Optimal Hedging with Basis Risk under Mean-Variance Criterion

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Abstract

Basis risk occurs naturally in a number of financial and insurance risk management problems. A notable example is in the context of hedging a derivative when the underlying security is either non-tradable or not sufficiently liquid. Other examples include hedging longevity risk using index-based longevity instrument and hedging crop yields using weather derivatives. These applications give rise to basis risk and it is imperative that such a risk needs to be taken into consideration for the adopted hedging strategy. In this paper, we consider the problem of hedging a European option using another correlated and liquidly traded asset and we investigate an optimal construction of hedging portfolio involving such an asset. The mean-variance criterion is adopted to evaluate the hedging performance, and a subgame Nash equilibrium is used to define the optimal solution. The problem is solved by resorting to a dynamic programming procedure and a change-of-measure technique. A closed-form optimal control process is obtained under a general diffusion model. The solution we obtain is highly tractable and to the best of our knowledge, this is the first time the analytical solution exists for dynamic hedging of general European options with basis risk under the mean-variance criterion. Examples on hedging European call options are presented to foster the feasibility and importance of our optimal hedging strategy in the presence of basis risk.

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1 Introduction

It is well-known in the financial theory that when an option is written on an asset that is tradable, it can be hedged by trading in the underlying asset. What if an option is written on an asset that is either illiquid or even non-tradable? In this case, a common hedging practice is to use another asset that is tradable, highly liquid, and also has the desirable property of being highly correlated to the underlying asset of the option. Because the hedged asset does not perfectly capture the behavior of the underlying asset, there is a mismatch between the risk exposure of the hedged portfolio and the option in question; this gives rise to the so-called basis risk. As shown in Davis (2006), the basis risk could be huge even though both assets have very high correlation. This implies that the basis risk can have a detrimental effect on the hedging performance and hence it needs to be prudently managed.

Basis risk does not just confine to hedging financial derivatives, it exists in many other settings, notably when an index-based security is used for hedging. For example, a pension plan sponsor may choose to hedge the plan’s longevity risk by resorting to standard longevity instrument that is traded in the capital market. While such “standard” instrument provides liquidity and transparency, its payoffs are typically determined by mortality indices based on one or more populations. As the longevity experience of the pension plan can deviate significantly from the reference populations, the basis risk, or more specifically, the population basis risk, is said to occur; see also Li and Hardy (2011), Coughlan et al. (2012). Another example is in the context of managing agricultural risk. In this application, using weather derivatives for hedging agricultural risk could give rise to variable basis risk and spatial basis risk (e.g., Brockett et al., 2005; Woodard and Garcia, 2008). Another situation for which basis risk arises is when a farmer purchases a crop insurance that is based on area yield, instead of individual yield. The area-yield crop insurance, which is known as the Group Risk Plan in the U.S., is an insurance scheme with indemnity depending on the aggregated county yields. The individual-yield crop insurance, which is known as the Annual Production History Insurance in the U.S., is another insurance scheme with payoff that is linked to individual farm yields. The discrepancy between yields at the county level and at the individual level gives rise to the basis risk; see for example Skees, et al (1997) and Turvey and Islam (1995).

A typical example in the financial market is that the hedging for an option written on a non-tradable asset is often conducted via trading over one liquidly traded asset which is closely
correlated with the non-tradable underlying asset. However, one should be very careful to use such a strategy since “close correlation” between the two underlying assets cannot guarantee the hedging performance to be as good as one may desire. Indeed, Davis (2006) showed that the unhedgeable noise, which is attributed to the mismatch between the two assets, may be huge even though the two underlying assets have very high correlation, and the “naive” hedging strategy may be ineffective.

In the existing literature, analytical results on optimal hedging in the presence of basis risk can broadly be classified into two streams. In order to ensure the model’s tractability, the first stream of investigation considers hedging general derivatives with basis risk under an exponential utility maximization framework. The pioneering closed-form optimal hedging strategies were obtained by Davis (2006). The basic model of Davis (2006) was subsequently extended by Monoyios (2004) and Musiala and Zariphopoulou (2004) in a few interesting directions including indifference pricing, perturbation expansions, etc. All of these generalizations are restricted to an exponential preference optimization framework. If we were to consider other optimization hedging frameworks such as under a mean-variance criterion, analytical optimal strategies with basis risk have been obtained but only for hedging futures. We classify this line of inquiry as the second stream. The main contribution is attributed to Duffie and Richardson (1991) who obtained the optimal continuous-time futures hedging policy under geometric Brownian motion assumptions. They demonstrated that the optimal hedging strategy can be derived from the normal equations for orthogonal projection in a Hilbert space. Their method, however, is not readily applicable to more general derivatives other than the futures contract. This is because their proposed method depends highly on the specific formulation of the problem and the trivial structure of the payoff function of the futures contract.

Motivated by the above two streams of investigation, this paper attempts to address each of their limitations by studying the dynamic hedging of general European options with basis risk under a mean-variance criterion. Since the seminal work of Markowitz (1952), the mean-variance criterion has been widely applied in finance. A key advantage of the mean-variance criterion over an utility maximization objective is that in practice it is typically challenging to accurately evaluate a hedger’s utility function while the mean-variance criterion provides a subjective measure. Furthermore, by comparing to the expected utility approach, MacLean et

\footnote{Note that the work of Davis (2006) was done in 2000 but it was not formally published until 2006.
al. (2011) concluded that, for less volatile financial market, the mean-variance criterion yields a better investment portfolio return.

It is important to emphasize that the optimal portfolio model of Markowitz (1952) is a one-period model. If we are interested in a dynamic portfolio selection strategy, it is important to distinguish optimal strategy that is “pre-commitment” from “time-consistent planning” because of the added possibility of re-optimizing and re-balancing the portfolio at intertemporal times. After a decision maker obtained his/her optimal dynamic strategy at time $t_1$, the decision maker might find that the adopted strategy from $t_1$ does not necessarily maximize his/her objective by the time he/she progresses to time $t_2$, where $t_2 > t_1$. In this situation, the decision maker can either continue to adopt the original plan or to devise a new plan that is “optimal” for him/her at $t_2$. Strotz (1955) referred formal strategy as the “precommitment” strategy and the latter as the “consistent planing” strategy. Strotz (1955) also shown that the best investment strategy should be a plan for which the investor will actually follow, i.e., a consistent planing strategy.

The analytical solutions provided by Zhou and Li (2000) and Li and Ng (2000) for, respectively, the continuous-time and multiperiod analogs of Markowitz (1952) are examples of precommitment optimal strategies. To derive the optimal strategies that are time consistent under the mean-variance criterion is considerably more challenging. The complication is driven by the fact that the mean-variance function is not separable so that the Bellman optimality principle cannot be directly applied for deriving an optimal dynamic solution. This problem was not solved until another decade later by Basak and Chabakauri (2010) who provided a novel approach of obtaining a “consistent planing” solution to the portfolio selection problem involving mean-variance objective. They used the total variance formula to derive an extended Hamilton-Jacobi-Bellman (HJB) equation and ingeniously obtain the optimal hedging strategy without directly solving the extended HJB equation as a partial differential equation. Subsequently Björk and Murgoci (2010, 2014) developed a more rigorous theory for general time-inconsistent problems by providing a formal way of defining a “consistent planing” solution using game theoretic approach and providing the verification theorem. In recent years, the time consistent planning strategies have also been widely studied for decision problems in insurance (e.g., Li et al., 2012; Wong et al., 2014).

In this paper, we aim to establish a “consistent planning” optimal hedging strategy in the sense of Björk and Murgoci (2010). The problem is solved by resorting to a dynamic pro-
gramming procedure and solving an extended HJB equation using a certain change-of-measure technique. The solution we obtain is highly tractable and to the best of our knowledge, this is the first time the analytical solution exists for dynamic hedging of general European options with basis risk under the mean-variance criterion. The solution we obtained also reduces to the classical delta hedging strategy when the two involved assets are indistinguishable and the risk aversion coefficient in the mean-variance objective goes to infinity. For plain vanilla call options, the calculation of the optimal strategy requires only a minimum amount of numerical procedure. Examples based on hedging futures and European call options are presented to highlight the importance of our proposed optimal strategy, relative to other commonly adopted hedging strategies that do not take into consideration the basis risk.

The rest of the paper proceeds as follows. The problem formulation is given in Section 2 and the consistent planning equilibrium solution is derived in Section 3. Discussions on some special cases are presented in Section 4. Some numerical examples are provided in Section 5 to highlight our theoretical results. Section 6 concludes the paper along with an Appendix that provides semi closed-form expressions for the equilibrium value functions of both futures and European call options.

2 Formulation of the optimal hedging problem

Let us begin by first introducing the following notations. For a function $F(t, s_1, s_2, x)$, we use $F_y$ to denote its first partial derivative with respect to (w.r.t.) variable $y$ where $y \in \{t, s_1, s_2, x\}$. Analogously, we use $F_{yz}$ to denote its second derivatives w.r.t. variables $y$ and $z$ where $y, z \in \{t, s_1, s_2, x\}$. Note that the function $F$ and its derivatives can be time-dependent processes. In this case, each of the notation will be indexed by an argument $t$. Similarly, if the arguments $s_1$, $s_2$ and $x$ are also processes, then they will be denoted by $S_1(t)$, $S_2(t)$ and $X(t)$, respectively. We also use $\mathbb{R}$ to denote the real line and $\mathbb{R}_+$ to mean the set of positive real numbers.

Consider a non-arbitrage market with two risky assets $\{S_1(t), t \geq 0\}$ and $\{S_2(t), t \geq 0\}$ as well as a risk-free asset earning at a constant rate of $r > 0$. The price processes of the two risk assets are defined over a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and they follow two general
diffusion processes under the physical measure $\mathbb{P}$ as below:

\[
\begin{cases}
  dS_i(t)/S_i(t) = \mu_i(t, S_i(t)) dt + \sigma_i(t, S_i(t)) dW_i(t), & i = 1, 2, \\
  dW_1(t) dW_2(t) = \rho(t) dt,
\end{cases}
\]

where $W_1 := \{W_1(t), t \geq 0\}$ and $W_2 := \{W_2(t), t \geq 0\}$ are two standard Brownian motions under $\mathbb{P}$. The coefficient $\rho(t)$ is a deterministic function of $t$, $\mu_i(t, s) : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ and $\sigma_i(t, s) : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}_+$, $i = 1, 2$. When there is no confusion about their dependence on $t$ and $s$, it is convenient to use the simplified notation of $\rho$, $\mu_i$, and $\sigma_i$, respectively. We assume that the drift and diffusion coefficients for both $S_i$ satisfy the global Lipschitz continuity and linear growth conditions so that the above stochastic differential equations admit a strong solution.

The above specification implies that the random sources between the two risky assets are correlated and the strength of correlation is governed by the coefficient function $\rho(t)$. Let $G = G(S_2(T))$ be the payoff at maturity $T > 0$ of a European option that is written on asset $S_2$. Here we assume that $S_2$ is either a non-tradable asset or a thinly traded asset so that it lacks the necessary liquidity to be used for hedging the option that is written on it. Instead, we assume that $S_1$ is a highly liquid and tradable asset so that together with the risk-free asset, a hedging portfolio can be constructed to hedge a short position of the above European option written on asset $S_2$. As $S_1$ is related to $S_2$ via the correlation parameter $\rho(t)$, using $S_1$ to hedge $G(S_2(T))$ gives rise to basis risk, unless in the special case $\rho(t) = 1 \ \forall t \in [0, T]$, and the coefficient functions $\mu_1(\cdot, \cdot) = \mu_2(\cdot, \cdot)$ and $\sigma_1(\cdot, \cdot) = \sigma_2(\cdot, \cdot)$.

At any time $t \in [0, T)$, the hedging portfolio is completely specified by the pair $\{X^\theta(t), \theta(t)\}$, where $\theta(t)$ denotes the time-$t$ investment in the risky asset $S_1$ and $X^\theta(t)$ represents the time-$t$ hedging portfolio value resulting from a strategy $\theta$. This implies that the time-$t$ investment in the risk-free asset is given by $X^\theta(t) - \theta(t)$. At the inception of the option contract, i.e., at $t = 0$, the hedging cost is given by $x_0 = X^\theta(0) > 0$. This also corresponds to the initial value of the hedging portfolio. Then, the value process of the hedging portfolio is governed by the following stochastic differential equation (SDE):

\[
\begin{align*}
  dX^\theta(t) &= \frac{\theta(t)}{S_1(t)} dS_1(t) + [X^\theta(t) - \theta(t)] r dt, \\
  &= [rx^\theta(t) + \theta(t)(\mu_1 - r)] dt + \theta(t) \sigma_1 dW_1(t), \quad t \in (0, T)
\end{align*}
\]
\[ X^\theta(0) = x_0, \]

where \( \theta(t) = \theta(t, S_1(t), S_2(t), X^\theta(t)), \ t \in [0, T) \). Note that \( X^\theta(t) \) is a controlled Markovian process.

Let \( \mathcal{F} := \{ \mathcal{F}_t, t \geq 0 \} \) be the filtration generated by \( \{(S_1(t), S_2(t)), t \geq 0\} \) and write the conditional expectation as \( E_{t,s_1,s_2,x}[^\cdot] = E[^\cdot | S_1(t) = s_1, S_2(t) = s_2, X^\theta(t) = x], \forall (t, s_1, s_2, x) \in [0, T] \times \mathbb{R}^2_+ \times \mathbb{R}. \) We are interested in the optimal hedging strategy among those admissible strategies in Definition 1 below.

**Definition 1.** An admissible strategy \( \theta(t) = \theta(t, X^\theta(t), S_1(t), S_2(t)), t \in [0, T] \) is defined as a progressively measurable process such that:

(a) \( E \left[ \int_0^T \theta(u)^2 \, du \right] < \infty. \)

(b) \( E_{t,s_1,s_2} \left[ \int_t^T |\theta(u)| \, du \right] \leq Ke^{K(s_1^2 + s_2^2)} \) for some constant \( K > 0, \forall (t, s_1, s_2) \in [0, T] \times \mathbb{R}^2_+. \)

We use \( \Theta \) to denote the set of all admissible strategies.

Both conditions (a) and (b) in Definition 1 are quite mild. The square integrability condition in (a) is almost the minimum requirement to ensure that the SDE (2) for \( X^\theta \) is well defined, and the exponentially growth condition (b) allows a wide class of admissible strategies. Imposing condition (b) ensures the uniqueness of the solution to the partial differential equation (PDE) (37), which is critical to deriving an explicit optimal strategy as we will see in Section 3.3.

Because of the basis risk and the market incompleteness, the hedging strategy involving \( \theta \) can not perfectly replicate the maturity value of the European option. The hedging error at expiration of the option is given by \( G(S_2(T)) - X^\theta(T). \) By defining \( V^\theta(T) \) as the profit-and-loss random variable for the hedger, then we have \( V^\theta(T) = X^\theta(T) - G(S_2(T)). \) For any time \( t < T \) and under mean-variance criterion, an optimal hedging strategy can be defined as one that solves the following optimization problem:

\[
\max_{\theta \in \Theta} U(t, s_1, s_2, x; \theta) := E_{t,s_1,s_2,x}[V^\theta(T)] - \frac{\gamma}{2} \text{Var}_{t,s_1,s_2,x}[V^\theta(T)]
\]

where \( \text{Var}_{t,s_1,s_2,x}[\cdot] = \text{Var}[\cdot | S_1(t) = s_1, S_2(t) = s_2, X^\theta(t) = x] \) and \( \gamma > 0 \) is a constant parameter capturing the risk aversion of the hedger. Note that the objective of the hedger is to
choose an optimal hedging strategy \( \theta \) that maximizes hedger’s (conditional) expected profit (i.e. \( E_t,s_1,s_2,x[V^\theta(T)] \)) subject to the penalty attributed to the (conditional) variance of the profit (i.e. \( \text{Var}_t,s_1,s_2,x[V^\theta(T)] \)). The degree of penalty is quantified by the parameter \( \gamma \), which is called risk aversion coefficient.

Let \( \{\hat{\theta}_0(t), t \in [t_1, T]\} \) be a pre-commitment solution of problem (3) derived by sitting at \( t = t_1 \). Then, it is well-known that the truncated strategy \( \{\hat{\theta}_0(t), t \in [t_2, T]\} \) is not generally optimal for the decision at a later time \( t_2 > t_1 \) (e.g., Basak and Chabakauri, 2010). Recall that this is the time-inconsistency issue associated with the mean-variance analysis.

To develop a time-consistent hedging strategy, we follow the game theoretic framework of Björk and Murgoci (2010) and Basak and Chabakauri (2010). More recently, similar approach has also been used by Li et al. (2012) and Wong, et al. (2014) to study some insurance related applications. Using the game theoretic formulation, the “optimality” is defined as a subgame perfect Nash equilibrium solution. The idea is to take the decision making process as a non-cooperative game among a continuum of players over the time horizon (who can be viewed as the future incarnations of the decision-maker), and each player can only influence the control process over an infinitesimal time interval. The formal mathematical definition is given as follows.

**Definition 2.** Consider a control process \( \theta^* \in \Theta \). For any arbitrary constant \( q \in \mathbb{R} \), \( \tau \in \mathbb{R}_+ \), and initial point \( (t, s_1, s_2, x) \) for \( (t, S_1(t), S_2(t), X^\theta(t)) \), define the control process \( \hat{\theta} \) as

\[
\hat{\theta}(v, s_1, s_2, x) = \begin{cases} 
q, & \text{for } t \leq v < t + \tau \\
\theta^*(v, s_1, s_2, x), & \text{for } t + \tau \leq v \leq T
\end{cases}
\]

(4)

Then \( \theta^* \) is an equilibrium control process if

\[
\liminf_{\tau \to 0} \frac{U(t, s_1, s_2, x; \theta^*) - U(t, s_1, s_2, x; \hat{\theta})}{\tau} \geq 0
\]

(5)

holds for all \( q \in \mathbb{R} \) where \( U \) is the objective function in problem (3). Furthermore, the equilibrium value function is defined by

\[
J(t, s_1, s_2, x) = U(t, s_1, s_2, x; \theta^*).
\]

(6)
3 Optimal time consistent hedging strategy

When only a controlled Markovian process is involved, there exists a standard procedure to derive the extended HJB equation for mean-variance optimization, as one can see in some recent applications such as Björk and Murgoci (2010) and Li et al. (2012). For our problem (3), the profit-and-loss random variable $V^\theta(T)$, however, depends on not only the controlled Markovian process $\{X^\theta(t), t \in [0, T]\}$ but also the price process $\{S_2(t), t \in [0, T]\}$. Explicit dependence of $G(S_2(T))$ in $V^\theta(T)$ distinguishes our model from other mean-variance based formulations and this complicates the derivation of an optimal solution. Subsection 3.1 will first establish an extended HJB equation for problem (3) in a heuristic way, subsection 3.2 will then formally justify our result by providing a verification theorem. Subsection 3.3 demonstrates that the proposed solution indeed satisfies all the conditions given in Subsection 3.2.

3.1 The extended HJB equation

We begin by first obtaining an alternate expression for the objective function in problem (3). We achieve this via the following total variance decomposition for an admissible hedging strategy $\theta \in \Theta$ and $\tau \in \mathbb{R}_+$:

$$\text{Var}_{t,s_1,s_2,x}(V^\theta(T)) = E_{t,s_1,s_2,x}[\text{Var}_{t+\tau}(V^\theta(T))] + \text{Var}_{t,s_1,s_2,x}[E_{t+\tau}(V^\theta(T))].$$

(7)

The objective function in problem (3) therefore can be written as

$$U(t, s_1, s_2, x; \theta)$$

$$= E_{t,s_1,s_2,x}[V^\theta(T)] - \frac{\gamma}{2}\text{Var}_{t,s_1,s_2,x}[V^\theta(T)]$$

$$= E_{t,s_1,s_2,x}[V^\theta(T)] - \frac{\gamma}{2}E_{t,s_1,s_2,x}[\text{Var}_{t+\tau}(V^\theta(T))] - \frac{\gamma}{2}\text{Var}_{t,s_1,s_2,x}[E_{t+\tau}(V^\theta(T))]$$

$$= E_{t,s_1,s_2,x}[U^\theta(t + \tau)] - \frac{\gamma}{2}\text{Var}_{t,s_1,s_2,x}[E_{t+\tau}(V^\theta(T))],$$

(8)

where $U^\theta(t) := U(t, S_1(t), S_2(t), X_t^\theta)$.

Recall that the random variable $V^\theta(T) = X^\theta(T) - G(S_2(T))$ represents the profit-and-loss
of the hedging strategy at maturity of the European option. For $t < T$, we define $V^\theta(t)$ as

$$V^\theta(t) \equiv V(t, S_2(t), X^\theta(t)) := X^\theta(t) - \Pi(t, s_2), \quad t \in [0, T],$$

where

$$\Pi(t, s_2) := E_{t,s_2} [e^{-r(T-t)} G(S_2(T))].$$

It should be emphasized that $\Pi(t, s_2)$ differs from the time-$t$ price of the European option $G(S_2(T))$ since the expectation in (10) is taken under the physical measure $\mathbb{P}$, as opposed to a risk neutral probability measure. To facilitate further development, we introduce the following functions:

$$\begin{align*}
m(t, s_1, s_2, x; \theta) &:= E_{t,s_1,s_2,x} [V^\theta(T)] \\
n(u, s_2, x) &:= [x - \Pi(u, s_2)] e^{r(T-u)} \\
l(t, s_1, s_2, x; \theta) &:= E_{t,s_1,s_2,x} \left[ \int_t^T e^{r(T-u)} \theta(u)(\mu_1 - r) du \right]
\end{align*}$$

for $(t, s_1, s_2, x) \in [0, T] \times \mathbb{R}_+^2 \times \mathbb{R}$, and adopt the following short-hand notation:

$$\begin{align*}
m^\theta(u) &= m(u, S_1(u), S_2(u), X^\theta(u); \theta), \\
n^\theta(u) &= n(u, S_2(u), X^\theta(u)), \\
l^\theta(u) &= l(u, S_1(u), S_2(u), X^\theta(u); \theta),
\end{align*}$$

for $u \in [t, T]$. It is well known that, with certain conditions on the payoff function $G$ (e.g. $\exists a > 0$ such that $\int_{-\infty}^\infty e^{-ax^2} |G(x)| dx < \infty$; see Lemma 5.1.3, P124, Musiela and Rutkowski, 1997), $\Pi(t, s_2)$ is smooth enough to apply Itô’s formula and obtain the following PDE:

$$r\Pi(t, s_2) = \Pi_t(t, s_2) + \Pi_{s_2}(t, s_2)s_2\theta + \frac{1}{2} \Pi_{s_2 s_2}(t, s_2)s_2^2\sigma^2_2. $$

Consequently, for $u \in (t, T)$, we apply Itô’s formula to $n^\theta(u)$ in conjunction with equations (1), (2) and (12) to obtain

$$dn^\theta(u) = d[X^\theta(u)e^{r(T-u)}] - d[\Pi(u)e^{r(T-u)}]$$
\[ e^{r(T-u)}dX^\theta(u) - X^\theta(u)e^{r(T-u)}rdu + \Pi(u)e^{r(T-u)}rdu - e^{r(T-u)}d\Pi(u) \]
\[ = e^{r(T-u)}(\mu_1 - r)du + e^{r(T-u)}[\theta(u)\sigma_1dW_1(u) - \Pi_{s_2}(u)S_2(u)\sigma_2dW_2(u)]. \] (13)

By noticing \( n^\theta(T) = V^\theta(T) \) and \( n^\theta(t) = V^\theta(t)e^{r(T-t)} \), we further get

\[ m(t, s_1, s_2, x; \theta) \equiv E_{t,s_1,s_2,x}[V^\theta(T)] \]
\[ = n(t, s_2, x; \theta) + E_{t,s_1,s_2,x}\left(\int_t^T e^{r(T-u)}\theta(u)(\mu_1 - r)du\right) \]
\[ = n(t, s_2, x; \theta) + l(t, s_1, s_2, x; \theta), \] (14)

and

\[ V^\theta(T) = V^\theta(t) + \int_t^T e^{r(T-u)}\theta(u)(\mu_1 - r)du \]
\[ + \int_t^T e^{r(T-u)}[\theta(u)\sigma_1dW_1(u) - \Pi_{s_2}(u)S_2(u)\sigma_2dW_2(u)]. \] (15)

Consequently, we can rewrite the objective function in problem (3) in a more compact expression as follows:

\[ U(t, s_1, s_2, x; \theta) = E_{t,s_1,s_2,x}[V^\theta(T)] - \frac{\gamma}{2}\text{Var}_{t,s_1,s_2,x}[V^\theta(T)] \]
\[ = n(t, s_2, x; \theta) + l(t, s_1, s_2, x; \theta) \]
\[ - \frac{\gamma}{2}\text{Var}_{t,s_1,s_2,x}\left[V^\theta(t)e^{r(T-t)} + \int_t^T d\left(e^{r(T-u)}V(u)\right)\right] \]
\[ = n(t, s_2, x; \theta) + \tilde{U}(t, s_1, s_2, x; \theta) \] (16)

where

\[ \tilde{U}(t, s_1, s_2, x; \theta) \]
\[ = l(t, s_1, s_2, x; \theta) - \frac{\gamma}{2}\text{Var}_{t,s_1,s_2,x}\left(\int_t^T e^{r(T-u)}\theta(u)(\mu_1 - r)du \right. \]
\[ + \left. \int_t^T e^{r(T-u)}[\theta(u)\sigma_1dW_1(u) - \Pi_{s_2}(u)S_2(u)\sigma_2dW_2(u)]\right). \] (17)
Substituting (16) into both sides of (8), we obtain
\[
\begin{align*}
n(t, s_2, x) + \tilde{U}(t, s_1, s_2, x; \theta) &= E_{t, s_1, s_2, x}[n^\theta(t + \tau)] + E_{t, s_1, s_2, x}[\tilde{U}^\theta(t + \tau)] - \frac{\gamma}{2} \text{Var}_{t, s_1, s_2, x}[m^\theta(t + \tau)], \quad (18)
\end{align*}
\]
where \(\tilde{U}^\theta(t) = \tilde{U}^\theta(t, S_1(t), S_2(t), X^\theta(t))\), and, by the definition of \(m^\theta(t)\),
\[
\begin{align*}
\text{Var}_{t, s_1, s_2, x}[m^\theta(t + \tau)] &= E_{t, s_1, s_2, x}[(m^\theta(t + \tau))^2 - (E_{t, s_1, s_2, x}(m^\theta(t + \tau)))^2] - [(E_{t, s_1, s_2, x}(m^\theta(t + \tau)))^2 - (E_{t, s_1, s_2, x}(m^\theta(t)))^2].
\end{align*}
\]

For any \(\tau \in \mathbb{R}_+\) and \(q \in \mathbb{R}\), we let \(\hat{\theta}\) to denote a hedging strategy with a generic admissible constant \(q \in \mathbb{R}\) applied over \([t, t + \tau)\) and the equilibrium strategy \(\theta^*\) applied over \([t + \tau, T)\), i.e., \(\hat{\theta}\) is as defined in equation (4). Thus, dividing by \(\tau\) and letting \(\tau \to 0\) in (18) gives the following extended HJB equation:
\[
0 = \max_{q \in \mathbb{R}} \left( \mathcal{A}^q F(t, s_1, s_2, x) - \xi^q(m(t, s_1, s_2, x; \theta^*)) \right), \quad (19)
\]
where \(\mathcal{A}^q\) is the infinitesimal generator for processes \(\{S_1, S_2, X^q\}\) and is given by
\[
\begin{align*}
\mathcal{A}^q F(t, s_1, s_2, x) &= F_t + F_x x + q F_x (\mu_1 - r) + F_{s_1} s_1 \mu_1 + F_{s_2} s_2 \mu_2 \\
&\quad + \frac{1}{2} F_{xx} (q \sigma_1)^2 + \frac{1}{2} F_{s_1 s_1} (s_1 \sigma_1)^2 + \frac{1}{2} F_{s_2 s_2} (s_2 \sigma_2)^2 \\
&\quad + F_{x s_1} s_1 \sigma_1 q_1 + F_{x s_2} s_2 \sigma_2 q_1 \rho + F_{s_1 s_1} s_1 \sigma_1 s_2 \sigma_2 \rho \quad (20)
\end{align*}
\]
and
\[
\xi^q(m(t, s_1, s_2, x)) = \frac{\gamma}{2} \left\{ \mathcal{A}^q \left[ m(t, s_1, s_2, x)^2 \right] - 2m(t, s_1, s_2, x) \mathcal{A}^q [m(t, s_1, s_2, x)] \right\}. \quad (23)
\]
Recall that \(F_x\) and \(F_{xx}\) respectively denote the first and second derivatives of \(F\) with respect to \(x\), and the other notations are similarly defined.
3.2 Verification theorem

Based on the extended HJB equation which was established in Section 3.1, the objective of this section is to develop a verification theorem, together with the required conditions, that guarantees a solution of the extended HJB equation, and to solve the mean-variance optimization problem (3).

**Theorem 1. (Verification Theorem).** If there exists a control process \( \theta^* \in \Theta \) and a function \( F \) such that

\[
\theta^*(t) = \arg \max_q \{ \mathcal{A}^q F(t, s_1, s_2, x) - \xi^q (g(t, s_1, s_2, x)) \} \quad (24)
\]

\[
0 = \mathcal{A}^{\theta^*} F(t, s_1, s_2, x) - \xi^{\theta^*} (g(t, s_1, s_2, x)) \quad (25)
\]

\[
F(T, s_1, s_2, x) = U(T, s_1, s_2, x) = x - G(s_2) \quad (26)
\]

\[
g(t, s_1, s_2, x) = E_{t,s_1,s_2,x} [V^{\theta^*}(T)] \quad (27)
\]

for any \( (t, s_1, s_2, x) \in [0, T] \times \mathbb{R}^3_+ \times \mathbb{R} \), then, \( \{ \theta^*(t) \}_{t \in [0, T]} \) is an equilibrium hedging strategy and \( F(t, s_1, s_2, x) \) is the equilibrium value function, i.e., \( F(t, s_1, s_2, x) = J(t, s_1, s_2, x) = U(t, s_1, s_2, x; \theta^*), \forall (t, s_1, s_2, x) \in [0, T] \times \mathbb{R}^3_+ \times \mathbb{R} \).

**Proof.** We will first show that \( J(t, s_1, s_2, x) = F(t, s_1, s_2, x) \). Indeed, By Dynkin’s Theorem,

\[
E_{t,s_1,s_2,x} [F(T, S_1(T), S_2(T), X^{\theta^*}(T))] = F(t, s_1, s_2, x) + E_{t,s_1,s_2,x} \left[ \int_t^T \mathcal{A}^{\theta^*} F(u, S_1(u), S_2(u), X^{\theta^*}(u)) du \right]. \quad (28)
\]

Then, it follows from equations (23) and (25) that

\[
E_{t,s_1,s_2,x} [F(T, S_1(T), S_2(T), X^{\theta^*}(T))] = F(t, s_1, s_2, x) + E_{t,s_1,s_2,x} \left[ \int_t^T \xi^{\theta^*} (g(u, S_1(u), S_2(u), X^{\theta^*}(u))) du \right]
\]

\[
= F(t, s_1, s_2, x) + E_{t,s_1,s_2,x} \left\{ \int_t^T \gamma \frac{\mathcal{A}^{\theta^*} [g(u)^2]}{2} - \gamma g(u) \mathcal{A}^{\theta^*} [g(u)] \right\} du \}
\]

\[
= F(t, s_1, s_2, x) + E_{t,s_1,s_2,x} \left\{ \int_t^T \frac{\gamma}{2} \mathcal{A}^{\theta^*} [g(u)^2] du \right\}
\]
\[
F(t, s_1, s_2, x) + \gamma \frac{1}{2} E_{t,s_1,s_2,x} \left[ g^2(T, S_1(T), S_2(T), X^{\theta^*}(T)) \right] - \gamma \frac{1}{2} g^2(t, s_1, s_2, x). \tag{29}
\]

The third equality follows from the fact that \( \mathcal{A}^{\theta^*}[g(u)] = 0 \) and the last equality can be obtained by applying Dynkin’s Theorem in conjunction with the definition of \( \xi^\theta \) given in (23). Moreover, by the boundary condition (26), we have
\[
E_{t,s_1,s_2,x} \left[ F(T, S_1(T), S_2(T), X^{\theta^*}(T)) \right] = E_{t,s_1,s_2,x} \left[ U(T, S_1(T), S_2(T), X^{\theta^*}(T)) \right] = E_{t,s_1,s_2,x} \left[ V^{\theta^*}(T) \right]. \tag{30}
\]

Combining equations (29) and (30) yields
\[
F(t, s_1, s_2, x) = E_{t,s_1,s_2,x} \left[ V^{\theta^*}(T) \right] - \gamma \frac{1}{2} E_{t,s_1,s_2,x} \left[ g^2(T, S_1(T), S_2(T), X^{\theta^*}(T)) \right] + \gamma \frac{1}{2} g^2(t, s_1, s_2, x)
= E_{t,s_1,s_2,x} \left[ V^{\theta^*}(T) \right] - \gamma \frac{1}{2} \text{Var}_{t,s_1,s_2,x} \left( V^{\theta^*}(T) \right)
= U(t, s_1, s_2, x; \theta^*), \tag{31}
\]

where the second equality follows from the definition of \( g \) given in (27). What we have established in equation (31) is that a function \( F(t, s_1, s_2, x) \) that satisfies conditions (24)-(27) is indeed the equilibrium value, provided that \( \theta^* \) is an equilibrium solution.

Next we show that \( \theta^* \) is indeed an equilibrium strategy. First note from equations (24) and (25) that
\[
\mathcal{A}^q F(t, s_1, s_2, x) - \xi^q(t, s_1, s_2, x) \leq 0, \quad \forall \ q \in \mathbb{R}.
\]

Then, discretizing the left-hand-side of the above inequality leads to
\[
E_{t,s_1,s_2,x} [F(t + \tau, S_1(t + \tau), S_2(t + \tau), X^{\theta^*}(t + \tau))] - F(t, s_1, s_2, x)
- \gamma \left( E_{t,s_1,s_2,x} [g(t + \tau, S_1(t + \tau), S_2(t + \tau), X^{\theta^*}(t + \tau))^2] - g(t, s_1, s_2, x)^2 \right)
+ \gamma \left( E_{t,s_1,s_2,x} [g(t + \tau, S_1(t + \tau), S_2(t + \tau), X^{\theta^*}(t + \tau))] - g(t, s_1, s_2, x)^2 \right) \leq o(\tau),
\]
where \( o(\tau)/\tau \to 0 \) as \( \tau \to 0 \). Furthermore, using the definition of \( g \) we obtain

\[
F(t, s_1, s_2, x) \\
\geq E_{t,s_1,s_2,x}[F(t+\tau, S_1(t+\tau), S_2(t+\tau), X^\theta(t+\tau))] \\
- \frac{\gamma}{2}(E_{t,s_1,s_2,x}[g(t+\tau, S_1(t+\tau), S_2(t+\tau), X^\theta(t+\tau))])^2 - \\
+ E_{t,s_1,s_2,x}[g(t+\tau, S_1(t+\tau), S_2(t+\tau), X^\theta(t+\tau))]) + o(\tau) \\
= E_{t,s_1,s_2,x}[F(t+\tau, S_1(t+\tau), S_2(t+\tau), X^\hat{\theta}(t+\tau))]) - \frac{\gamma}{2} \text{Var}_{t,s_1,s_2,x}[V^\hat{\theta}(T)] + o(\tau).
\]

The last equality holds because of equation (8). We have already showed that \( F(t, s_1, s_2, x) = U(t, s_1, s_2, x; \theta^*) \); hence,

\[
\liminf_{\tau \to 0} \frac{U(t, s_1, s_2, x; \theta^*) - U(t, s_1, s_2, x; \hat{\theta})}{\tau} \geq 0, \quad \forall \theta \in \mathbb{R},
\]

which implies that \( \theta^* \) is an equilibrium strategy.

\[\square\]

### 3.3 Optimal solution

In this subsection, we will show that the following \( \theta^* \) is an optimal solution:

\[
\theta^*(t, s_1, s_2) = e^{-r(T-t)} \left[ \frac{\mu_1 - r}{\gamma \sigma_1^2} - \eta_{s_1}(t) s_1 - \frac{s_2 \sigma_2 \rho}{\sigma_1} (\eta_{s_2}(t) - e^{r(T-t)} \Pi_{s_2}(t)) \right], \tag{32}
\]

where

\[
\eta(t, s_1, s_2) = E_{t,s_1,s_2}^* \left\{ \int_t^T \frac{1}{\gamma} \left( \frac{\mu_1 - r}{\sigma_1} \right)^2 + (\mu_1 - r) \frac{\sigma_2}{\sigma_1} S_2(u) e^{r(T-u)} \Pi_{s_2}(u) \right\} du \right. \tag{33}
\]

In the above formula \( \eta_{s_1} = \frac{\partial}{\partial s_1} \eta(t, s_1, s_2), \Pi_{s_2} = \frac{\partial}{\partial s_2} \Pi(t, s_2) \). \( E_{t,s_1,s_2}^* [\cdot] \) denotes conditional expectation under probability measure \( \mathbb{P}^* \), which is defined by the Radon-Nikodym derivative given as follows:

\[
\left. \frac{d \mathbb{P}^*}{d \mathbb{P}} \right|_{\mathcal{F}_t} = \exp \left( -\frac{1}{2} \int_0^t \left( \frac{\mu_1 - r}{\sigma_1} \right)^2 \, du - \int_0^t \frac{\mu_1 - r}{\sigma_1} dW_1(u) \right). \tag{34}
\]
By Girsanov's Theorem (Karatzas and Shreve, 1998), under \( \mathbb{P}^* \), the two risky assets \( S_1 \) and \( S_2 \) follow the following dynamics:
\[
\begin{align*}
\frac{dS_1(t)}{S_1(t)} &= r dt + \sigma_1 dW^*_1(t), \\
\frac{dS_2(t)}{S_2(t)} &= \left( \mu_2 - (\mu_1 - r) \frac{\rho \sigma_2}{\sigma_1} \right) dt + \sigma_2 dW^*_2(t),
\end{align*}
\]
where \( W^*_1 \) and \( W^*_2 \) are two standard Brownian motions under \( \mathbb{P}^* \).

The formal proof of the optimality of \( \theta^* \) is given in Theorem 2, which depends on the following technical lemma.

**Lemma 1.** If there exists a constant \( K > 0 \) such that \( |\eta(t, s_1, s_2)| \leq K \cdot e^{K(s_1^2 + s_2^2)}, \forall (t, s_1, s_2) \in [0, T] \times \mathbb{R}^2_+ \). Then, \( \eta \) satisfies the following recursion:
\[
\eta(t, s_1, s_2) = E_{t, s_1, s_2} \left[ \int_t^T e^{r(T-u)} \theta^*(u)(\mu_1 - r) du \right],
\]
where \( \theta^* \) is given by equation (32).

**Proof.** By applying Feynman-Kac Theorem to function \( \eta(t, s_1, s_2) \) along with equation (35), we obtain the following PDE:
\[
\begin{align*}
\eta_t + s_1 \mu_1 \eta_{s_1} + s_2 \mu_2 \eta_{s_2} + \frac{1}{2} \frac{s_1^2 \sigma_1^2}{\sigma_1} \eta_{s_1^2} + \frac{1}{2} \frac{s_2^2 \sigma_2^2}{\sigma_2} \eta_{s_2^2} + \rho s_1 s_2 \sigma_1 \sigma_2 \eta_{s_1 s_2} \\
+ \frac{1}{\gamma} \left( \frac{\mu_1 - r}{\sigma_1} \right)^2 - (\mu_1 - r) s_1 \eta_{s_1} - (\mu_1 - r) \frac{\rho \sigma_2}{\sigma_1} s_2 \left( \eta_{s_2} - e^{r(T-s)} \Pi_{s_2} \right) &= 0.
\end{align*}
\]

The given exponential growth condition in the lemma guarantees the uniqueness of solution to the second-order linear parabolic PDE (37) (see Chen, 2003; Lieberman, 1996). Thus, applying Feynman-Kac Theorem again, its solution is given by
\[
\begin{align*}
\eta(t, s_1, s_2) &= E_t \left[ \int_t^T \frac{1}{\gamma} \left( \frac{\mu_1 - r}{\sigma_1} \right)^2 du \right] \\
&= E_t \left[ \int_t^T (\mu_1 - r) \left( \eta_{s_1}(u)S_1(u) + \frac{\rho \sigma_2 S_2(u)}{\sigma_1} \left[ \eta_{s_2}(u) - e^{r(T-s)} \Pi_{s_2}(u) \right] \right) du \right] \\
&= E_{t, s_1, s_2} \left[ \int_t^T e^{r(T-u)} \theta^*(u)(\mu_1 - r) du \right].
\end{align*}
\]
in view of equation (32).

Let us denote $C^{1,2,2} = \{ f(t, s_1, s_2, x) : [0, T] \times \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R} \text{ s.t. } f \text{ is continuously differentiable in } t \text{ and twice continuously differentiable in } s_1, s_2 \text{ and } x \}$.

**Theorem 2.** Let $\theta^*$ defined in equation (32) is an equilibrium solution with the equilibrium value given by $U(t, s_1, s_2, x; \theta^*)$, provided that $U(t, s_1, s_2, x; \theta^*) \in C^{1,2,2}$.

**Proof.** We need to show $(\theta^*, F)$ with $F(\cdot) = U(\cdot, \cdot, \cdot, \cdot; \theta^*)$ solves the equation system (24)-(27). From equation (16), we obtain

$$J(t, s_1, s_2, x) = U(t, s_1, s_2, x; \theta^*) = n(t, x) + \tilde{U}(t, s_1, s_2; \theta^*)$$

with

$$\tilde{U}(t, s_1, s_2) = \eta(t, s_1, s_2) - \frac{\gamma}{2} \text{Var}_{t,s_1,s_2,x} \left( \int_t^T e^{r(T-u)} \theta^*(u) (\mu_1 - r) du \right.

\left. + \int_t^T e^{r(T-u)} [\theta^*(u) \sigma_1 dW_1(u) - \Pi_{s_2} S_2(u) \sigma_2 dW_2(u)] \right),$$

which is independent of $x$ because $\theta^*$ does not depend on $x$. Therefore, equation (26) holds, since

$$F(T, s_1, s_2, x) = n(T, s_2, x) + \tilde{U}(T, s_1, s_2) = (x - \Pi(T, s_2)) + \eta(T, s_1, s_2) = x - G(s_2).$$

Furthermore, equation (27), in conjunction with equation (14), implies

$$g(t, s_1, s_2, x) = E_{t,s_1,s_2,x}[V^{\theta^*}(T)] = n(t, s_2, x) + E_{t,s_1,s_2,x} \left( \int_t^T e^{r(T-u)} \theta^*(u) (\mu_1 - r) du \right)$$

$$= n(t, s_2, x) + \eta(t, s_1, s_2),$$

where the last equality follows from Lemma 1.

It remains to check equations (24) and (25). Regarding (25), we notice that $g(t, s_1, s_2, x) = E_{t,s_1,s_2,x}[V^{\theta^*}(T)]$ to obtain

$$s_{\theta^*} F(t, s_1, s_2, x) - \xi_{\theta^*} (g(t, s_1, s_2, x))$$
\[ \mathcal{A}^\theta t^* U(t, s_1, s_2, x; \theta^*) - \xi^\theta (g(t, s_1, s_2, x)) \]
\[ = \mathcal{A}^\theta v \left( g(t, s_1, s_2, x) - \frac{\gamma}{2} E_{t, s_1, s_2, x} [V^\theta (T)^2] + \frac{\gamma}{2} g^2(t, s_1, s_2, x) \right) - \xi^\theta (g(t, s_1, s_2, x)), \]

where the last equality is due to equation (8). Note that \( g(t, S_1(t), S_2(t), X^\theta (t)) \) is a martingale. Also, if we denote \( h(t, s_1, s_2, x) = E \left[ V^\theta (T)^2 \right] \), then \( h(t, S_1(t), S_2(t), X^\theta (t)) \) is also a martingale. Therefore, \( \mathcal{A}^\theta v g(t, s_1, s_2, x) = \mathcal{A}^\theta v h(t, s_1, s_2, x) = 0 \), and it follows from the definition of \( \xi^\theta \) that

\[ \mathcal{A}^\theta v F(t, s_1, s_2, x) - \xi^\theta (g(t, s_1, s_2, x)) \]
\[ = \frac{\gamma}{2} \mathcal{A}^\theta v \left( g^2(t, s_1, s_2, x) \right) - \gamma g(t, s_1, s_2, x) \mathcal{A}^\theta v \left( g(t, s_1, s_2, x) \right) - \xi^\theta (g(t, s_1, s_2, x)) \]
\[ = 0, \]

which implies condition (25).

Finally, we verify equation (24). By the definition of \( \xi^\theta \), we obtain

\[ \xi^\theta (g(t, s_1, s_2, x)) = \frac{\gamma}{2} \left[ g^2(s_1 \sigma_1)^2 + g^2(s_1 \sigma_1)^2 + g^2(s_2 \sigma_2)^2 \right. \]
\[ \left. + 2g_s g_x s_1 q \sigma_1^2 + 2g_s g_x s_2 q \sigma_1 \sigma_2 \rho + 2g_s g_x s_1 s_2 \sigma_1 \sigma_2 \rho \right]. \]

Therefore, we can use equation (39) to rewrite the right-hand-side of equation (24) as follows:

\[ \mathcal{A}^\theta v F(t, s_1, s_2, x) - \xi^\theta (g(t, s_1, s_2, x)) = \mathcal{A}^\theta v \tilde{U}(t, s_1, s_2) - \frac{\gamma}{2} \left( a_0 q^2 + a_1 q + a_2 \right), \quad (41) \]

where

\[
\begin{cases}
  a_0 = g^2 \sigma_1^2 \\
  a_1 = 2g_s g_x s_1 \sigma_1^2 + 2g_s g_x s_2 \sigma_1 \sigma_2 \rho - \frac{2}{\gamma} e^{r(T-t)} (\mu_1 - r) \\
  a_2 = g^2 \sigma_1^2 + g^2 \sigma_2^2 \sigma_1^2 + 2g_s g_x s_1 s_2 \sigma_1 \sigma_2 \rho - \frac{2}{\gamma} e^{r(T-t)} r + n s_2 \mu s_2 + \frac{1}{2} n s_2 s_2 \sigma_2^2.
\end{cases}
\]
Maximizing (41) with respect to \( q \) and using equations (40) and (32) yield

\[
\arg \max_q \left\{ \mathcal{A}^q F(t, s_1, s_2, x) - \xi^q (g(t, s_1, s_2, x)) \right\} = -\frac{a_1}{2a_0} \\
= e^{r(T-t)}(\mu_1 - r) - \gamma g_{s_1}g_x s_1 \sigma_1^2 - \gamma g_{s_2}g_x s_2 \sigma_2 \rho \\
\gamma g_x^2 \sigma_1^2 \\
= e^{-r(T-t)} \left[ \frac{\mu_1 - r}{\gamma \sigma_1^2} - \eta_{s_1}(t)s_1 - \frac{s_2 \sigma_2 \rho}{\sigma_1} \left( \eta_{s_2}(t) - e^{r(T-t)} \Pi_{s_2}(t) \right) \right] \\
= \theta^*(t, s_1, s_2).
\]

This confirms equation (24) and hence completes the proof. \( \square \)

4 Discussions

In this section, we provide additional analysis on some special cases for the general results derived in the preceding section. In particular, optimal trading strategies for variance minimization and/or the absence of basis risk are shown to be special cases of the general results. We shall use the notation \( Y \ind Z \) for two processes \( Y \) and \( Z \) to denote that they are indistinguishable, i.e., \( \mathbb{P}(Y(t) = Z(t), t \in [0, T]) = 1 \).

4.1 The case with no basis risk

A natural question one may ask is that under what conditions our problem degenerates to the case where the two stocks are perfectly correlated and there is no basis risk. It turns out that the case with no basis risk is indeed one special case of our problem (3), as shown in Proposition 1 below.

**Proposition 1.** In equation (1), if we let \( \rho = 1 \), \( \mu_1 \ind \mu_2 \ind \mu \) and \( \sigma_1 \ind \sigma_2 \ind \sigma \) for some progressively measurable processes \( \mu \) and \( \sigma \), then the two stochastic processes of stock price \( S_1(t) \) and \( S_2(t) \) are indistinguishable and therefore they can be viewed as the same stock.
Further, the equilibrium solution in this case is given by:

$$\theta^*(t, s) = s \cdot \Pi_s(t) + e^{-r(T-t)} \left[ \frac{\mu - r}{\gamma \sigma^2} - s\eta_s(t) \right]$$  \hspace{1cm} (42)

and

$$\eta(t, s) = \mathbb{E}_{t,s}^* \left\{ \int_t^T \left[ \frac{1}{\gamma} \left( \frac{\mu - r}{\sigma} \right)^2 + (\mu - r)S(u)e^{r(T-u)}\Pi_s(u) \right] du \right\},$$  \hspace{1cm} (43)

and $S(\cdot)$ is a progressively measurable process such that $S \overset{\text{ind}}{=} S_1 \overset{\text{ind}}{=} S_2$.

**Proof.** When $\rho = 1$, $\forall t_1, t_2 \in (0, T)$,

$$(W_1(t_2) - W_1(t_1), W_2(t_2) - W_2(t_1)) \sim N \left( (0, 0), \begin{bmatrix} t_2 - t_1 & t_2 - t_1 \\ t_2 - t_1 & t_2 - t_1 \end{bmatrix} \right).$$

Therefore $(W_1(t_2) - W_1(t_1), W_2(t_2) - W_2(t_1))$ is a degenerate bivariate normal random variable with $W_1(t_2) - W_1(t_1) = W_2(t_2) - W_2(t_1)$ a.s., and thus $\forall t \in (0, T)$, $W_1(t) = W_2(t)$ a.s., which along with the fact that both $W_1$ and $W_2$ have continuous paths almost surely, implies that $W_1(t)$ and $W_2(t)$ are indistinguishable, and so are $S_1(t)$ and $S_2(t)$. Consequently, equation (42) and equation (43) are obtained trivially from equation (32) and equation (33). \qed

### 4.2 The limiting case when $\gamma \to \infty$

When the risk aversion coefficient $\gamma$ in problem (3) becomes larger, this implies that the hedger is more risk averse and is more concerned with the variability of his/her hedging strategy. In the limiting case of $\gamma \to \infty$, the hedger can be perceived as one who is pre-dominantly concerned with the variability of the adopted hedging strategy and hence his/her objective boils down to minimizing $\text{Var}_{t,s_1,s_2,x}[V^\theta(T)]$. In this special case, it is of interest to investigate if the equilibrium solution given by equation (32) reduces to that of the variance minimization problem. The answer is affirmative as justified by following proposition:

**Proposition 2.** By denoting $\theta^*_\gamma$ as the equilibrium solution given in equation (32) to emphasize...
its dependence on the risk aversion coefficient $\gamma$ in problem (3), and letting

$$
\theta_0^*(t, s_1, s_2) = \lim_{\gamma \to \infty} \theta_0^*(t, s_1, s_2) = e^{-r(T-t)} \left[ -\eta_{s_1}(t)s_1 - \frac{s_2\sigma_2\rho}{\sigma_1} \left( \eta_{s_2}(t) - e^{r(T-t)}\Pi_{s_2}(t) \right) \right]
$$

(44)

and

$$
\eta(t, s_1, s_2) = E_{t,s_1,s_2}^* \left\{ \int_t^T \left[ (\mu_1 - r)\frac{\rho\sigma_2}{\sigma_1} S_2(u)e^{r(T-u)}\Pi_{s_2}(u) \right] du \right\},
$$

(45)

then $\theta_0^*$ defined in (44) is an equilibrium solution to the following variance minimization problem:

$$
\max_{\theta \in \Theta} \left\{ U(t, s_1, s_2, x; \theta) := -\text{Var}_{t,s_1,s_2,x}[V^\theta(T)] \right\}.
$$

(46)

**Proof.** The technique used to derive an equilibrium solution of problem (46) parallels to that of problem (3) in Section 3, hence we only highlight some key steps of the proof. First, by the total variance formula, the objective function of problem (46) satisfies the recursion:

$$
U(t, s_1, s_2, x; \theta) = -\text{Var}_{t,s_1,s_2,x}[V^\theta(T)] = -E_{t,s_1,s_2,x}[\text{Var}_{t+\tau}(V^\theta(T))] - \text{Var}_{t,s_1,s_2,x}[E_{t+\tau}(V^\theta(T))]
$$

(47)

Second, in parallel to equations (16) and (17),

$$
U(t, s_1, s_2, x; \theta) = -\text{Var}_{t,s_1,s_2,x} \left( \int_t^T e^{r(T-u)}\theta(u)(\mu_1 - r)du \\
+ \int_t^T e^{r(T-u)} \left[ \theta(u)\sigma_1 dW_1(u) - \Pi_{s_2}(u)S_2(u)\sigma_2 dW_2(u) \right] \right).
$$

Third, by a similar argument as in equations (18) and (19), we can use the recursion (47) to
establish the following extended HJB equation for equilibrium solution $\theta^*$ to satisfy:

$$0 = \max_{q \in \mathbb{R}} \left( \mathcal{A}^q F(t, s_1, s_2, x) - \xi^q(m(t, s_1, s_2, x; \theta^*)) \right),$$  \hspace{1cm} (48)

where the generator $\mathcal{A}^q$ is defined in equation (20), $m(t, s_1, s_2, x; \theta) := \mathbb{E}_{t, s_1, s_2, x}[V^\theta(T)]$ as defined in equation (11), and in parallel to equation (23),

$$\xi^q(m(t, s_1, s_2, x)) = \mathcal{A}^q \left[ m(t, s_1, s_2, x)^2 \right] - 2m(t, s_1, s_2, x)\mathcal{A}^q \left[ m(t, s_1, s_2, x) \right].$$

Finally it is straightforward to verify that Theorem 1 is still valid with $\xi^q$ replaced by the above definition. Then following a procedure similar to Theorem 2, we can prove $\theta^*_0$ is an equilibrium solution to problem (46).

\[\square\]

### 4.3 The limiting case when $\gamma \to \infty$ with no basis risk

We consider the case with no basis risk, i.e., $S_1$ and $S_2$ are indistinguishable, for which a sufficient conditions are $\rho = 1$, $\mu_1 \overset{\text{ind}}{=} \mu_2$ and $\sigma_1 \overset{\text{ind}}{=} \sigma_2$, as formally proved in Proposition 1. Let $S$ be a process which is indistinguishable from $S_1$ and $S_2$. So, we can equivalently view the European option as if it is written on $S$. By further letting $\sigma$ be a process indistinguishable from $\sigma_1$ and $\sigma_2$, it follows from equation (35) that, under the probability measure $\mathbb{P}^*$, $S$ follows a dynamic $dS(t) = rdt + \sigma dW^*(t)$, where $W^*$ is a standard Brownian motion under $\mathbb{P}^*$. Thus, the delta of the European option at time $t$ is given by $\Delta(t, s) = \frac{\partial}{\partial s} \Pi^*(t, s)$, where

$$\Pi^*(t, s) = \mathbb{E}_{t, s}^* \left[ e^{-r(T-t)} G(S(T)) \right].$$  \hspace{1cm} (49)

In this special case with no basis risk, it is well known that a dynamic delta hedging can fully replicate the payoff $G(S(T))$ of the European option. Therefore, a trading strategy $\tilde{\theta} = \{\tilde{\theta}(t, s), \ t \in [0, T], \ s \in \mathbb{R}_+ \}$ with $\tilde{\theta}(t, s) = s\Delta(t, s)$ is an equilibrium solution to the variance minimization problem (46), because the variance attains its minimum value, zero, with such a trading strategy at any time $t \in [0, T]$.

Recall that (42) of Proposition 1 gives an equilibrium solution when there is no basis risk.
With $\gamma \to \infty$, $\theta^*(t,s)$ in (42) reduces to
\[
\theta^*(t,s) = e^{-r(T-t)} \left[ -s \left( \eta_s(t,s) - e^{r(T-t)} \Pi_s(t,s) \right) \right], \quad t \in [0,T], \; s \in \mathbb{R}_+.
\] (50)

Therefore, in view of the above analysis, one may expect that $\theta^*$ given in equation (50) is the same as the delta hedging strategy $\tilde{\theta}$. Proposition 3 below confirms such a conjecture.

**Proposition 3.** $\tilde{\theta}(t,s) = \theta^*(t,s)$ for any $t \in [0,T]$ and $s \in \mathbb{R}_+$, where $\theta^*$ is given by equation (50).

**Proof.** Let us denote $y(t,s) := \mathbb{E}_{t,s}[G(S(T))] = e^{-r(T-t)} \Pi(t,s)$, $t \in [0,T]$ and $s \in \mathbb{R}_+$. Then, by Feynman-Kac Theorem,
\[
y_t + \mu s y_s + \frac{1}{2} \sigma^2 s^2 y_{ss} = 0.
\] (51)

Since $\Pi(t,s) = \mathbb{E}_{t,s}[e^{-r(T-t)}G(S(T))]$, it follows from (50) and Lemma 1 that
\[
\theta^*(t,s) = s \cdot e^{-r(T-t)} [y_s(t,s) - \eta_s(t,s)],
\]
with
\[
\eta(t,s) = \mathbb{E}_{t,s} \left[ \int_t^T e^{r(T-u)} \theta^*(u,S(u)) (\mu - r) du \right] = \mathbb{E}_{t,s} \left[ \int_t^T s(\mu - r) [y_s(u,S(u)) - \eta_s(u,S(u))] du \right].
\]

Applying the Feynman-Kac Theorem again yields
\[
\eta_t + \mu s \eta_s + \frac{1}{2} \sigma^2 s^2 \eta_{ss} + s(y_s - \eta_s)(\mu - r) = 0.
\] (52)

Combining equations (51) and (52) yields
\[
(y - \eta)_t + rs(y - \eta)_s + \frac{1}{2} \sigma^2 s^2 (y - \eta)_{ss} = 0.
\] (53)

We further define $y^*(t,s) := \mathbb{E}_{t,s}^*[G(S(T))]$, $t \in [0,T]$ and $s \in \mathbb{R}_+$, and use the Feynman-
Kac Theorem to obtain
\[ y^*_t + r s y^*_s + \frac{1}{2} \sigma^2 s^2 y^*_{ss} = 0. \] (54)

Note that \( y^*(t, s) \) and \( y(t, s) - \eta(t, s) \) satisfy the same PDE with the same boundary condition, i.e., \( y(T, s) - \eta(T, s) = G(S(T)) = y^*(T, s) \). Thus, \( y^*(t, s) = y(t, s) - \eta(t, s) \) and \( y^*_s(t, s) = y_s(t, s) - \eta_s(t, s) \). We further note \( \hat{\theta}(t, s) = s \Delta(t, s) = e^{-r(T-t)} y^*_s(t, s) \) and \( \theta^* = e^{-r(T-t)} [y_s(t, s) - \eta_s(t, s)] \) to conclude \( \hat{\theta}(t, s) = \theta^*(t, s) \).

4.4 Solutions under geometric Brownian motions

In this subsection, we investigate the hedging problem of two specific contingent claims – futures contract and the European call option – under the assumption that both asset price processes \( S_1 \) and \( S_2 \) are geometric Brownian motions. Under the assumption that \( S_1 \) is a geometric Brownian motion with constants \( \mu_1 \) and \( \sigma_1 \), the function \( \eta \) is independent of \( s_1 \). In this special case, we suppress the notation \( \eta(t, s_1, s_2) \) into \( \eta(t, s_2) \) throughout the section. Similarly for the optimal hedging strategy \( \theta^* \) that only depends on time \( t \) and the price of asset \( S_2 \), and we use the simplified notation \( \theta^*(t, s_2) \). Finally, we denote
\[ \mu_2^* = \mu_2 - (\mu_1 - r) \frac{\rho \sigma_2}{\sigma_1}, \] (55)
which is the drift coefficient of \( S_2 \) under the measure \( P^* \) as given in equation (35).

4.4.1 Futures contract

As pointed out in the introduction section that the payoff function of a futures contract is relatively simple. Its maturity value at time \( T \) given by \( G(S_2(T)) = S_2(T) - K \). Hence we have
\[ \Pi(t, s_2) = E_{t,s_2}[e^{-r(T-t)}(S_2(T) - K)] = s_2 e^{(\mu_2 - r)(T-t)} - Ke^{-r(T-t)} \]
and
\[
E^*_{t,s_2} [S_2(u) \Pi_{s_2}(u)] = E^*_{t,s_2} [S_2(t) e^{(\mu_2-r)(T-t)}] = s_2 e^{\mu_2(u-t)} e^{(\mu_2-r)(T-t)}.
\]

Substituting these into equation (33) yields
\[
\eta(t, s_2) = \int_t^T \left[ \frac{1}{\gamma} \left( \frac{\mu_1 - r}{\sigma_1} \right)^2 - \left( \mu_1 - r \right) \frac{\rho \sigma_2}{\sigma_1} s_2 e^{\mu_2(u-t)+\mu_2(T-u)} \right] du
\]
and
\[
\eta_{s_2} = \int_t^T \left[ (\mu_1 - r) \frac{\rho \sigma_2}{\sigma_1} e^{\mu_2(u-t)+\mu_2(T-u)} \right] du = e^{\mu_2(T-t)} - e^{\mu_2(T-t)}.
\]

Therefore, from equation (32) we obtain the optimal hedging strategy as follows
\[
\theta^*(t, s_2) = \frac{\mu_1 - r}{\gamma (\sigma_1)^2} e^{-r(T-t)} + \frac{\rho \sigma_2}{\sigma_1} s_2 e^{(\mu_2-r)(T-t)}, \quad 0 \leq t \leq T.
\]

Equilibrium value function \( J(t, s_1, s_2, x) \) does not have an explicit form, and thus need to be calculated numerically. A semi closed-form expression is given in Appendix A.1.

### 4.4.2 European call option

Recall that the key contribution of this paper is to provide an analytical optimal strategy to hedge derivative security. In this section, we provide an in-depth analysis by considering hedging a European call option under the Black-Scholes framework. The payoff of a European call option at maturity \( T \) is given by \( G(S_2(T)) = (S_2(T) - K)^+ \), where \( K \) is the pre-determined strike price and \((x)^+ = \max(x, 0)\). From the Black-Scholes formula, it can be shown that

\[
\Pi(t, s_2) = E_{t,s_2} [e^{-r(T-t)}(S_2(T) - K)^+]
\]
\[
= e^{(\mu_2-r)(T-t)} E_{t,s_2} [e^{-\mu_2(T-t)}(S_2(T) - K)^+]
\]
\[
= e^{(\mu_2-r)(T-t)} \left[ s_2 \Phi(d_{1,t}) - K e^{-\mu_2(T-t)} \Phi(d_{2,t}) \right]
\]
where $\Phi(\cdot)$ is the standard normal distribution function, and

\[
\begin{align*}
    d_{1,t} &= \frac{\ln(s_2/K) + (\mu_2 + \frac{1}{2} \sigma_2^2)(T - t)}{\sigma_2 \sqrt{T - t}}, \\
    d_{2,t} &= d_{1,t} - \sigma_2 \sqrt{T - t} \cdot \frac{\ln\left(\frac{s_2}{K}\right) + (\mu_2 - \frac{1}{2} \sigma_2^2)(T - t)}{\sigma_2 \sqrt{T - t}}.
\end{align*}
\]

Hence

\[
\Pi_{s_2} = \frac{\partial}{\partial s_2} \left( e^{(\mu_2 - r)(T - t)} [s_2 \Phi(d_{1,t}) - e^{-\mu_2(T - t)} K \Phi(d_{2,t})] \right)
\]

\[
= e^{(\mu_2 - r)(T - t)} \frac{\partial}{\partial s_2} [s_2 \Phi(d_{1,t}) - e^{-\mu_2(T - s)} K \Phi(d_{2,t})]
\]

By denoting

\[
c(u, s_2) := \ln s_2 + (\mu_2^* - \frac{1}{2} \sigma_2^2)(u - t) - \ln K + (\mu_2 + \frac{\sigma_2^2}{2})(T - u)
\]

we obtain, for $0 \leq t < u$,

\[
E^*_t, s_2 \left[ S_2(u) \Pi_{s_2}(u) \right] = E^*_t, s_2 \left[ S_2(u) e^{(\mu_2 - r)(T - u)} \Phi \left( \frac{\ln(S_2(u)/K) + (\mu_2 + \frac{\sigma_2^2}{2})(T - u)}{\sigma_2 \sqrt{T - u}} \right) \right]
\]

\[
= e^{(\mu_2 - r)(T - u)} \int_{-\infty}^{\infty} s_2 e^{(\mu_2^* - \frac{1}{2} \sigma_2^2)(u - t) + \sqrt{u - t} z} \Phi \left( \frac{\sqrt{u - t} + c(u, s_2)}{\sigma_2 \sqrt{T - u}} \right) e^{-\frac{z^2}{2}} dz dx
\]

\[
= s_2 e^{(\mu_2 - r)(T - u)} e^{(\mu_2^* - \frac{\sigma_2^2}{2})(u - t)} \int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{2\pi}} \right)^2 e^{\frac{u - t}{2}} e^{-\frac{z^2}{2}} \int_{-\infty}^{\frac{\sqrt{u - t}}{\sqrt{\sigma_2 \sqrt{T - u}}}} e^{\frac{x^2}{2}} dx
\]

\[
= s_2 e^{(\mu_2 - r)(T - u)} e^{(\mu_2^* - \frac{\sigma_2^2}{2})(u - t)} \int_{-\infty}^{\infty} \int_{-\infty}^{\frac{\sqrt{u - t}}{\sqrt{\sigma_2 \sqrt{T - u}}}} e^{\frac{x^2}{2}} dx dz
\]

\[
= s_2 e^{(\mu_2 - r)(T - u)} e^{(\mu_2^* - \frac{\sigma_2^2}{2})(u - t)} \int_{-\infty}^{\infty} \int_{-\infty}^{\frac{\sqrt{u - t}}{\sqrt{\sigma_2 \sqrt{T - u}}}} e^{\frac{(x - \sqrt{u - t})^2 + s_2^2}{2}} dx dz
\]

\[
= s_2 e^{(\mu_2 - r)(T - u)} e^{(\mu_2^* - \frac{\sigma_2^2}{2})(u - t)} \int_{-\infty}^{\infty} \int_{-\infty}^{\frac{\sqrt{u - t}}{\sqrt{\sigma_2 \sqrt{T - u}}}} e^{\frac{(x - \sqrt{u - t})^2 + s_2^2}{2}} dx dz
\]

\[
Z \leq \frac{\sqrt{u - t}}{\sigma_2 \sqrt{T - u}} (X + \sqrt{u - t} + c(u, s_2))
\]

26
where \(X\) and \(Z\) are two independent standard normal variables so that \(\tilde{Z} := Z - \frac{\sqrt{u-t}}{\sigma_2 \sqrt{T-u}} X\) is also a normal variable with \(E[\tilde{Z}] = 0\) and \(\text{Var}[\tilde{Z}] = \frac{\sigma_2^2(T-u)+(u-t)}{\sigma_2^2 (T-u)}\). Consequently,

\[
E_{t,s_2}^* [S_2(u) \Pi_{s_2}(u)] = s_2 e^{(-\mu_2+r+\mu_2-\frac{\sigma_2^2}{2}+\frac{1}{2}) u} e^{(-\mu_2+r+\mu_2-\frac{\sigma_2^2}{2}+\frac{1}{2}) u} \Phi \left( \frac{(u-t) + c(u,s_2) \sigma_2 \sqrt{T-u}}{\sqrt{(u-t) + \sigma_2^2 (T-u)}} \right).
\]

Using equation (33), we have

\[
\eta(t, s_2) = \frac{1}{\gamma} \left( \frac{\mu_1 - r}{\sigma_1} \right)^2 (T-t) + (\mu_1 - r) \frac{\rho \sigma_2}{\sigma_1} e^{\mu_2 t} e^{(-\mu_2-\frac{\sigma_2^2}{2}+\frac{1}{2}) u} \cdot s_2 \int_t^T e^{(\mu_2-\frac{\sigma_2^2}{2}+\frac{1}{2}-\mu_2) u} \Phi \left( \frac{(u-t) + c(u,s_2) \sigma_2 \sqrt{T-u}}{\sqrt{(u-t) + \sigma_2^2 (T-u)}} \right) du,
\]

and

\[
\eta_{s_2} = (\mu_1 - r) \frac{\rho \sigma_2}{\sigma_1} e^{\mu_2 T} e^{(-\mu_2-\frac{\sigma_2^2}{2}+\frac{1}{2}) u} \int_t^T e^{(\mu_2-\frac{\sigma_2^2}{2}+\frac{1}{2}-\mu_2) u} \Phi \left( \frac{(u-t) + c(u,s_2) \sigma_2 \sqrt{T-u}}{\sqrt{(u-t) + \sigma_2^2 (T-u)}} \right) du
\]

\[
+ (\mu_1 - r) \frac{\rho \sigma_2}{\sigma_1} e^{\mu_2 T} s_2 e^{(-\mu_2-\frac{\sigma_2^2}{2}+\frac{1}{2}) u} \int_t^T e^{(\mu_2-\frac{\sigma_2^2}{2}+\frac{1}{2}-\mu_2) u} \Phi \left( \frac{(u-t) + c(u,s_2) \sigma_2 \sqrt{T-u}}{\sqrt{(u-t) + \sigma_2^2 (T-u)}} \right) \frac{\sigma_2 \sqrt{T-u}}{\sqrt{(u-t) + \sigma_2^2 (T-u)}} \frac{\partial c(u,s_2)}{\partial s_2} du
\]

\[
= (\mu_1 - r) \frac{\rho \sigma_2}{\sigma_1} e^{\mu_2 T} e^{(-\mu_2-\frac{\sigma_2^2}{2}+\frac{1}{2}) u} \int_t^T e^{(\mu_2-\frac{\sigma_2^2}{2}+\frac{1}{2}-\mu_2) u} \Phi \left( \frac{(u-t) + c(u,s_2) \sigma_2 \sqrt{T-u}}{\sqrt{(u-t) + \sigma_2^2 (T-u)}} \right) du
\]

\[
+ \Phi \left( \frac{(u-t) + c(u,s_2) \sigma_2 \sqrt{T-u}}{\sqrt{(u-t) + \sigma_2^2 (T-u)}} \right) \frac{1}{\sqrt{(u-t) + \sigma_2^2 (T-u)}} \left[ \Phi \left( \frac{(u-t) + c(u,s_2) \sigma_2 \sqrt{T-u}}{\sqrt{(u-t) + \sigma_2^2 (T-u)}} \right) \right] du
\]

where \(\phi\) is the density function of standard normal distribution.

Consequently, equation (32) becomes

\[
\theta^*(t, s_2) = e^{-r(T-t)} \frac{\mu_1 - r}{\gamma \sigma_1^2} - e^{-r(T-t)} \frac{\rho \sigma_2}{\sigma_1} s_2 \eta_{s_2} + \frac{\rho \sigma_2}{\sigma_1} s_2 \Pi_{s_2}
\]
\[
= e^{-r(T-t)} \frac{\mu_1 - r}{\gamma \sigma^2_1} + \frac{\rho \sigma_2}{\sigma_1} s_2 e^{(\mu_2 - r)(T-t)} \Phi(d_{1,t}) \\
- e^{-(\mu_2 - \frac{\sigma_2^2}{2} + \frac{1}{2} - r)t} e^{(\mu_2 - r)T} \left( \frac{\rho \sigma_2}{\sigma_1} \right)^2 (\mu_1 - r) s_2 \\
\int_t^T e^{(\mu_2 - \frac{\sigma_2^2}{2} + \frac{1}{2} - \mu_2)u} \left[ \Phi(d_{*,u}) + \frac{\phi(d_{*,u})}{\sqrt{(u-t) + \sigma_2^2(T-u)}} \right] du \tag{57}
\]

where

\[
d_{*,u} = \frac{(u-t) + c(u,s_2) \sigma_2 \sqrt{T-u}}{\sqrt{(u-t) + \sigma_2^2(T-u)}}.
\]

Similar to futures contracts, equilibrium value function \( J(t, s_1, s_2, x) \) does not have an explicit form, and a semi closed-form expression is given in Appendix A.2.

5 Numerical examples

Based on the consistent planning equilibrium strategy derived in Subsection 4.4.2 for hedging a short position in European call option, this section provides some numerical evidences on highlighting the importance of the proposed equilibrium hedging strategy based on mean-variance criterion. Throughout the numerical examples, we assume that the strike price of the European call option is \( K = 100 \) with \( T = 1 \) year time to maturity, and that both asset prices follow geometric Brownian motions with the parameter values given in Table 1. Furthermore, the initial hedging cost is consistently set at \( x_0 = 20 \) for all hedging strategies.

<table>
<thead>
<tr>
<th>( S_1(0) )</th>
<th>( S_2(0) )</th>
<th>( K )</th>
<th>( r )</th>
<th>( \mu_1 )</th>
<th>( \sigma_1 )</th>
<th>( \mu_2 )</th>
<th>( \sigma_2 )</th>
<th>( T )</th>
<th>( x_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>100</td>
<td>100</td>
<td>0.05</td>
<td>0.1</td>
<td>0.25</td>
<td>0.12</td>
<td>0.3</td>
<td>1</td>
<td>20</td>
</tr>
</tbody>
</table>

For our first set of comparison, we numerically evaluate the performance of our proposed mean-variance equilibrium strategy against two other hedging strategies. The first strategy we benchmark against is known as the “naive delta hedge”, which is defined by

\[
\theta_{\text{naive}} = \frac{\sigma_2 S_2(t)}{\sigma_1} \frac{\partial}{\partial s} BS(S_2(t)),
\]
where \( BS(\cdot) \) denotes the Black-Scholes price and \( \frac{\partial}{\partial s} BS(\cdot) \) denotes the Black-Scholes delta. We refer \( \theta_{\text{naive}} \) as a naive delta hedge as such a strategy ignores the basis risk and delta hedge dynamically based on \( S_1 \) (i.e. assuming \( S_1 \) is the underlying asset). The naive delta hedge is not time consistent in terms of optimizing the mean-variance objective. The second strategy we benchmark against is simply the “no hedging” strategy, which merely investing the initial hedging amount of \( x_0 = 20 \) in the risk-free bond to earn the risk-free rate of \( r = 5\% \).

For each of the above strategies, we compute the mean and variance of the terminal wealth, along with the mean-variance objective value \( E[V(T)] - \frac{\gamma}{2} \text{Var}[V(T)] \). These are depicted in Table 2 by assuming the risk aversion coefficient \( \gamma = 1 \). For both equilibrium and naive delta hedging strategies, we further assume that the correlation parameter \( \rho \) increases from 0.5 to 1, at increment of 0.1. Based on these results, it is clear that simply investing in the risk-free bond is an inadequate strategy, as can be seen from the unacceptable large negative value of \( |E[V(T)] - \frac{\gamma}{2} \text{Var}[V(T)]| \). On the other hand, in the extreme case with \( \rho = 1 \), both equilibrium strategy and naive delta strategy are competitively effective for hedging the European call option. This is supported by the negligible variance of the terminal wealth of the hedger. PLS CONFIRM THIS: This is not surprising as in this particular case the basis risk does not exist. The advantage of the equilibrium strategy becomes more pronounced as the correlation decreases so that basis risk becomes more prominent. As \( \rho \) decreases, the variances of the terminal wealth of both strategies increase sharply (from perfect correlation case) but nevertheless the mean-variance objective values of the equilibrium strategy are consistently higher than the corresponding values from the naive delta hedging strategy. Hence indicating the importance of taking into account the basis risk and the superiority of the equilibrium hedging strategy.

**decimal places of all the values in the table are kept up to 2 are sufficient**

In the second set of comparison, we investigate the effect of risk aversion coefficient \( \gamma \) on the performance of the equilibrium hedging strategy and the naive delta hedging strategy. We similarly use the preceding example except by considering \( \rho \in \{0.9, 1\} \) and \( \gamma \in \{1/8, 1/4, 1/2, 1, 2, 4, \infty\} \). The numerical results are reported in Tables 3 and 4 for \( \rho = 1 \) and \( \rho = 0.9 \), respectively. Note that as shown in Proposition 2, the equilibrium strategy reduces to a solution for variance minimization as \( \gamma \to \infty \).

The advantage of the equilibrium hedging strategy is highlighted in this example in that it is
Table 2: Comparison of different strategies on hedging European call option (γ = 1)

<table>
<thead>
<tr>
<th></th>
<th>E[V(T)]</th>
<th>Var[V(T)]</th>
<th>E[V(T)] - \frac{1}{2} Var[V(T)]</th>
</tr>
</thead>
<tbody>
<tr>
<td>ρ = 1</td>
<td>equilibrium</td>
<td>5.494821658</td>
<td>0.913213176</td>
</tr>
<tr>
<td></td>
<td>naive delta</td>
<td>5.387401043</td>
<td>1.14021333</td>
</tr>
<tr>
<td>ρ = 0.9</td>
<td>equilibrium</td>
<td>4.724552507</td>
<td>131.9892846</td>
</tr>
<tr>
<td></td>
<td>naive delta</td>
<td>5.187382877</td>
<td>137.461826</td>
</tr>
<tr>
<td>ρ = 0.8</td>
<td>equilibrium</td>
<td>4.200962698</td>
<td>249.8859801</td>
</tr>
<tr>
<td></td>
<td>naive delta</td>
<td>5.209032423</td>
<td>270.244319</td>
</tr>
<tr>
<td>ρ = 0.7</td>
<td>equilibrium</td>
<td>3.695830639</td>
<td>356.143685</td>
</tr>
<tr>
<td></td>
<td>naive delta</td>
<td>5.254366872</td>
<td>402.8399208</td>
</tr>
<tr>
<td>ρ = 0.6</td>
<td>equilibrium</td>
<td>3.202118363</td>
<td>449.5889465</td>
</tr>
<tr>
<td></td>
<td>naive delta</td>
<td>5.307278533</td>
<td>534.6501187</td>
</tr>
<tr>
<td>ρ = 0.5</td>
<td>equilibrium</td>
<td>2.712917413</td>
<td>529.9371259</td>
</tr>
<tr>
<td></td>
<td>naive delta</td>
<td>5.358250607</td>
<td>664.9721419</td>
</tr>
<tr>
<td></td>
<td>no hedge</td>
<td>0.075854329</td>
<td>799.7683669</td>
</tr>
</tbody>
</table>

capable of reflecting the risk aversion of the hedger. When the hedger has a higher risk tolerance, he/she seeks an optimal strategy that has a higher mean value of the terminal wealth, though at the expense of higher terminal wealth variability. When the hedger becomes more and more risk averse, a greater penalty is imposed on the variability of the terminal wealth, which in turn also dampens the expected value of the terminal value. In contrast, the naive delta hedging strategy is invariant to the risk aversion of the hedger and hence produces the same set of mean and variance of the terminal wealth, irrespective of γ. Regardless of the risk aversion of the hedger, the mean-variance trade-off value of the equilibrium hedging strategy is consistently higher than the corresponding value from the naive delta hedging strategy, indicating the superiority of the former strategy. In the special case with ρ = 1 and irrespective of γ, both hedging strategies are very competitive and effective, as signalled by the small variance of the terminal wealth. This observation is consistent with the earlier example.
### Table 3: Hedging European call with $\rho = 1$: “equilibrium solution” vs “naive delta hedge”.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$E[V(T)]$</th>
<th>$Var[V(T)]$</th>
<th>$E[V(T)] - \frac{1}{2} Var[V(T)]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma = 1/8$</td>
<td>5.31613771</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma = 1/4$</td>
<td>5.244874377</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma = 1/2$</td>
<td>5.102347711</td>
<td></td>
<td></td>
</tr>
<tr>
<td>nave delta $\gamma = 1$</td>
<td>5.387401043</td>
<td>1.14021333</td>
<td>4.817294378</td>
</tr>
<tr>
<td>$\gamma = 2$</td>
<td></td>
<td></td>
<td>4.247187713</td>
</tr>
<tr>
<td>$\gamma = 4$</td>
<td></td>
<td></td>
<td>3.106974384</td>
</tr>
<tr>
<td>$\gamma \rightarrow \infty$</td>
<td></td>
<td></td>
<td>n/a</td>
</tr>
</tbody>
</table>

### Table 4: Hedging European call with $\rho = 0.9$: “equilibrium solution” vs “naive delta hedge”.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$E[V(T)]$</th>
<th>$Var[V(T)]$</th>
<th>$E[V(T)] - \frac{1}{2} Var[V(T)]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma = 1/8$</td>
<td>-3.403981248</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma = 1/4$</td>
<td>-11.99534537</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma = 1/2$</td>
<td>-29.17807362</td>
<td></td>
<td></td>
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<tr>
<td>naive delta $\gamma = 1$</td>
<td>-5.187382877</td>
<td>137.461826</td>
<td>-63.5435301</td>
</tr>
<tr>
<td>$\gamma = 2$</td>
<td></td>
<td></td>
<td>-132.2744431</td>
</tr>
<tr>
<td>$\gamma = 4$</td>
<td></td>
<td></td>
<td>-269.7362691</td>
</tr>
<tr>
<td>$\gamma \rightarrow \infty$</td>
<td></td>
<td></td>
<td>n/a</td>
</tr>
</tbody>
</table>

### Table 3: Hedging European call with $\rho = 0.9$: “equilibrium solution” vs “naive delta hedge”.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$E[V(T)]$</th>
<th>$Var[V(T)]$</th>
<th>$E[V(T)] - \frac{1}{2} Var[V(T)]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma = 1/8$</td>
<td>-3.403981248</td>
<td></td>
<td></td>
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<tr>
<td>$\gamma = 1/4$</td>
<td>-11.99534537</td>
<td></td>
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<td>$\gamma = 1/2$</td>
<td>-29.17807362</td>
<td></td>
<td></td>
</tr>
<tr>
<td>equilibrium $\gamma = 1$</td>
<td>4.724552507</td>
<td>131.9892846</td>
<td>-61.27008977</td>
</tr>
<tr>
<td>$\gamma = 2$</td>
<td></td>
<td></td>
<td>-127.2555196</td>
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<td>$\gamma = 4$</td>
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<td></td>
<td>-259.2125777</td>
</tr>
<tr>
<td>$\gamma \rightarrow \infty$</td>
<td></td>
<td></td>
<td>n/a</td>
</tr>
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</table>
6 Conclusion

The optimal dynamic hedging for a European-style derivatives is studied in this paper in the presence of basis risk where the underlying asset of the option is not traded in the market and is hedged by a traded asset. Under a general diffusion model setup, analytical hedging strategy is obtained from optimizing a mean-variance criterion and resorting to the Nash subgame equilibrium framework. The derivation is based on an extended HJB equation and change-measure techniques. The existent literature usually either focus on the hedging of futures contracts or follow an exponential preference optimization framework for mathematical convenience. In contrast, in this paper formal analysis is provided for the mean-variance optimal hedging strategy on hedging general European-style derivatives. The optimal hedging strategies in the absence of basis risk and/or for variance minimization can be recovered, as special cases, from the general results that were established in this paper.

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Appendix

A.1 Equilibrium value function for futures contract

A semi closed-form expression for the equilibrium value function is obtained by plugging equation (56) into equation (39) as follows:

\[
J(t, s_1, s_2, x) = (x - \Pi(t, s_2))e^{r(T-t)} + E_{t, s_1, s_2, x} \left( \int_t^T e^{r(T-u)}[\theta^*(u)(\mu_1 - r)]du \right)
\]
\[-\frac{\gamma}{2}E_{t,s_1,s_2} \left[ \left( \int_t^T e^{r(T-u)}[\theta^*(u)(\mu_1 - r)]du \right. \right. \right.
\left. \left. + \int_t^T e^{r(T-u)}\theta^*(u)\sigma_1 dW_1(u) - \int_t^T e^{\mu_2(T-u)}S_2(u)\sigma_2 dW_2(u) \right)^2 \right] \]

\[+ \frac{\gamma}{2} \left[ E_{t,s_1,s_2} \left( \int_t^T e^{r(T-u)}[\theta^*(u)(\mu_1 - r)]du \right)^2 \right] \]

\[= x e^{r(T-t)} + xK - s_2 e^{\mu_2(T-t)} + \frac{(\mu_1 - r)^2(T-t)}{2\gamma \sigma_1^2} \]

\[+ \frac{1}{2} s_2 \left[ e^{\mu_2(T-t)} - e^{\mu_2^*(T-t)} \right] - \frac{\gamma \rho^2 \sigma_2^2 s_2^2}{4\mu_2 + 2\sigma_2^2 - 4\mu_2^*} \left[ e^{(2\mu_2 + \sigma_2^2)(T-t)} - e^{2\mu_2^*(T-t)} \right] \]

\[-\frac{\gamma}{2} \left\{ \frac{(\mu_1 - r)^2(T-t)}{\gamma \sigma_1^2} + s_2 \left[ e^{\mu_2(T-t)} - e^{\mu_2^*(T-t)} \right] \right\} \]

\[-\frac{\gamma}{2} E_{t,s_1,s_2} \left[ \left( \int_t^T e^{r(T-u)}[\theta^*(u)(\mu_1 - r)]du \right)^2 \right] \]

\[-\gamma E_{t,s_1,s_2} \left[ \left( \int_t^T e^{r(T-u)}[\theta^*(u)(\mu_1 - r)]du \right. \right. \right.
\left. \left. \left( \int_t^T e^{r(T-u)}\theta^*(u)\sigma_1 dW_1(u) - \int_t^T e^{\mu_2(T-u)}S_2(u)\sigma_2 dW_2(u) \right) \right)^2 \right] \]

\[
A.2 \quad \text{Equilibrium value function for European call options}
\]

A semi closed-form expression for the equilibrium value function is obtained by plugging equation (57) into equation (39) as follows:

\[J(t, s_1, s_2, x) = (x - \Pi(t, s_2)) e^{r(T-t)} + E_{t,s_1,s_2} \left( \int_t^T e^{r(T-u)} [\theta^*(u)(\mu_1 - r)] du \right) \]

\[-\frac{\gamma}{2} E_{t,s_1,s_2} \left[ \left( \int_t^T e^{r(T-u)}[\theta^*(u)(\mu_1 - r)]du \right. \right. \right.
\left. \left. \left. + \int_t^T e^{r(T-u)}\theta^*(u)\sigma_1 dW_1(u) - \int_t^T e^{\mu_2(T-u)}S_2(u)\sigma_2 dW_2(u) \right)^2 \right] \]

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\[ \frac{\gamma}{2} \left[ E_{t,s_1,s_2,x} \left( \int_t^T e^{r(T-u)} [\theta^* (u) (\mu_1 - r)] du \right) \right]^2 \]

References


