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Abstract

In this paper, an optimal reinsurance problem is formulated from the perspective of an insurer, with the objective of minimizing the risk-adjusted value of its liability where the valuation is carried out by a cost-of-capital approach and the capital at risk is calculated by either the value at risk (VaR) or conditional value at risk (CVaR). In our reinsurance arrangement, we also assume that both insurer and reinsurer are obligated to pay more for a larger realization of loss as a way of reducing ex post moral hazard. A key contribution of this paper is to expand the research on optimal reinsurance by deriving explicit optimal reinsurance solutions under an economic premium principle. It is a rather general class of premium principles which includes many weighted premium principles as special cases. The advantage of adopting such a premium principle is that the resulting reinsurance premium depends not only on the risk ceded but also on an economic factor that reflects the market environment or the risk the reinsurer is facing. This feature appears to be more consistent with the reinsurance market. We show the optimal reinsurance policies are piecewise linear under both VaR and CVaR risk measures. While the structures of optimal reinsurance solutions are the same for both risk measures, we also formally show that there are some significant differences, particularly on the managing tail risk. Because of the integration of the market economic factor (via the reinsurance pricing) into the optimal reinsurance model, some new insights on the optimal design of reinsurance could be gleaned, which would otherwise be impossible for many of the existing models.
For example, the economic factor has a non-trivial effect on the optimal reinsurance which is greatly influenced by the changes of the joint distribution of the economic factor and the loss. Finally, under an additional assumption that the market factor and the loss have a copula with quadratic sections, we demonstrate that the optimal reinsurance policies admit relatively simple forms to foster the applicability of our theoretical results.

**Key-words:** Optimal reinsurance, Cost of capital; Conditional value at risk; Economic premium principles; Value at risk; Piecewise linear reinsurance.

## 1 Introduction

Reinsurance is one of the key risk management tools for insurance companies to manage its risk. Reinsurance could strategically reduce an insurance company’s risk exposure, underwrite more businesses, and stabilize earnings over time, just to name some of the potential benefits. As such, reinsurance should be integrated to the insurance company’s prudent risk management strategy and exploited fully and optimally. Since the seminal work of Borch (1960), the quest for optimal contracts has attracted significant attention from both academics and practitioners and a variety of optimal reinsurance policies have been devised by either minimizing some risk measure of an insurer’s risk exposure or maximizing the expected utility of the final wealth of a risk-averse insurer. See, for example, Borch (1960), Arrow (1963), Raviv (1979), Huberman et al. (1983), Young (1999), Kaluszka (2001), Kaluszka and Okolewski (2008), Bernard and Tian (2009), and references therein. In particular, optimal reinsurance models based on value-at-risk (VaR) and conditional value-at-risk (CVaR) have been extensively studied recently in part due to their prevalent use in finance and insurance. See Cai and Tan (2007), Cai et al. (2008), Balbás et al. (2009), Tan et al. (2009), Cheung (2010), Chi and Tan (2011, 2013), and Chi and Lin (2014), just to name a few.

In insurance economics, the expected value principle is the most commonly adopted assumption for calculating reinsurance premium. Its popularity stems from its simplicity and its tractability; optimal solutions usually can be derived explicitly for many reinsurance models. See, for example, Borch (1960), Arrow (1963) and Van Heerwaarden et al. (1989). As observed by Venter (1991), a potential drawback of using such a simple premium principle lies in its inconsistency with market practice, particularly with respect to (w.r.t.) property and casualty reinsurance pricing. As a result, many other premium principles have been introduced and analyzed. Young (2004), for example, catalogs as many as eleven of these premium principles. The generalization of these sophisticated premium principles stimulates more innovations and creates more challenges in optimal reinsurance problems. For instance, optimal reinsurance is studied by Kaluszka (2001) for mean-variance premium principles and by Young (1999) for Wang’s premium principle. More generally, Chi and Tan (2013) show that one-layer reinsurance is always optimal under VaR and CVaR risk measures if the reinsurance premium principle preserves the stop-loss order. Their results are further generalized by Chi (2012) who addresses optimal reinsurance with the reinsurance premium principles that are convex order preserving.
In the afore-mentioned studies, the reinsurance premium is calculated based only on the ceded loss. Bühlmann (1980), on the other hand, argues that the reinsurance market is incomplete so that the reinsurance premium should depend not only on the ceded loss itself but also on the existing market environment. For this reason, he advocates an economic pricing of reinsurance contracts that reflects the risk exposure not only attributed to the potential losses, but also from the market itself. Motivated by the equilibrium pricing model of Bühlmann (1980), Wang (2002) introduces a premium pricing framework based on the exponential tilting, which in turn has been extended to the economic weighted pricing by Furman and Zitikis (2009). Under an economic premium principle, not only that the reinsurance premium is typically no longer law-invariant, it also depends on the joint distribution of the ceded loss and the underlying economic factors reflecting the market environment. Another advantage of the economic premium principle is that it can be perceived as a general class of premium principles which encompasses many popular premium principles such as net, modified covariance, size-biased, Esscher, Aumann-Shalpey, Kamps, excess-of-loss, distorted, proportional hazard, modified tail covariance and conditional tail expectation (CTE) premium principles as special cases. It is also worth noting that the cost-of-capital premium calculation principle introduced by Merz and Wüthrich (2014) is a special case of the economic premium principle. Despite its viability, to the best of our knowledge there are very limited studies using the economic premium principle in the context of optimal reinsurance problems. It is precisely the objective of this paper to fill this gap.

In this paper, we study the design of an optimal reinsurance policy from the perspective of an insurer, where the reinsurance premium is calculated by an economic premium principle. Mathematically, the reinsurance premium is set as the expected ceded loss weighted by a market factor. In order to reduce ex post moral hazard, we assume that both parties in a reinsurance arrangement are obligated to pay more for a larger realization of loss, as suggested by Huberman et al. (1983). The adopted objective of the optimal reinsurance model is to minimize the value of the insurer’s risk-adjusted liability, where the valuation is carried out by a cost-of-capital approach proposed by Swiss Federal Office of Private Insurance (2006). When the capital at risk is calculated by VaR or CVaR risk measure, we show that strategically it is optimal for the insurer to purchase a piecewise linear reinsurance, that is, the derivative of whose ceded loss function is an indicator function. We find that while the optimal reinsurance policies under both of these two risk measures are alike in some ways, they are also different in many other aspects, particularly on ceding tail risk. This finding is novel and important, especially under the current regime of transitioning from VaR to CVaR for quantifying the regulatory capital. In addition, we find that the optimal reinsurance heavily relies not only on the distribution function of the market factor, but also on the dependence between the loss and the weighted economic factors. As a result, the effect of market factor on optimal reinsurance design is analyzed in detail by a comparative analysis. Finally, to illustrate the applicability of our theoretical results, explicit and relatively simple forms of the optimal contracts are derived under the assumption that the market factor and the loss have a bivariate copula with a quadratic section.

The optimal reinsurance model analyzed in this paper is closely related to Chi (2012) and Asimit et al. (2013). There are, however, some important differences. First, while the formulations of the optimal reinsurance models are similar in that all of them have the same objective function, the adopted reinsurance premium principles are vastly different. They usually assume that the
reinsurance premium principles are law-invariant and convex order preserving. In contrast, the economic premium principle that we are considering in this paper usually does not satisfy these two properties. Second, while all the optimal reinsurance policies are piecewise linear, our optimal contracts are usually much more complicated than the existing. Third and most importantly, our optimal reinsurance policies depend explicitly on a market factor via the economic premium principle. This, in turn, enables us to conduct a comparative analysis and provide important insights on the effects of the market factor and the dependence between the market factor and the loss on optimal reinsurance.

The rest of this paper is organized as follows. Section 2 introduces the optimal reinsurance model with an economic premium principle that we will investigate. The model is solved and optimal reinsurance is derived explicitly in Section 3. Section 4 carries out a comparative analysis of the optimal contract w.r.t. the market factor. To illustrate the applicability of our theoretical results, optimal reinsurance policies with simple forms are derived in Section 5 under a special dependence structure between the market factor and the loss. Section 6 concludes the paper and the Appendix collects the proofs to the theorems and propositions established in the papers.

2 The model

Let \( X \) be the loss an insurer faces over a given time period. It is natural to assume that \( X \) is a non-negative random variable on the probability space \((\Omega, \mathcal{F}, P)\) with cumulative distribution function (c.d.f.) \( F_X(x) \triangleq P(X \leq x) \) and \( E[X] < \infty \). The main concern of reinsurance design is how to optimally split \( X \) into \( f(X) \) and \( R_f(X) \), where \( f(X) \) represents the portion of loss that is ceded to a reinsurer while \( R_f(X) \) is the residual loss retained by the insurer (cedent). Thus, \( f(x) \) and \( R_f(x) \) are known as the insurer's ceded and retained loss functions, respectively. To reduce ex post moral hazard, we follow Huberman et al. (1983) to assume that both the insurer and the reinsurer are obligated to pay more for a larger loss in a typical reinsurance treaty. In other words, both ceded and retained loss functions are constrained to be increasing\(^1\) and we use \( \mathcal{C} \) to denote the set of admissible ceded loss functions:

\[
\mathcal{C} \triangleq \{ 0 \leq f(x) \leq x : \text{both } R_f(x) \text{ and } f(x) \text{ are increasing functions} \} .
\] (2.1)

The ceded loss functions in the above set have many nice properties. For instance, as shown in Chi and Tan (2011), \( f(x) \in \mathcal{C} \) is increasing and Lipschitz continuous, i.e.

\[
0 \leq f(x_2) - f(x_1) \leq x_2 - x_1, \quad \forall 0 \leq x_1 \leq x_2,
\] (2.2)

which is equivalent to \( f'(x) \in [0, 1], a.s. \) where \( f'(x) \) represents the derivative of \( f(x) \).

Because part of loss is ceded to a reinsurer, the insurer in a reinsurance arrangement incurring an additional cost in the form of reinsurance premium which is payable to the reinsurer. It is well-known that the reinsurance market is incomplete. Thus, Bühlmann (1980) advocates that the reinsurance premium should be calculated based not only on the ceded loss but also on the market environment. For simplicity, we use a non-negative random variable \( \mathcal{M} \) with \( E[\mathcal{M}] = 1 \)

\(^1\)Throughout the paper, “increasing” and “decreasing” mean “non-decreasing” and “non-increasing”, respectively.
to reflect the reinsurance market environment and other risks the reinsurer is facing, and let the reinsurance premium be calculated by an economic premium principle as

$$\pi_{\mathcal{M}}(Y) = \mathbb{E}[Y \mathcal{M}]$$

(2.3)

for all non-negative random variable $Y$. From the above equation, it is easy to see that the reinsurance premium is no longer law-invariant and relies heavily on the joint distribution of the ceded loss and the market factor. Moreover, $\pi_{\mathcal{M}}(.)$ usually does not preserve the convex order. Thus, such a premium principle is very different from that considered in Chi (2012). Indeed, this class of premium principles is quite general and includes many widely used premium principles such as weighted premium principles. More specifically, the economic premium principle becomes the weighted premium principle of Furman and Zitikis (2009) by setting the economic factor to be

$$\mathcal{M} = \frac{w(Z)}{\mathbb{E}[w(Z)]}$$

(2.4)

for a non-negative random variable $Z$ and a non-negative increasing function $w(.)$, provided that the expectation exists. By an appropriate specification of the weighting function $w(.)$, the weighted premium principle in turn recovers many popular premium principles; these include net, modified covariance, sized-biased, Esscher, Aumann-Shapley, Kamps, Excess-of-loss, distorted, proportional hazard, CTE and modified tail covariance premium principles, as shown Table 2.1, which is reproduced from Table 2 of Furman and Zitikis (2009) for convenience. Note that the Esscher premium principle is also known as the exponential tilting in Wang (2002) even if $Z$ is negative.

<table>
<thead>
<tr>
<th>Weighted premium principle</th>
<th>$w(z)$</th>
<th>$\Pi_{\mathcal{M}}(Y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Net</td>
<td>constant</td>
<td>$\mathbb{E}[Y]$</td>
</tr>
<tr>
<td>Modified covariance</td>
<td>$z$</td>
<td>$\mathbb{E}[Y] + \text{Cov}[Z, Y]/\mathbb{E}[Z]$</td>
</tr>
<tr>
<td>Size-biased</td>
<td>$z^t$</td>
<td>$\mathbb{E}[YZ^t]/\mathbb{E}[Z^t]$</td>
</tr>
<tr>
<td>Esscher</td>
<td>$e^{tz}$</td>
<td>$\mathbb{E}[Ye^{tz}]/\mathbb{E}[e^{tz}]$</td>
</tr>
<tr>
<td>Aumann-Shapley</td>
<td>$e^{tf_Z(z)}$</td>
<td>$\mathbb{E}[Ye^{tf_Z(Z)}]/\mathbb{E}[e^{tf_Z(Z)}]$</td>
</tr>
<tr>
<td>Kamps</td>
<td>$1 - e^{-tz}$</td>
<td>$\mathbb{E}[Y(1 - e^{-tz})]/\mathbb{E}[(1 - e^{-tz})]$</td>
</tr>
<tr>
<td>Excess-of-loss</td>
<td>$I_{{z \geq t}}$</td>
<td>$\mathbb{E}[Y</td>
</tr>
<tr>
<td>Distorted</td>
<td>$g'(S_Z(z))$</td>
<td>$\mathbb{E}[Yg'(S_Z(Z))]$</td>
</tr>
<tr>
<td>Proportional hazard</td>
<td>$pS_Z^{-1}(z)$</td>
<td>$p\mathbb{E}[YS_Z^{-1}(Z)]$</td>
</tr>
<tr>
<td>CTE</td>
<td>$I_{{z \geq \text{VaR}_p(Z)}}$</td>
<td>$\mathbb{E}[Y</td>
</tr>
<tr>
<td>Modified tail covariance</td>
<td>$I_{{z \geq \text{VaR}_p(Z)}}$</td>
<td>$\mathbb{E}[YZ</td>
</tr>
</tbody>
</table>

In the table, $t \in [0, \infty)$, $p \in [0, 1)$, $S_Z(z)$ is the survival function of a random variable $Z$ and $I_A$ is an indicator function of an event $A$. Moreover, $g(.)$ is an increasing concave distortion function with $g(0) = 0$ and $g(1) = 1$, and $\text{VaR}_p(Z)$, representing the VaR of $Z$ at the confidence level $1 - p$, is formally defined in Definition 2.1.
In the presence of reinsurance, the liability or the total risk exposure of the insurer is now given by the sum of the retained loss and the incurred reinsurance premium instead of $X$. Using $T_f(X)$ to denote the resulting liability of the insurer, we have

$$T_f(X) = R_f(X) + \pi_M(f(X)). \tag{2.5}$$

To evaluate the liability of the insurer, we use a cost-of-capital approach that is recently introduced by Swiss Federal Office of Private Insurance (2006) and has been widely adopted by the insurance companies in Europe. Specifically, the best estimate of the insurer’s liability is usually represented by $\mathbb{E}[T_f(X)]$. From a regulatory risk capital requirement point of view, it is prudent to also account for the unexpected loss $T_f(X) - \mathbb{E}[T_f(X)]$. If the risk measure $\varphi$ is used to quantify the risk associated with the unexpected loss, then the capital at risk corresponds to

$$\varphi\left(T_f(X) - \mathbb{E}[T_f(X)]\right).$$

In practice, the return from the capital investment is much smaller than that required by shareholders. By denoting $\delta \in (0, 1)$ as the return difference, which is known as the cost-of-capital rate, the risk margin is then given by the product of the cost-of-capital rate and the capital at risk. According to Risk Margin Working Group (2009), the risk-adjusted value or market-consistent price of the insurer’s liability, $L^\varphi_f(X)$, is defined as

$$L^\varphi_f(X) = \mathbb{E}[T_f(X)] + \delta \varphi\left(T_f(X) - \mathbb{E}[T_f(X)]\right). \tag{2.6}$$

For more details on evaluating the insurer’s liability with a cost-of-capital approach, we refer to Swiss Federal Office of Private Insurance (2006) and Risk Margin Working Group (2009).

In the insurance and other financial industry, VaR and CVaR risk measures have been widely used for quantifying risk and setting regulatory capital. Formally, these risk measures are defined as follows:

**Definition 2.1.** The VaR of a random variable $Y$ at a confidence level $1 - \alpha$ where $0 < \alpha < 1$ is defined as

$$VaR_\alpha(Y) \triangleq \inf\{t \in \mathbb{R} : S_Y(t) \leq \alpha\}, \tag{2.7}$$

where $\inf \emptyset = \infty$ by convention. Based upon the definition of VaR, CVaR of $Y$ at confidence level $1 - \alpha$ is defined as

$$CVaR_\alpha(Y) \triangleq \frac{1}{\alpha} \int_0^\alpha VaR_s(Y)ds. \tag{2.8}$$

It follows from the definition of VaR that

$$VaR_\alpha(Y) \leq z \iff S_Y(z) \leq \alpha \tag{2.9}$$

holds for any $z \in \mathbb{R}$. Therefore, for any non-negative $X$, we have $VaR_\alpha(X) = 0$ for $\alpha \geq S_X(0)$. For this reason, we assume in this paper that the parameter $\alpha$ satisfies $0 < \alpha < S_X(0)$ to avoid trivial cases. Another important property associated with $VaR_\alpha(Y)$ is that for any increasing continuous function $m(x)$, we have (see Theorem 1 in Dhaene et al. (2002))

$$VaR_\alpha(m(Y)) = m(VaR_\alpha(Y)). \tag{2.10}$$
The risk measure CVaR is also known as the “average value at risk” and “expected shortfall”. A key advantage of CVaR over VaR is that the former is a coherent risk measure while the latter is not as it fails to satisfy the subadditivity property. More detailed discussions on the properties of VaR and CVaR risk measures can be found in Artzner et al. (1999) and Föllmer and Schied (2004).

By specifying $\varphi$ to be either VaR or CVaR risk measure at the confidence level $1 - \alpha$, the respective optimal reinsurance models can be formulated as follows:

$$
\text{VaR-related optimization: } L_{VaR}^{f^*} (X) = \min_{f \in \mathcal{C}} L_{f}^{VaR}(X),
$$
(2.11)

and

$$
\text{CVaR-related optimization: } L_{CVaR}^{f^*} (X) = \min_{f \in \mathcal{C}} L_{f}^{CVaR}(X),
$$
(2.12)

where $f^*$ is the resulting optimal ceded loss function among $\mathcal{C}$. The rest of this paper is devoted to analyzing the solutions to these two optimal reinsurance models assuming that the reinsurance premium is calculated by an economic premium principle.

### 3 Optimal reinsurance policies

In this section, we solve the optimal reinsurance models as formulated in (2.11) and (2.12). As pointed out earlier one of the key features of our optimal reinsurance models is that the reinsurance premium is determined by an economic premium principle, which in turn takes into consideration the market factor $\mathcal{M}$. Here we assume that the weighted c.d.f. of $X$ w.r.t. the market factor $\mathcal{M}$ is represented by

$$
F_{\mathcal{M}}^{X} (t) \triangleq \mathbb{E}[\mathcal{M} \times I_{\{X \leq t\}}], \forall t \geq 0,
$$
(3.1)

with its corresponding survival function

$$
S_{\mathcal{M}}^{X} (t) \triangleq 1 - F_{\mathcal{M}}^{X} (t), \text{for any } t \geq 0.
$$

The theorem below solves the VaR-related reinsurance model (2.11). The proof to this theorem can be found in the appendix.

**Theorem 3.1.** An optimal solution to the VaR-related reinsurance model (2.11) is given by

$$
f^*_{\varphi} (x) = \int_{0}^{x} I_{\{\varphi(t) \leq 0\}} dt,
$$
(3.2)

where

$$
\varphi(t) \triangleq S_{\mathcal{M}}^{X} (t) - (1 - \delta)S_{X} (t) - \delta I_{\{S_{X} (t) > \alpha\}}, \forall t \geq 0.
$$
(3.3)

Note that the optimal ceded loss function (3.2) is expressed in the form of an integration over an indicator function. While the optimal ceded loss function is fully determined by the sign of the function $\varphi(t)$, $\varphi(t)$ is only known to be right-continuous, and its monotonicity, which relies heavily on the dependence between $\mathcal{M}$ and $X$, is unclear.

The optimal reinsurance under CVaR risk measure can be derived analogously and is formally stated in the following theorem.

**Theorem 3.2.** A ceded loss function that optimally solves the CVaR-related reinsurance problem (2.12) is given by

$$
f_{cv}^* (x) = \int_{0}^{x} I_{\{\psi(t) < 0\}} dt,
$$
(3.4)
where

\[
\psi(t) \triangleq S^M_X(t) - (1 - \delta)S_X(t) - \delta \min\{1, S_X(t)/\alpha\}.
\]  (3.5)

The proof to the above theorem is relegated to the appendix. Here we remark the following:

• First it is of interest to note that from both Theorems 3.1 and 3.2, the piecewise linear reinsurance treaty is always optimal regardless of which risk measure, Var or CVaR, is adopted. Second, both optimal ceded loss functions differ by the terms \(I_{\{S_X(t) > \alpha\}}\) and \(\min\{1, S_X(t)/\alpha\}\). Third, it is easy to see that

\[
I_{\{S_X(t) > \alpha\}} \leq \min\{1, S_X(t)/\alpha\}, \forall t \geq 0,
\]

with equality for \(0 \leq t < VaR_\alpha(X)\). Thus, it follows from (3.2) and (3.4) that \(f^*_c(x) = f^*_v(x)\) for any \(x \in [0, VaR_\alpha(X)]\) and \(f^*_c(x) \leq f^*_v(x)\) for each \(x > VaR_\alpha(X)\). In other words, if we change the regulatory risk measure from Var to CVaR, the impact is for the insurer to cede more tail loss at the same confidence level. This is intuitive since CVaR, as opposed to VaR, demands more regulatory capital on risk. This finding is also consistent with that obtained by Chi (2012) under the expected value principle and Wang’s premium principle.

• By further comparing our results with Chi (2012), we can see that optimal reinsurance under the economic reinsurance premium principle appears to be more complicated than that under most of widely used premium principles listed in Young (2004). More specifically, Chi (2012) shows that it is optimal for the insurer to purchase a one-layer reinsurance of the form

\[
f^*(x) = \int_0^x I_{\{t \in (a^*, b^*)\}} \, dt, \forall x \geq 0
\]

for some \(0 \leq a^* \leq b^*\).

• The piecewise linear optimal reinsurance policies under both VaR and CVaR differ only in how the tail risk is ceded. The difference is further highlighted in their effect on the cost-of-capital rate \(\delta\). More specifically, it is easy to see that \(\psi(t)\) is decreasing in \(\delta\), and this in turn implies that the insurer would buy more reinsurance under CVaR when the capital becomes more costly. On the other hand, while \(\phi(t)\) is also decreasing in \(\delta\) for \(t \leq VaR_\alpha(X)\), it becomes increasing for each \(t > VaR_\alpha(X)\). The consequence is that under VaR it may be optimal for the insurer to retain more tail risk for a higher cost-of-capital rate.

4 The effects of the market factor on optimal reinsurance

In addition to the critical role of the underlying loss distribution of \(X\) in determining the optimal reinsurance policies (3.2) and (3.4), the market factor \(M\) also has a non-trivial effect on the optimality, as reflected through the weighted survival function \(S^M_X(t)\). Thus, it is of interest to study its impact on the optimal reinsurance policies. In particular, in this section we will analyze the effects of the market factor on optimal reinsurance design from two aspects: (1) special optimal reinsurance is derived by assuming some stochastic dependence between the market factor \(M\) and the loss \(X\) (Subsection 4.1); (2) optimal reinsurance policies are compared for different market factors (Subsection 4.2).
4.1 Expectation dependence

This subsection analyzes the effect of stochastic dependence between the market factor $M$ and the loss $X$ on the optimal reinsurance design under both VaR and CVaR risk measures. To facilitate our analysis, the stochastic dependence is expressed in the form of the expectation dependence, which is defined formally as follows:

**Definition 4.1.** A random variable $Y_1$ is said to be positively expectation dependent (PED) w.r.t a random variable $Y_2$ if

$$E[Y_1|Y_2 \leq y] \leq E[Y_1], \forall y \in \mathbb{R}.$$  

Similarly, $Y_1$ is said to be negatively expectation dependent (NED) w.r.t $Y_2$ if

$$E[Y_1|Y_2 \leq y] \geq E[Y_1], \forall y \in \mathbb{R}.$$  

The above two notions of expectation dependence are attributed to Wright (1987). The average estimate of $Y_1$ depends on the relative magnitude of $Y_2$. In particular, when we discover that $Y_2$ is smaller, or more precisely $Y_2 \leq y$, then the (conditional) expectation of $Y_1$ is adjusted downward for PED but upward for NED. These forms of stochastic dependence have been applied by Wright (1987) to analyze a risk-averse investor’s portfolio investment decision. These notions of stochastic dependence have also been used by Hong et al. (2011) to generalize Mossin’s Theorem to a setting with random initial wealth. Given a fair premium, Hong et al. (2011) establish that any risk averse individual purchases less than full or more than full insurance if and only if the random loss $X$ is PED or NED w.r.t. random initial wealth $W$, respectively.

Here we similarly use the above stochastic dependence to analyze its impact on the optimal decision to cede losses. We show that the resulting optimal ceded loss functions have very simple structures, as formally established in the following proposition and Proposition 4.2 for optimal reinsurance models (2.12) and (2.11), respectively. Their proofs are collected in the appendix.

**Proposition 4.1.** If the market factor $M$ is NED w.r.t. the loss $X$, then the optimal solution to the CVaR-related reinsurance model (2.12) is full reinsurance in that the entire loss is transferred to the reinsurer.

**Proposition 4.2.** For the VaR-related reinsurance model (2.11), it is optimal for the insurer to retain the risk over $\text{VaR}_\alpha(X)$ if $M$ is PED w.r.t. $X$. On the other hand, it is optimal for the insurer to cede all the risk less than $\text{VaR}_\alpha(X)$ if $M$ is NED w.r.t. $X$.

Proposition 4.1 is very intuitive once we understand the implication of $M$ being NED w.r.t. $X$. More precisely, when $M$ is NED w.r.t. $X$, it follows from (A.2) and (A.6) that

$$\pi_M(f(X)) \leq E[f(X)]$$

for any admissible ceded loss function $f(x) \in \mathcal{C}$. Hence the reinsurance premium is less than the net premium and thus is not surprising that full reinsurance is optimal. This result is also consistent with Mossin’s theorem.

Comparing both Propositions 4.1 and 4.2, the effect of the stochastic dependence on the optimal reinsurance design can be different depending on the choice of the risk measure. In
particular, when the market factor is NED w.r.t. the loss, the optimal solution under VaR risk measure is full reinsurance only if
\[ S^M_X(t) \leq (1 - \delta)S_X(t), \forall t \in [\text{VaR}_\alpha(X), \text{Supp}(X)), \]
where \( \text{Supp}(X) \triangleq \inf\{t \geq 0 : P(X > t) = 0\} \). Obviously, this condition is not always satisfied. For instance, when \( X \) is independent of \( M \), we have \( S^M_X(t) = S_X(t) \) and hence the above condition fails to be satisfied. For this case, the optimal ceded loss function under VaR risk measure is \( f^*_c(x) = \min\{x, \text{VaR}_\alpha(X)\} \).

4.2 Comparative analysis

In this subsection, we provide some comparative analysis on the effects of different market factors on the optimal reinsurance policies. We analyze these effects through copula. The copula function, which is formally defined below, can be a very useful tool to capture the stochastic dependence of random vectors.

**Definition 4.2.** A bivariate copula is a function \( C(u,v) \) with domain \([0,1]^2\) satisfying the following properties

1. for any \( u, v \in [0,1] \),
\[ C(u,0) = C(0,v) = 0 \quad \text{and} \quad C(u,1) = u, \quad C(1,v) = v; \] (4.1)

2. \( C(u,v) \) is 2-increasing, i.e.,
\[ C(u_2,v_2) - C(u_1,v_2) - C(u_2,v_1) + C(u_1,v_1) \geq 0 \]
for any \( 0 \leq u_1 \leq u_2 \leq 1 \) and \( 0 \leq v_1 \leq v_2 \leq 1 \).

By Sklar’s Theorem, there exists a copula \( C_{M,X}(u,v) \) such that
\[ P(M \leq t, X \leq x) = C_{M,X}(F_M(t), F_X(x)) \]
for any \((t,x) \in \mathbb{R}^2\). See Nelsen (2006) for detailed discussions on copula.

For our first set of comparative analysis, we examine the optimal reinsurance policies for different copulas of \((M,X)\) by fixing the distribution function of the market factor. The results are summarized in the following proposition.

**Proposition 4.3.** For two market factors \( M_1 \) and \( M_2 \) with the same distribution function \( F_M(t) \), if \( C_{M_1,X}(u,v) \geq C_{M_2,X}(u,v) \) for any \((u,v) \in [0,1]^2\), then the insurer would cede less risk under \( M_1 \) than that under \( M_2 \) for both VaR and CVaR risk measures.

The result established in the above proposition is intuitive. As shown in the proof of the above proposition in the appendix, if \( C_{M_1,X}(u,v) \geq C_{M_2,X}(u,v) \) for any \((u,v) \in [0,1]^2\), we must have \( S^M_{X_1}(t) \geq S^M_{X_2}(t), \forall t \geq 0 \). This result, together with \((A.2)\) and the assumptions in the above proposition, imply that the reinsurance premium is higher under the market factor \( M_1 \) than that under \( M_2 \). Thus, it is intuitive that the insurer would cede less risk when the reinsurance is more costly.
Next, we compare optimal reinsurance design for different distribution functions of market factors by fixing the copula of the market factor and the loss $X$. The notion of “convex order”, as formally defined below, will facilitate the comparison of distribution functions.

Definition 4.3. A random variable $Y_1$ is said to be smaller than $Y_2$ in the sense of convex order (denoted by $Y_1 \leq_{cx} Y_2$) if

$$E[Y_1] = E[Y_2] \quad \text{and} \quad E[(Y_1 - d)_+] \leq E[(Y_2 - d)_+], \quad \forall d \in \mathbb{R},$$

provided that the expectations exist.

For more details on stochastic orders, we refer to Müller and Scarsini (2001) and Shaked and Shanthikumar (2007).

By imposing a weak condition on the copula, the comparison of optimal reinsurance for different distributions of the market factor are given in the following proposition and is proved in the appendix.

Proposition 4.4. For two market factors $M_1$ and $M_2$, assume that $(M_i, X)$ for $i = 1, 2$ have the same copula $C(u, v)$ and $M_1 \leq_{cx} M_2$. If $\frac{\partial C(u, v)}{\partial u}$ is decreasing in $u$ for every $v \in [0, 1]$, then we have $S_{X}^{M_1}(t) \leq S_{X}^{M_2}(t)$ for any $t \geq 0$ such that the insurer would cede more risk under $M_1$ than that under $M_2$; otherwise, if $\frac{\partial C(u, v)}{\partial u}$ is increasing in $u$ for every $v \in [0, 1]$, then the opposite conclusion is true.

Recall that the economic premium principle is a rather general premium principle and it encompasses many widely used premium principles as special cases. In particular, depending on the specification of the weighting function of the market factor, we can recover at least eleven premium principles as listed in Table 2.1. These weighting functions are determined by a parameter, which is denoted by either $t$ or $p$. Thus, it is of interest to compare the effects of these parameters on the optimal design of the reinsurance. These results are summarized in the following corollary with its proof relegated to the appendix.

Corollary 4.1. Under both VaR and CVaR risk measures, if the loss $X$ is stochastically increasing w.r.t. $Z$, which is denoted by $X \uparrow_{SI} Z$, then for a larger premium parameter, the insurer would cede less risk for Esscher, size-biased, excess-of-loss and Aumann-Shapley premium principles but retain less risk for Kamps, proportional hazard, CTE and modified tail covariance premium principles.\(^2\) Otherwise, the opposite results hold if $X \downarrow_{SI} -Z$.

Let $C(u, v)$ be the copula function for the random vector $(Z, X)$. If $\frac{\partial C(u, v)}{\partial u}$ is decreasing in $u$ for each $v \in [0, 1]$, then it follows from Proposition 2.12 in Cai and Wei (2012) that $X \uparrow_{SI} Z$, which in turn implies $X \uparrow_{SI} M$. Otherwise, if it becomes increasing, then we have $X \downarrow_{SI} -Z$. Further, we can see from (A.2) and the above proof that if $X \uparrow_{SI} Z$, then the reinsurance becomes more costly for Esscher, size-biased, excess-of-loss and Aumann-Shapley premium principles with a larger parameter, and for Kamps, proportional hazard, CTE and modified tail covariance premium principles with a smaller parameter.

\(^2\)A random variable $Y_1$ is said to be stochastically increasing w.r.t a random variable $Y_2$ (denoted by $Y_1 \uparrow_{SI} Y_2$) if $P(Y_1 > y_1 | Y_2 = y_2)$ is increasing over the support of $Y_2$ for any $y_1 \in \mathbb{R}$. See Shaked and Shanthikumar (2007) for more details.
5 Optimal reinsurance for bivariate copulas with quadratic sections

Motivated by the critical role of the bivariate copula of the market factor $M$ and the loss $X$ on the optimal reinsurance design, this section derives explicitly the optimal ceded loss functions under the additional assumption of bivariate copula with quadratic sections. Formally, a bivariate copula with quadratic sections is represented by

$$C(u, v) = uv + \lambda(u)v(1 - v),$$  \hspace{1cm} (5.1)

where $\lambda(u)$ is a function over $[0, 1]$. According to Theorem 3.2.4 in Nelsen (2006), $C(u, v)$ defined above is a copula if and only if $\lambda(u)$ satisfies

$$|\lambda(u_1) - \lambda(u_2)| \leq |u_1 - u_2|, \forall 0 \leq u_1, u_2 \leq 1.$$  \hspace{1cm} (5.2)

In the special case where $\lambda(u) = \theta u(1 - u)$ for a $\theta \in [-1, 1]$, the above copula recovers the famous Farlie-Gumbel-Morgenstern copula.

The above assumption, together with (A.7) and $\mathbb{E}[M] = 1$, leads to

$$S^M_X(t) = S_X(t)(1 + F_X(t)Q^M_M),$$  \hspace{1cm} (5.3)

where $Q^M_M \triangleq \int_0^\infty \lambda(F_M(y))dy$. Furthermore,

$$|Q^M_M| \leq \int_0^\infty |\lambda(F_M(y)) - \lambda(1)|dy \leq \int_0^\infty S_M(y)dy = 1,$$

which follows from the Lipschitz condition (5.2) and $\lambda(1) = 0$. Hence, we obtain from (5.3) that

$$\mathbb{E}[M|X \leq t] = 1 - S_X(t)Q^M_M.$$ 

Thus, the notion of expectation dependence can be characterized by $Q^M_M$, as stated below. The proof to this proposition is omitted.

**Proposition 5.1.** If the market factor $M$ and the loss $X$ have a bivariate copula of the form (5.1), then $M$ is PED (NED) w.r.t. $X$ if and only if $Q^M_M$ is non-negative (non-positive).

Armed with these assumptions and results, we are able to show that the resulting optimal ceded loss functions have relatively simple structures, as presented in the following proposition. The proof is collected in the appendix.

**Proposition 5.2.** Under the same assumption of Proposition 5.1 and that $\delta \geq \alpha$, the optimal ceded loss functions of the VaR-related and CVaR-related reinsurance models (2.11) and (2.12) are given, respectively, by

$$f^*_c(x) = \begin{cases} 
\min\{x, VaR_\alpha(X)\} + \left(x - VaR_{\min\{\alpha, 1 + \frac{\delta}{Q^M_M}\}}(X)\right)_+, & Q^M_M \in [-1, -\delta); \\
\min\{x, VaR_\alpha(X)\}, & Q^M_M \in [-\delta, \delta]; \\
\min\left\{\left(x - VaR_{\frac{\delta}{Q^M_M}}(X)\right)_+, VaR_\alpha(X) - VaR_{\frac{\delta}{Q^M_M}}(X)\right\}, & Q^M_M \in (\delta, 1]. 
\end{cases}$$
and

\[ f^*_{cv}(x) = \begin{cases} 
    x, & Q^M_{\lambda} \in [-1, 0]; \\
    \min \left\{ (x - Var_{\frac{1}{\lambda}}(X))_+, Var_{\frac{1}{\lambda}}(1/\alpha - 1)(X) - Var_{\frac{1}{\lambda}}(X) \right\}, & \text{otherwise},
\end{cases} \]

where \((x)_+ = \max(x, 0)\).

When the market factor \(M\) and the loss \(X\) have a copula with quadratic sections, the above proposition asserts that the optimal contract is either one-layer or two-layer. Moreover, ceding the very tail risk for \(Q^M_{\lambda} \in [-1, -\delta]\) can be rather surprising since under the VaR risk measure it is generally optimal to retain the tail risk.

To conclude this section, it should be pointed out that \(\delta \geq \alpha\) is a reasonable assumption in practice.

6 Conclusion

In this paper, we push the boundary on the research of optimal reinsurance by considering a reinsurance model that minimizes the risk-adjusted value of an insurer’s liability subject to the reinsurance premium being calculated by an economic premium principle and that both insurer and reinsurer are obligated to pay more for a larger realization of loss. Here the valuation of the insurer’s liability is based on a cost-of-capital approach with the capital at risk being quantified by VaR or CVaR risk measure, as recommended by Swiss Federal Office of Private Insurance (2006) and Risk Margin Working Group (2009).

While the underlying formulation of the optimal reinsurance model has appeared in the literature (see Chi, 2012; Asimit et al., 2013), the assumption of an economic premium principle on reinsurance pricing is, however, new and important. In addition to the desirable property that it is a rather general class of premium principles which encompasses many popular premium principles as special cases, more importantly the economic premium principle has the capability of integrating the market factor into the reinsurance pricing framework. As the reinsurance market is generally perceived to be incomplete, in practice the reinsurance premium depends not only on the loss ceded, but also on the market economic factor. The economic premium principle is able to reflect this feature and hence is more consistent with market practice.

Because of the integration of the market economic factor (via the reinsurance pricing) into the optimal reinsurance model, some new insights on the optimal design of reinsurance could be gleaned, which would otherwise be impossible for the existing models. For example, it was formally shown that the optimality is greatly influenced by the changes of the joint distribution of the market factor and the loss. Comparative study was also conducted to investigate the effect of the market factor on the optimal reinsurance policies. Finally, it was shown that optimal reinsurance policies could admit relatively simple forms under an additional assumption that the market factor and the loss have a copula with quadratic sections.

A Proofs

This appendix collects the proofs to the theorems and propositions discussed in the papers.
A.1 Proof of Theorem 3.1

For an admissible ceded loss function $f(x) \in \mathcal{C}$ and by the definition of $L_{f}^{VaR_{\alpha}}(X)$ as given in (2.6), we have

$$L_{f}^{VaR_{\alpha}}(X) = (1 - \delta)E[R_{f}(X)] + \delta VaR_{\alpha}(R_{f}(X))$$

$$= (1 - \delta)E[X] + \delta VaR_{\alpha}(X) + \mathbb{E}[f(X) \times \mathcal{M}] - \delta f(VaR_{\alpha}(X)). \quad (A.1)$$

The first equality follows from the liability decomposition (2.5), the definition of the economic premium principle (2.3), and the translation invariant property of VaR. The second equality follows from (2.10) and $X = f(X) + R_{f}(X)$.

Since $f(x)$ is an increasing Lipschitz-continuous function with $f(0) = 0$, then we can see from Fubini’s theorem that

$$E[f(X) \times \mathcal{M}] = \mathbb{E} \left[ \mathcal{M} \times \int_{0}^{\infty} I_{\{X > t\}} df(t) \right] = \int_{0}^{\infty} S_{X}^{\mathcal{M}}(t) df(t). \quad (A.2)$$

Similarly, it is easy to show that

$$E[f(X)] = \int_{0}^{\infty} S_{X}(t) df(t) \quad (A.3)$$

and

$$f(VaR_{\alpha}(X)) = \int_{0}^{\infty} I_{\{VaR_{\alpha}(X) > t\}} df(t) = \int_{0}^{\infty} I_{\{S_{X}(t) > \alpha\}} df(t), \quad (A.4)$$

where the last equality can be justified using (2.9).

Consequently, (A.1) simplifies to

$$L_{f}^{VaR_{\alpha}}(X) = (1 - \delta)E[X] + \delta VaR_{\alpha}(X) + \int_{0}^{\infty} \phi(t) df(t),$$

where $\phi(t)$ corresponds to (3.3). Note that $f'(t) \in [0, 1]$ for any $t \geq 0$. Thus, the admissible ceded loss function that minimizes $L_{f}^{VaR_{\alpha}}(X)$ is given by (3.2) and this completes the proof.

A.2 Proof of Theorem 3.2

Using the same arguments for establishing (A.1), we have, for an admissible ceded loss function $f(x) \in \mathcal{C}$,

$$L_{f}^{CVaR_{\alpha}}(X)$$

$$= (1 - \delta)E[R_{f}(X)] + \delta \times CVaR_{\alpha}(R_{f}(X)) + \Pi_{\mathcal{M}}(f(X))$$

$$= (1 - \delta)E[X] + \delta CVaR_{\alpha}(X) + \mathbb{E}[\mathcal{M} f(X)] - (1 - \delta)\mathbb{E}[f(X)] - \delta CVaR_{\alpha}(f(X)). \quad (A.5)$$
Again, using (2.10) we have

\[
CVaR_\alpha(f(X)) = \frac{1}{\alpha} \int_0^\alpha Var_s(f(X)) ds \\
= \frac{1}{\alpha} \int_0^\alpha f(VaR_s(X)) ds \\
= \frac{1}{\alpha} \int_0^\alpha \int_0^\infty \mathbb{I}_{\{S_X(t) > s\}} df(t) ds \\
= \int_0^\infty \min\{1, S_X(t)/\alpha\} df(t),
\]

where the third and the last equalities follow, respectively, from (A.4) and the Fubini’s theorem. Substituting (A.2), (A.3) and the above equation into (A.5) leads to

\[
L_f^{CVaR_\alpha}(X) = (1 - \delta)E[X] + \delta CVaR_\alpha(X) + \int_0^\infty \psi(t) df(t),
\]

where \(\psi(t)\) is defined in (3.5). As a consequence, \(f^{*}_c(x)\) given in (3.4) is a solution to the optimal reinsurance problem (2.12) and the proof is complete.

### A.3 Proof of Proposition 4.1

If \(\mathcal{M}\) is NED w.r.t. \(X\), it follows from Definition 4.1 that

\[
S_X^M(t) = 1 - E[\mathcal{M}\mathbb{I}_{\{X \leq t\}}] = 1 - P(X \leq t) \times E[\mathcal{M}|X \leq t] \leq S_X(t), \quad (A.6)
\]

which in turn implies

\[
\psi(t) \leq -\delta \min\{F_X(t), S_X(t)\} \leq 0, \quad \forall t \geq 0.
\]

Consequently, using Theorem 3.2, we get that the optimal ceded loss function under CVaR risk measure is given by \(f_c^*(x) = x\), as required.

### A.4 Proof of Proposition 4.2

Note that similar to establishing (A.6), we have \(S_X^M(t) \geq S_X(t)\) for any \(t \geq 0\) if \(\mathcal{M}\) is PED w.r.t. \(X\). Moreover, it follows from (2.9) that \(\mathbb{I}_{\{S_X(t) > \alpha\}} = 0\) for any \(t \geq VaR_\alpha(X)\). Consequently, using (3.3), we have

\[
\phi(t) = S_X^M(t) - (1 - \delta)S_X(t) \geq 0, \quad \forall t \geq VaR_\alpha(X),
\]

which in turn implies \(f^*_c(x) = f^*_c(VaR_\alpha(X))\) for all \(x \geq VaR_\alpha(X)\).

On the other hand, if \(\mathcal{M}\) is NED w.r.t. \(X\), both (2.9) and (A.6) can again be used to show that

\[
\phi(t) = S_X^M(t) - (1 - \delta)S_X(t) - \delta \leq -\delta F_X(t) \leq 0, \quad \forall t < VaR_\alpha(X).
\]

Consequently, it follows from Theorem 3.1 that \(f^*_c(x) = x\) for any \(x \leq VaR_\alpha(X)\). The proof is thus complete.

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A.5 Proof of Proposition 4.3

By simple algebra, we can rewrite $S_X^M(t)$ as

$$S_X^M(t) = \mathbb{E}\left[ \int_0^\infty I\{M>y\}dy \times I\{X>t\} \right] = \int_0^\infty \mathbb{P}(M>y, X>t)dy = \int_0^\infty S_X(t) - F_M(y) + C_{M,X}(F_M(y), F_X(t))dy,$$

where the second equality is due to Fubini’s theorem. Thus, if $C_{M_1,X}(u,v) \geq C_{M_2,X}(u,v)$ for any $(u,v) \in [0,1]^2$, we must have

$$S_X^M_1(t) \geq S_X^M_2(t), \forall t \geq 0. \quad (A.8)$$

Thus the required result can be easily obtained from (3.2) and (3.4) and the proof is complete.

A.6 Proof of Proposition 4.4

Since it is assumed that $(M_i, X)$ for $i = 1, 2$ have the same copula $C(u,v)$, it follows from Sklar’s Theorem that there must exist uniformly distributed random variables $U$ and $V$ such that $(M_i, X)$ has the same distribution with $(F_{M_i}^{-1}(U), F_X^{-1}(V))$, where $F_Y^{-1}(t)$ is a generalized inverse function of $F_Y(t)$. Thus, $S_X^{M_i}(t)$ can be rewritten by

$$S_X^{M_i}(t) = \mathbb{E}\left[ F_{M_i}^{-1}(U)I\{F_X^{-1}(V)>t\} \right] = \mathbb{E}[F_{M_i}^{-1}(U)\kappa_t(U)], \quad (A.9)$$

where

$$\kappa_t(u) \triangleq \mathbb{P}(F_X^{-1}(V) > t | U = u) \text{ for all } u \in [0,1], t \geq 0.$$

If $\frac{\partial C(u,v)}{\partial u}$ is decreasing in $u$ for every $v \in [0,1]$, and noting that $F_Y^{-1}(y)$ is an increasing and right-continuous function, we have $\kappa_t(u)$ is increasing in $u$. Further, it is easy to see that $F_{M_i}^{-1}(U)$ and $\kappa_t(U)$ are comonotonic. Since it is assumed that $M_1 \leq_{dcx} M_2$, then Theorem 4.5 in Müller and Scarsini (2001) implies

$$(F_{M_1}^{-1}(U), \kappa_t(U)) \leq_{dcx} (F_{M_2}^{-1}(U), \kappa_t(U)),$$

where $(X_1, Y_1) \leq_{dcx} (X_2, Y_2)$ is defined by

$$\mathbb{E}[h(X_1, Y_1)] \leq \mathbb{E}[h(X_2, Y_2)]$$

for any directionally convex function $h(x, y)$, provided that the expectations exist.\(^3\) Fortunately, it is easy to verify that $h(x, y) = xy$ is directionally convex, so that we have

$$\mathbb{E}[F_{M_1}^{-1}(U)\kappa_t(U)] \leq \mathbb{E}[F_{M_2}^{-1}(U)\kappa_t(U)],$$

\(^3\)A two-dimensional function $h(x, y)$ is called to be directionally convex if it satisfies

$$h(x_1, y_1) + h(x_2, y_2) \leq h(\min\{x_1, x_2\}, \min\{y_1, y_2\}) + h(\max\{x_1, x_2\}, \max\{y_1, y_2\})$$

and is convex in each coordinate when the other coordinate is fixed.
which leads to the final result by using (A.9).

Now consider \( \frac{\partial C(u,v)}{\partial u} \) is increasing in \( u \) for every \( v \in [0,1] \) so that \( \kappa_t(u) \) is a decreasing function for any \( t \geq 0 \). Using similar arguments, we obtain \( \mathbb{E}[F^{-1}_{M_1}(U)(1-\kappa_t(U))] \leq \mathbb{E}[F^{-1}_{M_2}(U)(1-\kappa_t(U))] \) for each \( t \geq 0 \). Further, using (A.9) and \( \mathbb{E}[M_t] = 1 \), we have

\[
S^{M_1}_X(t) = 1 - \mathbb{E}[F^{-1}_{M_1}(U)(1-\kappa_t(U))],
\]

which in turn implies \( S^{M_1}_X(t) \geq S^{M_2}_X(t) \). As a result, the insurer would cede less risk under the market factor \( M_1 \) than under \( M_2 \). The proof is thus complete.

### A.7 Proof of Corollary 4.1

If \( Z \) is a constant, the market factor is always equal to one for any parameter and hence the result is trivial. Thus, we assume \( \mathbb{P}(Z = \mathbb{E}[Z]) < 1 \) in the remaining of the proof.

We first verify the result for Esscher premium principle. Specifically, we define

\[
h_t(z) \equiv e^{t z} / \mathbb{E}[e^{t Z}], \forall z \geq 0,
\]

which is obviously a strictly increasing function for any \( t \geq 0 \). For any \( 0 \leq t_2 < t_1 \) with \( \mathbb{E}[e^{t_1 Z}] < \infty \), let

\[
A(z) \equiv h_{t_1}(z)/h_{t_2}(z) = e^{(t_1 - t_2)z} \times \frac{\mathbb{E}[e^{t_2 Z}]}{\mathbb{E}[e^{t_1 Z}]},
\]

which is an increasing continuous function with \( A(0) < 1 \) and \( \lim_{z \to \infty} A(z) = \infty \). Thus, there must exist a \( z_0 > 0 \) such that \( h_{t_1}(z) \leq h_{t_2}(z) \) for any \( 0 \leq z \leq z_0 \) and \( h_{t_1}(z) \geq h_{t_2}(z) \) for each \( z > z_0 \). In other words, the function \( h_{t_1}(z) \) up-crosses the function \( h_{t_2}(z) \). Noting that \( \mathbb{E}[h_t(Z)] = 1 \), we can get \( h_{t_1}(Z) \leq_{ct} h_{t_2}(Z) \) by using Lemma 3 in Ohlin (1969). If \( X \uparrow_{SI} Z \), then \( B_x(z) \equiv \mathbb{P}(X > x | Z = z) \) is increasing over the support of \( Z \) for any \( x \geq 0 \). Using the same arguments as that in the proof of Proposition 4.4, we have

\[
\mathbb{E}[h_{t_1}(Z) I_{\{X > x\}}] = \mathbb{E}[h_{t_1}(Z) B_x(Z)] \geq \mathbb{E}[h_{t_2}(Z) B_x(Z)] = \mathbb{E}[h_{t_2}(Z) I_{\{X > x\}}],
\]

which together with Theorems 3.1 and 3.2 leads to that the insurer would cede less risk for a larger \( t \). Otherwise, if \( X \uparrow_{SI} -Z \), then \( B_x(z) \) is decreasing over the support of \( Z \). Similarly, we have

\[
\mathbb{E}[h_{t_1}(Z) I_{\{X > x\}}] = 1 - \mathbb{E}[h_{t_1}(Z)(1 - B_x(Z))] \leq 1 - \mathbb{E}[h_{t_2}(Z)(1 - B_x(Z))] = \mathbb{E}[h_{t_2}(Z) I_{\{X > x\}}],
\]

which implies that optimal ceded loss functions are larger for a higher \( t \).

The proofs for size-biased, Aumann-Shapley and proportional hazard premium principles are similar and hence are omitted. We now proceed to consider the excess-of-loss premium principle. It is easy to see that the function \( I_{\{z \geq t_1\}}/\mathbb{P}(Z \geq t_1) \) up-crosses the function \( I_{\{z \geq t_2\}}/\mathbb{P}(Z \geq t_2) \) for any \( 0 \leq t_2 < t_1 \) with \( \mathbb{P}(Z \geq t_1) > 0 \). Thus, it follows from Lemma 3 in Ohlin (1969) that

\[
I_{\{z \geq t_2\}}/\mathbb{P}(Z \geq t_2) \leq_{ct} I_{\{z \geq t_1\}}/\mathbb{P}(Z \geq t_1).
\]

Noting that \( I_{\{z \geq t\}}/\mathbb{P}(Z \geq t) \) is increasing in \( z \) for each \( t \geq 0 \), we obtain the result following the arguments similar to the Esscher premium principle. Furthermore, from the definition of VaR,
we can see that \( \text{Var}_p(Z) \) is decreasing in \( p \). Hence the analysis for the excess-of-loss premium principle can similarly be applied to CTE and modified tail covariance premium principles so that their proofs are not presented.

Finally, we show the result for Kamps premium principle. In this case it is only necessary to prove \( k_1(Z) \leq_{cx} k_2(Z) \) for any \( 0 \leq t_2 < t_1 \), where

\[
k_1(z) \triangleq (1 - e^{-tz})/E[1 - e^{-tZ}], \forall z \geq 0.
\]

Obviously, \( k_1(z) \) is an increasing continuous function for any \( t \geq 0 \), and

\[
\gamma(t) \triangleq k_1(z)/k_2(z) = \frac{\int_0^z e^{-ty} dy}{\int_0^z e^{-ty} dy} \times \frac{E \left[ \int_0^z e^{-ty} dy \right]}{E \left[ \int_0^z e^{-ty} dy \right]}.
\]

Thus, we have

\[
\gamma(0) = \frac{E \left[ \int_0^z e^{-ty} dy \right]}{E \left[ \int_0^z e^{-ty} dy \right]} > 1,
\]

and hence \( k_1(z) \) up-crosses \( k_2(z) \). Note that \( E[k_1(Z)] = 1 \), it follows from Lemma 3 in Ohlin (1969) that \( k_1(Z) \leq_{cx} k_2(Z) \). The proof is now complete.

A.8 Proof of Proposition 5.2

If \( M \) and \( X \) have a copula (5.1), it follows from (3.3) and (5.3) that

\[
\phi(t) = \begin{cases} 
F_X(t)(Q^M_X(t) - \delta), & 0 \leq t < \text{Var}_\alpha(X); \\
S_X(t)(Q^M_X(t) + \delta), & \text{Var}_\alpha(X) \leq t < \text{Var}_\alpha(X). 
\end{cases}
\]

If \( Q^M_X \in [-1, -\delta] \), we can see from the above equation that \( \phi(t) \leq 0 \) for \( 0 \leq t < \text{Var}_\alpha(X) \). Moreover, it follows from (2.9) that \( Q^M_X F_X(t) + \delta \leq 0 \) for \( t \geq \text{Var}_\alpha(X) \) is equivalent to \( t \geq \text{Var}_{\min(\alpha, 1 + \frac{\delta}{Q^M_X})}(X) \). Thus, the result follows by using (3.2). Else if \( Q^M_X \in [-\delta, \delta] \), it is easy to see that \( \phi(t) \) is negative for \( 0 \leq t < \text{Var}_\alpha(X) \) and is non-negative for \( t \geq \text{Var}_\alpha(X) \). Thus, it follows from (3.2) that \( f^*_\alpha(x) = \min\{x, \text{Var}_\alpha(X)\} \). Otherwise, if \( Q^M_X \in (\delta, 1] \), it is trivial that \( \phi(t) \geq 0 \) for each \( t \geq \text{Var}_\alpha(X) \). Moreover, using (2.9) and the assumption of \( \delta > \alpha \), we obtain that \( Q^M_X X(t) - \delta \leq 0 \) for \( 0 \leq t \leq \text{Var}_\alpha(X) \) is equivalent to \( t \in [\text{Var}_\alpha(\frac{\delta}{Q^M_X})(X), \text{Var}_\alpha(X)] \). Thus, \( f^*_\alpha(x) \) is a one-layer reinsurance.

Similarly, (3.5) and (5.3) lead to

\[
\psi(t) = \begin{cases} 
F_X(t)(Q^M_X(t) - \delta), & 0 \leq t < \text{Var}_\alpha(X); \\
S_X(t)(Q^M_X(t) + \delta (\frac{1}{\alpha} - 1)), & \text{Var}_\alpha(X) \leq t < \text{Var}_\alpha(X). 
\end{cases}
\]

If \( Q^M_X \in [-1, 0] \), it is obvious that \( \psi(t) \leq 0 \) for \( t \geq 0 \) such that \( f^*_\alpha(x) = x \). Otherwise, if \( Q^M_X > 0 \), we can see from the above analysis that \( Q^M_X X(t) - \delta \leq 0 \) for \( 0 \leq t < \text{Var}_\alpha(X) \) if and only if \( t \in [\text{Var}_\alpha(\frac{\delta}{Q^M_X})(X), \text{Var}_\alpha(X)] \). Moreover, similarly using (2.9), we obtain \( Q^M_X F_X(t) - \delta (\frac{1}{\alpha} - 1) < 0 \) is equivalent to \( t < \text{Var}_\alpha(\frac{\delta}{Q^M_X}(\frac{1}{\alpha} - 1))(X) \). Noting that \( \delta > \alpha \), we have \( 1 - \frac{\delta}{Q^M_X(\frac{1}{\alpha} - 1)} < \alpha \). Consequently, the result follows from (3.4) for this case and the proof is thus complete.
References


