Liability Driven Dynamic Hedging Strategies for Cash Balance Pension Plans

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Abstract

Cash balance pension plans with crediting rates linked to long bond yields are relatively common in the US, but their liabilities are proving very challenging to hedge. In this paper we consider dynamic hedge strategies using the one-factor and two-factor Hull White models, based on results for the liability valuation from Hardy et al. (2014). The strategies utilise simple hedge portfolios combining one or two zero-coupon bonds, and a money market account. We assess the effectiveness of the strategies by considering how accurately each one would have hedged a 5-year CB liability through the past two decades, using real world returns and crediting rates, and assuming parameters calibrated using the information available at the time. We show that there is considerable impact of model and parameter uncertainty, with additional, less severe impact from discrete hedging error and transactions costs. Despite this, the dynamic hedge strategies do manage to stabilize surplus substantially, even through the turbulence of the past decade.
1 Introduction

Cash Balance (CB) pension plans are the fastest growing pension design in the US, according to Kravitz (2015). The number of CB plans has increased from around 3% to 28% of all Defined Benefit (DB) plans over the past decade. In 2014 there were 9,648 CB plans in the US, with a total of over 12 million participants.

The original motivation for the CB design was to provide sponsors with more predictable, less volatile costs within a DB plan. CB plans were supposed to preserve the lower risk properties of Defined Contribution (DC) plans for pension sponsors, with relatively predictable benefits for participants. CB plans remained rare for nearly a decade and started gaining popularity in the mid-1990s. Paulson (2004) explains that from 1995 to 2001 companies seeking ways to avoid the reversion tax applied to the excess pension assets for DB Plans frequently switched to CB plans. Coronado and Copeland (2003) on the other hand conclude that the conversions from DB to CB were primarily driven by the labor market conditions. In 2015, new regulations came into effect clarifying and extending the permitted range of crediting rate formulae\(^1\).

CB plans are classified and regulated as DB plans but ostensibly share some features of DC plans. Participants have individual accounts showing their up-to-date accrued benefits. However, the important difference between CB and DC plans is that employee accounts in the CB plan are notional. The assets are not allocated to individuals and the amounts paid into the accounts need not be equal to the notional contributions. The total funds invested are generally not equal to the sum of the employee accounts, and are often substantially smaller because of actuarial funding methods used, and because DB plans are not required to be fully funded (see Hardy et al. (2014)). Accounts are notionally accumulated at the specified plan crediting rate. Popular choices for the crediting rates are fixed rates of return, 30-Year Treasury Rates with no floor, which account for 22.4% of plans and the 30-Year Treasury Rate with a Floor, which accounts for 21.7% (based on 553 Kravitz Cash Balance clients, Kravitz (2015)). Another option gaining popularity with more recently established plans is the actual rate of return of the investments (4.4% of plans). This option shifts the investment risk back to employees but with some protection such as the preservation of capital, indicating that a floor rate of 0% is expected to be applied from the notional contribution date to the exit date. In this paper we focus on


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the 30-year Treasury crediting rate, still one of the most common options, and one of the hardest to hedge.

The traditional actuarial valuation approach to CB plans has been studied by Murphy (2001), who concluded that traditional methods may understate the actual liability, and consequently generate a loss on early termination. Brown, Dybvig, and Marshall (2001) adopted a market value approach, using a Vasicek model to forecast future yields and evaluate the costs of CB liabilities. Hardy, Saunders, and Zhu (2014) followed a similar approach, but with more sophisticated arbitrage-free interest rate models. Moreover, Hardy *et al.* (2014) derived explicit valuation formulae, and considered the sensitivity of valuation with respect to changes in the economic conditions, the interest rate model and the parameters. In Hardy *et al.* (2014), crediting rates are assumed to be paid continuously. Part of this work extends the results to the discrete case. In practice, crediting rates are usually applied annually or semi-annually.

Using a market consistent approach, as in Hardy *et al.* (2014) or Brown, Dybvig, and Marshall (2001), accrued CB benefits are viewed as financial contracts. Under a complete and arbitrage-free market, the price of transferring the pension liability to a third party would be the same as the market liability value if retained. Although this technique involves subjective selection of an interest rate model and parameters, Hardy *et al.* (2014) show that their results are relatively insensitive to the choice of the interest rate model and the parameters, even using longer time horizons and long-term crediting rates. Market consistent valuation techniques have been studied for other pension and annuity liabilities. Some examples can be found in Marshall (2011) and Chen and Hardy (2009). Despite the fact that the US public and private sector pension plans have not yet adopted market consistent valuation, much of the literature has discussed the benefit of using market values to assess objectively the funding status of pension plans. Novy-Marx and Rauh (2009) and Biggs (2010) recalculate the funding ratios of US public pensions and conclude that traditional actuarial techniques significantly overstate the funding ratio (by understating liabilities, and potentially overstating asset values). Moreover, Biggs and Smetters (2013) outlined common misconceptions about using a long-term market rate as the discount rate for pension liabilities. In this paper, we reinforce the advantage of market consistent valuation from a risk management perspective, as it provides a natural approach to hedging the accrued benefit.

Hedging the liability is as important as pricing, and is more sensitive to the model selected.
The CB crediting rate guarantee is a form of interest rate option. In the broad area of interest rate derivatives, a frequently asked question is how many stochastic drivers are required to construct an effective hedging portfolio. Fan et al. (2001) have studied up to four factor models in the swaption market and conclude that low-dimensional models are capable of accurately pricing swaptions, but are insufficient for hedging purposes. Gupta and Subrahmanyam (2005) and Driessen et al. (2000) draw a similar conclusion for the cap/floor market. Their work is closer to ours, as they used delta hedging strategies, whereas Fan et al. (2001) focus on bucket hedging. In the case of CB plans, Hardy et al. (2014) show that the difference in pricing using one and two factor Hull-White (HW) models is insignificant. However, the hedging of the CB liability is still unexplored. Brown et al. (2001) studied the duration of the CB plans, which leads to a model-independent duration hedging strategy. On the other hand, Harvey (2012) discussed the difficulties in calculating the duration of a CB plan, and outlined several practical investment strategies including credit default swaps, Treasury futures and swaptions, but without supporting theoretical or empirical analysis. Motivated by the results from the swaption and cap/floor market, and to match our pricing paradigm, we consider delta hedging in one-factor and two-factor Hull-White models.

The remainder of the paper is structured as follows. Section 2 introduces the assumptions, notation and models we adopt in this paper, and also the valuation formulae. Section 3 presents the construction of hedging portfolios. In Section 4 we analyze the effectiveness of dynamic hedging strategies under both simulations and real data. Section 5 concludes.

2 Model and Assumptions

In this section we outline the formulae and assumptions used for valuating the cash balance pension liability. The results are from Hardy et al. (2014) and Zhu (2015).

2.1 CB Benefit

This paper focuses solely on the hedging of interest rate risk. We ignore mortality and other demographic considerations. Treating the individual CB account balance at retirement as a contingent payment, the market value of the liability should be the risk neutral expectation of the discounted account value at termination. Our valuation approach is
accrual based, and so are all the hedging strategies we construct. Under the accrual principle, the accrued liability is based only on past contributions. Future contributions into the plan can be each regarded as a separate derivative which will be evaluated and hedged in the same way as the existing account value at the time of payment.

We denote by $F_t$ the notional account value at time $t$, and let $i^c(t)$ and $r^c(t)$ be the discrete and continuous crediting rates at time $t$, respectively. The frequency of interest crediting varies between plans, but the most common choice is annual crediting. We present the calculations under both continuous crediting and discrete, annual crediting, which, in most cases, display similar numerical results.

The starting date of the plan is $t = 0$, and we value the liability at arbitrary dates between 0 and $T$. The lump-sum payment at the retirement date is

$$F_T = F_0 \prod_{t=0}^{T-1} (1 + i^c(t/n))$$

for discrete crediting, where $n$ is the crediting frequency per year, and

$$F_T = F_0 e^{\int_0^T r^c(t) dt}$$

for continuous crediting.

Kravitz (2015) and Hill et al. (2010) report that a large proportion of CB plans are using the 30-year Treasury rate as the crediting rate; the second most popular choice is a fixed rate. Since the liability using a fixed rate can be perfectly replicated using zero coupon bonds, this paper will only consider Treasury based crediting rates, which we approximate with spot rates, as in Hardy et al. (2014).

2.2 The Yield Curve

Let $r(t)$ denote the continuously compounded short rate of interest at time $t$, $p(t, T)$ denote the price of a zero coupon bond at time $t$ with face value of $1$, maturing at $T$, and $r_k(t)$ denote the $k$-year spot rate at time $t$. The relations between spot rates, bond prices and short rates are as follows:

$$r(t) = \lim_{k \to 0} r_k(t)$$
\[ p(t, T) = e^{-(T-t)r_{T-t}(t)} = E_t^Q \left[ e^{-\int_t^T r(s)ds} \right] \]

where Q denotes that the expectation is over the risk neutral measure, and \( E_t \) indicates that the expectation uses all relevant information up to time \( t \).

**The Valuation Formula**

Using option pricing theory, the value at time \( t \) of the payoff \( F_T \) due at time \( T \geq t \) is

\[ V_t = E_t^Q \left[ F_T e^{-\int_t^T r(s)ds} \right] \]

where both \( F_s \) and \( r(s) \) can be random processes, for \( t < s \leq T \). In the CB liability context, the valuation formula becomes

\[ V_t = F_t E_t^Q \left[ e^{\int_t^T r^c(s)-r(s)ds} \right] \]

for continuous crediting, with a similar expression for discrete crediting.

In this paper we use joint models for \( r^c(t) \) and \( r(t) \) to value \( V_t \), as they are naturally dependent. For the rest of the paper, we refer to \( V(t, T) \) as the ‘valuation factor’. This is the market value at time \( t \) per $1 in the participant’s account balance at time \( t \), where the benefit matures at time \( T \geq t \).

### 2.3 Short Rate Models

We use both the One-Factor and Two-Factor versions of the Hull-White (HW) model. Both of these maintain analytical tractability and allow a perfect match between the model and market starting yield curve. One disadvantage of these models is that the short rate has a Gaussian distribution, with the possibility of being negative, but Hardy *et al.* (2014) showed that this has a rather insignificant impact on the valuation factor. For the one-factor HW model, the instantaneous short rate under the risk-neutral measure has the following SDE:

\[ dr(t) = (\theta(t) - ar(t)) + \sigma dW(t), \quad r(0) = r_0 \]

where \( a > 0, \sigma > 0 \) are constant, and \( \theta(t) \) is a deterministic function of \( t \) chosen to match the market term structure at the starting date, \( W(t) \) is a standard Brownian motion under the \( Q \) measure, and \( r_0 \) is the observed short rate at time 0.
There are two common parameterizations for the two-factor HW model. The one we adopt here is commonly referred to as G2++ (Brigo and Mercurio (2001)). The dynamics of the instantaneous short rate under $Q$ are

$$r(t) = x(t) + y(t) + \varphi(t), \quad r(0) = r_0$$

$$dx(t) = -a_1x(t)dt + \sigma_1dW_1(t), \quad x(0) = 0$$

$$dy(t) = -a_2y(t)dt + \sigma_2dW_2(t), \quad y(0) = 0$$

$$dW_1(t)dW_2(t) = \rho dt$$

where $r_0$ is the initial observed short rate, $a_1, a_2, \sigma_1, \sigma_2$ are all positive constants, $\varphi(t)$ is the deterministic function of $t$ to match the initial term structure, and $(W_1(t), W_2(t))$ is a two-dimensional Brownian motion under the $Q$ measure, with correlation $\rho, -1 \leq \rho \leq 1$.

As mentioned above, the most common crediting rate is the par yield on the 30-year Treasury bond, with no margin. However, there is no explicit solution to the valuation factor equation, and numerical methods are required for the valuation. Hardy et al. (2014), and Zhu (2015) used the $k$-year spot rate as the crediting rate, leading to analytical solutions for the one and two-factor HW models, under both discrete and continuous crediting. This approximation is valuable as it offers some insight into the valuation and hedging results, and in most cases is close to the value using par yields. The valuation formulae for both continuous and discrete crediting are provided in the Appendix; interested readers can refer to Hardy et al. (2014) and Zhu (2015) for more detailed derivations.

3 Constructing the Dynamic Hedging Portfolio

In recent years, driven by the new regulatory landscape, pension sponsors have increased their focus on risk management strategies, using traditional immunization techniques, perhaps with some derivative overlay. The Cash Balance Pension liability differs from the traditional DB pension liability in that the retirement benefit is dependent on the path of crediting rates, and in most cases (other than fixed rate or ‘own fund return’ cases) no marketable asset exists with identical returns to use as a simple static hedge. The CB benefit is a form of interest rate derivative, and so we use techniques from mathematical finance to construct a dynamic hedging strategy.

In this article, we consider delta and delta-gamma hedging strategies for a CB plan with crediting rate equal to the 30-year spot rate on treasuries. We consider both the one-factor
and two-factor HW models. The hedging instruments we use are a money market account, with return equal to the short rate, and treasury STRIPs of varying length. In practice, hedging with treasury STRIPs may be impractical due to the size of the STRIPs market. In December 2014, the outstanding principal was only about 213 billion. However, the dynamic hedging techniques may be applied with other fixed income securities, such as treasury bonds. We re-calibrate the parameters using Swaptions data from the date of valuation, and re-balance the portfolio. Although re-calibration of parameters at portfolio re-balancing dates is inconsistent with our valuation paradigm which implicitly assumes constant parameters, it is more realistic. Also, since the drift term of the Hull-White model is implied by the observed term structure, it is natural to update the parameters to reflect yield curve changes.

3.1 Greeks

The delta and gamma for a particular security are defined as the first and second order derivatives respect to the underlying risky process. Here we provide the delta and gamma for the CB liability (subscript V) and for the zero coupon bond price (subscript B) under the one-factor HW model:

\[
\Delta_B(t) = \frac{\partial P(t, T)}{\partial r(t)} = -B(a, T - t)P(t, T)
\]

\[
\Delta_V(t) = \frac{\partial V(t, T)}{\partial r(t)} = -\gamma(a, k)B(a, T - t)V(t, T)
\]

\[
\Gamma_B(t) = \frac{\partial^2 P(t, T)}{\partial r(t)^2} = B(a, T - t)^2P(t, T)
\]

\[
\Gamma_V(t) = \frac{\partial^2 P(t, T)}{\partial r(t)^2} = \gamma(a, k)^2B(a, T - t)^2V(t, T)
\]

\[
B(a, T) = \frac{1 - e^{-aT}}{a}
\]

\[
\gamma(a, k) = 1 - \frac{B(a, k)}{k}
\]

Similarly under the two-factor HW model, the deltas for the CB liability and for the zero coupon bond are the first partial derivatives respect to each stochastic driver (x and y).

\[
\Delta^x_B(t) = -B(a_1, T - t)P(t, T)
\]

\[
\Delta^y_B(t) = -B(a_2, T - t)P(t, T)
\]
\[ \Delta_x V(t) = -\gamma(a_1, k) B(a_1, T - t) V(t, T) \]
\[ \Delta_y V(t) = -\gamma(a_2, k) B(a_2, T - t) V(t, T) \]

where \( B(a, T) \) and \( \gamma(a, k) \) are defined the same as in the one-factor model. The Greeks are defined above for the continuously crediting CB plans, but the same procedure applies for discrete crediting.

3.2 Position of hedging instruments

The hedging instruments we use for delta hedging under the one-factor HW model are the money market account, with return equal to the short rate, and a zero-coupon bond with maturity \( T_1 \). For delta-gamma hedging under the one-factor HW model and delta hedging under the two-factor HW model, we add another zero-coupon bond with maturity \( T_2 > T_1 \). The position in each instrument can be obtained by solving linear equation(s).

For example, for delta hedging under the one-factor HW model, we solve for the position in the \( T_1 \)-year zero coupon bond (\( \Lambda_{1}^{HW}(t) \)) as:

\[ \Delta V(t) = \Delta B(t, T_1) \Lambda_{1}^{HW}(t) \]

For delta hedging with the one-factor HW model, the position in the \( T_1 \)-year zero coupon bond is

\[ \Lambda_{1}^{HW} = \frac{\gamma(a, k) B(a, T - t) V(t, T)}{B(a, T_1 - t) B(a, T_1 - t) - B(a, T_2 - t) P(t, T_1)} \]

For delta-gamma hedging with the one-factor HW model, the position in the \( T_1 \)-year zero coupon bond (\( \Lambda_{1}^{HW} \)) and the position in the \( T_2 \)-year zero coupon bond (\( \Lambda_{2}^{HW} \))

\[ \Lambda_{1}^{HW} = \frac{\gamma(a, k) B(a, T - t)(\gamma(a, k) B(a, T - t) - B(a, T_2 - t)) V(t, T)}{B(a, T_1 - t)(B(a, T_1 - t) - B(a, T_2 - t)) P(t, T_1)} \]
\[ \Lambda_{2}^{HW} = \frac{\gamma(a, k) B(a, T - t)(\gamma(a, k) B(a, T - t) - B(a, T_1 - t)) V(t, T)}{B(a, T_2 - t)(B(a, T_2 - t) - B(a, T_1 - t)) P(t, T_2)} \]

For delta- hedging with the two-factor HW model, the position in the \( T_1 \)-year zero coupon bond (\( \Lambda_{1}^{G2} \)) and the position in the \( T_2 \)-year zero coupon bond (\( \Lambda_{2}^{G2} \)) of the delta hedging strategy under the two-factor HW model are

\[ \Lambda_{1}^{G2} = \frac{(\gamma(a_1, k) B(a_1, T - t) B(a_2, T_2 - t) - \gamma(a_2, k) B(a_2, T - t) B(a_1, T_2 - t)) V(t, T)}{(B(a_1, T_1 - t) B(a_2, T_2 - t) - B(a_2, T_1 - t) B(a_1, T_2 - t)) P(t, T_1)} \]
\[ \Lambda^G_2 = \frac{(\gamma(a_1, k)B(a_1, T - t)B(a_2, T_1 - t) - \gamma(a_2, k)B(a_2, T - t)B(a_1, T_1 - t))V(t, T)}{(B(a_1, T_2 - t)B(a_2, T_1 - t) - B(a_1, T_1 - t)B(a_2, T_2 - t))P(t, T)} \]

In each case, the amount invested in the bank account \( S(t) \) equals the difference between the liability value and the total value of the position in zero coupon bonds.

For simplicity, we select the first zero coupon bond duration, \( T_1 \) to be the same maturity as the horizon of the liability, and select the longest maturity zero coupon bond available for \( T_2 \) (30-year for the U.S. Treasury Strip market).

### 3.3 Estimating the Parameters

In Section 4, we will assess the effectiveness of delta and delta-gamma hedging strategies by applying them to CB liabilities maturing in the period 2005-2015. In order to do that, we need to estimate parameters for the models.

In both Hardy et al. (2014) and Zhu (2015), the same parameters are used. For the one-factor HW model, we used \( \alpha = 0.02, \sigma = 0.006 \), and for the two-factor model, we used

\[
\begin{align*}
    a_1 &= 0.055, \\
    a_2 &= 0.108, \\
    \sigma_1 &= 0.032, \\
    \sigma_2 &= 0.044, \\
    \rho &= -0.9999
\end{align*}
\]

These parameter values imply a long-term unconditional standard deviation for the short rate that is close to experience over the past 30-40 years; however, they should not be used to examine the effectiveness of the hedging strategies using the historical data, as they were derived using that same data. Intuitively, the parameters used at each valuation date should depend on the past data only. Therefore, in order to accurately analyze the performance of the dynamic hedging strategy through the period 2000-2015, we need to estimate the parameters at each valuation date (or re-balancing date).

The most common choices of market derivatives used for calibration of interest rate models are caps and swaptions. We choose swaptions as they contain more information on the correlations between forward rates (see Brigo and Mercurio (2001)). The choices for the number of swaptions used in the calibration strongly affects the parameter values. For example, Gurrieri et al. (2009) compare the calibration results using three sets of swaptions, for the one-factor HW with time-varying mean reversion and volatility. We use at-the-money swaptions with expiration dates between one year and the maturity of
the CB liability. The tenor of the swaptions are chosen to be 30 years, with quarterly payments\(^2\). We set the parameters by minimizing the sum of the relative squared difference between the theoretical swaption price and the market price, that is, minimizing

\[ \text{obj} = \sum_{i=1}^{n} \left( \frac{\text{HW implied swaption price}_i - \text{market swaption price}_i}{\text{market swaption price}_i} \right)^2 \]

For the two-factor HW model, to distinguish between the parameters of the two stochastic drivers, we set constraints as

\[ 0 < a_2 < a_1 < 1, \quad 0 < \sigma_1, \sigma_2 < 0.5, \quad -1 < \rho < 1 \]

\[ \text{Figure 1: Calibrated Parameters for one-factor HW} \]

Figures 1 and 2 present the calibrated parameters. At first glance, the estimated parameters for the two-factor HW model do not match with the estimated parameters for the one-factor HW model. Indeed, the variance of future short rates under these two sets of parameters differs significantly. The reason is that the swaptions we used for calibration are focused on the long rates over the next five years. In fact, the two sets of parameters

\(^2\)The swaption data are quoted as a Black-volatility matrix and are obtained from Bloomberg.
imply a close match for the variance of spot rates with long maturity in the next one to five years. Also, notice that all calibrated parameters are very unstable, especially for the two-factor HW model (where parameters are often close to the constraints). This phenomenon has been observed elsewhere in the literature (for example, Enev (2011) and Gurrieri et al. (2009)).

Figure 2: Calibrated Parameters for Two-Factor HW (volatility and correlation only)

4 Hedging Performance

Under certain theoretical assumptions, a dynamic hedging strategy with no transaction costs can perfectly replicate the liability. However, this is impossible in practice, and the difference between the liability and the replicating instrument value is the hedge error. The hedge error comes from three sources: the discretization of hedge periods, the mismatch between the model and the real movements of interest rates, and parameter uncertainty. There are also additional costs, such as transaction fees, which can be significant, especially for frequent re-balancing.
Hedging error from discrete rebalancing arises because in practice, only periodical rebalancing is possible, compared with the continuous rebalancing assumed in the theory. In Section 4.2, we quantify the discrete hedge error by considering weekly, monthly and annual re-balancing, where the monthly case serves as the benchmark. We use simulations, based on the assumed model and parameters, so that only the effect of discrete rebalancing is measured.

The model and parameter error are added to the discrete hedge error in section 4.3, where we examine how well the hedge strategy performs when applied to historical paths of crediting rates and yield curves. We compare the performance using the different models and hedge strategies (one-factor HW with delta hedging, one-factor HW with delta-gamma hedging, and two-factor HW with delta hedging), using (i) constant parameters, (ii) calibrated parameters for valuation only at the beginning of the term, and (iii) re-calibrated parameters, assuming adjusting the parameters throughout the hedge period.

In Section 4.4 we quantify the potential transactions costs under the three strategies.

### 4.1 Hedging Error

We denote $H(t)$ as the value of the hedge portfolio at time $t$, given the participant remains in the plan. By definition, on rebalancing dates we have $H(t) = V(t)$. Let $H(t^-)$ be the value of the hedge portfolio at time $t$ immediately before re-balancing, and we let $E(t)$ denote the hedge error at time $t$ before re-balancing.

$$E(t) = V(t) - H(t^-)$$

The funding ratio is often cited as the basis for the financial health of a pension plan. Assuming that at each re-balancing date, we invest/borrow the amount equal to the hedge error at the risk-free rate, the maturity hedge error (MHE) is defined as

$$MHE = \sum_{t=1}^{T} E(t)e^{\int_t^T r(s)ds}$$

The final hedge value, $H(T)$ is equal to the fund value at maturity, $F_T$. The maturity funding ratio (MFL) of the plan is

$$MFR = \frac{F_T - MHE}{F_T}$$
which represents the proportion of the final liability funded by the hedge, after allowing for the accumulated hedge errors.

A transaction cost is incurred whenever the hedge portfolio is rebalanced. Let $\kappa$ be the transaction cost rate, which is set to be 10 basis points. We assume no transaction costs for the bank account. The cumulative transaction cost ($C$) of $T_1$-year zero coupon bond for delta hedging, using the one-factor model is

$$C = \sum_{t=1}^{T} \kappa \left| \Lambda^{HW}(t) - \Lambda^{HW}(t-1) \right| P(t, T_1)$$

where $\Lambda(t)$ is the number of $T_1$-year zero coupon bonds in the portfolio at time $t$. Similar equations hold for delta-gamma hedging, and delta hedging using the Two-Factor model.

### 4.2 Evaluating discrete hedging errors through simulation

The distribution of hedge errors under simulation can be obtained by generating a large number of sample paths for the future crediting rate under the real-world probability measure, then calculating the maturity hedge error for each sample path. Notice that, under simulation, we do not have any other external information to adjust the estimate of the parameter values, thus we assume the same parameter values throughout the liability horizon. Moreover, the parameters used to value the liability are under the risk-neutral measure. To simulate the real-world path, we need to shift the drift according to the market price of risk. For details see Björk (2009). For simplicity, we follow the work of PricewaterhouseCoopers (2014), which assumed a constant market price of risk $\lambda$, estimated through historical zero rates of maturities between fifty and twelve years from 2/01/2001 to 2/01/2013. The dynamics of the short rate under the real world measure are

$$dr(t) = (\theta(t) - ar(t) + \lambda \sigma)dt + \sigma dW^P(t)$$

The estimates of $\lambda$ lie between 0.46 and 0.55. We assume 0.5 would be a reasonable approximation for $\lambda$, and since we use 0.006 for the volatility term, the drift adjustment would be 0.003.

As the real world and risk neutral models are consistent, there is no source of hedging error in this example other than the difference between discrete and continuous rebalancing.
Figures 3 and 4 display the distribution of maturity hedge errors under simulation using the one-factor HW model, assuming the initial fund value is $1000. The starting date of the plan (or the initial term structure) is chosen as 2009-02-27, but we find that the maturity hedge error is not very sensitive to the initial term structure. Immediately, we observe that increasing the frequency of the re-balancing will reduce the overall hedge loss, as we expect. Also, for delta hedging, the effect of switching from annual re-balancing to monthly is much greater than from monthly to weekly (in both the magnitude and the shape of the distribution). Most importantly, even for annual re-balancing, the maturity hedge error for a 5-year plan is less than 0.2%, which is insignificant compared with other sources of error (as we illustrate in Section 4.3). In terms of terminal funding level, a monthly hedge under simulation perfectly replicates the maturity benefit with error less than 0.01%. Cumulative transaction costs exhibit similar patterns as shown in Zhu (2015), where transaction costs increase with re-balancing frequency and the number of hedging instruments, but all less than $0.2 for a $1000 CB plan. Combining the transaction costs with hedging errors, re-balancing monthly and weekly both result in a very close hedge. The outstanding performance of the delta hedging strategy under simulation is to be expected, as the simulation implicitly assumes that the model and parameters are correct. In the following section, we apply a more robust test, based on the hypothetical performance of each hedge strategy through the period 2000-2015.

4.3 Hedging Performance using Historical Data

Although Hardy et al. (2014) demonstrated that the valuation factor is not very sensitive to the model we use, the same cannot be said for the hedging positions.

In this section we consider a Cash Balance plan with annual crediting rate that is settled in advance. Due to the limitations of the data (with swaptions data only for 15 years), we consider a 5-year horizon plan. That is, we assume the plan trustees hedge the accrued liability (i.e. past notional accumulated contributions) assuming all members leave at the end of 5-years\(^3\). To compare with the results from the previous subsection, we assume monthly re-balancing. To illustrate the parameter effect, we display the results using (i) constant parameters, (ii) parameters calibrated at the plan starting date, and then maintained through the five year term, and (iii) parameters re-calibrated at each hedging

\(^3\)This is analogous to a partly projected approach to traditional pension valuation.
Figure 3: Simulation Results for Delta Hedging (One Factor HW), plan started at 2009-02-27, initial account value $F_0 = 1000$
Figure 4: Simulation Results for Gamma Hedging (One Factor HW), plan started at 2009-02-27, initial account value $F_0 = 1000$
Figure 5: Valuation Factor evaluated using different parameters

date. Recalibrating will, in principle, affect both the hedge and the valuation factors. Figure 5 presents the valuation factors under different parameters. This demonstrates that the valuation factors are not heavily dependent on the parameters, since the differences are barely noticeable, with the largest difference at around 0.6% (during the crisis). The robustness of valuation factors to the parameters allows us to focus solely on the hedge errors.

To illustrate the need for hedging strategies, Figure 6 demonstrates the performance when there is no hedge, and assets are invested in one of the strategies:

(a) All invested in stocks (represented by the S&P 500 index returns)

(b) All invested in 30-year zero-coupon bonds

(c) Invested 60%/40% in stocks and bonds.

Despite the fact that there is potentially a large gain at the maturity date for both strategies, the key finding is that the strategies can also lead to huge losses. At first glance, investing in the bond seems an ideal investment option since it has generated some very large gains and only a few losses in the period studied. However, this is due
to the fact that over this time bond rates have experienced an overall downward trend, recently arriving at historical lows. This has generated consistent gains on long bonds that cannot continue once interest rates achieve their minimum level. An upward trend in interest rates would lead to large losses on CB plans using the long bond investment strategy. More importantly, the volatility of the hedge error is huge, and a simple mixed portfolio of stocks and bonds can not achieve a satisfactory level of risk diversification.

We note that the purpose of pension risk management is not to generate windfall gains, but to minimize the losses as much as possible. To illustrate the potential stabilizing impact of hedging, in Figure 7 we include the performance of dynamic hedging strategies using constant parameters. This clearly shows that risks would have been greatly reduced with a dynamic hedging strategy.

In Figures 8 to 10 we show results for each of the different hedging portfolios, with different approaches to parameter recalibration. Figure 8 shows the results for the delta strategy, using the one-factor HW model. Figure 9 shows the delta-gamma strategy hedging result, still with the one-factor HW model, and Figure 10 shows the delta strategy hedging result using the two-factor HW model.
First notice that for the one-factor HW model, using calibrated parameters does generally decrease the volatility of the hedge errors compared with using constant parameters. Recalibration of parameters at each re-balancing date gives the best performance. It is less significant for delta-hedging, but for delta-gamma hedging, re-calibration of parameters effectively controls most terminal hedge errors within $-5\%$ to $5\%$ of the initial notional amount. Things get reversed for the two-factor HW model. Using calibrated parameters introduces extra volatility to the maturity hedge errors. Increasing the number of recalibration dates during the hedge period does not lead to significant improvements. However, it is interesting to note that the two-factor HW model does not perform worse than gamma hedging in the one-factor model, with less volatility hedge errors. As shown in the previous section, estimated parameters fluctuated wildly during the crisis. Clearly, parameter uncertainty is a key factor. All three cases demonstrate that even though the valuation factors are not sensitive to the parameters, the performance of the hedge clearly is.

Figures 11 and 12 more clearly illustrate the relative performance of the different strategies. In Figure 11 we show results with constant parameters, and in Figure 12 we show
Figure 8: Maturity Hedge Loss for 5-year CB plan maturing at 2005-2015, initial account value $F_0 = 1000$, Delta-Hedging with the one-factor HW Model

Figure 9: Maturity Hedge Loss for 5-year CB plan maturing at 2005-2015, initial account value $F_0 = 1000$, Gamma-Hedging with the one-factor HW Model
Figure 10: Maturity Hedge Loss for 5-year CB plan maturing at 2005-2015, initial account value $F_0 = 1000$, Delta-Hedging with the two-factor HW Model

the same comparison with the parameters re-calibrated monthly.

There are three important points we observe from these two graphs. Firstly, the delta-hedging under the one-factor can not achieve the best performance regardless of the choice of the parameters. Secondly, although the parameter uncertain greatly reduces the effectiveness of delta-hedging under the two-factor model, the volatility of its maturity hedge error is still smaller than the delta-hedging under one-factor model. Lastly, the best hedging performance under both cases can keep most of the maturity hedge loss between -5% and 5% of the initial account value.

4.4 Transaction Costs

We follow Marshall (2011) by choosing 10 basis points as the transaction cost for treasury strips, which is slightly larger than the median of the bid-ask spread in the recent STRIPs market. Figures 13, 14 and 15 display the cumulative transaction costs under different hedging strategies with different sets of parameters. As expected, frequently changing the parameters will incur more transactions costs. From the graphs, we observe that the transaction or the position of each instrument is sensitive to changes in the parameters. Using the constant parameters that matches the entire history will result in more stable
Figure 11: Maturity Hedge Loss for 5-year CB plan maturing at 2005-2015, initial account value $F_0 = 1000$, comparing different hedging strategies, using constant parameters.

Figure 12: Maturity Hedge Loss for 5-year CB plan maturing at 2005-2015, initial account value $F_0 = 1000$, comparing different hedging strategies, parameters re-calibrated monthly.
cumulative transaction costs, but necessarily lower. In addition to the conclusion in Zhu (2015) that the hedging portfolio constructed by the Two-Factor model is more sensitive to the changes in interest rates, here we see the opposite for the changes in the parameters. It is also clear that transaction costs are relatively small in general compared to the hedge loss. Using less frequently calibrated parameters would result in a difference in the cumulative transaction cost of less than $2 for most cases.

Figure 13: Cumulative Transaction Cost for 5-year CB plan maturing at 2005-2015, initial account value $F_0 = 1000$, Delta-Hedging with the one-factor HW Model
Figure 14: Cumulative Transaction Cost for 5-year CB plan maturing at 2005-2015, initial account value $F_0 = 1000$, Gamma-Hedging with the one-factor HW Model

Figure 15: Cumulative Transaction Cost for 5-year CB plan maturing at 2005-2015, initial account value $F_0 = 1000$, Delta-Hedging with the Two-Factor HW Model


5 Conclusion

In this article, we have studied dynamic hedging strategies for Cash Balance Pension plans. Dynamic hedging strategies should be considered when the pension sponsor's investment objective is to match the terminal liability. Dynamic hedging is consistent with financial valuation techniques, it is relatively simple to construct, and it offers significant protection against interest rate risk.

We have restricted the crediting rate to the 30-year spot rate. This assumption differs from current practice which uses par rates, but the main driver of uncertainty is still the interest rate risk. Also, for simplicity, we include only zero coupon bonds in our asset portfolio, but the principles can be easily extended to other interest rate derivatives. Our analysis is based on both simulations and historical data.

Clearly, performance under real world data is worse than the simulation result. This is a measure of model and parameter risk. However, dynamic strategies do provide a satisfactory hedge against interest rate risk in our real world analysis. We show that the maturity hedge error could be well controlled within $\pm 10\%$ of the initial funding value (funding ratio greater than 90%). However, in contrast to the analysis of valuation factors, we have shown that the final performance of the hedging strategy greatly depends on the parameters adopted.

The results from dynamic hedging strategies are not perfect. In future work, we will explore other types of hedging strategies, for instance, semi-static hedging. See Marshall (2011) and Liu (2010) for discussions of semi-static hedging for different guaranteed options embedded in variable annuities.

References and Notes


A Continuous Crediting Based on the k-year spot rate: one-factor HW Model

This section and the following three sections provide formulae for valuation factors. Detailed proofs can be found in Hardy et al. (2014) and Zhu (2015). For CB plans with the k-year spot rate as the continuous crediting rate, the valuation factor using the one-factor HW model is

\[
V(t, T) = E^Q_t \left[ e^{\int_0^T r_k(s) ds - \int_t^T r(s) ds + m(k)T} \right] = e^{\int_0^T r_k(s) ds} e^{\int_t^T A(s, s+k) ds} P_\gamma(t, T) e^{m(k)T}
\]

where

\[
P_\gamma(t, T) = \exp \left\{ A_\gamma(t, T) - B_\gamma(a, T - t)r(t) \right\}
\]
\[ A_\gamma(t, T) = \gamma \log \left\{ \frac{P^M(0, T)}{P^M(0, t)} \right\} + \left\{ \frac{\sigma^2 \gamma}{4a} \left( \frac{2}{a} (\gamma - 1)(T - t) + (e^{-2at} - \gamma)B^2(a, T - t) \right) + \frac{1}{a} (2 - 2\gamma)B(a, T - t) \right\} \]

\[ B_\gamma(a, T - t) = \gamma B(a, T - t) \]

\[ B(a, k) = \frac{1 - e^{-ak}}{a} \]

\[ A(t, t + k) = \log \left\{ \frac{p(0, t + k)}{p(0, t)} \right\} + f(0, t)B(a, k) - \frac{\sigma^2}{4a}B(a, k)^2(1 - e^{-2at}) \]

\[ \gamma = 1 - \frac{B(a, k)}{k} \]

\section*{B \ Discrete Crediting Based on the k-year spot rate: one-factor HW Model}

A CB plan with the k-year spot rate as the crediting rate and crediting frequency n times per year, has the valuation factor using the one-factor HW model

\[ V(t, T) = e^{\sum_{i=1}^{T} \frac{r_N(i/n)}{n} e^{A(t, T - i) - B(a, T - i)r(t)}e^{EH(t, T, n)} + \frac{1}{2} VH(t, T, n)} \]  

where

\[ EH(t, T, n) = \sum_{i=tn+1}^{T} \frac{-A \left( \frac{i}{n}, \frac{i}{n} + k \right)}{nk} + \frac{B(a, k)}{nk} \sum_{i=tn+1}^{T} \left[ r(t)e^{-a(\frac{i}{n} - t)} + \varphi \left( \frac{i}{n} \right) - \varphi(t)e^{-a(\frac{i}{n} - t)} \right] \]

\[ -\frac{\sigma^2}{a^2} \left( 1 - \frac{1}{2} e^{-a(T - \frac{i}{n})} - e^{-a(\frac{i}{n} - t)} + \frac{1}{2} e^{-a(T + \frac{i}{n} - 2t)} \right) \]

\[ VH(t, T, n) = \left( \frac{\sigma B(a, k)}{nk} \right)^2 \sum_{j=tn+1}^{T} \left( \frac{e^{-a(j - 1)} - e^{-aT}}{e^{\frac{j}{n}} - 1} \right)^2 \frac{e^{2a\frac{j}{n}} - e^{2a\frac{j-1}{n}}}{2a} \]

\[ \varphi(t) = f^M(0, t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2 \]

\( f^M(0, t) \) is the observed instantaneous forward rate at time t, and \( A(t, T), B(a, k) \) are defined in the previous section.
C Continuous Crediting Based on the k-year spot rate: two-factor HW Model

For a CB plan with the k-year spot rate as the continuous crediting rate, the valuation factor using the two-factor HW model is

\[ V(t, T) = \exp \left\{ \int_0^t r_k(s) ds \right\} \exp \left\{ m(k)T \right\} \exp \left\{ -\int_t^T \frac{A^G(s, s + k)}{k} ds \right\} \]

\[ \exp \left( A^*(t, T) - \gamma(a_1, k)B(a_1, T - t)x(t) - \gamma(a_2, k)B(a_2, T - t)y(t) \right) \]

where

\[ A^G(t, T) = \log \frac{p(0, T)}{p(0, t)} + \frac{1}{2}(\nu(T - t) + \nu(t) - \nu(T)) \]

\[ \nu(k) = \frac{\sigma_1^2}{a_1^2}(k - 2B(a_1, k) + B(2a_1, k)) + \frac{\sigma_2^2}{a_2^2}(k - 2B(a_2, k) + B(2a_2, k)) \]

\[ + \frac{2\rho\sigma_1\sigma_2}{a_2a_2}(k - B(a_1, k) - B(2a_2, k) + B(a_1 + a_2, k)) \]

\[ B(a, k) = \frac{1}{a}(1 - e^{-ak}) \]

\[ A^*(t, T) = \log \frac{p(0, T)}{p(0, t)} + \frac{1}{2}(\nu^*(T - t) + \nu(t) - \nu(T)) \]

\[ \nu^*(k) = \frac{\gamma(a_1, k)^2\sigma_1^2}{a_1^2}(k - 2B(a_1, k) + B(2a_1, k)) + \frac{\gamma^2(a_2, k)\sigma_2^2}{a_2^2}(k - 2B(a_2, k) + B(2a_2, k)) \]

\[ + \frac{2\rho\gamma(a_1, k)\gamma(a_2, k)\sigma_1\sigma_2}{a_1a_2}(k - B(a_1, k) - B(2a_2, k) + B(a_1 + a_2, k)) \]

\[ \gamma(a_j, k) = 1 - \frac{B(a_j, k)}{k}, \quad j = 1, 2 \]

D Discrete Crediting Based on the k-year spot rate: two-factor HW Model

The CB plan with the k-year spot rate as the crediting rate and crediting frequency n times per year, has the valuation factor using the two-factor HW model

\[ V(t, T) = e^{\sum_{i=1}^{n} \frac{t_i + \frac{1}{2} \rho \gamma(a_j, k) \sigma_1 \sigma_2}{k} \gamma(a_j, k)}} e^{A^G(t, T - B(a_1, T - t)x(t) - B(a_2, T - t)y(t)} e^{E(t, T, n) + \frac{1}{2} V(t, T, n)} \]

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where

\[
EG(t, T, n) = \sum_{i=tn+1}^{Tn} \frac{-A^G(\frac{i}{n}, \frac{i}{n} + k)}{nk} + \frac{B(a_1, k)}{nk} \left[ x(t)e^{-a_1(\frac{i}{n} - t)} - M^T_x(t, \frac{i}{n}) \right] \\
+ \frac{B(a_2, k)}{nk} \left[ y(t)e^{-a_2(\frac{i}{n} - t)} - M^T_y(t, \frac{i}{n}) \right]
\]

\[
VG(t, T, n) = \left( \frac{\sigma_1 B(a_1, k)}{nk} \right)^2 \sum_{j=tn+1}^{Tn} \left( \frac{e^{-\frac{a_1}{n}(j-1)} - e^{-a_1 T}}{e^{\frac{a_1}{n}} - 1} \right)^2 \frac{e^{2a_1 \frac{j}{n} - e^{2a_1 \frac{i-1}{n}}}}{2a_1} \\
+ \left( \frac{\sigma_2 B(a_2, k)}{nk} \right)^2 \sum_{j=tn+1}^{Tn} \left( \frac{e^{-\frac{a_2}{n}(j-1)} - e^{-a_2 T}}{e^{\frac{a_2}{n}} - 1} \right)^2 \frac{e^{2a_2 \frac{j}{n} - e^{2a_2 \frac{i-1}{n}}}}{2a_2} \\
2\rho \frac{\sigma_1 B(a_1, k)}{nk} \left( \frac{\sigma_2 B(a_2, k)}{nk} \right) \sum_{j=tn+1}^{Tn} \left( \frac{e^{-\frac{a_1}{n}(j-1)} - e^{-a_1 T}}{e^{\frac{a_1}{n}} - 1} \right) \left( \frac{e^{-\frac{a_2}{n}(j-1)} - e^{-a_2 T}}{e^{\frac{a_2}{n}} - 1} \right) \frac{e^{(a_1 + a_2) \frac{j}{n} - e^{(a_1 + a_2) \frac{i-1}{n}}}}{a_1 + a_2}
\]

\[
M^T_x(s, t) = \left( \frac{\sigma_1^2}{a_1} + \frac{\sigma_1 \sigma_2}{a_2} \right) B(a_1, t - s) - \frac{\sigma_1^2}{a_1} e^{-a_1(T-t)} B(2a_1, t - s) - \frac{\rho \sigma_1 \sigma_2}{a_2} e^{-a_2(T-t)} B(a_1 + a_2, t - s)
\]

\[
M^T_y(s, t) = \left( \frac{\sigma_2^2}{a_2} + \frac{\sigma_1 \sigma_2}{a_1} \right) B(a_2, t - s) - \frac{\sigma_2^2}{a_2} e^{-a_2(T-t)} B(2a_2, t - s) - \frac{\rho \sigma_1 \sigma_2}{a_1} e^{-a_1(T-t)} B(a_1 + a_2, t - s)
\]