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Construction of Symmetric Balanced Squares with Blocksize More than One

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Introduction By a *symmetric balanced square* with *blocksize* k, *order* v and *side* s we mean an $s \times s$ array in which every cell contains a subset of cardinality k from a set of elements V of cardinality v satisfying the following properties:

1. every element occurs in $\lfloor \frac{ks}{v} \rfloor$ or $\lceil \frac{ks}{v} \rceil$ cells of each row or column,
2. every element occurs in $\left\lfloor \frac{ks^2}{v} \right\rfloor$ or $\left\lceil \frac{ks^2}{v} \right\rceil$ cells of the array, and
3. the array is symmetric.

Note that it is inherent in our definition that $k \leq v$. Let, $m = \lfloor \frac{ks}{v} \rfloor$ (i.e. the integer part of $\frac{ks}{v}$) and $n = \lfloor \frac{ks^2}{v} \rfloor$. We shall use the notation $SBS_k(s, v)$ to denote such a symmetric balanced square. Observe that an $SBS(s, s)$ is a symmetric Latin square of order s. We will use the notation $SBS(s, v)$ to denote an $SBS_k(s, v)$ when $k = 1$. Dutta and Roy [?] have completely resolved the existence problem when $k = 1$. (The case $k = 1$ is also a special case of Theorem **??** and Theorem **??**, which we prove later.)

Clearly there is an $SBS_k(1, v)$ for every positive integer k and every integer $v \geq k$. Suppose A is an $SBS_k(s, v)$. Dividing ks^2 by v, we obtain unique nonnegative integers n and r such that

$$ks^2 = vn + r \quad \text{where} \quad 0 \leq r < v,$$

or equivalently,

$$ks^2 = r(n + 1) + (v - r)(n).$$

This implies that A has r elements of frequency $n + 1$ and $v - r$ elements of frequency n. Let d, e, δ and ϵ be integers such that

$$ks^2 = r(n + 1) + (v - r)(n) = \delta(d) + \epsilon(e), \tag{1}$$

where e is an even integer, $\{d, e\} = \{n, n+1\}$, and $\{\delta, \epsilon\} = \{r, v-r\}$. Then A has δ elements of odd frequency d and ϵ elements of even frequency e. An element of odd frequency d is defined to be an *odd element*, and an element of even frequency e is defined to be an *even element*. Since A is symmetric, every odd element is contained in an odd number of cells of the main diagonal. Thus, the number of odd elements cannot exceed ks; that is, $\delta \leq ks$. This observation is recorded in the following lemma.

Lemma 1 *A necessary condition for the existence of an $SBS_k(s, v)$, where $k \leq v$, is that the number of odd frequency elements in the array is at most ks.* □

Lemma 1 and the discussion preceeding it motivates the following definition.

Definition 2 *We say that an $SBS_k(s, v)$ is* feasible *if $k \leq v$ and there exist nonnegative integers d, e, δ, ϵ satisfying Equation (1), such that d is odd, $\{d, e\} = \{n, n+1\}$, $\{\delta, \epsilon\} = \{r, v-r\}$ and $\delta \leq ks$.*

The following result is an immediate application of Lemma 1.

Lemma 3 *If $1 < s$, $1 \leq k \leq v$ and $SBS_k(s, v)$ is feasible, then $v \leq \frac{ks(s+1)}{2}$.*

Proof Suppose $SBS_k(s, v)$ is feasible. We use Definition 2 to show that this necessarily implies $v \leq \frac{ks(s+1)}{2}$. Let the parameters r, δ and n be as defined in Definition 2. By the feasibility condition, we must have $\delta \leq ks$.

If $\delta = r$, then n is even and $ks^2 - vn = r \leq ks$. Since $s > 1$, $0 < ks^2 - ks \leq vn$ and hence $n > 0$. Since n is even,

$$2 \leq n = \left\lfloor \frac{ks^2}{v} \right\rfloor \leq \frac{ks^2}{v}.$$

This gives $v \leq \frac{ks^2}{2} < \frac{ks(s+1)}{2}$.

If $\delta = v - r$, then n is odd and $v(n+1) - ks^2 = v - r = \delta \leq ks$. Since n is odd, $n \geq 1$ and

$$2v \leq (n+1)v \leq ks^2 + ks = ks(s+1).$$

Therefore, $v \leq \frac{ks(s+1)}{2}$.

This completes the proof. □

It is possible to prove Lemma 3 directly by counting the maximum number of distinct elements possible in a symmetric $s \times s$ square where each cell can accommodate at most k elements. However, the proof we have provided shows that Lemma 3 is dependent on Lemma 1. Thus Lemma 1 is an independent necessary condition. The rest of the paper is devoted to providing evidence that this is also sufficient.

Lemma 4 *There is an $SBS_k(s, v)$ if and only if there is an $SBS_{v-k}(s, v)$.*

Proof Let A be an $SBS_k(s, v)$. If we replace the k-subset $A_{i,j}$ in row i and column j of A, for $1 \leq i, j \leq s$, by its complement, the result is an $SBS_{v-k}(s, v)$. □

Remark 1 *In light of Lemma 4, we assume throughout this paper that $k \leq \left\lfloor \frac{v}{2} \right\rfloor$.*