Determination of approximate point load weight functions for embedded elliptical cracks

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ABSTRACT

This paper presents the application of weight function method for the calculation of stress intensity factors in embedded elliptical cracks under complex two-dimensional loading conditions. A new general mathematical form of point load weight function is proposed based on the properties of weight functions and the available weight functions for two-dimensional cracks. The existence of this general weight function form has simplified the determination of point load weight functions significantly. For an embedded elliptical crack of any aspect ratio, the unknown parameters in the general form can be determined from one reference stress intensity factor solution. This method was used to derive the weight functions for embedded elliptical cracks in an infinite body and in a semi-infinite body. The derived weight functions are then validated against available stress intensity factor solutions for several linear and non-linear stress distributions. The derived weight functions are particularly useful for the fatigue crack growth analysis of planer embedded cracks subjected to fluctuating non-linear stress fields resulting from surface treatment (shot peening), stress concentration or welding (residual stress).

1. Introduction

A problem frequently encountered in applied fracture and fatigue analysis is the calculation of the stress intensity factors for defective components subjected to a complex stress distribution. Weight function method [1,2] has been widely used in the determination of stress intensity factors for its distinctive advantage of separating the loading and the geometry. Once the weight function is known for a given cracked geometry, the stress intensity factor due to any load system applied to the body can be determined by using the same weight function. Therefore it is especially suited when a large number of stress intensity factor solutions are desired for complex stress distributions. The success of the weight function method depends on the accurate determination of the weight function itself. Because the concept of "weight function" was originally introduced [1,2] for one-dimensional edge cracks or through cracks, most of work in the literature has been concentrated on one-dimensional cracks. The methods of obtaining the weight functions for one-dimensional cracks have been well developed, see [3–5] for example. However, to date, the method of obtaining weight functions for two-dimensional cracks is still not well developed. For two-dimensional cracks, close-form exact weight functions are available only for very limited cases: the circular crack and the half plane crack in an infinite body [6].

Embedded elliptical crack is one of the most used models for two-dimensional embedded cracks in many engineering components (Fig. 1). Fig. 1a shows an embedded elliptical crack in an infinite body (i.e., b, t and $h \gg a$ and c); and Fig. 1b illustrates an embedded elliptical crack in a semi-infinite body, where the embedded crack in close to one free surface represented by the distance d. The notation of the ellipse geometry is shown in Fig. 2. For embedded elliptical crack in an infinite body (Fig. 1a), stress intensity factor solutions are available for limited simple loading conditions. When the uncracked stress distribution in the area to be occupied by the elliptical crack is a simple one such as uniform (tension) or one-dimensional linearly varying (bending), the well-known explicit solutions [7,8] can be used to determine the stress intensity factor at any point, P', along the crack front. When the stress distribution is two-dimensional depending on two variables, $\sigma(x, y)$, which is the case in many engineering applications, more complicated calculations have to be made. Exact stress intensity factor solutions for 2D polynomial stress distributions of the order of three were provided by Shah and Kabayashi [9], and for polynomial of any order n were provided by Vijaykumar and Atluri [10] and Nishioka and Atluri [11], where tedious labour intensive evaluation of elliptic integrals are involved.

For embedded elliptical cracks in a semi-infinite body (Fig. 1b), the effect of free boundary needs to be included and there is no analytical solution available. Using finite element method, the stress intensity factors were obtained by Shiratori et al [12] for polynomial stress distributions of order 3. For complex stress

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Nomenclature

A, B, C	three points along the crack front for embedded crack in semi-infinite body	*r*	radius of polar coordinate
a	half length of minor axis for embedded elliptical crack; crack length	*S*	entire crack face area
		R	radius of polar coordinate for points along crack front
b, t, h	dimensions of the cracked body	*s*	shortest distance between point load and the boundary of crack front
c	half length of major axis for embedded elliptical crack		
d	distance of embedded crack in semi-infinite body to free surface	u_r	crack face displacement for reference stress intensity factor
E	Young's modulus; shape factor for an ellipse	*x, y*	Cartesian coordinates for point *P*
F	normalized stress intensity factor	$w(x,y;P')$	regular function in weight function expression
H	generalized elastic modulus	θ	angle of polar coordinate for crack front point *P′*
K	mode I stress intensity factor	ρ	distance between load point *P* and point along crack front under consideration *P′*
K_r	reference stress intensity factor		
l	small crack front segment variation	ϕ	parametric angle of ellipse for point *P′*
$m(x,y;P')$	point load weight function	v	Poisson's ratio
M_i, M	parameters in point load weight function expression	σ_0	nominal stress
P	point load location on crack face	α	aspect ratio of elliptical crack = *a/c*
P′	point along the crack front under consideration	Γ	boundary of crack front
		φ	angle of polar coordinate for load point *P*

distributions which can be approximated using polynomials of order 3, the stress intensity factor can be then calculated using the superposition method. However, there are stress distributions which cannot be easily represented by polynomials. It is therefore of practical significance to develop weight function solutions for both embedded elliptical cracks in infinite body (Fig. 1a) and in semi-infinite body (Fig. 1b), which will enable the determination of stress intensity factor for any complex stress distributions.

Several attempts have been made to derive the weight functions for embedded elliptical crack by solving the problem analytically, see recent development in [13,14]. However, these solutions are in a series expansion form and cannot be easily used for engineering applications. Numerical methods were developed that can be used to determine the weight functions numerically, for example using finite element method [15,16]. However, the amount of work needed to determine the weight function for two-dimensional cracks with numerical method can be very large, and the weight functions in numerical form are often inconvenient to use.

One of the important features of weight function method is that the weight functions for a wide variety of crack configurations can be represented using the same mathematical form, which has been demonstrated extensively from weight function theories for one-dimensional crack problems [5]. The existence of a general mathematical form simplified the determination of weight functions significantly. Several attempts have been made to find the general weight function form for two-dimensional crack. The most commonly used method is the O-integral method [17,18]. However, it has been realised that the O-integral functional form works well for some geometries but its accuracy is not as good for other geometries, particular for low aspect ratio embedded elliptical cracks [18]. Various analytical methods have also been used by several authors to develop approximate weight functions specifically for embedded elliptical cracks [19–21]. However, no general accepted form has been established for 2D cracks.

The objective of the present paper is the development of approximate weight functions for embedded elliptical cracks based on a newly proposed general form for two-dimensional crack problems. The paper is organized as follows. First, the background of weight function method is reviewed, and a general form for point load weight function is proposed and the method of finding the parameters of the weight function is discussed. Based on the developed method, weight functions for embedded elliptical cracks in an infinite body (Fig. 1a) and in a semi-infinite body (Fig. 1b) for a wide range of aspect ratios of *a/c* (see Fig. 2) is derived and validated against available stress intensity factor solutions for several linear and non-linear stress distributions.

2. Approximate point load weight functions

2.1. Theoretical background

The weight function technique for calculating stress intensity factors is based on the principle of superposition. For one-dimensional cracks, it can be shown [1] that the stress intensity factor for a cracked body (Fig. 3a) subjected to the external loading system, *S*, is the same as the stress intensity factor in a geometrically identical body (Fig. 3c) with the local stress field $\sigma(x)$ applied to the crack faces. The local stress field, $\sigma(x)$, induced in the prospective crack plane by preload *S*, is determined from an uncracked body (Fig. 3b). The stress intensity factor for a cracked body with loading applied to the crack faces can be calculated by integrating the product of the weight function, $m(x,a)$, and the stress distribution, $\sigma(x)$, in the crack plane:

$$K = \int_0^a \sigma(x)m(x,a)\mathrm{d}x \tag{1}$$

The weight function $m(x,a)$ depends only on the geometry of the crack and the cracked body. Once the weight function has been determined, the stress intensity factor for this geometry can be obtained from Eq. (1) for any stress distribution, $\sigma(x)$. Mathematically, the weight function, $m(x,a)$, is the Green's function for the present boundary value problem scaled with respect to the crack dimension *a*. It represents the stress intensity factor at the crack tip for a pair of unit point loads acting on the surface at the location *x*.

For a two-dimensional crack, the stress intensity factors vary along the crack front, as shown in Fig. 4. The counterpart to Eq. (1) for two-dimensional cracks is a double integral over the crack surface

$$K(P') = \int\int_S \sigma(x,y)m(x,y;P')\mathrm{d}S \tag{2}$$
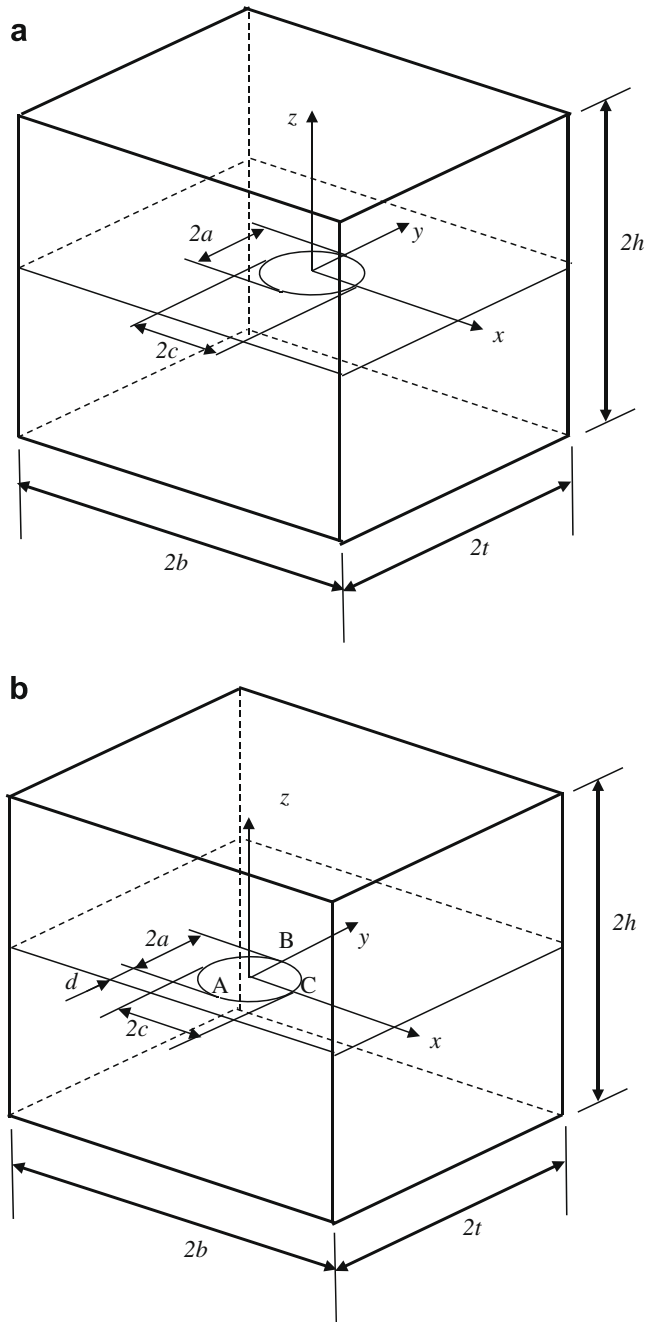
a

b

Fig. 1. Elliptical crack in an infinite solid (a) and in a semi-infinite body (b).

where $m(x,y;P')$ is the point load weight function, it represents stress intensity factor at point P' on the crack front for a pair of unit point loads acting on the crack surface at point (x,y) as shown in Fig. 4, and $\sigma(x,y)$ is a general two-dimensional stress distribution as shown in Fig. 4. If the stress distribution $\sigma(x,y)$ is one-dimensional, for example, only a function of x, then Eq. (2) can be simplified to

$$K(P') = \int_a \sigma(x) \left(\int m(x,y;P') \mathrm{d}y \right) \mathrm{d}x = \int_a \sigma(x) M(x;P') \mathrm{d}x \qquad (3)$$

where $M(x;P')$ represents the stress intensity factor at point P' for unit line load at position x as shown in Fig. 4; and a is the crack depths in the x-direction. In other words, $M(x;P')$ is the line load weight function for two-dimensional cracks.
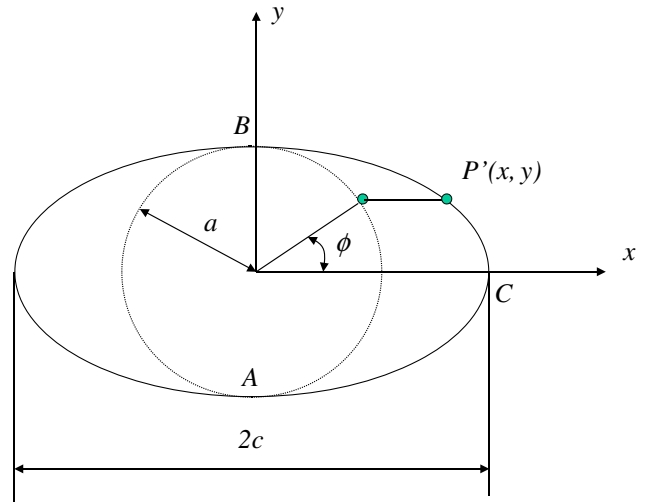

Fig. 2. Notation for embedded elliptical crack.

For any one-dimensional or two-dimensional cracks, if the weight functions $m(x,a)$, $m(x,y;P')$ or $M(x;P')$ are obtained, the stress intensity factors for other loading conditions can be calculated using Eqs. (1)–(3).

For one-dimensional cracks, the determination of the weight function, $m(x,a)$, in Eq. (1) can be simplified by using the relation between stress intensity factor under consideration and a reference stress intensity factor solution and the corresponding crack face displacement, as derived in [2]

$$K \times K_r = H \int_0^a \sigma(x) \frac{\partial u_r(x,a)}{\partial a} \mathrm{d}x \qquad (4)$$

where H is the generalized elastic modulus which equals E for plane stress or $E/(1 - v^2)$ for plane strain, and K and u_r are the stress intensity factor and corresponding crack face displacement for one reference stress distribution. From Eq. (4), the weight function $m(x,a)$ can be obtained as

$$m(x,a) = \frac{H}{K_r} \frac{\partial u_r(x,a)}{\partial a} \qquad (5)$$

Eq. (5) provides an efficient way to determine weight function from a reference stress intensity factor solution and the corresponding displacement fields. An appropriate reference stress intensity factor K_r can often be found either in the literature or by numerical calculation. Although the corresponding analytical expression for the crack opening displacement function $u_r(x,a)$ is more difficult to obtain, because it is seldom published together with stress intensity factor solutions, several authors (Petroski and Achenbach, [22]; Wu and Carlsson [3]; Fett and Munz [4]; Glinka and Shen [5]) have proposed approximate expressions for the displacement, $u_r(x,a)$, or the weight function, $m(x,a)$. Glinka and Shen [5] have found that the mode I weight functions for a variety of 1D crack geometries can be accurately approximated using the following expression

$$m(x,a) = \frac{2}{\sqrt{2\pi(a-x)}} \left[1 + M_1 \left(1 - \frac{x}{a} \right)^{1/2} \right.$$
$$\left. + M_2 \left(1 - \frac{x}{a} \right) + M_3 \left(1 - \frac{x}{a} \right)^{3/2} \right] \qquad (6)$$

As shown in Eq. (6), the weight function has the same singular term and M_1–M_3 are parameters of the non-singular term and they depend on the specific crack geometry. The existence of a general weight function form simplified the determination of weight
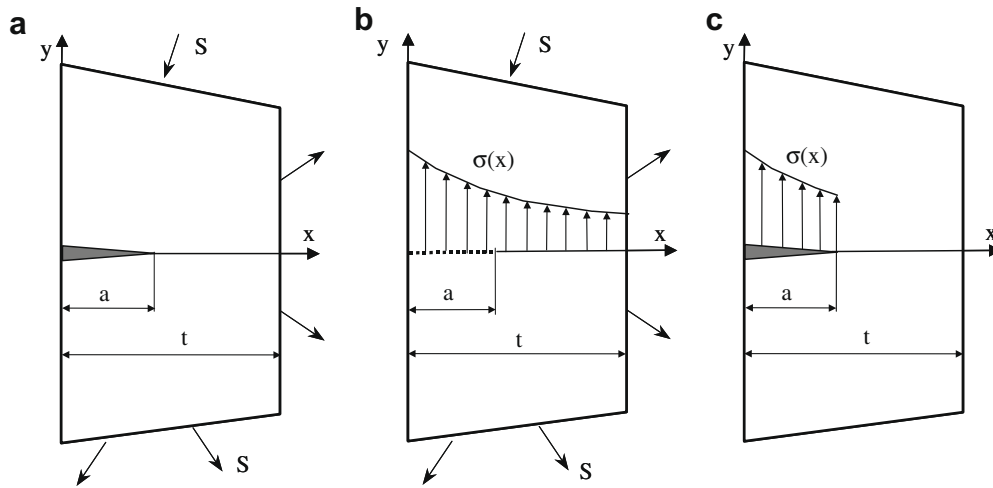
Fig. 3. Weight function for one-dimensional cracks, (a), (b) and (c).

functions; the derivation of weight function for a particular geometrical configuration of cracked body is reduced to the determination of parameters M_1–M_3.

For two-dimensional cracks (see Fig. 5), the counter-part relationship between two stress intensity factor solutions (Eq. (4)) are also derived by Rice [2,6]

$$\int_\Gamma [K \times K_r \delta l] \mathrm{d}\Gamma = H \int \int_\delta \sigma(x,y)\delta u_r(x,y)\mathrm{d}S \tag{7}$$

where Γ represents the crack front, δl is a smooth function of position along Γ making the advance of the crack in a direction locally normal to Γ. $\delta u(x,y)$ is the first order variation of displacement corresponding to δl. If we define the δl is local to point P', and the corresponding area change is $\delta F_{P'} = \delta l \times \mathrm{d}\Gamma$, the weight function $m(x,y,P')$ can be obtained

$$m(x,y,P') = \frac{H}{K_r(P')} \frac{\delta u_r(x,y)}{\delta F_{P'}} \tag{8}$$

That is if the three-dimensional solutions of any reference load system is known so that the first order variation $\delta u(x,y)$ can be determined, at any point along the crack front P', corresponding to variation δF_P, the weight function for point P' can be obtained from Eq. (8).

Several methods were developed to apply Eqs. (7) and (8) to derive weight functions for two-dimensional cracks. However, the complete solution for $\delta u(x,y)$ for arbitrary δF_P is much more diffi-
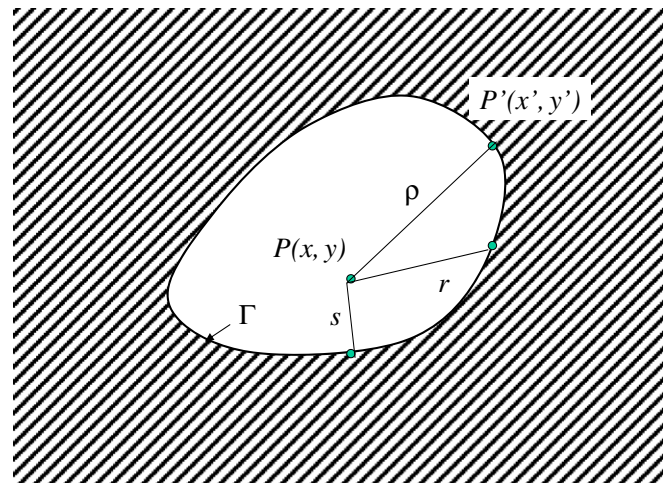

Fig. 5. Weight function notation for general two-dimensional crack.

culty to obtain than $\partial u_r/\partial a$ in the one-dimensional crack problem, several simplifications were used (See Fett [23], for example).

On the other hand, the line load weight function for two-dimensional cracks $M(x;P')$ in Eq. (3) can be determined in a similar way as deriving weight function $m(x,a)$ for one-dimensional cracks. For surface cracks and corner cracks, the general weight function forms of $M(x,P')$ have been obtained for the deepest point $M(x,A')$, surface point $M(x,B')$ and general point $M(x,P')$ [24,25]. Using these general forms, the line load weight functions were derived for a variety of two-dimensional crack geometries [24–28].

In spite of the high efficiency and usefulness of the line load weight function in engineering applications, they cannot be used if the stress field is of a two-dimensional nature, i.e., when the stress field $\sigma(x,y)$ depends on both x and y coordinates. Therefore, a general method for the determination of point load weight function $m(x,y;P')$ is needed.

2.2. Properties of point load weight functions for two-dimensional cracks

By analyzing the properties of weight functions for two-dimensional cracks, Rice [6] pointed out that, for an embedded planar crack in an infinite body (see Fig. 5), there are two key parameters s and ρ in the weight function expression, $m(x,y;P')$. Here s is the
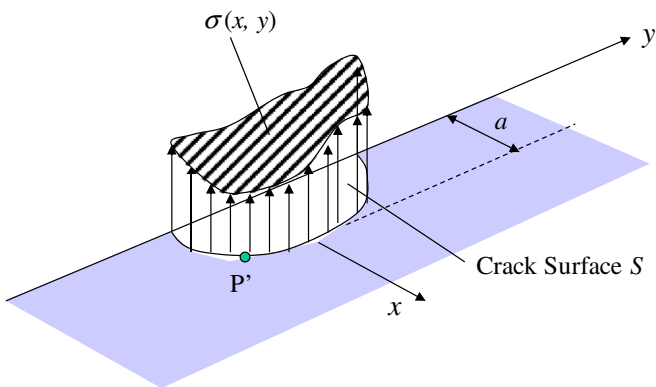

Fig. 4. Two-dimensional crack under two-dimensional stress distribution.
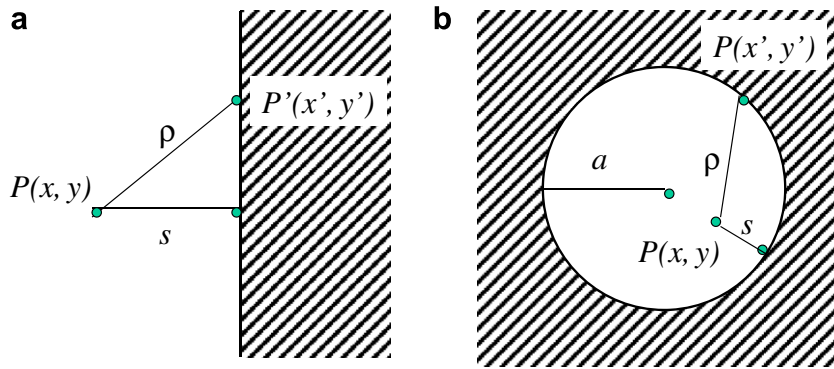
Fig. 6. Weight function for (a) half plane crack and (b) circular crack.

shortest distance between the load point $P(x,y)$ and the boundary of the crack front Γ, and ρ is the distance between the load P and the point P' under consideration (see Fig. 5). These two parameters can be used to describe available analytical weight functions. For the half-plane crack in an infinite body as shown in Fig. 6a, the weight function is

$$m(x,y,P') = \frac{\sqrt{2s}}{\pi^{3/2}\rho^2} \tag{9}$$

For the penny shaped crack as shown in Fig. 6b

$$m(x,y,P') = \frac{\sqrt{2s}}{\pi^{3/2}\rho^2}\sqrt{1-\frac{s}{2a}} \tag{10}$$

where a is the radius of the penny shaped crack.

For an arbitrary planar crack embedded in an infinite body (Fig. 5), the weight function can be represented using the following general expression [6]

$$m(x,y,P') = \frac{\sqrt{2s}}{\pi^{3/2}\rho^2}w(x,y;P') \tag{11}$$

It is apparent from Eq. (11) that the singularity term in the point load weight function is of the order of $\frac{\sqrt{s}}{\rho^2}$, and weight function tends to infinite when ρ approaches zero. When s equals zero and ρ is not zero, the weight function value is zero.

The function $w(x,y;P')$ describes the effect of the embedded crack geometry configuration. For any given crack geometry, if the function $w(x,y;P')$ can be obtained, then the point load weight function can be obtained from Eq. (11).

For the half plane crack

$$w(x,y,P') = 1 \tag{12}$$

and for the penny shape crack

$$w(x,y,P') = \sqrt{1-\frac{s}{2a}} \tag{13}$$

It was also pointed out by Rice [6] that the function $w(x,y;P')$ has a well-defined limit when point $P(x,y)$ approached the crack boundary, i.e., s approaches 0. For both cases of a half plane or penny shaped crack

$$\lim_{s\to 0}[w(x,y,P')] = 1 \tag{14}$$

That is the function $w(x,y;P')$ in Eq. (11) is a regular function (without any singularity), and the singular term has already been represented by the term $\frac{\sqrt{s}}{\rho^2}$.

Note that as shown in Eq. (8), the weight function is closely related to the crack opening displacement fields. It has also been shown by Bueckner [1] that weight functions are in fact singular
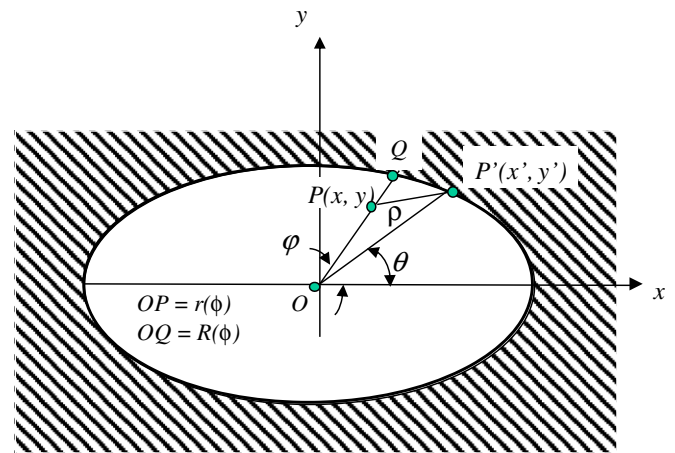

Fig. 7. Weight function for embedded elliptical crack.

displacements of so-called "fundamental fields," which produce an infinite energy in a small volume.

2.3. Proposed general form for point load weight function

Now consider the embedded elliptical crack as shown in Fig. 7. The objective is to find the weight function $m(x,y,P')$. Based on the preceding discussion, it is reasonable to further represent the point load weight function using the following form:

$$m(x,y;P') = \frac{\sqrt{2s}}{\pi^{3/2}\rho^2}\left[1+\sum_{i=1}^{n}M_i(\theta,\alpha)\left(1-\frac{r(\varphi)}{R(\varphi)}\right)^i\right] \tag{15}$$

i.e.,

$$w(x,y;P') = \left[1+\sum_{i=1}^{n}M_i(\theta,\alpha)\left(1-\frac{r(\varphi)}{R(\varphi)}\right)^i\right] \tag{16}$$

That is: the weight function can be represented by the summation of two parts; the first part is the singular term, and the second part accounts for the effect of crack configurations. Here θ represents the location of P'; and r, φ are the polar coordinates of point $P(x,y)$. And $R(\varphi)$ is the corresponding point on the crack front (See Fig. 7). Note that the weight function parameter M_i depends on the aspect ratio of the ellipse, $\alpha = a/c$.

It can be easily shown it satisfies the condition represented by Eq. (14). In addition, for the special case of half plane crack, from Eq. (12), we have

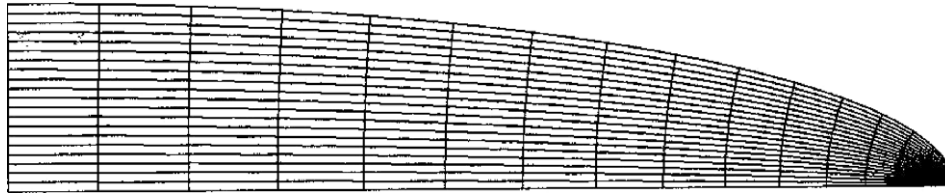$$M_1 = M_2 = M_3 \cdots = 0 \tag{17}$$

Fig. 8. Typical mesh used for numerical integration ($a/c = 0.2$).

for circular crack, the exact weight function Eq. (13) gives

$$w(x, y, P') = \left(1 - \frac{s}{2a}\right)^{1/2} = \left(1 - \frac{1}{2}\left(1 - \frac{r}{R}\right)\right)^{1/2} \quad (18)$$

It can be represented using a following series expansion (i.e., Eq. (15))

$$w(x, y, P') = 1 + M_1\left(1 - \frac{r}{R}\right) + M_2\left(1 - \frac{r}{R}\right)^2 + M_3\left(1 - \frac{r}{R}\right)^3$$
$$+ \cdots \quad (19)$$

Table 1
Weight function parameter $M(\phi, \alpha)$ for $\alpha = 0.1, 0.2, 0.4, 0.6, 0.8$ and 1.

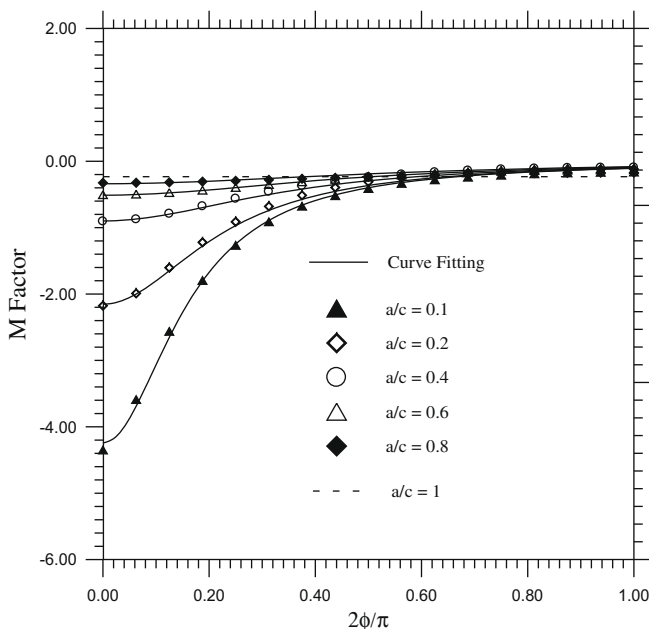| $2\phi/\pi$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$
0	−4.35	−2.1772	−0.9053	−0.5042	−0.3262	−0.2321
0.0625	−3.5898	−1.9917	−0.8735	−0.4958	−0.3241	−0.2321
0.125	−2.5633	−1.6044	−0.7895	−0.4721	−0.3178	−0.2321
0.1875	−1.7968	−1.2214	−0.6784	−0.4371	−0.3079	−0.2321
0.25	−1.2669	−0.9121	−0.5634	−0.3954	−0.2952	−0.2321
0.3125	−0.9127	−0.6811	−0.4587	−0.3518	−0.2806	−0.2321
0.375	−0.6774	−0.5144	−0.3701	−0.3096	−0.2651	−0.2321
0.4375	−0.5192	−0.3959	−0.2986	−0.2711	−0.2494	−0.2321
0.5	−0.4107	−0.3117	−0.2423	−0.2374	−0.2343	−0.2321
0.5625	−0.3346	−0.2515	−0.199	−0.2089	−0.2202	−0.2321
0.625	−0.2801	−0.2083	−0.166	−0.1853	−0.2076	−0.2321
0.6875	−0.2404	−0.177	−0.1412	−0.1664	−0.1966	−0.2321
0.75	−0.2114	−0.1544	−0.123	−0.1516	−0.1875	−0.2321
0.8125	−0.1904	−0.1384	−0.11	−0.1407	−0.1803	−0.2321
0.875	−0.1757	−0.1277	−0.1013	−0.1331	−0.1751	−0.2321
0.9375	−0.1668	−0.1214	−0.0963	−0.1287	−0.172	−0.2321
1	−0.1636	−0.1192	−0.0946	−0.1271	−0.1709	−0.2321

Fig. 9. Results of $M(\phi, \alpha)$ for $\alpha = 0.1, 0.2, 0.4, 0.6, 0.8$ and 1.

In fact, further studies indicate, which will be discussed in the following Section 2.4, that only two-terms of Eq. (19) are required to provide good approximation of Eq. (18) with $M_1(\theta, \alpha) = -0.23213$ and all other $M_i = 0$ ($i = 2, 3$ and . . .).

Another available weight function for cracks in an infinite body is the tunnel crack geometry. Through an approximate analysis, the weight function was presented in [18] for tunnel crack in an infinite body, which can be further accurately approximated using Eq. (19).

In summary, our analyses have indicated that the general form of Eq. (15), with one term, i.e., $n = 1$, can approximate the point load weight functions with good accuracy for a wide range of embedded crack configurations. That is

$$m(x, y; P') = \frac{\sqrt{2s}}{\pi^{3/2}\rho^2}\left[1 + M(\theta, \alpha)\left(1 - \frac{r(\varphi)}{R(\varphi)}\right)\right] \quad (20a)$$

$$\text{or} \quad w(x, y; P') = \left[1 + M(\theta, \alpha)\left(1 - \frac{r(\varphi)}{R(\varphi)}\right)\right] \quad (20b)$$

In addition, for the case of embedded elliptical crack in a semi-infinite body (Fig. 1b), the weight functions will also depend on the distance of the crack to the free surface, d. This effect can be accounted for by the M factor in Eq. (20). It is also a function of the distance. It is therefore reasonable to further represent the corresponding weight function as

$$m(x, y; P') = \frac{\sqrt{2s}}{\pi^{3/2}\rho^2}\left[1 + M\left(\theta, \alpha, \frac{d}{a}\right)\left(1 - \frac{r(\varphi)}{R(\varphi)}\right)\right] \quad (21a)$$

$$\text{or} \quad w(x, y; P') = \left[1 + M\left(\theta, \alpha, \frac{d}{a}\right)\left(1 - \frac{r(\varphi)}{R(\varphi)}\right)\right] \quad (21b)$$

The derivation of point load weight function can then be simplified using this general expression. Here d/a is the non-dimensional distance.

2.4. Determination of weight function parameters

Knowing the general weight function form, Eq. (21) or (20) which is a specific case of (21)), the derivation of weight function for a particular embedded elliptical crack has been reduced to the derivation of parameters $M(\theta, \alpha, d/a)$ along the entire crack front.

The parameter $M(\theta, \alpha, d/a)$ can be determined using Eq. (2) provided that one reference stress intensity factors solution K_r is known. The stress distribution expression and the general weight function expression Eq. (21) can be substituted for $\sigma(x, y)$ and $m(x, y; P')$ into Eq. (2). This leads to the equation for the determination of the unknown parameter $M(\theta, \alpha, d/a)$

$$K_r(P') = \int\int_s \sigma_r(x, y)\frac{\sqrt{2s}}{\pi^{3/2}\rho^2}\left[1 + M\left(\theta, \alpha, \frac{d}{a}\right)\left(1 - \frac{r(\varphi)}{R(\varphi)}\right)\right]dS \quad (22)$$

After integration, Eq. (22) can be used to solve for $M(\theta, \alpha, d/a)$. Note that this calculation needs to be carried out at every point along the crack front. The $M(\theta, \alpha)$ in Eq. (20) can be determined in the exact same way, except it is independent of d/a.

3. Point weight function for embedded elliptical cracks

In this section, Eq. (22) is applied to derive weight functions for embedded elliptical cracks in both infinite and semi-infinite bodies. The M factors are determined for each crack configurations. The derived weight functions are then validated using stress intensity factor solutions for other loading conditions. For any point P' along the crack front, it can either be identified by the polar angle θ or parametric angle ϕ (see Figs. 2 and 7), and they are related simply by: $\tan\theta = \alpha\tan\phi$. In the following sections, parametric angle ϕ is used to represent point P'.

3.1. Embedded elliptical cracks in an infinite body

The weight functions for embedded elliptical crack in an infinite body (Fig. 1a) in the format of Eq. (20) are derived and validated in this section. The solutions are derived for the entire crack front.

3.1.1. Reference stress intensity factor solutions

For embedded elliptical crack as shown in Fig. 1a, the stress intensity factor for uniform stress field is used as reference solution. The uniform stress is applied directly on to the crack face

$$\sigma(x, y) = \sigma_0 \tag{23}$$

The exact stress intensity factor solution is [7]

$$K(\phi) = \frac{\sigma_0\sqrt{\pi a}}{E}\left[\sin^2\phi + \frac{1}{\alpha^2}\cos^2\phi\right]^{1/4} \tag{24}$$

where ϕ is the parametric angle, $\alpha = a/c$ and E is the complete elliptical integral of second kind, given by the following empirical equation [29]

$$E = \sqrt{1.0 + 1.464(\alpha)^{1.65}} \tag{25}$$

3.1.2. Determination of weight functions

By substituting Eqs. (23) and (24) into Eq. (2), an equation with unknown $M(\phi, \alpha)$ is established. Numerical integration is required to solve for $M(\phi, \alpha)$.

A computer program was developed to perform the numerical integration based on the standard Gauss-Legendre quadrature technique. Curved eight-nodes elements are used to distretize the entire elliptical areas. The integration algorithm was verified using the analytical weight function for a penny shaped crack ($a/c = 1$). For uniform stress field, the maximum difference between the exact stress intensity factor solution and the calculation based on the present integration routine was less than 0.8% along the whole crack front. Fig. 8 shows a typical mesh used in the present calculations for $a/c = 0.2$.

The results for the parameters $M(\phi, \alpha)$ are obtained and presented in Table 1 for points along the crack front. The aspect ratios considered are $a/c = 0.1, 0.2, 0.4, 0.6$ and 0.8. The results are plotted in Fig. 9. Note the function $M(\phi, \alpha)$ is symmetric about both x and y axis, therefore only a quarter ($0 \leqslant \phi \leqslant \pi/2$) is plotted. Using this method, the $M(\phi, \alpha)$ for penny shaped crack is found to be -0.23213. For comparison, the results for penny shaped crack ($a/c = 1$) are also presented. For engineering applications, the $M(\phi, \alpha)$ factors are fitted using empirical formulas, and it is summarised in Appendix A1.

3.1.3. Validation of weight functions

Six different loading cases were applied to the surface of the elliptical crack to validate the derived weight functions in the form of Eq. (20). Applying Eq. (2), stress intensity factors along the crack front of an embedded elliptical crack of aspect ratio $\alpha = 0.1, 0.2, 0.4, 0.6$ and 0.8 were calculated for the following stress fields

Uniform stress field

$$\sigma(x, y) = \sigma_0 \tag{26}$$

One-dimensional linear stress field depending on coordinate x

$$\sigma(x, y) = \sigma_0 \frac{x}{a} \tag{27}$$

One-dimensional linear stress field depending on coordinate y

$$\sigma(x, y) = \sigma_0 \frac{y}{c} \tag{28}$$

Two-dimensional non-linear stress field

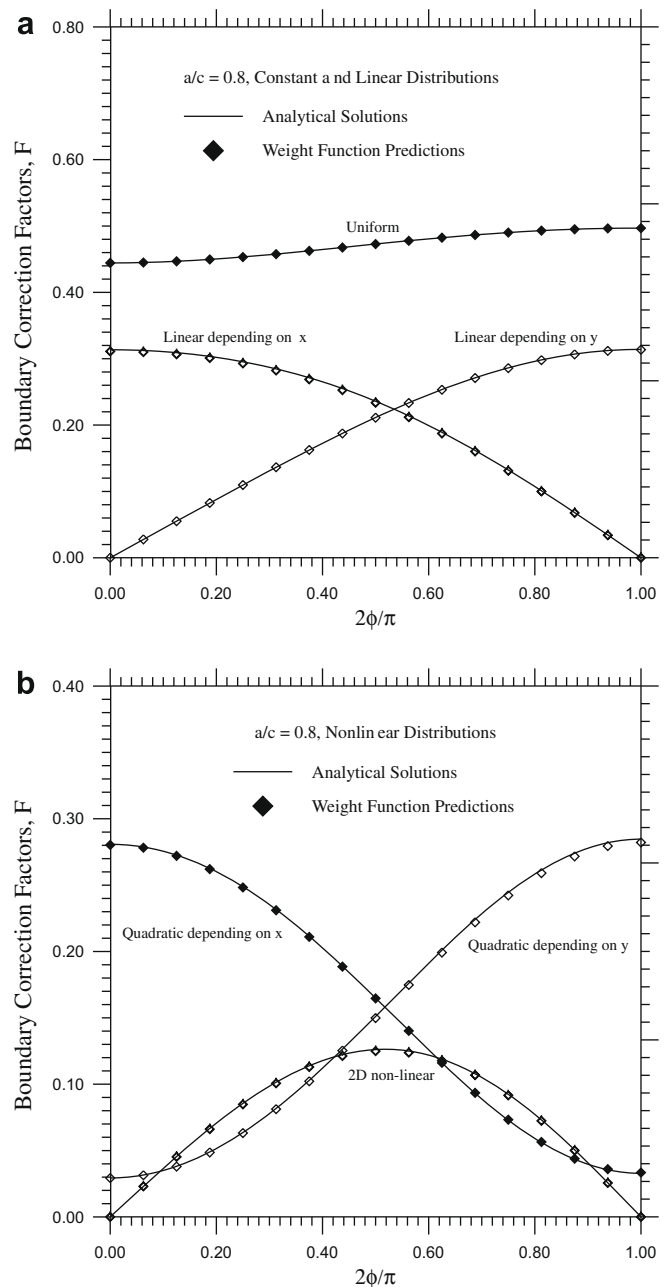$$\sigma(x, y) = \sigma_0 \frac{xy}{ac} \tag{29}$$

Fig. 10. Comparison of the weight function based stress intensity factor and exact solution for $a/c = 0.8$. (a) Uniform and linear stress distributions. (b) Two-dimensional non-linear and parabolic stress distributions.

One-dimensional quadratic stress field depending on coordinate x

$$\sigma(x,y) = \sigma_0 \left(\frac{x}{a}\right)^2 \tag{30}$$

and one-dimensional quadratic stress field depending on coordinate y

$$\sigma(x,y) = \sigma_0 \left(\frac{y}{c}\right)^2 \tag{31}$$

The resulting stress intensity factors were normalized as follows:

$$F(\phi) = \frac{K(\phi)}{(\sigma_0 \sqrt{\pi a}/E)} \tag{32}$$

where F is the boundary correction factor, and E is given in Eq. (25).

The boundary correction factors F from the weight function predictions were compared with the exact solutions for these six loading conditions [9]. The results are shown in Figs. 10–14 for aspect ratios α = 0.1, 0.2, 0.4, 0.6 and 0.8, respectively. Note that the uniform stress distribution is the reference case. For other linear and non-linear loadings, the differences were generally within 5% for all the aspect ratios. These accuracies are in the same range as the weight functions developed in [21]. But the present weight function is much easier to implement.

3.2. Embedded elliptical cracks in an semi-infinite body

In this section the weight functions for embedded elliptical crack in a semi-infinite body (Fig. 1b) are derived and validated.
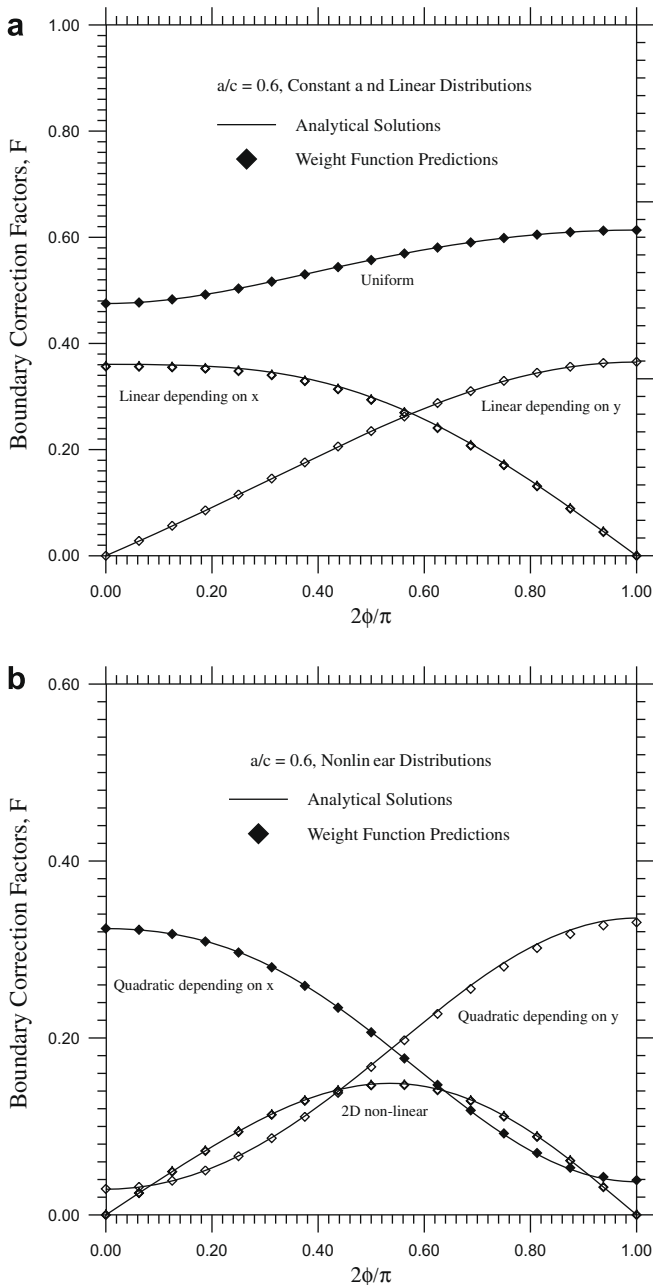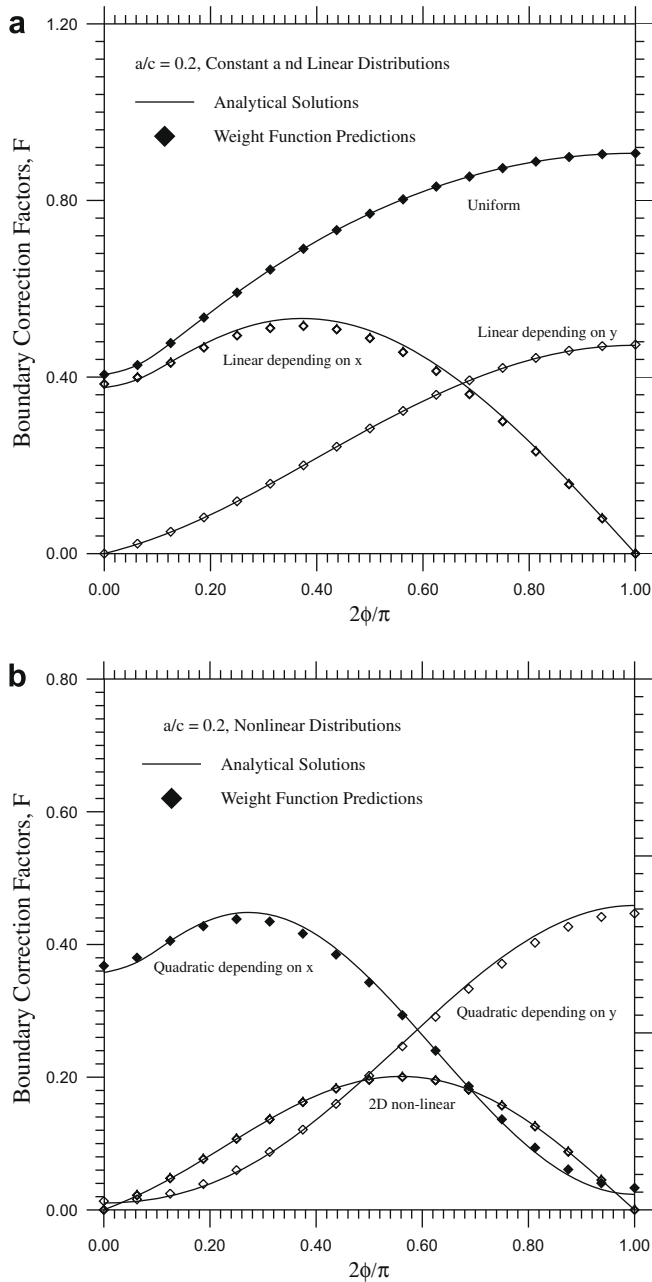

Fig. 11. Comparison of the weight function based stress intensity factor and exact solution for $a/c = 0.6$. (a) Uniform and linear stress distributions. (b) Two-dimensional non-linear and parabolic stress distributions.
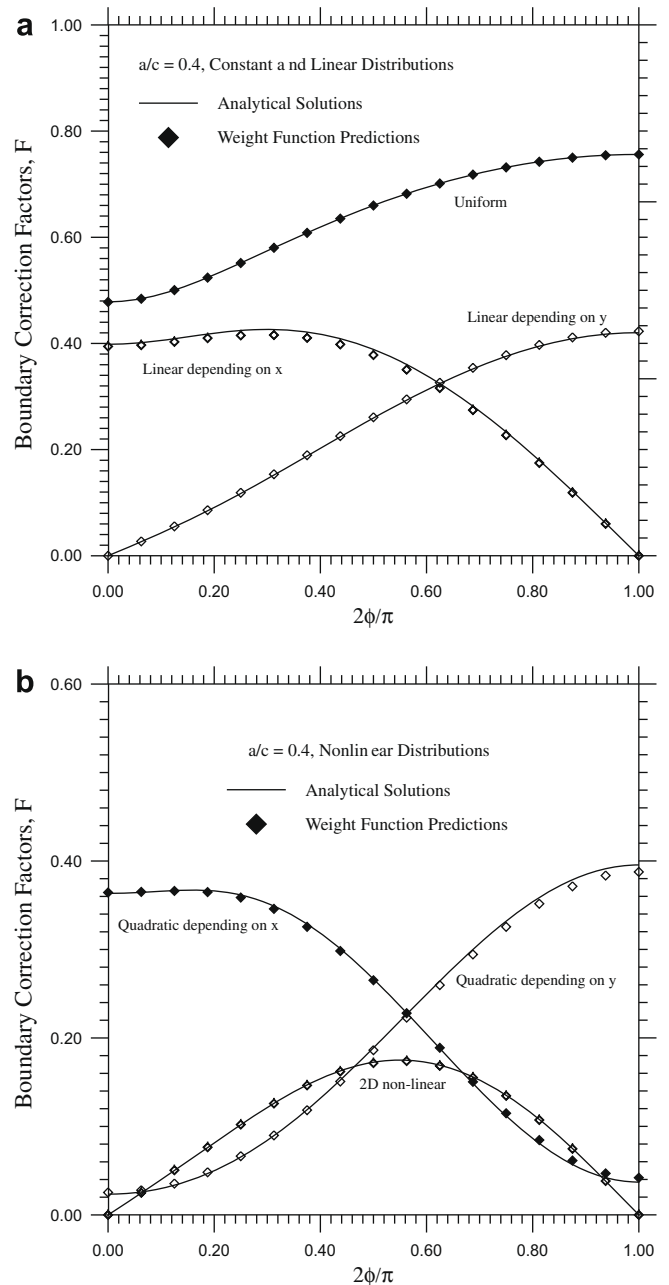
Fig. 12. Comparison of the weight function based stress intensity factor and exact solution for $a/c = 0.4$. (a) Uniform and linear stress distributions. (b) Two-dimensional non-linear and parabolic stress distributions.

Fig. 13. Comparison of the weight function based stress intensity factor and exact solution for $a/c = 0.2$. (a) Uniform and linear stress distributions. (b) Two-dimensional non-linear and parabolic stress distributions.

Fig. 14. Comparison of the weight function based stress intensity factor and exact solution for $a/c = 0.1$. (a) Uniform and linear stress distributions. (b) Two-dimensional non-linear and parabolic stress distributions.
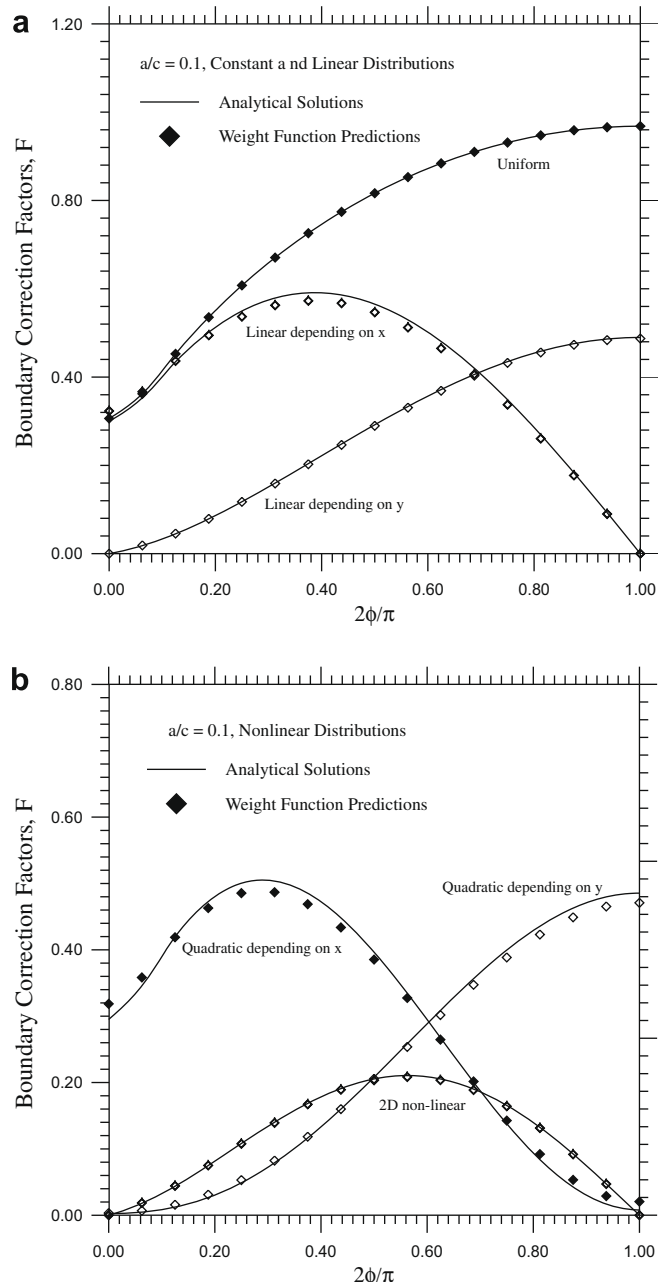
The weight functions are derived using one reference stress intensity factor solution corresponding to uniform loading condition, and validated using several linear and non-linear stress distributions.

3.2.1. Reference stress intensity factor solutions

The stress intensity factors at the three points A, B and C along the crack front (see Figs. 1b and 2) were obtained using finite element method in Ref. [12]. The uniform stress was applied directly on to the crack face (Eq. (23)). The stress intensity factor was normalised following Eq. (32). The resulting F for $a/c = 0.2$, 0.4, 0.6 and 1.0 and $d/a = 0.25$, 0.4, 0.5 and 0.625 are summarised in Table 2. They are used as the reference stress intensity factor solutions.

3.2.2. Determination of weight functions

Following the same procedure as Section 3.1.2, the weight function is now in the format of Eq. (21). The factor $M(\theta, \alpha, d/a)$ is solved at point A (corresponds to $\phi = -\pi/2$), B ($\phi = \pi/2$) and C ($\phi = 0$). The results for the parameters $M(\phi, \alpha, d/a)$ are obtained and presented in Table 3. Empirical formulas of $M(\phi, \alpha, d/a)$ are given in Appendix A2. Note the geometry is only symmetric about y axis, and M at point A and B are therefore have different values.
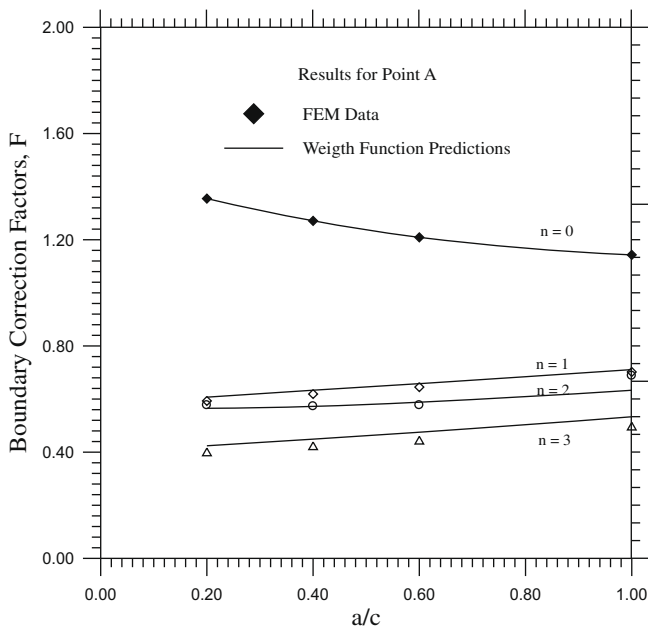
3.2.3. Validation of weight functions

The derived weight functions for point A, B and C are validated using finite element results for various linear and non-linear stress distributions. Using Eq. (2), stress intensity factors are calculated for the following stress fields applied to the crack face

Table 2
Reference stress intensity factor solutions at point A ($\phi = -\pi/2$), Point B ($\phi = \pi/2$) and Point C ($\phi = 0$), from Ref. [12].

	a/c	d/a = 0.25	d/a = 0.4	d/a = 0.5	d/a = 0.625
Point A	0.2	1.355	1.238	1.195	1.157
	0.4	1.271	1.174	1.138	1.108
	0.6	1.209	1.131	1.102	1.086
	1.0	1.143	1.089	1.071	1.055
Point B	0.2	1.108	1.088	1.079	1.071
	0.4	1.078	1.063	1.056	1.048
	0.6	1.053	1.043	1.038	1.033
	1.0	1.041	1.038	1.037	1.035
Point C	0.2	0.467	0.464	0.462	0.460
	0.4	0.685	0.676	0.672	0.668
	0.6	0.815	0.819	0.814	0.810
	1.0	1.050	1.040	1.036	1.031

Table 3
Weight function parameter $M(\phi, \alpha, d/a)$ for α = 0.2, 0.4, 0.6 and 1 and d/a = 0.25, 0.4, 0.5 and 0.625.
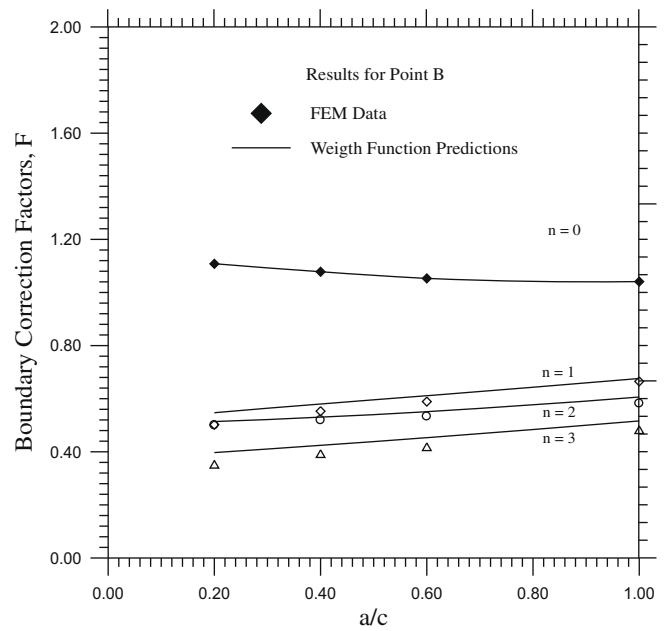
	a/c	d/a = 0.25	d/a = 0.4	d/a = 0.5	d/a = 0.625
Point A	0.2	0.8941	0.5601	0.4373	0.3289
	0.4	0.7890	0.4727	0.3553	0.2575
	0.6	0.6169	0.3392	0.2360	0.1790
	1.0	0.3386	0.1187	0.0454	−0.0196
Point B	0.2	0.1890	0.1319	0.1062	0.0834
	0.4	0.1597	0.1108	0.0879	0.0618
	0.6	0.0616	0.0259	0.0081	−0.009
	1.0	−0.0766	−0.0888	−0.0929	−0.1174
Point C	0.2	−1.8246	−1.8780	−1.9137	−1.9493
	0.4	−0.4171	−0.4727	−0.5379	−0.5750
	0.6	−0.2463	−0.2207	−0.2526	−0.2782
	1.0	−0.0400	−0.0808	−0.0971	−0.1011

Fig. 16. Comparison of the weight function based stress intensity factor and finite element data [12] at Point B for polynomial stress distributions, n = 0, 1, 2 and 3.
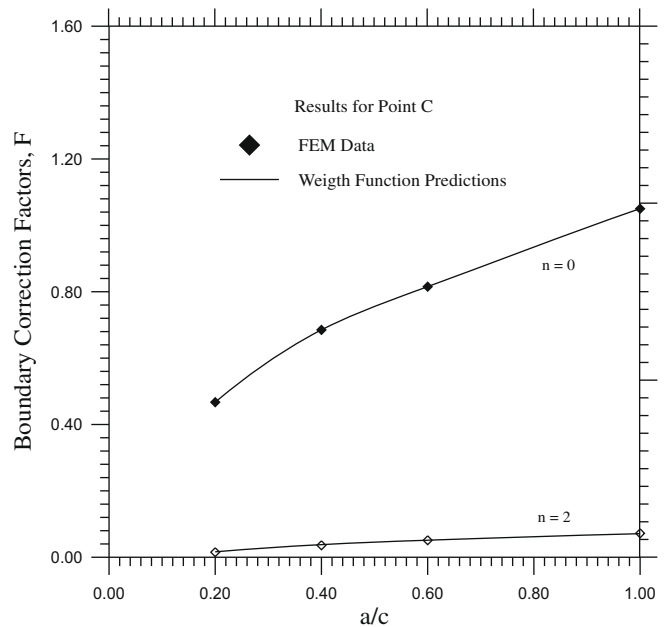

Fig. 15. Comparison of the weight function based stress intensity factor and finite element data [12] at Point A for polynomial stress distributions, n = 0, 1, 2 and 3.

Fig. 17. Comparison of the weight function based stress intensity factor and finite element data [12] at Point C for polynomial stress distributions, n = 0 and 2.

$$\sigma(x, y) = \sigma_0 \left(1 - \frac{x}{a}\right)^n \qquad n = 0, 1, 2, 3 \qquad (33)$$

The stress intensity factors calculated from weight functions are compared to the finite element results from [12] for the same stress

distributions. Very good agreements were achieved for all the aspect ratios, $a/c = 0.2$, 0.4, 0.6 and 1.0 and $d/a = 0.25$, 0.4, 0.5 and 0.625 at all these points. The maximum differences are within 5%. The boundary correction factor results F for $d/a = 0.25$ are shown in Figs. 15–17 for points A, B and C, respectively.

4. Conclusions

An approximate general mathematical form of weight function is proposed which has simplified the determination of weight functions for embedded elliptical cracks. Based on this general form, the point load weight functions are derived for embedded elliptical cracks in an infinite body and in a semi-infinite body. One reference stress intensity factor solution is used to derive these weight functions. It is demonstrated that this method gives very accurate weight functions for the wide range of geometric configurations for embedded elliptical cracks (for the aspect ratio range $0.1 \leqslant a/c \leqslant 1.0$). The derived weight functions are suitable for calculating stress intensity factors for embedded elliptical cracks under any complex two-dimensional stress distributions. They are particularly useful for the fatigue crack growth analysis of planer embedded cracks subjected to fluctuating non-linear stress fields resulting from surface treatment (shot peening), stress concentration or welding (residual stress).

The new weight function form can also serve as the foundation for the further development of weight functions for two-dimensional surface cracks, corner cracks and other part-through cracks in engineering structures.

Acknowledgements

Appendix A1

Weight function parameters $M(\phi, \alpha)$ for an embedded elliptical crack in an infinite body presented in Table 1 have been fitted using empirical formulas with the maximum differences generally within 3% or better.

Applicable range: $0 \leqslant \frac{2\phi}{\pi} \leqslant 1$, and $0.1 \leqslant \alpha < 0.8$.

M factor

$$M(\phi, \alpha) = \frac{1}{\left[-0.1104 - 9.0633 \left(\frac{2\phi}{\pi}\right)^2 - 3.9542(\alpha)^{1.5} \right]} \quad \text{(A1)}$$

$$\alpha = \frac{a}{c}$$

Comparison of the curve fitting predictions and the numerical data is shown in Fig. 9.

Appendix A2

Weight function parameters $M(\phi, \alpha, d/a)$ for an embedded elliptical crack in a semi-infinite body presented in Table 3 at three points, point A (corresponds to $\phi = -\pi/2$), B ($\phi = 0$) and C ($\phi = \pi/2$), have been fitted into empirical formulas with the maximum differences generally within 4% or better.

Applicable range: $0.25 \leqslant \frac{d}{a} \leqslant 0.625$, and $0.2 \leqslant \alpha < 1.0$.

M factor at Point A ($\phi = -\pi/2$)

$$M_A\left(\alpha, \frac{d}{a}\right) = E_A + F_A\alpha + G_A\left(\frac{d}{a}\right) + H_A\alpha^2 + I_A\left(\frac{d}{a}\right)^2 + J_A\alpha\left(\frac{d}{a}\right) \quad \text{(A2)}$$

$E_A = 1.8495$
$F_A = -0.7826$
$G_A = -3.9999$
$H_A = -0.07223$
$I_A = 2.6881$
$J_A = 0.7203$
$\alpha = \dfrac{a}{c}$

M factor at Point B ($\phi = \pi/2$)

$$M_B\left(\alpha, \frac{d}{a}\right) = E_B + F_B\alpha^{-1} + G_B\left(\frac{d}{a}\right) + H_B\alpha^{-2} + I_B\left(\frac{d}{a}\right)^2$$
$$+ J_B\alpha^{-1}\left(\frac{d}{a}\right) \quad \text{(A3)}$$

$E_B = -0.2257$
$F_B = 0.2606$
$G_B = -0.2548$
$H_B = -0.03145$
$I_B = 0.1671$
$J_B = -0.03906$
$\alpha = \dfrac{a}{c}$

M factor at Point C ($\phi = 0$)

$$M_C\left(\alpha, \frac{d}{a}\right) = E_C + F_C\alpha^{-1} + G_C\left(\frac{d}{a}\right) + H_C\alpha^{-2} + I_C\left(\frac{d}{a}\right)^2$$
$$+ J_C\alpha^{-1}\left(\frac{d}{a}\right) \quad \text{(A4)}$$

$E_C = 0.07905$
$F_C = -0.02209$
$G_C = -0.1046$
$H_C = -0.06776$
$I_C = -0.02864$
$J_C = -0.05068$
$\alpha = \dfrac{a}{c}$

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