between the risks \(X_i\) such that the bounds \(A\) and \(B\) are both sharp, then there exists a rearrangement of the matrix \(M\) such that the sum of each row \(i\), \(\sum_{j=1}^{n+1} x_{ij} = 0\) exactly. The algorithm that we describe below attempts to obtain this situation of obtaining a sum that exhibits no variability as much as possible\(^8\).

The basic observation to be made is that rearranging values within each column of the matrix \(M\) has no impact on the marginal distributions involved, but rather only affects the allocation of the outcomes of the different risks to the different economic scenarios. Hence, consistent with (8), the algorithm effectively consists in rearranging the values within each column such that the rearranged matrix, denoted by \(M^* := (x_{ij}^*)\), satisfies the condition that all columns are antimonotonic with the sum of all other columns; for this observation, see Puccetti and Rüschendorf (2012a, Theorem 2.1)).\(^9\) Specifically,

**Algorithm 1** Consider \(M = (x_{ij})\) as in (11).

1. Rearrange the values in each column such that the column becomes antimonotonic to the sum of all other columns and denote the matrix after rearrangement by \(M^*\).

2. for \(i = 1, 2, \ldots, d\), consider the values \(s_i^* := \sum_{j=1}^{n} x_{ij}^*\) and rank them in increasing order, \(s_{[1]}^* \leq s_{[2]}^* \leq \cdots \leq s_{[d]}^*\).

3. The approximation for the lower bound \(A\) is then given by \(s_{[d-k-1]}^*\) and the approximation for the upper bound \(B\) is given by \(s_{[d-k]}^*\).

We formulate the following two remarks.

\(^{8}\)More precisely, in this context the appropriate notion to indicate when is a sum is flatter than another one is the so-called convex order. In Bernard et al. (2013) it is shown that maximum VaR bounds are obtained if one can create a dependence among the risks (restricted to the domain \([q, 1]\)) that makes the sum minimum in the sense of convex order. For \(n = 2\), this result goes back to Rüschendorf (1982).

\(^{9}\)Note indeed that minimizing the variance of a sum requires that each component has minimum correlation with the sum of all other components.
Remark 3.2 (connection with the original RA). Embrechts et al. (2013) propose the RA as a suitable approach to approximate the unconstrained VaR bounds (i.e., where there is no dependence information). These authors show that adding an extra column, as we do, is not needed. In fact, in order to find an approximation for the sharp VaR upper bound the original RA amounts to removing the last column of $M$ and next applying Algorithm 1 on the last $n - k$ rows (similarly, in order to approximate the VaR lower bound one applies Algorithm 1 on the first $k$ rows of the matrix $M$ after removing its last column). We show in Section 5 that Algorithm 1 can also be applied when there is additional information on the portfolio moments\(^{10}\), whereas the original RA is not suitable in these instances.

Remark 3.3 (block RA). As suggested in Remark 4.1 of Bernard et al. (2013), it is possible to improve the algorithm by rearranging “blocks of columns” instead of one column at a time. The basic idea is that minimizing the variance among the row sums actually requires that for any decomposition of \{1, 2, \ldots, n\} into two disjoint sets $A$ and $B$, the sum of columns with index in $A$ must be oppositely ordered to the sum of the remaining columns (with index in $B$). In the basic RA the first set $A$ is a singleton but in the block RA this is no longer necessary. As compared to the RA, the block RA makes it possible to further decrease the variance among the row sums.

3.3 Sharp VaR bounds for portfolios with a homogeneous structure of exposures

In this subsection, we assume that all exposures are equal, that is for all $i$, $v_i = EAD_iLGD_i = v$. In this case, we are able to give explicit sharp bounds. As each loss $X_i$ takes value zero or $v_i$, it is clear that the portfolio sum $S$ can only take values that are multiples of $v$ and between zero (no loss) and $nv$ (all loans default). Therefore, the bounds $A$ and $B$ established in (4) cannot be sharp (attainable by a potential dependence between the loans) as soon as they are not a multiple of $v$.

For credit risk portfolios that are homogeneous in the composition of the exposures the problem of finding a dependence structure that makes it possible to attain the lower and upper VaR of the portfolio of loans can be done without using the algorithm presented above. It is closely related to solving the problem of finding the dependence structure that minimizes the variance. First, we show that the problem

\[
(P) \quad \min \text{var}(Y_1 + Y_2 + \cdots + Y_n) \\
\text{subject to} \quad \forall j = 1, 2, \ldots, n \quad Y_j \sim v_j B(p_j).
\]  

\[(12)\]

can be solved exactly and an algorithm\(^{11}\) is not needed. Armed with this result we provide sharp VaR bounds (Proposition 3.4). In this regard, it is convenient to introduce the notation $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) to reflect the smallest (resp. largest) integer number that is larger (resp. smaller) than $x$.

Lemma 3.1 (Minimum variance portfolio). Consider the problem $(P)$ in (12). Assume

\(^{10}\)See also Remark 4.2 in Bernard et al. (2013) where the same idea appears in the context of a constraint on the portfolio variance.

\(^{11}\)Algorithm 1 of Section (3.2) can essentially be applied too (i.e., after removing the last column of $M$), but doing so is unnecessary and does not yield the sharp bounds in general. By contrast, when the portfolio is inhomogeneous it makes sense to apply Algorithm 1 to obtain the approximate minimum variance portfolio.
that the exposures are identical, i.e. \( v := v_j \) \( (j = 1, 2, \ldots, n) \). Define for \( j = 1, \ldots, n, \)

\[
a_j = \left( \sum_{i=1}^{j} p_i \right) \mod 1,
\]

and the sets

\[
I_j = \begin{cases} 
[a_{j-1}, a_j] & \text{if } a_j > a_{j-1} \\
[0, a_j] \cup [a_{j-1}, 1] & \text{if } a_j < a_{j-1}
\end{cases},
\]

where we define \( a_0 = 0 \). Then, the solution to \((P)\) in (12) is obtained for \( Y_j^* \) given as

\[
Y_j^* = v 1_{U \in I_j},
\]

(13)

where \( U \) is a standard uniformly distributed random variable. Furthermore,

\[
\text{var}(Y_1^* + Y_2^* + \cdots + Y_n^*) = v^2 p^*(1 - p^*),
\]

where \( p^* = \frac{\mu}{v} - \left\lfloor \frac{\mu}{v} \right\rfloor \).

**Proof.** See Appendix A.1.

We make the following three observations.

(i) The minimum variance portfolio \((Y_1^*, Y_2^*, \ldots, Y_n^*)\) has the property that its sum is concentrated on two values around the mean. Precisely,

\[
Y_1^* + Y_2^* + \cdots + Y_n^* = \begin{cases} 
v \left\lfloor \frac{\mu}{v} \right\rfloor & \text{with probability } 1 - p^* \\
v \left\lfloor \frac{\mu}{v} \right\rfloor & \text{with probability } p^*
\end{cases},
\]

(14)

(ii) The variance is a traditional measure for comparing variability ("degree of riskiness") among risks. A more general concept to discuss and compare variability of risks is the so-called convex order. One says that a risk \( X \) is smaller than a risk \( Y \) in the sense of convex order if and only if \( E(v(X)) \leq E(v(Y)) \) for all convex functions \( v \) such that the expectations exist. Convex order is consistent with the preferences of risk averse decision makers (who maximize the expected utility of wealth with a concave utility function). Consequently, it is often argued in the literature that when measuring risk one should use risk measures that are consistent with convex order such as the variance or the TVaR (and not VaR). From the proof of Lemma 3.1, one can see that the minimum variance portfolio \((Y_1^*, Y_2^*, \ldots, Y_n^*)\) is also a convex minimum (among all portfolios with fixed marginal distributions). In other words, let \( \rho \) be a risk measure that is consistent with convex order, then the problem

\[
\min \rho(Y_1 + Y_2 + \cdots + Y_n) \\
\text{subject to } Y_j \sim v_j B(p_j),
\]

(15)

has the same solution as problem \((P)\).

(iii) A more specific algorithm to find the minimum variance portfolio (or more generally, the convex minimum) in the context of heterogeneous credit risk portfolios is available in Appendix A.5. An interesting feature of this specific algorithm is that it converges after \( n - 1 \) steps \( (n \) is the number of loans in the portfolio) to a (local) minimum, a feature that the original algorithm of Puccetti and Rüschendorf (2012a) nor our modification (Algorithm 1) is having.

The next proposition gives exact sharp bounds for the VaR of the portfolio sum in (1) and (2) without moment constraints. It shows that there exists a dependence structure among the risks \( X_1, X_2, \ldots X_n \) such that these bounds are attainable.
Proposition 3.4 (Unconstrained VaR bounds). Consider the problems (1) and (2) in which $c_k = \infty$ $(k = 2, 3, \ldots, K)$. Assuming that all exposures are identical $v_i = v$ for all $i = 1, \ldots, n$, then
\[ v \left( \frac{A}{v} \right) \leq \text{VaR}_q[S] \leq \text{VaR}_q^+[S] \leq v \left( \frac{B}{v} \right), \tag{16} \]
where $A := \text{LTVaR}_q[S']$ and $B := \text{TVaR}_q[S']$ (from (4)). Furthermore, these bounds are sharp.

Proof. See appendix A.2. \qed

The analytical bounds $A$ and $B$, the approximate sharp bounds obtained by the RA or the exact bounds from Proposition 3.4 can be very wide. This feature will also be confirmed later in the examples and the distance between the largest possible value for VaR and its lowest value can only be reduced by considering possible available information on the dependence among the loans. Therefore, in order to reduce the uncertainty on the estimate of the VaR of a portfolio of loans, we consider different possibilities for incorporating dependence information.

4 VaR bounds when default probabilities and pairwise correlations are known

It is possible to improve the approximations for the VaR bounds when the pairwise correlations among the credit losses $X_i$ $(i = 1, 2, \ldots, n)$ are available.

In this regard, we can assume that $n$ is even (possibly by adding a risk with zero exposure, i.e., by taking $v_n = 0$). When the correlation between the $X_i$ are known, the distribution of each pair $(X_i, X_j)$ and thus also of each partial sum $S_{i,j} = X_i + X_j$ $(i \neq j = 1, 2, \ldots, n)$ is known. In particular when $n = 2$, the VaRs of $S$ can be exactly determined exactly. It is then also clear that VaR bounds of $S$ that are only based on the marginal distributions of the $X_i$ have to be wider than the ones that are based on the marginal distributions of the $S_{i,j}$.

4.1 Reducing the VaR bounds

For every permutation $\pi$ of $\{1, 2, \ldots, n\}$, using a similar reasoning as in (4), we find that for every portfolio sum $S = X_1 + X_2 + \cdots + X_n$,
\[ \text{VaR}_q^+[S] \leq \sum_{i=1}^{n/2} \text{TVaR}_q[S_{\pi(i), \pi(i+1)}] \leq B. \tag{17} \]
Any permutation $\pi$ leads to a reduction of the unconstrained bound $B$. Let us denote by $\Pi$ the set of permutations. We obtain the following proposition.

12The basic reason is that in the case of (scaled) Bernoulli distributions, knowledge of pairwise correlations and single default probabilities implies knowledge of the (pairwise) joint default probabilities, and thus also knowledge of the distribution of partial sums that only involve two components. Note that the probabilities that three or more loans default together cannot be determined based on single default probabilities and default correlations alone.
Proposition 4.1 (Optimal reduction). For any portfolio sum \( S = X_1 + X_2 + \cdots + X_n \) it holds:

\[
\inf_{\pi \in \Pi} \left\{ \sum_{i=1}^{n/2} \text{LTVaR}_q[S_{\pi(i),\pi(i+1)}] \right\} \leq \text{VaR}_q[S] \leq \text{VaR}_q^+[S] \leq \sup_{\pi \in \Pi} \left\{ \sum_{i=1}^{n/2} \text{TVaR}_q[S_{\pi(i),\pi(i+1)}] \right\}.
\]

Note that it can be hard to compute the infimum and supremum exactly because of the too large number of possible permutations. However, it is possible to obtain some approximate bounds by using specific permutations. Knowledge of pairwise distributions and some related examples on the magnitude of reduction has also been considered in Puccetti and Rüschendorf (2012b) and Embrechts and Puccetti (2009).

4.2 Approximations for sharp VaR bounds

As before, the bounds \( C \) and \( D \) from Proposition 4.1 are not sharp (attainable) as it is typically not possible to change the dependence among the risks such that the quantile function of the portfolio sum \( S \) becomes flat on \([0,q]\) and \([q,1]\), respectively. However, we can apply Algorithm 1 that we described in Section 3.2. In this regard, we make use of the auxiliary (extra) variable \( S_{n+1} \),

\[
S_{n+1} = \begin{cases} 
-D & \text{with probability } 1 - q \\
-C & \text{with probability } q 
\end{cases}
\] (18)

It is convenient to use the shorthand notation \( S_i \) to denote \( S_{\pi^*(i),\pi^*(i+1)} \). Note that each \( S_j \) can take four values namely, 0, \( v_{\pi^*(i)} \), \( v_{\pi^*(i+1)} \) and \( v_{\pi^*(i)} + v_{\pi^*(i+1)} \), occurring with the appropriate probabilities that are derived from the marginal PDs and the correlations. With no loss of practical generality we can assume that all probabilities are rational numbers. As before, we sample each risk \( S_j \) \((j = 1, 2, \ldots, n/2)\) into \( d \) equiprobable values that are ordered from low to high (i.e., we start again with a portfolio that exhibits comonotonic dependence). Hence, every \( S_j \) takes \( d \) values \( s_{ij} \) \((i = 1, 2, \ldots, d)\), all occurring with probability \( 1/d \). The \( d \times n/2 \) matrix \( (s_{ij}) \) can then be seen as a representation of the multivariate vector \( (S_1, S_2, \ldots, S_n) \). Next we add a column, reflecting the variable \( S_{n+1} \), to this matrix to obtain the \( d \times (n/2 + 1) \) matrix \( S \)

\[
S := \begin{bmatrix}
  s_{1,1} & s_{1,2} & \cdots & s_{1,n/2} & s_{1,n/2+1} \\
  s_{2,1} & s_{2,2} & \cdots & s_{2,n/2} & s_{2,n/2+1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  s_{k,1} & s_{k,2} & \cdots & s_{k,n/2} & s_{k,n/2+1} \\
  s_{k+1,1} & s_{k+1,2} & \cdots & s_{k+1,n/2} & s_{k+1,n/2+1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  s_{d,1} & s_{d,2} & \cdots & s_{d,n/2} & s_{d,n/2+1}
\end{bmatrix},
\]

in which

\[
s_{1,n/2+1} = s_{2,n/2+1} = \cdots = s_{k,n/2+1} = -D \quad \text{and} \quad s_{k+1,n+1} = s_{k+2,n+1} = \cdots = s_{d,n+1} = -C
\]

Hence, as before we rearrange the values in the columns of \( S \) such that the rearranged matrix \( S^* \) has the property that all columns are antimonotonic with the sum of all other columns; see Algorithm 1 in Section 3.2.
5 VaR bounds when default probabilities and bounds on higher order moments are known

Next, we consider the general constrained VaR maximization and minimization problems (1) and (2) in which \( c_k < \infty \) for some \( k = 2, 3, \ldots, K \).

5.1 Constrained VaR Bounds

It is clear that the unconstrained bounds \( A \) and \( B \) are also bounds of the general problem if \( S^* \) (as in (5)) satisfies the moment constraints. In the opposite case, it means that the two-point variable \( S^* \) exhibits too much spread. In this case, in order to obtain VaR bounds, the idea is to construct another (less-dispersed) two-point variable that is still as dispersed as possible while satisfying the constraints. To this end, let us define \( A(\alpha) \) and \( B(\alpha) \) (0 \( \leq \alpha \leq q \)) as

\[
B(\alpha) := \frac{1}{1 - q} \int_{q-\alpha}^{1-\alpha} \text{VaR}_u[S^*] du, \quad A(\alpha) := \frac{E(S) - B(\alpha)(1 - q)}{q},
\]

(19)

and note that \( B(0) = B \) and \( A(0) = A \). Consider variables \( X_{n+1}(\alpha) \) (0 \( \leq \alpha \leq q \)),

\[
X_{n+1}(\alpha) = \begin{cases} 
  A(\alpha) & \text{with probability } 1 - q \\
  B(\alpha) & \text{with probability } q
\end{cases}
\]

(20)

and note that \( X_{n+1}(0) = X_{n+1} \). For ease of exposition, we further denote \( X_{n+1}(\alpha) \) by \( X(\alpha) \). The moments of \( X(\alpha) \) are given by

\[
E[(X(\alpha))^k] = A^k(\alpha)q + B^k(\alpha)(1 - q),
\]

(21)

and note that \( E[X(\alpha)] = E[S] = \mu \). For each \( k, \alpha \to E[(X(\alpha))^k] \) is continuous on \([0, q] \). Precisely, the function first decreases and has minimum value \( \mu^k \) for \( \gamma \in [0, q] \) with the property that \( A(\gamma) = B(\gamma) \). Note that \( \mu^k \leq c_k \) (see (3)). Hence, there exists

\[
\alpha^* := \min \left\{ \alpha \in [0, \gamma] \mid E \left[ (X(\alpha))^k \right] \leq c_k, \; k = 2, 3, \ldots, K \right\}.
\]

(22)

In Theorem 5.1 we will show that the variable \( X(\alpha^*) \) yields the upper VaR bound \( B(\alpha^*) \) and the lower VaR bound \( A(\alpha^*) \) for the constrained VaR problems (1) and (2). Here, we point out that as expected adding dependence information makes it possible to strengthen the unconstrained bounds \( B \) and \( A \). First, when adding more dependence constraints, the value for \( \alpha^* \) can obviously only increase, which implies that \( B(\alpha^*) - A(\alpha^*) \) decreases (see (19)). Second, when the dependence constraints \( c_k \) take smaller values we effectively reduce the set of admissible dependence structures among the risks, which translates into a increasing value for \( \alpha^* \) and hence a decrease in \( B(\alpha^*) - A(\alpha^*) \). In particular, when there are no dependence constraints or when they take very high values then \( \alpha^* = 0 \), and we obtain the unconstrained bounds \( A \) and \( B \). By contrast, when the \( c_k \) are minimum (i.e., \( c_k = \mu^k \)) we only allow for a dependence that would make the portfolio sum constant. In this case \( \alpha^* = \gamma \) and \( B(\gamma) = A(\gamma) = E(S) \).
Proposition 5.1 (moment-constrained VaR bounds). Consider the problems (1) and (2) and let \( \alpha^* \) be defined by (22). We have that

\[
A(\alpha^*) \leq \text{VaR}_q[S] \leq \text{VaR}_q^+[S] \leq B(\alpha^*).
\]

Proof. See Appendix A.3.

This proposition can be seen as a generalization of Theorem 3.3 in Bernard et al. (2013) who considered the case \( K = 2 \). A two-point distribution provides the best bounds in all cases, which at first may seem counterintuitive. The reason is that we have inequality on moments and therefore all moment constraints are not binding. Note that Proposition 5.1 also covers the unconstrained case (in this case \( \alpha^* = 0 \) so that \( A(\alpha^*) = A \) and \( B(\alpha^*) = B \).

5.2 Approximating sharp VaR Bounds

Proposition 5.1 shows that best-possible sharp upper and lower VaR bounds are obtained if one can construct a dependence among the risks \( X_i \) (\( i = 1, 2, \ldots, n \)) such that \( S = X_1 + X_2 + \cdots + X_n \) takes values \( A(\alpha^*) \) and \( B(\alpha^*) \). Hence, we can use Algorithm 1, which we described in Section 3.2 in the context of the unconstrained problem, also for approximating VaR bounds in the given constrained situation. The only difference is that the last column in the \( d \times (n+1) \) matrix \( M = (x_{ij}) \) contains the realizations of the random variable \( -X_{n+1}(\alpha^*) \) instead of \( -X_{n+1}, \) i.e.,

\[
\begin{align*}
x_{1,n+1} &= x_{2,n+1} = \cdots = x_{k,n+1} = -B(\alpha^*) \quad \text{and} \\
x_{k+1,n+1} &= x_{k+2,n+1} = \cdots = x_{d,n+1} = -A(\alpha^*).
\end{align*}
\]

Similarly, as in Section 3.3, the algorithm is not needed in the case of a portfolio that exhibits homogeneity in its exposures, since explicit sharp bounds can be computed (Proposition 5.3 below).

Remark 5.2. After running Algorithm 1 we ideally observe that the row sums are all equal to zero, and thus the approximations for sharp VaR bounds correspond with the theoretical bounds. However, in general, the row sums will not be equal to zero and one cannot readily expect that the sum over the first \( n \) columns (depicting the portfolio sum \( X_1 + X_2 + \cdots + X_n \) after rearrangement) meets the moment constraints. When the moment constraints are not after running Algorithm 1, we consider for some small \( \varepsilon > 0 \) the new target values \( B(\alpha^* + \varepsilon) \), \( A(\alpha^* + \varepsilon) \) in the \( (n + 1) \)-th column, and repeat the procedure. By gradually increasing \( \varepsilon \) one obtains a situation in which \( X_1 + X_2 + \cdots + X_n \) meets the moment constraints and approximate VaR bounds are obtained.

5.3 Portfolios with a homogeneous structure of exposures

In this section we assume that \( v := v_j \) (\( j = 1, 2, \ldots, n \)). To discuss sharp bounds it is convenient to consider the following auxiliary variable \( Y \) taking three values and explicitly given as

\[
Y = \begin{cases} 
 lv & \text{with probability } qz, \\
 (l + 1)v & \text{with probability } q(1 - z), \\
 \lfloor \frac{B(\alpha^*)}{v} \rfloor v & \text{with probability } 1 - q,
\end{cases}
\]

(23)
in which $0 \leq z \leq 1$ and $l \in \mathbb{N}$ are the unique\textsuperscript{13} values such that $E(Y) = E(S)$. Furthermore, $\alpha^*$ is defined by (22). The following proposition provides a sharp upper VaR bound for a homogeneous portfolio. The proof is completely similar to the proof in the unconstrained case (Proposition 3.4) and therefore omitted.

**Proposition 5.3** (Sharp moment-constrained bounds for a homogeneous portfolio). Consider the problems (1) and (2) and define $\alpha^*$ by (22). Assume that the variable $Y$ as defined in (23) satisfies the moment constraints (i.e., $E(Y^k) \leq c_k$), then,

$$\text{VaR}_q^+[S] \leq v \left\lfloor \frac{B(\alpha^*)}{v} \right\rfloor.$$  \hfill (24)

Furthermore, this bound is sharp.

Similarly, one gets the lower bound

$$\text{VaR}_q^-(S) \geq v \left\lceil \frac{A(\alpha^*)}{v} \right\rceil,$$  \hfill (25)

which is also attained by a corresponding three point distribution assuming that the moment constraints are satisfied.

The variable $Y$ satisfies the moment constraints in particular when $\frac{A(\alpha^*)}{v}$ and $\frac{B(\alpha^*)}{v} \in \mathbb{N}$ (see also the analysis in Section 5.1). In general, the moment constraints are approximately satisfied by this construction of $Y$ and (24) gives approximate best bounds.

The maximum portfolio VaR is typically strictly larger than the VaR that is obtained when assuming the risks are fully dependent (comonotonic). It has a VaR larger than the sum of the individual VaRs. In practice, however, it is often believed that comonotonic dependence among the risks should yield the maximum possible VaR in the sense that portfolios that give rise to a VaR that is higher than the comonotonic VaR (sum of individual VaRs) are considered as being unrealistic. However, it is not so clear whether comonotonic scenarios are more realistic (i.e., occur more often) than other extreme scenarios. In addition, this feature of having risk bounds that go beyond the comonotonic bounds does not occur when using a measure that is consistent with the convex order (unlike VaR). Nevertheless, if one goes “deep enough in the tail”, we still have that the worst case VaR occurs in the case of full dependence.

**Proposition 5.4** (Maximum VaR = comonotonic VaR). Consider the problem (1) and assume there exists an admissible portfolio $(X_1, X_2, \ldots, X_n)$ that strictly satisfies the moment constraints. Then, there exists $1 > q^* > 0$ and a portfolio $(X_1^*, X_2^*, \ldots, X_n^*)$ satisfying the constraints such that for $q \in [q^*, 1]$, \hfill (26)

$$\text{VaR}_q^+(X_1^* + X_2^* + \cdots + X_n^*) = \text{VaR}_q^+(X_1^*) + \text{VaR}_q^+(X_2^*) + \cdots + \text{VaR}_q^+(X_n^*) = nv.$$  

Proof. For the proof see Appendix A.4. \hfill \Box

The proposition is consistent with Theorem 8 of Cheung and Vanduffel (2013) where it is shown that when the marginal distributions of the risky components and variance of the sum are known, it is always possible to construct a dependence among the risks that yields no diversification in the sense that “deep enough in the tail” the VaR of the portfolio sum is equal to the sums of the VaRs (comonotonic VaR).

\textsuperscript{13}The first moment requirement amounts to a condition of the type $l + (1 - q) = \text{constant}$, which implies that $l$ and $q$ are uniquely determined.
6 Model risk of credit risk portfolio models

In this section, we discuss two main models that are used in the financial industry to assess VaRs of credit risk portfolios, namely the KMV model and CreditRisk+ (see Gordy (2000), Vanderdorpe et al. (2008)). Next, we will analyze to which extent these industry standards are robust with respect to model misspecification.

6.1 Credit risk portfolio models

6.1.1 KMV model (Merton’s model of the firm)

Many financial institutions as well as Basel III and Solvency II regulation rely on “Merton’s model of the firm” when computing the VaR of a portfolio of loans (see also the survey of McKinsey (2009)). The basic idea is very simple: a default is an event in which the asset value drops below a threshold value (a liability that is due). Formally, after normalization, default of the $i$-th risk occurs when \( N_i < c_i \) where \( N_i \) is the normalized asset return and \( c_i \) is the threshold value such that \( p_i = P(N_i < c_i) \). Merton’s model further assumes that the joint asset (log-)returns are multivariate normally distributed. Hence, for a loan portfolio, the loss \( S \) writes as

\[
S = \sum_{i=1}^{n} v_i 1_{N_i < c_i},
\]

in which \((N_1, N_2, \ldots, N_n)\) is multivariate normally distributed with some correlation matrix \( \rho \). Each asset return \( N_i \) (driving the randomness of the i-th loan) can be expressed as a linear combination of independent factors that are standard normally distributed. Specifically,

\[ N_i = \sum_{j=1}^{r} \sqrt{\rho_{ij}} M_j + \varepsilon_i \sqrt{1 - \sum_{j=1}^{r} \rho_{ij}}, \]

in which \( M_j \) is the explaining factor of the asset return \( N_j \), and in which \( \varepsilon_i \) represents the idiosyncratic (individual) risk. The factor weights \( \sqrt{\rho_{ij}} \) can also be interpreted as the correlation between the i-th return \( N_i \) and the \( j \)-th factor factor \( M_j \). It is natural to assume that there is always a strictly positive portion of idiosyncratic risk that remains inherent in \( N_i \), i.e., \( 1 - \sum_{j=1}^{r} \rho_{ij} > 0 \) and \( r < n \). Hence, we can also write the portfolio loss \( S \) as

\[ S = \sum_{i=1}^{n} v_i I_i, \]

in which \( I_i \) is a Bernoulli random variable with (stochastic) probability \( p_i(M_1, M_2, \ldots, M_r) \) given as

\[
p_i(M_1, M_2, \ldots, M_r) = \Phi \left( \frac{\Phi^{-1}(p_i) - \sum_{j=1}^{r} \sqrt{\rho_{ij}} M_j}{\sqrt{1 - \sum_{j=1}^{r} \rho_{ij}}} \right)
\]

and where \( \Phi \) is the distribution of the standard normal random variable. It is then clear that \( \text{VaR}_q [S] \) can be obtained for instance using Monte Carlo simulations.
Single factor model

Assuming a homogeneous portfolio \((v_i = v, \rho_{ij} = \rho)\) and asset returns that are driven by one single factor \(M\) only, then when the number of loans \(n \to \infty\), we find that

\[
\lim_{n \to \infty} \text{VaR}_q \left[ \frac{S}{nv} \right] = \Phi \left( \Phi^{-1}(p) + \sqrt{\rho} \cdot \Phi^{-1}(q) \right),
\]

(31)

see also Vasicek (2002). Note that in this case the (squared) weight \(\rho\) is the correlation between different asset return \(N_i\) and \(N_j\). This model is then an example of a one-factor mixture model in which the default event of the obligor is assumed to be driven by a common economic factor \(M\). It can also be seen as the one-factor version of the KMV model that is highly used in the industry and also appears in regulatory frameworks. For example, the Basel III standard framework relies on formula (31) to determine the required capital that banks need to hold for their credit portfolios; see the Basel Committee on Banking Supervision 2010). The Solvency II framework also uses this formula to decide the amount of capital that insurers need to hold as a buffer if reinsurance or derivative counterparts fail.

6.1.2 Credit Risk

Starting from the expression of the loss of a portfolio of loans in (29), it is clear that other assumptions for the default probabilities can be made (reflecting other choices for dependence among the risks), and (30) merely reflects only one such possibility. Note that we can rewrite the portfolio loss (29) as

\[
S = \sum_{i=1}^{n} \sum_{j=1}^{I_i} v_i
\]

(32)

in which \(I_i\) is a Bernoulli random variable with (stochastic) probability \(p_i(M_1, M_2, \ldots, M_r)\) given by (30). Since the dependent Bernoulli r.v.’s \(I_i\) are too difficult to work with, one substitutes them by other dependent r.v.’s \(N_i\) that are “close” to the \(I_i\) but that are more tractable. Hence, we consider

\[
S_* = \sum_{i=1}^{n} \sum_{j=1}^{N_i} v_i
\]

(33)

in which \(N_i\) is a Poisson random variable with (stochastic) intensity \(p_i(\Gamma_1, \Gamma_2, \ldots, \Gamma_r)\) given as

\[
p_i(\Gamma_1, \Gamma_2, \ldots, \Gamma_r) = p_i \left( w_i + \sum_{j=1}^{r} w_{ij} \Gamma_j \right)
\]

(34)

The coefficient \(w_i \geq 0\) reflects the portion of idiosyncratic risk that can be attributed to the \(i\)-th risk whereas \(w_{ij} \geq 0\) reflects its affiliation to the \(j\)-th common factor. The random variables \(\Gamma_i\) are assumed to be independent Gamma distributed variables with respective variances \(\sigma_i^2\). Since for any \(r\) and \(a > 0\), the r.v. \(a \Gamma_i\) will be distributed like a Gamma r.v. we can assume without loss of generality that \(E[\Gamma_i] = 1\). Assuming that conditionally on \((\Gamma_1 = \gamma_1, \Gamma_2 = \gamma_2, \ldots, \Gamma_r = \gamma_r)\), the random variables \(N_i\) are mutually independent, we find after some computations for the moment generating function \(S_*\),

\[
m_{S_*}(t) = \exp \left( \sum_{i=1}^{n} w_i p_i (\exp(tv_i) - 1) - \sum_{j=1}^{r} \frac{1}{\sigma_j^2} \ln \left[ 1 - \sigma_j^2 \sum_{i=1}^{n} w_{ik} p_i (\exp(tv_i) - 1) \right] \right)
\]

(35)
Using the Fast Fourier Transform, one can easily derive an algorithm that can be used to find the probability distribution function of $S_*$; see e.g. Haaf et al. (2003).

**Single factor model**

Similarly as for the KMV model, let us consider the single factor model and assume that there exists a single random variable $\Gamma = \gamma$ representing the “global state of the economy” with variance $\sigma^2 (= 1/\beta)$ such that, conditionally given $\Lambda = \lambda$, the random variables $N_i$ are Poisson distributed with parameters $p_i \lambda$. Then, the expression of the moment generating function in (35) can be simplified to

$$m_{S_*}(t) = \left[ \beta - \sum_{i=1}^n p_i (\exp(tv_i) - 1) \right]^\beta,$$

which is the moment generating function of a Compound Negative Binomially distributed variable, i.e.,

$$S_* = \sum_{i=1}^N Y_i$$

with

$$N \sim NB \left( \beta, \frac{\beta}{\beta + \sum_{i=1}^n p_i} \right)$$

and where the $Y_i \overset{d}{=} Y$ are i.i.d. and independent of $N$, with moment generating function given as

$$m_Y(t) = \frac{\sum_{i=1}^n p_i \exp(tv_i)}{\sum_{i=1}^n p_i}.$$

The Compound Negative Binomial distribution can be computed using (35) or by using the recursion of Panjer (1981). Observe that this model formally allows that a credit loan defaults more than once. However, a realistic model calibration based on the given default probabilities and the default correlations generally ensures that the probability that this occurs is very small. For more details; see also Credit Suisse (1997).

**6.1.3 Beta model**

The Beta distribution has always been influential in modeling credit portfolio risk. In this case, one assumes that

$$S \sim Beta(a, b),$$

where the parameters $a > 0$ and $b > 0$ are typically determined using moment matching techniques. In Section 7 of Dhaene et al. (2003) one shows that for large portfolios the Beta distribution provides a very good approximation for the distribution that is obtained when using the single factor version of the CreditRisk+ model.

**6.2 Assessing model risk**

In this section we analyze the robustness of VaR computations for several portfolios. We first compute the VaR numbers under a given model. Next, we assume that the model is not
completely correct, i.e., we only trust the marginal distributions of the risky loans and the first $K$ portfolio moments ($K = 1, 2, 3, 4$). We compare the VaR numbers obtained under the model with the maximum and minimum possible values when $K = 1$ (unconstrained bounds) and when $K = 2, 3$ or 4 (constrained bounds). The bounds are computed using Algorithm 1 (see Section 3.2 and Section 5.2).

**Large corporate portfolio (KMV model)**

We consider a corporate portfolio of a major European Bank. The portfolio contains 4495 loans mainly to medium sized and large corporate clients, but there are also some loans that were granted to (semi-)public entities. The total exposure (EAD) is 18642.7 (million Euros), and the top 10% of the portfolio (in terms of EAD) accounts for 70.1% of it. In Table 1 we provide some further summary statistics for the portfolio, which confirm that the portfolio exhibits some heterogeneity. The bank also has models in place for getting estimates for the PDs, LGDs, EADs and these will be used for the further analysis. We then compute for various levels of the asset correlation $\rho$ and confidence levels $q$ the true VaRs under the KMV framework. Then we assume that either no dependence information (no constraints) or some limited dependence information (constrained bounds) is available and compute the upper and lower bounds for VaR. The results can be found in Table 2.

<table>
<thead>
<tr>
<th>Summary statistics of a corporate portfolio</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>Default probability</td>
<td>0.0001</td>
<td>0.15</td>
<td>0.0119</td>
</tr>
<tr>
<td>EAD</td>
<td>0</td>
<td>750.2</td>
<td>116.7</td>
</tr>
<tr>
<td>LGD</td>
<td>0</td>
<td>0.90</td>
<td>0.41</td>
</tr>
</tbody>
</table>

Table 1: Summary statistics of a corporate portfolio containing 4495 loans of a major European bank.

We make the following observations. First, if we only trust the marginal distributions (unconstrained bounds), then the VaR upper bound is very large and very different from the KMV VaR. The VaR upper bound is typically significantly higher than the comonotonic VaR (=sum of marginal VaRs). However, when the confidence level $q$ is very high then both numbers tend to become close to each other (consistent with Theorem 5.4). By adding dependence information, the VaR bounds sharpen considerably. In particular, the upper bound reduces a lot and becomes typically much smaller than the comonotonic VaR. Furthermore, the higher the confidence level used the more dependence information appears to be required to obtain bounds that are in a reasonable range. Finally, from this table we also observe that even if one completely trusts all PDs, LGDs, EADs as well as the first four moments of the portfolio then there is still significant model risk in the KMV model.

---

14We discretized the originally provided PDs by expressing them in basis points (e.g., 0.0312% became 0.03%).
## VaR assessment of a corporate portfolio

<table>
<thead>
<tr>
<th>$q$</th>
<th>K MV</th>
<th>Comon.</th>
<th>Unconstrained</th>
<th>$K = 2$</th>
<th>$K = 3$</th>
<th>$K = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>95%</td>
<td>99%</td>
<td>99.5%</td>
<td>99.9%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho$</td>
<td>95%</td>
<td>99%</td>
<td>99.5%</td>
<td>99.9%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.05</td>
<td>281.3</td>
<td>393.3</td>
<td>(34.0 ; 2083.3)</td>
<td>(111.8 ; 483.1)</td>
<td>(111.8 ; 433.0)</td>
<td>(111.8 ; 412.8)</td>
</tr>
<tr>
<td>0.10</td>
<td>398.7</td>
<td>2374.1</td>
<td>(56.5 ; 6973.1)</td>
<td>(115.0 ; 943.9)</td>
<td>(117.4 ; 713.3)</td>
<td>(118.2 ; 610.9)</td>
</tr>
<tr>
<td>0.15</td>
<td>448.5</td>
<td>5088.5</td>
<td>(89.4 ; 10119.9)</td>
<td>(116.9 ; 1285.9)</td>
<td>(118.9 ; 889.5)</td>
<td>(119.8 ; 723.2)</td>
</tr>
<tr>
<td></td>
<td>573.1</td>
<td>12905</td>
<td>(111.8 ; 14784.9)</td>
<td>(120.2 ; 2718.1)</td>
<td>(121.2 ; 1499.6)</td>
<td>(121.8 ; 1075.9)</td>
</tr>
</tbody>
</table>

Table 2: We report for various asset correlation levels $\rho$ and confidence levels $q$ the VaRs under the KMV framework (second column), the comonotonic VaRs (third column) and the VaR bounds in the unconstrained and the constrained case (in the last four columns between brackets - $K$ reflects the number of moments of the portfolio sum that are known). The VaR bounds are obtained using Algorithm 1.

In Table 3 we provide the theoretical VaR bounds $B(\alpha^*)$ and $A(\alpha^*)$. It turns out that they closely correspond to the approximations for sharp VaR bounds that were presented in Table 2 indicating that the structure of the portfolio is such that a dependence can be created that yields a portfolio sum that is essentially concentrated on two values. This observation also indicates that the RA can be trusted as a reliable method to approximate sharp VaR bounds; for more numerical evidence we refer to Embrechts et al. (2013) for the unconstrained case and Bernard et al. (2013) for the variance-constrained setting.

## Theoretical VaR bounds of a corporate portfolio

<table>
<thead>
<tr>
<th>$q$</th>
<th>Unconstrained</th>
<th>$K = 2$</th>
<th>$K = 3$</th>
<th>$K = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>95%</td>
<td>(19.2 ; 2085.5)</td>
<td>(103.5 ; 483.5)</td>
<td>(106.5 ; 111.8)</td>
<td>(107.2 ; 413.8)</td>
</tr>
<tr>
<td>99%</td>
<td>(53.1 ; 6986.6)</td>
<td>(114.1 ; 946.5)</td>
<td>(116.5 ; 715.2)</td>
<td>(117.3 ; 713.3)</td>
</tr>
<tr>
<td>99.5%</td>
<td>(72.1 ; 10140.1)</td>
<td>(116.6 ; 1290.8)</td>
<td>(118.6 ; 893.3)</td>
<td>(119.5 ; 725.9)</td>
</tr>
<tr>
<td>99.9%</td>
<td>(107.8 ; 14801.0)</td>
<td>(119.8 ; 2740.1)</td>
<td>(121.1 ; 1511.1)</td>
<td>(121.5 ; 1082.6)</td>
</tr>
</tbody>
</table>

Table 3: We report for various asset correlation levels $\rho$ and confidence levels $q$ the theoretical lower and upper VaR bounds $A(\alpha^*)$ and $B(\alpha^*)$ (between brackets).

We also apply the single factor version of the CreditRisk$^+$ model as well as the Beta model to this portfolio. To this end, we determine the value of the parameter $\beta$ in (36) by matching the variance of $S_*$ (as in (36)) with the one computed under the KMV specification. Similarly, we determine the parameters $a, b$ of the Beta distribution by moment matching.
In Table 4 we report the VaR numbers of the different models. One observes that the different industry models do not differ a lot when the VaR is computed at a rather moderate confidence level, but exhibit more divergence when further increasing the confidence level. This observation is also consistent with findings in McNeil et al. (2005) and Bernard et al. (2013). In each of the following examples we consider only one particular portfolio model as a benchmark.

| VaRs of a corporate portfolio under different industry models |
|-------------------|-----------------|-----------------|----------------|----------------|
|                   | q =             | CoHon.          | KMV             | Credit Risk+   | Beta           |
|                   | 95%             | 393.5           | 281.3           | 281.8          | 282.5          |
|                   | 99%             | 2374.1          | 398.7           | 385.0          | 390.4          |
|                   | 99.5%           | 5088.5          | 448.5           | 428.4          | 435.1          |
|                   | 99.9%           | 12905.1         | 573.1           | 521.6          | 536.3          |
|                   | 95%             | 393.5           | 340.6           | 346.2          | 347.4          |
|                   | 99%             | 2374.1          | 539.4           | 513.4          | 520.2          |
|                   | 99.5%           | 5088.5          | 631.5           | 582.9          | 593.5          |
|                   | 99.9%           | 12905.1         | 862.4           | 744.3          | 762.0          |
|                   | 95%             | 393.5           | 388.4           | 405.2          | 406.4          |
|                   | 99%             | 2374.1          | 675.8           | 645.3          | 649.7          |
|                   | 99.5%           | 5088.5          | 816.1           | 750.1          | 754.9          |
|                   | 99.9%           | 12905.1         | 1178.4          | 994.6          | 998.6          |

Table 4: We report for various asset correlation levels $\rho$ and confidence levels $q$ the comonotonic VaRs (first column) as well as the VaRs under different industry standards.

**Small portfolio (CreditRisk+ model)**

In large portfolios a correspondence between the theoretical bounds and the (approximate) sharp bounds is to be expected implying that significant reductions of the unconstrained VaR bounds can be obtained. The following example shows that for smaller portfolios one can still observe strong reductions. We analyze a small concentrated portfolio of 25 exposures as described in Appendix B of Credit Suisse (1997). The exposures range from 0.4 to 20.4 and the total exposure is 130.5 (all numbers mentioned are in million Euros). As for the default rates, they range between 1.5% and 30% (Credit Suisse (1997), Page 61, Table 9). The portfolio expected loss is then equal to 14.2. Also the portfolio standard deviation is assumed to be known (it is derived from default statistics) and is equal to 12.7; see Credit Suisse (1997, Page 62). Assuming the single factor version of the CreditRisk$^+$ model, it is then straightforward to compute the value of the remaining parameter $\beta$ in (36) (by simple moment matching). Next, the VaRs can be computed using Panjer’s recursion for instance. The results of the CreditRisk$^+$ model are given in the second column of Table 5; see also page 63 in Credit Suisse (1997). For example, the 99.5%-VaR is equal to 62.

However, other modeling assumptions could be made. For instance, if we only trust the marginal distributions and the portfolio variance (moments up to $K = 2$), we observe from Table 5 that the true 99.5%-VaR can actually be any value between 13.6 and 130.5 showing that the 99.5%-VaR can easily be underestimated by a factor 2. In Table 5, we also provide VaR bounds assuming that more information on the higher order moments

\footnote{They have been computed under the specification of the Credit Risk$^+$ model.}
is available. This table makes clear that adding higher order information reduces the gap between the upper and lower bounds for VaR significantly, thus reduces model uncertainty on VaR assessment. When the probability level \( q \) that is used to assess the VaR is not too big (e.g., \( q = 95\% \)) then this has a positive impact on model uncertainty in the sense that the range of possible VaRs becomes “reasonable” as soon as second and/or third moment information is added. At high probability levels (e.g., \( q = 99.9\% \)), significant model risk is present even when higher order information is available.

<table>
<thead>
<tr>
<th>( q )</th>
<th>VaR assessment of a small portfolio (Appendix B of Credit Suisse (1997))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CreditRisk(^+)</td>
</tr>
<tr>
<td>75%</td>
<td>20.5</td>
</tr>
<tr>
<td>95%</td>
<td>38.9</td>
</tr>
<tr>
<td>99%</td>
<td>55.3</td>
</tr>
<tr>
<td>99.5%</td>
<td>62.0</td>
</tr>
<tr>
<td>99.9%</td>
<td>77.1</td>
</tr>
</tbody>
</table>

Table 5: Column 2 contains VaRs under the Credit Risk\(^+\) model. They can be compared to comonotonic VaRs and the VaR bounds (displayed in brackets) in the unconstrained and the constrained case (\( K \) reflects the number of moments of the portfolio sum that are known). The VaR bounds are obtained using Algorithm 1.

### 6.2.1 Homogeneous portfolio (Beta model)

We use this last example to obtain further insight into the VaR when including higher order information. We consider a homogeneous portfolio presented on page 365 in McNeil et al. (2005). Using the same parameters as in Table 8.6 page 365 from McNeil, Frey and Embrechts (2005), we set the default probability of all loans equal to \( p = 0.049 \) and the default correlation is equal to 0.0157. The variance of the portfolio sum of \( n \) correlated loans (all with exposure that is equal to \( 1/n \)) can be easily calculated. Next, the two parameters of the beta distribution can be inferred (by moment matching) and one can compute higher order moments. We use the associated higher moments as the moment constraints. We are interested in the VaR of the portfolio of loans at confidence levels 95\%, 99\%, 99.5\% and 99.9\%.

<table>
<thead>
<tr>
<th>( n )</th>
<th>VaR assessment of an homogeneous portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td>10,000</td>
<td>VaR(_{95%})</td>
</tr>
<tr>
<td>Beta</td>
<td>10%</td>
</tr>
<tr>
<td>Comon.</td>
<td>0%</td>
</tr>
<tr>
<td>( K=2)</td>
<td>16.72%</td>
</tr>
<tr>
<td>( K=3)</td>
<td>14.95%</td>
</tr>
<tr>
<td>( K=4)</td>
<td>14.00%</td>
</tr>
</tbody>
</table>

Table 6: We report VaRs as a percentage of total exposure assuming that the portfolio loss follows a Beta distribution (Beta). We provide the comonotonic VaRs which here coincide with the unconstrained bounds. We also report the maximum VaR when higher order information is available (obtained using Proposition 5.3).

According to the numerical results, we observe that taking additional moments con-
7 Conclusions

In order to assess credit portfolio risk one needs to model the marginal risks as well as the way they interact. Dependence modeling in a credit context usually focuses on finding the economic dimensions that influence the default behavior of the different loans. Apart from factors that describe the global state of the economy such as, for example, interest rates, the default drivers typically considered are asset size, industry sector and geographical situation; see Lopez (2002), Duellmann and Scheule (2003), Duellmann et al. (2008) or Dietsch and Petey (2004). As a result, companies of similar size, industry activity and geographical situation will be grouped together meaning that they behave similarly, which is akin to saying that they are positively dependent. However, all these dimensions together still do not fully capture all sources of dependence. In this paper we assess the model error due to incomplete information on dependence. Specifically, we study model risk on the estimation of VaR of a portfolio of loans in which the individual default probabilities and possibly the pairwise correlation or bounds on some moments of the portfolio are known. We develop an algorithm to estimate VaR bounds in the given context and provide some explicit results for homogeneous portfolios.

Under the internal model approach of Basel III and Solvency II, the financial institutions are allowed to use their own model for setting their credit capital requirements. However, in the presence of incomplete information there are several statistically indistinguishable models, and we are able to find the worst-case (yielding the VaR upper bound) and the best-case model (for the VaR lower bound). Numerical illustrations show that these two extreme models may provide very different VaR estimates, in particular when these VaR calculations are performed using very high confidence levels (e.g., 99.5%). In other words, while the inclusion of higher order moment information helps to reduce the unconstrained VaR bounds at all confidence levels, model risk remains an issue especially at the high confidence level.

The observation that the VaRs of credit portfolios are hard to estimate raises two concerns: First, it casts doubt on the use of internal models to set regulatory capital requirements. When the correct model cannot be identified, two institutions having a similar portfolio can use different, yet statistically indistinguishable, internal models and thus obtain very different capital charges. However, the two institutions should be required to hold a similar amount of regulatory capital because they have the same portfolio. As already mentioned in the introduction, the Basel Committee 2010) insists that a solvency framework must exhibit comparability among institutions. Second, it shows that setting capital requirements for credit risk portfolios is a difficult exercise and it is highly questionable whether a computed regulatory capital charge based on a 99.5% confidence level actually corresponds to a real 99.5% confidence level.

Here are some suggestions to deal with these issues. Regulators may impose on financial
institutions that the capital needed to cover for credit risk is no longer based on the output of a certain internal model, but rather by the worst case model that is given with the set of information available (typically, these are the default probabilities, EADs, LGDs and correlations). This solution enhances fair competition among the institutions and ensures that banks hold sufficient capital. However, the total amount of capital would typically become much higher than it is currently the case, and may prevent a proper functioning of the economy (i.e., banks may refrain from lending money). A compromise could be to compute VaR bounds based on the data and dependence information that are available and to scale down the numbers obtained with some factor that is acceptable for the industry.

The main conclusions and suggestions are specific to credit risk models. The high degree of model uncertainty for credit risk models is a natural phenomenon in the sense joint defaults are rarely observed, but Credit VaRs are driven by joint defaults. Hence, as it stands, models that assess VaRs of credit risk portfolios require to a great extent ad-hoc assumptions making them vulnerable to model risk. In order to reduce the model uncertainty, more attention could go to a better analysis and a more precise description of the interaction among credit risks in times of stress. For example, it might be reasonable to assume that the default of a mother company yields the default of all its subsidiaries (see also Vanduffel et al. (2008) for an extension of the CreditRisk model that accounts for this feature), or, on the contrary that the default of a company decreases the probability that its main competitors default. By contrast, banks have more market data available and thus acquire more statistical insight in the properties of a portfolio of market instruments than in their credit portfolio. It is expected that this additional information has a positive impact on the quality of Market VaR assessments, and might lead to acceptable risk bounds even when the confidence level used is high.
A Appendix

A.1 Proof of Lemma 3.1

Proof. Let us first observe that \( Y_j^* \sim vB(p_j) \). Furthermore, one can easily verify that \( S_n^* = Y_1^* + Y_2^* + \cdots + Y_n^* \) only takes values \( \ell v \) with probability \((1 - p^*)\) or \((\ell + 1) v \) with probability \(p^*\). Note that \( p^* = 0 \) may hold. It is straightforward to show that

\[
\text{var}(S_n^*) = v^2 p^*(1 - p^*),
\]

and we only need to show that any other sum \( S_n = Y_1 + Y_2 + \cdots + Y_n \) with \( Y_j \sim vB(p_j) \) has a larger variance. Consider any sum \( S_n \). In particular, \( S_n \) takes values in \( \{0, v, 2v, \ldots, nv\} \).

It is clear that \( \forall x \in ]0, \ell v[, F_{S_n}(x) \geq F_{S_n^*}(x) = 0 \text{ and } \forall x \in [(\ell + 1)v, +\infty[, F_{S_n}(x) \leq F_{S_n^*}(x) = 1 \). Since \( F_{S_n}(x) \) and \( F_{S_n^*}(x) \) are constant on the interval \([\ell v, (\ell + 1)v]\) one has,

\[
\exists c \geq 0, \quad \begin{cases}
\forall x \in (0, c), & F_{S_n}(x) \geq F_{S_n^*}(x) \\
\forall x \in (c, +\infty), & F_{S_n}(x) \leq F_{S_n^*}(x)
\end{cases}
\]

namely, \( c = (\ell + 1)v \) if \( F_{S_n}(\ell v) > F_{S_n^*}(x) \) and \( c = \ell v \) if \( F_{S_n}(\ell) \leq F_{S_n^*}(x) \). In other words, the distribution function \( F_{S_n} \) crosses \( F_{S_n^*} \) exactly once from above. Since \( \mathbb{E}(S_n) = \mathbb{E}(S_n^*) \) this implies the well-known fact that \( \mathbb{E}(h(S_n^*)) \leq \mathbb{E}(h(S_n)) \) for all convex functions \( h(x) \) (see, for example, Müller and Stoyan (2002)). Taking \( h(x) = x^2 \) ends the proof.

A.2 Proof of Proposition 3.4

Proof. The proof follows from Lemma 3.1 essentially. Consider variables \( Y_i \) given as \( Y_i = vI_{U_i \leq q} V_i + vI_{U_i > q} W_i \) in which \( V_i \) and \( W_i \) are Bernoulli distributed random variables that are independent of the uniform random variable \( U \) such that \( Y_i \sim vB(p_i) \) \((i = 1, 2, \ldots, n)\), \( \mathbb{E}(\sum_{i=1}^n V_i) = \frac{q}{v} \), \( \mathbb{E}(\sum_{i=1}^n W_i) = \frac{p}{v} \). Applying Lemma 3.1, they can be chosen such that the portfolio sum \( \sum_{i=1}^n Y_i \) takes four values, namely \( v \left[ \frac{2}{v} \right] \), \( v \left[ \frac{1}{v} \right] \), \( v \left[ \frac{0}{v} \right] \) and \( v \left[ \frac{-1}{v} \right] \). One observes that for this dependence among \( V_i \) and \( W_i \), \( \text{VaR}_q[Y_1 + Y_2 + \cdots + Y_n] = v \left[ \frac{2}{v} \right] \) and \( \text{VaR}_q[Y_1 + Y_2 + \cdots + Y_n] = v \left[ \frac{0}{v} \right] \), which ends the proof.

A.3 Proof of Proposition 5.1

Proof. We show that \( B(\alpha^*) \) is an upper bound of \( M \). To this end, assume that there exists \( T = \sum_{i=1}^n X_i \) that satisfies all moment constraints, i.e. \( \mathbb{E}[T^k] \leq c_k \), \( k = 2, 3, \ldots, K \) and such that \( \text{Var}_\alpha[T] > B(\alpha^*) \). Denote the distribution function of \( X(\alpha^*) \) by \( G \). Then \( \forall a \leq x < b \), \( F_T(x) \leq G(x) = q \). When \( b \leq x \), \( F_T(x) \leq G(x) = 1 \). Since \( G(x) = 0 \) for \( x < a \), this implies that

\[
\begin{cases}
\forall x < a, & F_T(x) \geq G(x) \\
\forall x \geq a, & F_T(x) \leq G(x).
\end{cases}
\]

In other words, the distribution function \( F_T \) crosses \( G \) once from above. Since \( \mathbb{E}[T] = \mathbb{E}(X[\alpha^*]) \), this implies that \( X(\alpha^*) \leq_{ct} T \) (cut criterion of Karlin and Novikoff (1963)). Moreover, since \( T \) and \( X^k(\alpha^*) \) are positive, it holds that \( \mathbb{E}[X^k(\alpha^*)] \leq \mathbb{E}[T^k] \) for all \( k = 2, 3, \ldots, K \). On the other hand, there exists \( k \geq 2 \) such that \( \mathbb{E}[X^k(\alpha^*)] = c_k \) (by definition of \( \alpha^* \)). This implies that \( \mathbb{E}[T^k] = \mathbb{E}[X^k(\alpha^*)] \) (because \( \mathbb{E}[T^k] \leq c_k \) because of the moment.
constraint). As \( \phi(x) = x^k (k \geq 2) \) is strictly convex, it follows that \( T \overset{d}{=} X(\alpha^*) \) must hold (Shaked and Shanthikumar (2007, Theorem 3.A.43)). This ends the proof. The proof that \( A(\alpha^*) \) is an absolute lower bound can be done in a similar way. 

### A.4 Proof of Proposition 5.4

**Proof.** Without loss of generality, we can express \( X_i \) as \( X_i = F_i^{-1}(V_i) \) for uniformly distributed \( V_i \) \((i = 1, 2, \ldots, n)\). Next, we consider variables \( X_i^* = \mathbb{I}_{U \leq u} F_i^{-1}(u V_i) + \mathbb{I}_{U > u} F_i^{-1}(U) \) in which \( 0 < u < 1 \) is chosen such that \( (X_1^*, X_2^*, \ldots, X_n^*) \) also satisfies the moment constraints. One observes then that for \( q > q^* := \max(1 - \min(p_1, p_2, \ldots, p_n), u) \), \( \text{VaR}_q^+(X_1^* + X_2^* + \cdots + X_n^*) = nv \).

### A.5 Specific Algorithm for getting minimum variance portfolio of risky loans

**Algorithm**

Recall that \( v_1 \geq v_2 \geq \ldots \geq v_d > 0 \).

1. Put in the first column of the matrix \( M^* \) the vector \([0, \ldots, 0, v_1, \ldots, v_1]^T\) where \( v_1 \) appears \( d_1 \) times.

2. For \( j = 2, 3, \ldots, n \), add in the \( j \)-th column, the vector \([0, \ldots, 0, v_1, \ldots, v_1]^T\) where \( v_j \) appears \( d_j \) times, such that it is antimonotonic to the sum of the \( j-1 \) first vectors of the matrix.

The matrix \( M^* \) that we obtain as an output of this algorithm is a representation of a random vector \( (X_1^*, X_2^*, \ldots, X_n^*)^T \) satisfying \( X_i^* \sim F_i \) and we only need to show that it is a possible solution to the minimum variance problem \( P \) given by (12).

**Proposition A.1** (convergence of algorithm). The random vector \( (X_1^*, X_2^*, \ldots, X_n^*)^T \) as constructed above gives rise to a candidate solution for Problem \( P \) in (12). That is, for all \( l = 1, 2, \ldots, n \), \( X_l^* \) is antimonotonic with \( \sum_{k=1, k \neq l}^n X_k^* \).

**Proof.** Fix some elements \( x_{kj} \) and \( x_{\ell j} \). By construction,

\[
(x_{kj} - x_{\ell j}) \sum_{r=1}^{j-1} (x_{kr} - x_{\ell r}) \leq 0. \tag{40}
\]

We first want to prove that

\[
(x_{kj} - x_{\ell j}) \sum_{r=1, r \neq j}^n (x_{kr} - x_{\ell r}) \leq 0. \tag{40}
\]

Clearly, the difference \( |x_{kr} - x_{\ell r}| \) is either equal to 0 or \( v_r \) \((r = 1, 2, \ldots, n)\). If \( x_{kj} - x_{\ell j} = 0 \), then property (40) is obvious. Without loss of generality, assume then that \( x_{kj} - x_{\ell j} > 0 \), i.e. \( x_{kj} - x_{\ell j} = v_j \). We want to prove that

\[
\sum_{r=1, r \neq j}^n (x_{kr} - x_{\ell r}) \leq 0. \tag{41}
\]
We know already that \( \sum_{r=1}^{j-1} (x_{kr} - x_{\ell r}) \leq 0 \) because by construction the \( j \)-th vector 
\([v_j, \ldots, v_j, \ldots, 0]^T\) is antimonotonic with the sum of \( j - 1 \) first vectors. Hence, if for all 
\( s > j, x_{ks} - x_{\ell s} \leq 0 \) then it is clear that (41) holds true. Let thus \( s > j \) be the smallest 
element such that \( x_{ks} - x_{\ell s} > 0 \) and observe that \( x_{ks} - x_{\ell s} = v_s \). Then,

\[
\sum_{r=1}^{s-1} (x_{kr} - x_{\ell r}) = \sum_{r=1}^{j-1} (x_{kr} - x_{\ell r}) + v_j + \sum_{r=j+1}^{s-1} (x_{kr} - x_{\ell r}) \leq 0.
\]

Since \( v_s \leq v_j \), we obtain

\[
\sum_{r=1, r \neq j}^{s} (x_{kr} - x_{\ell r}) = \sum_{r=1}^{s-1} (x_{kr} - x_{\ell r}) - v_j + v_s \leq 0.
\]

Next, we consider the first element \( s' > s \) such that \( (x_{ks'} - x_{\ell s'}) = v_{s'} > 0 \). Then,

\[
\sum_{r=1}^{s'-1} (x_{kr} - x_{\ell r}) \leq 0.
\]

Therefore,

\[
\sum_{r=1}^{s'-1} (x_{\ell j} - x_{\ell i}) = \sum_{r=1}^{j-1} (x_{kr} - x_{\ell r}) + v_{s'} + \sum_{r=s+1}^{s'-1} (x_{kr} - x_{\ell r}) \leq 0,
\]

and thus, as \( v_{s'} \leq v_j \),

\[
\sum_{r=1, r \neq j}^{s'} (x_{kr} - x_{\ell r}) \leq 0.
\]

We repeat this until we reach the last element with the property that \( (x_{kh} - x_{\ell h}) > 0 \). We obtain

\[
\sum_{r=1, r \neq j}^{h} (x_{kr} - x_{\ell r}) \leq 0.
\]

All remaining elements are non-positive, therefore

\[
\sum_{r=1, r \neq j}^{n} (x_{kr} - x_{\ell r}) \leq 0.
\]

We thus have proven that (41) holds and since this holds for all \( j, k \) and \( \ell \) thus also that 
every \( X_j^* \) is antimonotonic with \( \sum_{r=1, r \neq j}^{n} X_r^* \). This ends the proof. \( \square \)
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