Wrong-Way Risk Bounds in Counterparty Credit Risk Management *

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Abstract

We study the problem of finding the worst-case joint distribution of a set of risk factors given prescribed multivariate marginals and a nonlinear loss function. We show that when the risk measure is CVaR, and the distributions are discretized, the problem can be conveniently solved using linear programming. The method has applications to any situation where marginals are provided, and bounds need to be determined on total portfolio risk. In this paper we emphasize applications to counterparty credit risk including the assessment of wrong-way risk. A suitable algorithm for counterparty risk measurement of a real portfolio is also presented.

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1 Introduction

Counterparty credit risk management has become an important topic for both regulators and participants in over-the-counter derivatives markets. Even before the global financial crisis, the Counterparty Risk Management Policy Group noted that counterparty risk is “probably the single most important variable in determining whether and with what speed financial disturbances become financial shocks, with potential systemic traits” (CRMPG [2005]). This concern over counterparty credit risk as a source of systemic stress has been reflected in the historical developments in the Basel Capital Accords (BCBS [2006], BCBS [2011], see also Section 3 below).

Counterparty Credit Risk (CCR) is defined as the risk of loss due to default or the change in creditworthiness of a counterparty before the final settlement of the cash flows of a contract. An examination of the problems of measuring and managing this risk reveals a number of key features. Firstly, the risk is bilateral in nature, and current exposure can lie either with the institution or its counterparty. Secondly, evaluation of exposure must be done at the portfolio level, and take into account relevant credit mitigation arrangements, such as netting and the posting of collateral, which may be in place. Thirdly, exposure is stochastic and contingent on current market risk factors, as well as the creditworthiness of the counterparty, and credit mitigation. In addition, the possible dependence between credit risk and exposure, known as wrong-way risk, is an important modelling consideration. (Since in general the risk is bilateral, in the case of pricing contracts subject to counterparty credit risk (i.e. computing the credit valuation adjustment), the creditworthiness of both parties to the contract is relevant. In this paper, we take a unilateral perspective, focusing exclusively on the creditworthiness of the counterparty.) Finally, the required computation is highly intractable. To calculate risk measures for counterparty credit risk, we need a joint distribution of all market risk factors affecting the portfolio of (possibly tens of thousands of) contracts with the counterparty, as well as the creditworthiness of both counterparties, and values of collateral instruments posted. It is difficult to estimate this joint distribution accurately. We are faced with a problem of risk management under uncertainty, where at least part of the probability distribution needed to evaluate the risk measure is unknown.

But usually we have partial information to aid in the calculation of counterparty credit risk. Most financial institutions have in place models for simulating the joint distribution of counterparty exposures, created (for example) for the purpose of enforcing exposure limits. Additionally, internal models for assessing default probabilities, and credit models (both internal and regulatory) for assessing the joint distribution of counterparty defaults are available at our disposal. We can
view this case as one where we are given the (multi-dimensional) marginal distributions of certain risk factors, and are required to evaluate the portfolio risk for a loss variable that depends on their joint distribution. The Basel Accord (BCBS [2006]) has employed a simple adjustment based on the “alpha multiplier” to address this problem. A stress-testing approach, employing different copulas and financially relevant “directions” for dependence between the market and credit factors is presented in Garcia-Cespedes et al. [2010] and Rosen and Saunders [2010]. This method allows for a computationally efficient evaluation of counterparty credit risk, as it leverages pre-computed portfolio exposure simulations. (Generally speaking, the computational cost of the algorithms is dominated by the time required for evaluating portfolio exposures - which involves pricing thousands of derivative contracts under at least a few thousand scenarios at multiple time points - rather than from the simulation of portfolio credit risk models.)

In this paper we investigate the problem of determining bounds on risk by finding the worst-case joint distribution, i.e. the distribution that has the given marginals, and produces the highest risk measure. This approach is motivated by a desire to have conservative measures of risk and to provide a standard of comparison against which other methods are evaluated. While in this paper we focus on the application to counterparty credit risk, as mentioned earlier, the problem formulation is completely general, and can be applied to other situations in which marginals for risk factors are known, but the joint distribution is unknown. We note that we work with Conditional Value-at-Risk (CVaR), rather than Value-at-Risk (VaR), which is the risk measure that currently determines regulatory capital charges for counterparty credit risk in the Basel Accords (BCBS [2006], BCBS [2011]).

The motivation for this choice is twofold. First, it yields a computationally more tractable optimization problem for the worst-case joint distribution, which can be solved using linear programming. Secondly, although not specifically in the context of CCR, the Basel Committee is actively considering the possibility of replacement of VaR with CVaR as the risk measure for determining capital requirements for the trading book (See BCBS [2012] and Basel Committee on Banking Supervision (BCBS) [2014] for more details.).

Model uncertainty, and problems with given marginal distributions or partial information have been studied in many financial contexts. One example is the pricing of exotic options, where no-arbitrage bounds may be derived based on observed prices of liquid instruments. Related studies include Bertsimas and Popescu [2002], Hobson et al. [2005a], Hobson et al. [2005b], Laurence and Wang [2004], Laurence and Wang [2005], and Chen et al. [2008], for exotic options written on multiple assets ($S_1, \ldots, S_T$) observed at the same time $T$. Deriving integrals of piecewise constant functions with respect to copulas lies at the heart of the aforementioned problems; Hofer and Iacò [2014] investigate integration of two-dimensional, piecewise constant functions and their connection to linear as-
ignment problems and illustrate the numerical effectiveness of their method in model-independent pricing of first-to-default swaps.

The approach closest to the one we take in this paper is that of Beiglbock et al. [2011], in which the marginals \((\Psi(S^T_1), \ldots, \Psi(S^T_k))\) are assumed to be given, and an infinite-dimensional linear programming technique is employed to derive price bounds. There is also a large literature on deriving bounds on VaR for sums of random variables with given marginals. See Makarov [1982], Williamson and Downs [1990], Denuit et al. [1999], Firpo and Ridder [2010] and Embrechts et al. [2003] for the theoretical treatment of this problem and Embrechts and Puccetti [2006], Puccetti and Rüschendorf [2012], Puccetti and Rüschendorf [2012] and Hofert et al. [2015] for numerical methods developed for solving this problem.

Haase et al. [2010] propose a model-free method for a bilateral credit valuation adjustment; in particular their proposed approach does not rely on any specific model for the joint evolution of the underlying risk factors. Talay and Zhang [2002] treat model risk as a stochastic differential game between the trader and the market, and prove that the value function is the viscosity solution of the corresponding Isaacs equation. Avellaneda et al. [1995], Denis et al. [2011] and Denis and Martini [2006] consider pricing under model uncertainty in a diffusion context. Recent works on risk measures under model uncertainty include Kervarec [2008] and Bion-Nadal and Kervarec [2012]. Ghamami and Zhang [2014] present a more efficient Monte Carlo simulation approach for calculating expected positive exposure (EPE) and effective expected positive exposure (eEPE) and analyze the merits and drawbacks of path dependent simulation and direct jump simulation in computational problems. A more detailed treatment of the fundamental concepts and methodologies of counterparty credit risk management can be found in Gregory [2012], Cesari et al. [2009] and Brigo et al. [2013].

The problems considered in this paper can be characterized by three important aspects; (i) we use an alternative risk measure (CVaR), (ii) we are provided with multivariate (non-overlapping) marginal distributions, and (iii) we have losses that are a non-linear (and non-standard) function of the underlying risk factors. After the completion of this work we became aware of the independent work of Glasserman and Yang [2015], who derive bounds on CVA (expressed as an expectation) using a similar approach. Our results can be viewed as an extension of their approach to the case of deriving bounds on CVaR. One of the primary goals of our paper is to specifically address the numerical challenges which arise from the worst-case joint distribution problem.

The remainder of the paper is structured as follows. Section 2 frames the problem of finding bounds on risk based on the worst-case joint distributions for risk factors with given marginals, and shows how this can be reduced to a linear programming
problem when the risk measure is given by CVaR and the distributions are discrete. Section 3 outlines the application of this general approach to counterparty credit risk in the context of the model underlying the CCR capital charge in the Basel Accord. In section 4 a numerical example using a real portfolio is provided, and section 5 presents conclusions and directions for future research.

2 Bounds on Risk Measures and the Worst-Case Joint Distribution Problem

Let \( Y \) and \( Z \) be two vectors of risk factors. We assume that the multi-dimensional marginal distributions of \( Y \) and \( Z \), denoted by \( F_Y(y) \) and \( F_Z(z) \) respectively, are known, but that the joint distribution of \( (Y,Z) \) is unknown (in the context of counterparty credit risk management discussed in the next section, \( Y \) and \( Z \) will be vectors of the market and systematic credit factors respectively). Portfolio losses are defined to be \( L = L(Y,Z) \), where in general this function can be non-linear. We are interested in determining the joint distribution of \( (Y,Z) \) that maximizes a given risk measure \( \rho \):

\[
\max_{\mathcal{F}(F_Y, F_Z)} \rho(L(Y,Z)) \quad (1)
\]

where \( \mathcal{F}(F_Y, F_Z) \) is the set of all possible joint distributions of \( (Y,Z) \) matching the previously defined marginal distributions \( F_Y \) and \( F_Z \). More explicitly, for any joint distribution \( F_{YZ} \in \mathcal{F}(F_Y, F_Z) \) we have \( \Pi_y\{F_{YZ}\} = F_Y \) and \( \Pi_z\{F_{YZ}\} = F_Z \), where \( \Pi\{\cdot\} \) denote the projections that take the joint distribution to its (multi-variate) marginals. In particular, for any functions \( f(y) \) and \( g(z) \), and \( F_{YZ} \in \mathcal{F}(F_Y, F_Z) \), we have:

\[
\int f(y) \, dF_{YZ}(y,z) = \int f(y)\,dF_Y(y) \quad (2)
\]

\[
\int g(z) \, dF_{YZ}(y,z) = \int g(z)\,dF_Z(z) \quad (3)
\]

While we are mainly interested in its application to risk management, we note that the problem of deriving bounds on instrument prices can subsumed within the above framework by taking the risk measure to be the expectation operator (see Glasserman and Yang [2015] for the case of CVA).

Given a time horizon and confidence level \( \beta \), Value-at-Risk (VaR) is defined as the \( \beta \)-percentile of the loss distribution over the specified time horizon. An alternative risk measure that is also popular is Conditional Value at Risk (CVaR), also known
as tail VaR or Expected Shortfall. If the loss distribution is continuous, CVaR is the expected loss given that losses exceed VaR. More formally, we have the following definition of CVaR.

**Definition 2.1.** For the confidence level $\beta \in (0, 1)$ and loss random variable $L$, the Conditional Value at Risk at level $\beta$ is defined by

$$\text{CVaR}_\beta(L) = \frac{1}{1 - \beta} \int_\beta^1 \text{VaR}_u(L) \, du$$

We will use the following result, from Schied [2008] (using our notation). Here, $L$ is a bounded a random variable, defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$.

**Theorem 2.1.** $\text{CVaR}_\beta(L)$ can be represented as

$$\text{CVaR}_\beta = \sup_{G \in G_\beta} \mathbb{E}_G[L]$$

where $G_\beta$ is the set of all probability measures $G \ll \mathbb{F}$ whose density $dG/d\mathbb{F}$ is $\mathbb{F}$-a.s. bounded by $1/(1 - \beta)$.

$G \ll \mathbb{F}$ means $G$ is absolutely continuous with respect to $\mathbb{F}$, i.e. for any $B \in \mathcal{B}$ with $\mathbb{F}(B) = 0$, we have $G(B) = 0$.

Applying the above result, with $\rho = \text{CVaR}_\beta$, the worst-case joint distribution problem stated in (1) can be conveniently reformulated as:

$$\sup_{F, G} \mathbb{E}_G[L] \quad (4)$$

$$\Pi_y\{F\} = F_Y$$

$$\Pi_z\{F\} = F_Z$$

$$\frac{dG}{d\mathbb{F}} \leq \frac{1}{1 - \beta}$$

Note that the final constraint assumes explicitly that the corresponding density exists.

In many practical cases the marginal distributions will be discrete, either due to a modelling choice, or because they arise from the simulation of separate continuous models for $Y$ and $Z$. In this case, the marginal distributions can be represented by $F_Y(Y = y_m) = p_m, m = 1, \ldots, M$, and $F_Z(Z = z_n) = q_n, n = 1, \ldots, N$. Any joint distribution of $(Y, Z)$ is then specified by the quantities $F_{YZ}(Y = y_m, Z = z_n) = \psi_{mn}$, and the problem of finding an upper bound on
CVaR can be further simplified to:

\[
\max_{\psi, \mu} \quad \frac{1}{1 - \beta} \sum_{n, m} L_{mn} \cdot \mu_{mn} \\
\sum_{n} \psi_{mn} = p_m, \quad m = 1, \ldots, M \\
\sum_{m} \psi_{mn} = q_n, \quad n = 1, \ldots, N \\
\sum_{n, m} \mu_{mn} = 1 - \beta \\
0 \leq \mu_{mn} \leq \psi_{mm}
\] (5)

An alternative derivation of Problem (5) based on the characterization of CVaR due to Rockafellar and Uryasev [2000] is also possible, see Memartoluie [2016]. Evidently, Problem (5) is a linear programming problem, and has the general form of a mass transportation problem with multiple constraints. Note that since the sum of each marginal distribution is equal to one, we do not have to include the additional constraint that the total mass of \(\psi\) equals one.

Excluding the bounds, this linear program has \(2MN\) variables and \(M+N+1+NM\) constraints. Consequently, the above formulation can lead to very large linear programs. An important aspect of implementing any linear program in practice is the construction of the coefficient matrix of the constraints. In general size of this matrix is \((\text{total number of variables}) \times (\text{total number of constraints})\). Table 1 gives the percentage of the number of 1s in the coefficient matrix of the linear program (5) for different values of \(M\) and \(N\), leading to a sparse matrix as \(M\) and \(N\) increase.

In the numerical examples presented in section 4 we employ marginal distributions with numbers of market scenarios and credit scenarios ranging from 1,000 to 5,000, yielding joint distributions with on the order or \(10^7\) market-credit scenarios. Any improvement resulting in a reduction in the size of the LP can have a significant impact on the time required to find a solution. Specialized algorithms for linear programs that take advantage of the structure of the transportation problem (see, e.g. Bazaraa et al. [2011]) may be appropriate when solving problems with a large
number of scenarios.

3 Wrong-way Risk and Counterparty Credit Risk

The internal ratings based approach in the Basel Accord (BCBS [2006]) provides a formula for the charge for counterparty credit risk capital for a given counterparty that is based on four numerical inputs: the probability of default (PD), exposure at default (EAD), loss given default (LGD) and maturity (M).

\[
\text{Capital} = \text{EAD} \times \text{LGD} \times \left[ \Phi \left( \frac{\Phi^{-1}(0.999) \cdot (1 - \rho) \cdot \Phi^{-1}(PD)}{\sqrt{1 - \rho}} \right) - PD \right] \times \text{MA}(M,PD)
\]

Here \( \Phi \) is the cumulative distribution function of a standard normal random variable, and MA is a maturity adjustment (see BCBS [2006]). (In the most recent version of the charge, exposure at default may be reduced by current CVA, and the maturity adjustment may be omitted, if migration is accounted for in the CVA capital charge. See Basel Committee on Banking Supervision (BCBS) [2014] for details.) The probability of default is estimated based on an internal rating system, while the LGD is the estimate of a downturn loss given default for the counterparty based on an internal model. The correlation (\( \rho \)), is essentially determined as a function of the probability of default.

The exposure at default in the above formula is a constant. However, as noted above, counterparty exposures are inherently stochastic in nature, and potentially correlated with counterparty defaults (thus giving rise to wrong-way risk). The Basel accord addresses this issue by setting EAD = \( \alpha \times \text{Effective EPE} \), where Effective EPE is a functional of a given simulation of potential future exposures (see BCBS [2006], De Prisco and Rosen [2005] or Garcia-Cespedes et al. [2010] for detailed discussions). The multiplier \( \alpha \) defaults to a value of 1.4; however it can be reduced through the use of internal models (subject to a floor of 1.2).

Using internal models, a portfolio’s alpha is defined as the ratio of CCR economic capital from a joint simulation of market and credit risk factors (\( \text{EC}^{\text{Total}} \)) and the economic capital when counterparty exposures are deterministic and equal to expected positive exposure (EPE). (EPE is the average of potential future exposure, where averaging is done over time and across all exposure scenarios.)

\[
\alpha = \frac{\text{EC}^{\text{Total}}}{\text{EC}^{\text{EPE}}}
\]

The numerator of \( \alpha \) is economic capital based on a full joint simulation of all market and credit risk factors (i.e. exposures are treated as being stochastic, and they are not treated as independent of the credit factors). The denominator is...
economic capital calculated using the Basel credit model with all counterparty exposures treated as constant and equal to EPE. For infinitely granular portfolios in which PFEs are independent of each other and of default events, we can assume that exposures are deterministic and given by the EPE. Calculating $\alpha$ tells us how far we are from such an ideal case.

3.1 CVaR Bounds and Worst-Case Joint Distribution in the Basel Credit Model

In this section, we demonstrate how the worst-case joint distribution problem can be applied to the Basel portfolio credit risk model for the purpose of calculating the worst-case alpha multiplier.

In order to calculate the total portfolio loss, we have to determine whether each of the counterparties in the portfolio has defaulted or not. To do so, we define the creditworthiness index of each counterparty $k, 1 \leq k \leq K$, using a single factor Gaussian copula as:

$$\text{CWI}_k = \sqrt{\rho_k \cdot Z + \sqrt{1 - \rho_k^2}} \cdot \varepsilon_k$$  \hspace{1cm} (8)

where $Z$ and $\varepsilon_k$ are independent standard normal random variables and $\rho_k$ is the factor loading giving the sensitivity of counterparty $k$ to the systematic factor $Z$. The systematic risk factor (denoted by $Z$) represents macroeconomic or industry level events which influence the performance of each counterparty while idiosyncratic risk factors (denoted by $\varepsilon_k$) reflect the risk that is unique to a counterparty.

We choose the Gaussian copula solely for purposes of illustration. It could be easily replaced by other copulas in the simulation algorithm, as long as systematic and idiosyncratic risk can be identified, and conditional probabilities of default given systematic scenarios computed. While we use a single factor model, one can also easily employ a multi-factor structure to capture more sophisticated dependence structures.

If $\text{PD}_k$ is the default probability of counterparty $k$, then that counterparty will default if:

$$\text{CWI}_k \leq \Phi^{-1}(\text{PD}_k)$$

Assuming that we have $M < \infty$ market scenarios in total, if $y_{km}$ is the (loss given default adjusted) exposure to counterparty $k$ under market scenario $m$, the total loss under each market scenario is:

$$L_m = \sum_{k=1}^{K} y_{km} \cdot 1 \{ \text{CWI}_k \leq \Phi^{-1}(\text{PD}_k) \}$$  \hspace{1cm} (9)
Below we focus on the co-dependence between the market factors $Y$ and the systematic credit factor $Z$. In particular, we assume that the market factors $Y$ and the idiosyncratic credit risk factors $\varepsilon_k$ are independent. This amounts to assuming that there is systematic wrong-way risk, but no idiosyncratic wrong-way risk (see Garcia-Cespedes et al. [2010] for a discussion). Define the systematic losses under market scenario $m$ to be:

$$L_m(Z) = \sum_{k=1}^{K} y_{km} \Phi \left( \frac{\Phi^{-1}(PD_k) - \sqrt{\rho_k} \cdot Z}{\sqrt{1-\rho_k}} \right)$$

(10)

with probability $P(Y = y_m) = p_m$. Next we discretize the systematic credit factor $Z$ using $N$ points and define $L_{mn}$ as:

$$L_{mn}(Z) = \sum_{k=1}^{K} y_{km} \Phi \left( \frac{\Phi^{-1}(PD_k) - \sqrt{\rho_k} \cdot Z_n}{\sqrt{1-\rho_k}} \right)$$

$$P(Z = z_n) = q_n \quad \text{for} \quad 1 \leq n \leq N$$

where $L_{mn}$ represents the losses under market scenario $m$, $1 \leq m \leq M$, and credit scenario $n$, $1 \leq n \leq N$.

For illustration, we employ a naive discretization of the standard normal marginal of the systematic credit factor $Z$:

$$P_Z(Z = z_n) = q_n = \Phi(z_n) - \Phi(z_{n-1}) , \quad j = 1, \ldots, N$$

(11)

where $z_0 = -\infty$ and $z_{N+1} = \infty$. We set $N = 1000$, and take $z_j$ to be equally spaced points in the interval $[-5, 5]$. This enables us to consider the entire portfolio loss distribution under the worst-case joint distribution. There is certainly much scope for improvement of our strategy by choosing a finer discretization of $Z$ in the left tail, or by applying importance sampling techniques.

For a given confidence level $\beta$, the worst-case joint distribution of market and credit factors, $\psi_{mn}, m = 1, \ldots, M, n = 1, \ldots, N$ can be obtained by solving the LP stated in (5). Having found the discretized worst-case joint distribution, we can simulate from the full (not just systematic) credit loss distribution using the following algorithm in order to generate portfolio losses:

1. Simulate a random market scenario $m$ and credit state $N$ from the discrete worst-case joint distribution $\psi_{mn}$.

2. Simulate the creditworthiness index of each counterparty. Supposing that $z_n$ is the credit state for the systematic credit factor from Step 1, simulate $Z$ from the distribution of a standard normal random variable conditioned to
be in \((z_{n-1}, z_n)\). Then generate \(K\) i.i.d. standard normal random variables \(\varepsilon_k\), and determine the creditworthiness indicators for each counterparty using equation (8).

3. Calculate the portfolio loss for the current market/credit scenario: using the above simulated creditworthiness indices and the given default probabilities and asset correlations, calculate either systematic credit losses using (10) or total credit losses using (9).

4 Application to Counterparty Credit Risk

In this section we consider the use of the worst-case joint distribution problem to calculate an upper bound on the alpha multiplier for counterparty credit risk using a real-world portfolio of a large financial institution.

Results calculated using the algorithm described in section 3.1 are compared to those using the ordered scenario copula algorithm correlating the systematic credit factor to total portfolio exposure, as described in Garcia-Cespedes et al. [2010] and Rosen and Saunders [2010]. More specifically, we begin by solving the worst-case CVaR linear program (5) for a given, pre-computed set of exposure scenarios and the discretization of the (systematic) credit factor in the single factor Gaussian copula credit model described above. In the linear program, exposures at default are set to be time-averaged exposures based on a multi-step simulation using a model that assumes mean reversion for the underlying stochastic factors. We then simulate the full model based on the resulting joint distribution, under the assumption of no idiosyncratic wrong-way risk (so that the market factors and the idiosyncratic credit risk factors remain independent of each other).

The market scenarios are derived from a standard Monte-Carlo simulation of portfolio exposures, so that we have:

\[
P_Y(Y = y_m) = p_m = \frac{1}{M}, \quad i = 1, \ldots, M
\]  

(12)

4.1 Portfolio Characteristics

The analysis presented in this section is based on a large portfolio of over-the-counter derivatives including positions in interest rate swaps and credit default swaps with approximately 4,800 counterparties; the counterparties are sensitive to many risk factors, including interest rates and exchange rates. We focus on two cases, the largest 220 and largest 410 counterparties as ranked by exposure
(EPE); these two cases account for more than 95% and 99% of total portfolio exposure respectively.

Figures 1 and 2 present exposure concentration reports, giving the number of effective counterparties among the largest 220 and 410 counterparties respectively. Counterparty exposures (EPEs) are sorted in decreasing order. Let \( w_n \) be the \( n^{th} \) largest exposure; then the Herfindahl index of the \( N \) largest exposures is defined as:

\[
H_N = \frac{\sum_{n=1}^{N} w_n^2}{\left(\sum_{n=1}^{N} w_n\right)^2}
\]

The effective number of counterparties among the \( N \) largest counterparties with respect to total portfolio exposures is \( H_N^{-1} \). The effective number of counterparties for the entire portfolio is shown in Figure 3. As can be seen in these figures restricting our attention to the largest 220 and 410 counterparties is justified as the number of effective counterparties for the entire portfolio is 31 in each case.

Figure 1: Effective number of counterparties for the largest 220 counterparties.

Figure 2: Effective number of counterparties for the largest 410 counterparties.
The exposure simulation uses $M = 1000$ and $M = 2000$ market scenarios, while the systematic credit risk factor is discretized with $N = 1000$, $N = 2000$ and $N = 5000$ using the method described above. For CVaR calculations, we employ the 95% and 99% confidence levels. Importance sampling methods would be more suitable at higher confidence levels, such as the regulatory level of 99.9% prescribed in the Basel Accord. There would be a significant increase in the computational cost of simulating a larger number of exposure scenarios in this case.

The ranges of individual counterparty exposures are plotted in figure 4. The 95th and 5th percentiles of the exposure distribution are given as a percentage of the mean exposure for each counterparty. The volatility of the counterparty exposure tends to increase as the mean exposure of the respective counterparties decreases. In other words, counterparties with higher mean exposure tend to be less volatile compared to counterparties with lower mean exposure. Given the above characteristics, we would expect that wrong-way risk could have an important impact on portfolio risk, and that the contribution of idiosyncratic risk will also be significant. The distribution of the total portfolio time-averaged exposures from the exposure simulation is given in Figure 5. The excess kurtosis and the skewness of the exposure distribution are 117.21 and 9.42 respectively, indicating extreme leptokurtosis and a highly skewed distribution.

4.2 Numerical Results

To assess the severity of the worst-case joint distribution, and to determine the degree of conservativeness in earlier methods, we compare risk measures calculated using the worst-case joint distribution to those computed based on the ordered scenario copula algorithm presented in Garcia-Cespedes et al. [2010] and Rosen
Figure 4: 95%, mean and 5% percentiles of the exposure distributions of individual counterparties, expressed as a percentage of counterparty mean exposure (here counterparties are sorted in order of decreasing mean exposure).

Figure 5: Histogram of total portfolio exposures from the exposure simulation.

and Saunders [2010]. In this method, exposure scenarios are sorted in an economically meaningful way, and then a two-dimensional copula is applied to simulate the joint distribution of exposures (from the discrete distribution defined by the exposure scenarios) and the systematic credit factor. The algorithm is efficient, and preserves the (simulated) joint distribution of the exposures. Here, we apply a Gaussian copula, and sort exposure scenarios by the value of total portfolio exposure. This intuitive sorting method has proved to be conservative in many tests, conducted in Rosen and Saunders [2010]. However, see Glasserman and Yang [2015] for a case in which it is not conservative in estimating bounds for CVA. For each level of correlation in the Gaussian copula, we calculate the ratio of risk (as measured by 95% and 99% CVaR) estimated using the sorting method to risk estimated using the worst-case loss distribution.

We present the results of three discretizations of the worst-case joint distribution.
<table>
<thead>
<tr>
<th>Case</th>
<th>Market Scenarios</th>
<th>Credit Scenarios</th>
<th>α</th>
<th>min(CVaR_{sys}/CVaR_{wcc})</th>
<th>max(CVaR_{sys}/CVaR_{wcc})</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Case I</strong></td>
<td>M = 10^6</td>
<td>N = 1000</td>
<td>0.95</td>
<td>52.2%</td>
<td>95.1%</td>
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<td>51.8%</td>
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<tr>
<td><strong>Case II</strong></td>
<td>M = 4 × 10^6</td>
<td>N = 2000</td>
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<td>50.3%</td>
<td>96.0%</td>
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<tr>
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<td></td>
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<td><strong>Case III</strong></td>
<td>M = 10^7</td>
<td>N = 5000</td>
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</tbody>
</table>

Table 2: Minimum and maximum of ratio of systematic CVaR using the ordered scenario copula algorithm described in Rosen and Saunders [2010] to systematic CVaR using the worst-case joint distribution for the largest 220 counterparties at the 95% and 99% confidence levels.

<table>
<thead>
<tr>
<th>Case</th>
<th>Market Scenarios</th>
<th>Credit Scenarios</th>
<th>α</th>
<th>min(CVaR_{tot}/CVaR_{wcc})</th>
<th>max(CVaR_{tot}/CVaR_{wcc})</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Case I</strong></td>
<td>M = 10^6</td>
<td>N = 1000</td>
<td>0.95</td>
<td>52.2%</td>
<td>96.8%</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.99</td>
<td>51.6%</td>
<td>96.5%</td>
</tr>
<tr>
<td><strong>Case II</strong></td>
<td>M = 4 × 10^6</td>
<td>N = 2000</td>
<td>0.95</td>
<td>49.6%</td>
<td>97.2%</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.99</td>
<td>48.9%</td>
<td>97.4%</td>
</tr>
<tr>
<td><strong>Case III</strong></td>
<td>M = 10^7</td>
<td>N = 5000</td>
<td>0.95</td>
<td>45.8%</td>
<td>98.1%</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.99</td>
<td>44.7%</td>
<td>98.2%</td>
</tr>
</tbody>
</table>

Table 3: Minimum and maximum ratio of CVaR for total losses using the ordered scenario copula algorithm described in Rosen and Saunders [2010] to CVaR for total losses using the worst-case joint distribution for the largest 410 counterparties at the 95% and 99% confidence levels.

Case I employs \(M = 1,000\) market scenarios and \(N = 1,000\) credit scenarios; Case II doubles the number of market and credit scenarios. Lastly in Case III we use \(2,000\) market scenarios and \(5,000\) credit scenarios. Note that Case I yields a discretized worst-case distribution with \(10^6\) market-credit scenarios, while Case II has \(4 \times 10^6\) scenarios and Case III has \(10^7\) scenarios.

The graphs in the left column of figure 6 show the ratio of systematic CVaR using the ordered scenario copula algorithm described in Rosen and Saunders [2010] to systematic CVaR using the worst-case joint distribution method described in section 2 for the largest 220 counterparties at the 95% confidence level. The ratios of the CVaR of the systematic portfolio loss to the CVaR calculated using worst-case joint distribution across various levels of market-credit correlation and discretization scenarios indicate the worst-case joint distribution has a higher
Figure 6: Left column: Ratio of systematic CVaR using the ordered scenario copula algorithm in Rosen and Saunders [2010] to systematic CVaR using the worst-case joint distribution for the largest 220 counterparties at the 95% confidence level. Right column: Ratio of CVaR for total losses using the ordered scenario copula algorithm in Rosen and Saunders [2010] to CVaR for total losses using the worst-case joint distribution for the largest 410 counterparties at the 99% confidence level. See tables 2 and 3 for the description of each case.
CVaR compared to previous simulation methods for the largest 220 counterparties by 4.9%, 3.1% and 3.6% at $\beta = 0.95$ respectively when the systematic risk factor and market risk factor are perfectly correlated. The difference is larger for lower levels of market-credit correlation in the ordered scenario copula algorithm. It is evident that the sorting methods produce relatively conservative numbers (at high levels of market-credit correlation) for this portfolio.

In addition to the results presented in figure 6, table 2 shows the minimum and maximum ratios of systematic CVaR to CVaR calculated from the worst-case joint distribution at the 99% confidence level. Note that the results are consistent with what we observed at the 95% level.

Similar results for calculating CVaR ratios using the total portfolio loss for the largest 410 counterparties, which constitute more than 99.6% of total portfolio exposure are shown in the right column of figure 6. The graphs are based on a higher confidence level, $\beta = 99\%$, compared to the graphs on left hand side in figure 6. Table 3 shows comparable results to those presented in table 2 when we use total portfolio loss instead of systematic loss.

5 Conclusion and Future Work

In this paper, we studied the problem of finding the bounds on risk measures based on the worst-case joint distribution of a set of risk factors given prescribed multivariate marginals and a nonlinear loss function. We showed that when the risk measure is CVaR, and the distributions are discretized, the problem can be solved conveniently using linear programming. The method has applications to any situation where marginals are provided, and bounds need to be determined on total portfolio risk. This arises in many financial contexts, including pricing and risk management of exotic options, analysis of structured finance instruments, and aggregation of portfolio risk across risk types. Applications to counterparty credit risk measurement were emphasized in this paper. An application of the algorithm to find CVaR bounds for counterparty credit risk losses on a real portfolio was also presented and discussed.

The method presented in this paper will be of interest to risk managers, who can employ it to stress test their assumptions regarding dependence in risk measurement calculations. It will also be of interest to regulators, who are interested in determining how conservative dependence structures estimated (or assumed) by risk managers in industry actually are.
References


